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On Corrections to the Nambu Action

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Abstract

We analyze the thickness-corrections to the Nambu walls, focussing on recent discussions on the subject. The presence of corrections depending on the Gaussian curvature and its implications are reviewed. We also highlight the consistency of the calculations, its limitations and the connection between alternative derivations.

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This letter is addressed to the recent controversy in the literature concerning the thickness corrections to the Nambu action for topological defects. The thickness correction problem was originally analyzed by Gregory et. all [1], with the conclusion that the only corrections to the Nambu action are of second order in the thickness and proportional the Ricci curvature, R. In a subsequent paper, we discussed the problem in a slightly more general case and we showed that, if some properties of the static and plane solution are not assumed at the beginning, (as the original papers implicitly do), then additional contributions to the Nambu action, related to the mean curvature, arise [2]. More recently, another paper [3] addressed the same problem, asserting the correctness of Gregory's result. On the light of this recent claim, we want to clarify our results, stressing its basic assumptions and consistency, and also improving the analysis of an obscure point in our previous work.

As in [2], we consider topological solutions of a scalar field ϕ with potential $V(\phi)$ and degenerate vacuum states. In the case of walls, ϕ will assume different vacuum states, ϕ^1 and ϕ^2, on each side of the wall. We consider the solution concentrated on a surface, and, for $V(\phi) = \lambda(\phi^2 - v^2)^2$, this surface may be characterized by $\phi = 0$. In a more general case, this surface may be identified by $\phi = (\phi^1 + \phi^2)/2$ and $V(\phi)$ does not need to be symmetric around this point.

A Gaussian coordinate system is constructed, based on this surface. Points in the space-time are localized by:

$$Z^\mu(\sigma^A, \xi^i) = X^\mu(\sigma^A) + \xi^i N_i^\mu(\sigma^A) \tag{1}$$

where σ^A are coordinates on the surfece, $X^\mu(\sigma^A)$ describes the wall surface,

$N_i^\mu(\sigma^A)$ are the normal vectors to the surface and ξ^i are coordinates along these normal directions. The derivatives of $N_i^\mu(\sigma^A)$ are given by the Gauss-Weingarten equation

$$N^\mu_{i,A} = g^{CD} b_{ACi} X^\mu_{,D} + g^{kj} A_{kiA} N^\mu_j$$

where b_{ACi} is the second fundamental form and A_{kiA} is the twisting vector. The metric is projected in this new coordinate system to give:

$$G_{\mu\nu} dZ^\mu dZ^\nu = g_{AB} d\sigma^A d\sigma^B + 2g_{Aj} d\sigma^A d\xi^j + g_{ij} d\xi^i d\xi^j \qquad (2)$$

with

$$g_{AB} = \tilde{g}_{AB} + \xi^i N^\mu_{i,A} X_{\mu,B} + \xi^j N^\nu_{j,B} X_{\nu,A} + \xi^i \xi^j N^\mu_{i,A} N_{j\mu,B} \qquad (3)$$

$$g_{iB} = \xi^j N^\mu_i N_{\mu j,B} \quad g_{Aj} = \xi^i N^\mu_{i,A} N_{\mu j} \quad g_{ij} = -\delta_{ij} \qquad (4)$$

In this new system,the action may be written as

$$S = \int \sqrt{-g} \mathcal{L}\, d^{p+1}\sigma\, d^m\xi \qquad (5)$$

with $\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)$ and the equation of motion is:

$$\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g} g^{ij}\partial_j\phi) + \frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g} g^{iA}\partial_A\phi) + \frac{1}{\sqrt{-g}}\partial_A(\sqrt{-g} g^{Aj}\partial_j\phi) +$$

$$+ \frac{1}{\sqrt{-g}}\partial_A(\sqrt{-g} g^{AB}\partial_B\phi) + \frac{\partial V}{\partial\phi} = 0 \qquad (6)$$

1 Approximation procedures

Having in mind only solutions concentrated around $\xi = 0$, we consider ϕ in the form:

$$\phi = \phi_0(\xi^i) + \phi_1(\xi^i) + \phi_2(\xi^i, \sigma^A) \tag{7}$$

with ϕ_1 of the order of ϵ and ϕ_2 of the order of ϵ^2.

