HIGHER-ORDER SQUEEZING OF QUANTUM FIELD
AND THE GENERALIZED UNCERTAINTY RELATIONS
IN NON-DEGENERATE FOUR-WAVE MIXING

Xi-zeng Li  Bao-xia Su
Department of Physics, Tianjin University, Tianjin 300072, P.R.China

Abstract

It is found that the field of the combined mode of the probe wave and the phase-conjugate wave in the process of non-degenerate four-wave mixing exhibits higher-order squeezing to all even orders. And the generalized uncertainty relations in this process are also presented.

With the development of techniques for making higher-order correlation measurement in quantum optics, the new concept of higher-order squeezing of the single-mode quantum electromagnetic field was first introduced and applied to several processes by Hong and Mandel in 1985. Lately Xi-zeng Li and Ying Shan have calculated the higher-order squeezing in the process of degenerate four-wave mixing and presented the higher-order uncertainty relations of the fields in single-mode squeezed states. As a natural generalization of Hong and Mandel's work, we introduced the theory of higher-order squeezing of the quantum fields in two-mode squeezed states in 1993. In this paper we study for the first time the higher-order squeezing of the quantum field and the generalized uncertainty relations in non-degenerate four-wave mixing (NDFWM) by means of the above theory.

1 Definition of higher-order squeezing of two mode quantum fields

The real two mode output field \( \hat{E} \) can be decomposed into two quadrature components \( \hat{E}_1 \) and \( \hat{E}_2 \), which are canonical conjugates

\[
\hat{E} = \hat{E}_1 \cos(\Omega t - \phi) + \hat{E}_2 \sin(\Omega t - \phi),
\]

\[
[\hat{E}_1, \hat{E}_2] = 2i C_0.
\]

Then the field is squeezed to the \( N \)th-order in \( \hat{E}_1 (N = 1, 2, 3, \cdots) \) if there exists a phase angle \( \phi \) such that \( \langle (\Delta \hat{E}_1)^N \rangle \) is smaller than its value in a completely two-mode coherent state of the field, viz.,

\[
\langle (\Delta \hat{E}_1)^N \rangle < \langle (\Delta \hat{E}_1)^N \rangle_{\text{two-mode coh.s.}}.
\]

This is the definition of higher-order squeezing of two mode quantum fields.
2 Scheme for generation of higher–order squeezing via NDFWM

The scheme is shown in the following figure:

Where two strong, classical pump waves of complex amplitude \( v_1 = |v_1|e^{i\varphi_1} \) and \( v_2 = |v_2|e^{i\varphi_2} \) with the same frequency \( \Omega \) are incident on a nonlinear crystal possessing a third–order (\( \chi^{(3)} \)) nonlinearity. The length of the medium is \( L \). \( \hat{a}_4 \) is the annihilation operator of the transmitted –probe wave with frequency \( \omega_4 \), \( \hat{a}_s \) is the annihilation operator of the phase–conjugate wave with frequency \( \omega_s \), and

\[
\Omega = \frac{\omega_s + \omega_4}{2} \quad (4)
\]

The effective Hamiltonian of this interaction system has the form of

\[
\hat{H} = \hbar\omega_0 \hat{a}_s^+ \hat{a}_s + \hbar\omega_4 \hat{a}_4^+ \hat{a}_4 + \hbar g_0 (v_1 v_2 \hat{a}_3^+ \hat{a}_4^+ e^{-2\Omega t} + H.C) \quad (5)
\]

where \( g_0 \) is the coupling const, \( t \) is the time propagation of light in NL crystal.

By solving the Heisenberg Equation of motion we get the output mode

\[
\hat{a}_s(t) = \mu \hat{a}_s(L) + \nu \hat{a}_4^+(0) e^{-i\omega_t}, \quad (z = L - ct \text{ for } \hat{a}_s) \quad (6)
\]

\[
\hat{a}_4(t) = \mu \hat{a}_4(0) + \nu \hat{a}_s^+(L) e^{-i\omega_4 t}, \quad (z = ct \text{ for } \hat{a}_4) \quad (7)
\]

where

\[
\begin{align*}
\mu &= \sec|k|L, \\
\nu &= -ie^{i(\varphi_1 + \varphi_2)} \tan|k|L, \\
|k| &= \text{sgn}(\varphi_1, \varphi_2).
\end{align*}
\]
3 Combined mode and its quadrature components

It can be verified that the field of either \( \hat{a}_3(0) \) or \( \hat{a}_4(L) \) mode does not exhibit higher-order squeezing.

We consider the field of the combined mode of \( \hat{a}_3(t) \) and \( \hat{a}_4(t) \)

\[
\hat{E}(t) = \sqrt{\frac{\omega_3}{2}} \hat{a}_3(t) - i \sqrt{\frac{\omega_4}{2}} \hat{a}_4(t) + (H.C)
\]

\[
= \sqrt{\frac{\Omega}{2}} \lambda_3 \hat{a}_3(t) - i \sqrt{\frac{\Omega}{2}} \lambda_4 \hat{a}_4(t) + (H.C)
\]

where

\[
\lambda_3 = \sqrt{\frac{\omega_3}{\Omega}}, \lambda_4 = \sqrt{\frac{\omega_4}{\Omega}}
\]

and \(-i\) denotes the phase delay. The units are chosen so that \( h = c = 1 \).

\( \hat{E}(t) \) can be decomposed into two quadrature components \( \hat{E}_1 \) and \( \hat{E}_2 \), which are canonical conjugates

\[
\hat{E}(t) = \hat{E}_1 \cos(\Omega t - \phi) + \hat{E}_2 \sin(\Omega t - \phi),
\]

where

\[
\Omega = \frac{\omega_3 + \omega_4}{2},
\]

and \( \phi \) is an arbitrary phase angle that may be chosen at will.