Because of the fast decreasing behavior of $\partial_i\phi$ and $V(\phi)$ for $\xi > \epsilon$, terms like $\xi\partial_i\phi$ and $\xi V(\phi)$ are one order of correction higher than $\partial_i\phi$ and $V(\phi)$. This is all we will be assuming about the solution for ϕ. There is no reason forcing us to consider $\partial_i\partial^i\phi \gg \partial^i\phi$ (or $\partial_i\partial^i\phi \sim \epsilon^{-2}$, $\partial^i\phi \sim \epsilon^{-1}$). Had we in mind only a static plane wall solution, this would certainly be true, but there is no way to show that a solution like (7), restricted to satisfy $\xi\partial_i\phi \ll \partial_i\phi$, must obey this extra requirement, and we must be allowed to proceed, consistently, without taking it.

From (3)-(4) and (7) replaced in (6), we obtain an equation of motion which may be separated into the zero order and the first order equations:

$$\partial_i\partial^i\phi_0 + K_i^0\partial^i\phi_0 + \left.\frac{\partial V}{\partial\phi}\right]_0 = 0 \tag{8}$$

and

$$\partial_i\partial^i\phi_1 + K_i^0\partial^i\phi_1 + K_{ij}^1\xi^i\partial^j\phi_0 + \phi_1\left.\frac{\partial^2 V}{\partial\phi^2}\right]_0 = 0 \tag{9}$$

where $]_0$ indicates evaluation at $\phi = \phi_0$, $K_i^0 = g^{AB}b_{ABi}$ is the mean curvature and $K_{ij}^1 = -b_{ABi}b^{AB}{}_j$ is the gaussian curvature. From the Gauss-Weingarten equation, we see that K_i^0 and K_{ij}^1 are related to the gradient of the normal vectors, N_i^μ. Since we have not yet solved the evolution equation for the surface, we do not know, at this point, the values of K_i^0 and K_{ij}^1, so that we can not yet solve (8) and (9). This was missing in our first analysis of ref.[2], but, in fact, it is not essential to know ϕ_0 and ϕ_1 at this stage as it is possible to proceed without the use of these explicit solutions.

An important issue here is whether or not the expansion in the equation of motion should agree with the expansion in the action. As we understand, the expansion must be made in the equation of motion, so that it is consistent with the fact that the fundamental object of the theory is the scalar ϕ. The wall, and all aproximations that come with it, appear as features of some solutions of the equations of motion, rather than a feature of the action. Besides, to make the expansion in the action requires a change on the dynamical variables in the variational principle, which must be used as $\frac{\delta S_0}{\delta \phi_0} = 0$ and $\frac{\delta S_2}{\delta \phi_1} = 0$. A high price must be paid to do this, as parts of the original action (7) are completely ignored in this procedure [1]. In the effort to make the expansion in the action consistent, one ends up throwing away parts of the action that would otherwise affect the evolution of the system. This means that the expansion in the action is, in fact, inconsistent, and must be avoided.

So, back to the equations (8), once the solution for ϕ_0 is, formally, identified, the next step is to obtain an effective Lagrangean to describe the evolution of the surface $\xi = 0$. Since we now want to explore only the evolution $X^\mu(\sigma^A)$ (we are not looking for solution $\phi_0(\xi^i)$ and $\phi_1(\xi^i)$), the equation will include only derivatives in σ^A, which do not interfere with the expansion in ξ. At this stage, we may safely expand the action in powers of ϵ,

[1] Like the last term in the r.h.s. of (20) ref. [3] which neither contributes to the first order equation, because it does not depend on ϕ_1, nor it appears in the ϕ_0 equation because it is of higher order.

and using (7-9), we separate the ϵ-dependence from the σ-dependence and, formally, integrate out the ξ-coordinates. This procedure leaves behind only constant (σ-independent) factors which depend on ϕ_0 and ϕ_1. Without going into all the details, we just write down the final result for walls, as in [2], eq.(31):

$$S = \mu_0 \int \sqrt{-\tilde{g}} \left[1 + \frac{\mu_1}{\mu_0} K^0 - \frac{\mu_2}{\mu_0} R + \frac{\tilde{\mu}_2}{\mu_0} K \right] \; d^{p+1}\sigma. \tag{10}$$

where

$$\mu_1 = \int d^m\xi \; \xi \left[\frac{1}{2} \partial_j \phi_0 \partial^j \phi_0 - V(\phi_0) \right],$$

$$\mu_2 = \frac{1}{2} \int d^m\xi \; \xi^i \xi^j \left[\frac{1}{2} \partial_l \phi_0 \partial^l \phi_0 - V(\phi_0) \right]$$

$$\tilde{\mu}_2 = \frac{1}{2} \int d^m\xi \; \xi^i \partial^j \phi_0 \phi_1$$

and $K^0 = \tilde{g}^{AB} b_{AB}$, $K = b_{AB} b^{AB}$. Since there is only one normal direction in the case of walls, the indices i, j in μ and K were omitted.

Another important point must be stressed here. We could use partial integration to obtain equivalent expressions for μ. However, we must also notice that the use of partial integrations never changes any integral; it just provides alternative expressions for which the power counting in ϵ can not be immediately readable with the only assumptions we have, namely (7) and $\xi \partial_i \phi \sim \epsilon \partial_i \phi$.

The zero order term in (10) reproduces the Nambu equation, giving:

$$K^0 = \tilde{g}^{AB} b_{AB} = 0 \tag{11}$$

With this result, we may now go back to (8-9) and effectively solve for ϕ_0 and ϕ_1, a procedure similar to the one advocated by R. Gregory et all [1]. Since K^0 does not depend on σ^A, at least up to lower order, the requirement (7) is self-consistent. Note that, at this point, the zero order equation, which can now be solved, agrees with the equations found before in the literature [1,3]. The difference remains only in the ϕ_1 equation, and it arises because of the weaker conditions we start with. As compared with [3], the exclusion of the last term of the action, (eq. (20) of [3]) from the equations of motion, as mentioned before, is also related to this difference.

As for the first order contribution, we must note that if $V(\phi)$ is symmetric, the equation (8), with the apropriate boundary conditions, has an odd solution for which $\mu_1^i = 0$ and no first order correction appears. Besides, even when $\mu_1 \neq 0$, this first order contribuition in ϵ will only be important for walls that are not spatially flat. As an axample, we compare the evolution of a plane and a cylindrical wall in the presence of the first order term.

2 Plane and the cylindrical walls

We will consider walls produced by a potential $V(\phi)$ which is not symmetric between ϕ^1 and ϕ^2, the vacuum states on each side of the wall. In this case, $\mu_1^i \neq 0$, and a first order contribution to the Nambu action will be present.

For spatially flat walls, we consider $X^\mu = (t, x, y, z(t))$, the σ^A coordinates are identified with τ, x, y and $d\tau = \sqrt{1-\dot{z}^2}dt$. The only normal vector is given by:

$$N^\mu = (\frac{\dot{z}}{\sqrt{1-\dot{z}^2}}, 0, 0, \frac{1}{\sqrt{1-\dot{z}^2}})$$

Using $K^0 = \tilde{g}^{AB} b_{AB}$ and the definition of b_{AB}, $b_{AB} = X^\mu_{,A} N^\nu_{,B} G_{\mu\nu}$, we have:

$$K^0 = \frac{\ddot{z}}{(1-\dot{z}^2)^{3/2}}$$

The Nambu action requires the plane wall to move with $\ddot{z} = 0$, as expected. When the first order correction is considered, we get:

$$S = \mu_0 \int \sqrt{-\tilde{g}} \left[1 + \frac{\mu_1}{\mu_0} K^0\right] \, d\tau dx dy =$$

$$= \mu_0 \int dt dx dy \left[\sqrt{1-\dot{z}^2} + \frac{\mu_1}{\mu_0} \frac{\ddot{z}}{(1-\dot{z}^2)}\right]. \tag{12}$$

The first order contribution turns out to be a total derivative which gives no contribuition to the equation of motion. So, any possible correction to the Nambu action for plane walls will be at least of order ϵ^2.

To study cylindrical walls, we use $X^\mu = (t, r(t)\cos(\theta), r(t)\sin(\theta), z)$, the σ^A-coordinates are identified with τ, z, θ and $d\tau = \sqrt{1-\dot{r}^2}dt$. The only normal vector is given by:

$$N^\mu = \frac{1}{\sqrt{1-\dot{r}^2}}(\dot{r}, \cos(\theta), \sin(\theta), 0)$$

Following the same steps used for the plane wall, we compute K^0:

$$K^0 = \frac{\ddot{r}}{(1-\dot{r}^2)^{3/2}} + \frac{1}{r(1-\dot{r}^2)^{1/2}}$$

Making $K^0 = 0$, we have the usual Nambu equation for cylindric walls. By including the first order correction , we now obtain:

$$S = \mu_0 \int dt d\theta dz \; r\sqrt{1-\dot{r}^2} \left[1 + \frac{\mu_1}{\mu_0} \frac{\ddot{r}}{(1-\dot{r}^2)^{3/2}}\right] \tag{13}$$

and the equation of motion with first order correction is:

$$\frac{\ddot{r}}{(1-\dot{r}^2)} + \frac{1}{r} + \mu_1 \frac{2\ddot{r}}{r(1-\dot{r}^2)^{3/2}} = 0 \tag{14}$$

Using the zero order equation, we may say that, roughly, $\ddot{r} \sim r^{-1}$. So, the first order contribution will be basically $\sim r^{-2}$, to be compared with the other terms in (14), which are $\sim r^{-1}$. This new correction, which is already first order in ϵ, becames increasingly small when the limit $r \to \infty$ is also considered.

Finally we would like to comment on the limitations of any of these methods. Right from the start, the evolution of a field configuration, ϕ, is artificially splitted into two pieces: the evolution of $\phi(\xi)$ and the evolution of the surface $\phi = 0$. These pieces are, in fact, deeply interlocked and the split is promoted by the assumption that both ϕ_0 and ϕ_1 depend only on ξ^i. By construction, we may only be sure that $\phi]_{\xi=0} = 0$ and there is no way, a priori, to make sure that the σ^A-independence may be extended to $\xi \neq 0$. As it turns out, the lowest order equation for the evolution of the defect makes $K^0 = 0$, which is enough to garantee that $\phi_0 = \phi_0(\xi)$ is a self-consistent choice. Any correction to the Nambu action will be of higher order and will not affect (8). For ϕ_1, solution of (9), the prospect is not so good. Since ϕ_1 only affects the second order corrections to the Nambu action, we may safely consider $K^0 = 0$ to solve (9). Even so, K^1_{ij} is not necessarily σ^A-independent, thus invalidating the general use of $\phi_1 = \phi_1(\xi)$. However, in the same way that the static and plane walls are considered as good local approximation for more general solutions, we may also consider, as a better approximation, that the defect is locally described by a wall with constant K^0 and K, a higher order tangent manifold to the defect surface. It must be kept in mind that this is just an approximation, whose domain of validity depend on each case. For plane walls, $K^0 = 0 \Longrightarrow K = 0$, and the case is trivial. For cylindric and spheric walls, we have

$$K = \frac{\ddot{r}^2}{(1-\dot{r}^2)^3} + \frac{C}{r^2(1-\dot{r}^2)}$$

with $C = 1$ for cylindric walls and $C = 2$ for spherical walls. So, the approximation that assumes both K^0 and K as σ-independent may be consider a good approximation when r is large enough or when $\dot{r}$ is small. The larger r or the smaller $\dot{r}$, the better the approximation will do. Even though this procedure does not cover all possible cases of interest, it may provide important information about the evolution of the defect. The existence of a rigidity term [4], which would affect the evolution of a defect originally with $\dot{r} \sim 0$, may be analyzed within this framework, predicting whether or not the defect will straighten.

In conclusion, the derivation of the effective action for defects must be seen as an approximation and, as such, must be used with discrimination. Used correctly, it may provide answer for some questions concerning the evolution of the defect. However, to be useful, it is important that the derivation is done with the least possible number of assumptions to avoid the influence from the trivial case of plane and static defects. With the assumption that $\phi = \phi_0(\xi^i) + \phi_1(\xi^i) + \phi_2(\xi^i, \sigma^A)$ and for the specific potential $V(\phi) = \lambda(\phi^2 - v^2)^2$, this result states that there is no first order correction and there are two second order corrections: one proportional to the Ricci scalar R and another proportional to the Gaussian curvature K.

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