\( \hat{E}_1 \) can be expressed in term of initial modes \( \hat{a}_3(L) \) and \( \hat{a}_4(0) \),

\[
\hat{E}_1 = g \hat{a}_3(L) + h \hat{a}_4(0) + g^* \hat{a}_3^+(L) + h^* \hat{a}_4^+(0),
\]

where

\[
g = \sqrt{\frac{\Omega}{2}}[\lambda_3 e^{-i\phi} + \lambda_4 \nu e^{i(\phi + \pi/3)}] e^{i\epsilon t},
\]

\[
h = \sqrt{\frac{\Omega}{2}}[\lambda_4 e^{-i(\phi + \pi/3)} + \lambda_3 \nu^* e^{i\phi}] e^{-i\epsilon t},
\]

\[
\epsilon = \Omega - \omega_3 = \omega_4 - \Omega,
\]

\( \epsilon \) is the modulation frequency.

Now we define

\[
\hat{B} = g \hat{a}_3(L) + h \hat{a}_4(0),
\]

\[
\hat{B}^+ = g^* \hat{a}_3^+(L) + h^* \hat{a}_4^+(0),
\]

then

\[
\hat{E}_1 = \hat{B} + \hat{B}^+,
\]

where \( B^+ \) is the adjoint of \( \hat{B} \).
4 Higher-order noise moment \(< (\Delta \hat{E}_1)^N >\) and higher-order squeezing

By using the Campbell–Baker–Hausdorff formula, we get the Nth-order moment of \(\Delta \hat{E}_1\),

\[
< (\Delta \hat{E}_1)^N > = \sum_{\gamma=0}^{N} \left[ \frac{N^{(r)}}{1!} \left( \frac{1}{2} C_0 \right) \right] < (\Delta \hat{E}_1)^{N-\gamma} > + \sum_{\gamma=1}^{N-1} C_0^{N^2/2} \frac{1}{2} C_0^2.
\]

where

\[
N^{(r)} = N(N-1)\cdots(N-r+1), \quad C_0 = \frac{1}{2i}[\hat{E}_1, \hat{E}_2] = [\hat{B}, \hat{B}^+],
\]

and \(\sum\sum\) denotes normal ordering with respect to \(\hat{B}\) and \(\hat{B}^+\).

We take the initial quantum state to be \(|\alpha > 4|0 > 8\), which is a product of the coherent state \(|\alpha > 4\) for \(\hat{A}_4(0)\) mode and the vacuum state for \(\hat{A}_8(L)\) mode. Since \(|\alpha > 4|0 > 8\) is the eigenstate of \(\hat{B}\), we get

\[
< (\Delta \hat{E}_1)^N > = (N-1)! C_0^{N^2/2}.
\]

Then from (20),

\[
< (\Delta \hat{E}_1)^N > = (N-1)! C_0^{N^2/2},
\]

where

\[
C_0 = [\hat{B}, \hat{B}^+] = |\sigma|^2 + |h|^2,
\]

\[
= \frac{\Omega}{2} \{ (\lambda_1^2 + \lambda_3^2)(|\mu|^2 + |\nu|^2) + 2\lambda_3 \lambda_4 \mu^* \nu e^{i(\theta_3 + \theta_2)} + \mu \nu e^{i(\theta_3 + \theta_2)} \}.
\]

Substituting eqs. (8), (10), (24) into (23), we get the Nth-order moment of \(\Delta \hat{E}_1\),

\[
< (\Delta \hat{E}_1)^N > = (N-1)! \Omega^{N^2/2} |\sec^3|k|L + \tan^2|k|L
\]

\[
- 2\sqrt{1 - \frac{\epsilon^2}{\Omega^2} \sec|k|L \tan|k|L \cos(2\phi - \theta_1 - \theta_2)} \right)^{N/2}.
\]

If \(\phi\) is chosen to satisfy

\[
2\phi - \theta_1 - \theta_2 = 0, \quad \text{or} \quad \cos(2\phi - \theta_1 - \theta_2) = 1,
\]

then the above eq. (25) leads to the result

\[
< (\Delta \hat{E}_1)^N > = (N-1)! \Omega^{N^2/2} |\sec^3|k|L + \tan^2|k|L
\]

\[
- 2\sqrt{1 - \frac{\epsilon^2}{\Omega^2} \sec|k|L \tan|k|L} \right)^{N/2}.
\]
When \(0 < k|L| < \pi\), the right-hand side is less than \((N - 1)!\Omega^{N/2}\), which is the corresponding \(N\)-th order moment for two-mode coherent states. It follows that the field of the combined mode of the probe wave and the phase conjugate wave in NDFWM exhibits higher-order squeezing to all even orders.

The squeeze parameter \(q_N\) for measuring the degree of \(N\)-th order squeezing is

\[
q_N = \frac{< (\Delta \hat{E}_1)^N > - < (\Delta \hat{E}_1)^N >_{\text{coh.}}}{< (\Delta \hat{E}_1)^N >_{\text{two-mode coh.a}}} \quad (27)
\]

\[
= [\sec^2 |k|L + \tan^2 |k|L - 2 \sqrt{1 - \frac{\epsilon^2}{\Omega^2} \sec |k|L \tan |k|L}]^{N/2} - 1. \quad (28)
\]

We find that \(q_N\) is negative, and \(q_N\) increases with \(N\). This gives out the conclusion that the degree of higher-order squeezing is greater than that of the second order.

5 Generalized uncertainty relations in NDFWM

\(\hat{E}_2\) can be regarded as a special case of \(\hat{E}_1\) if \(\phi\) is replaced by \(\phi + \pi/2\). Then if \(\phi\) is chosen to satisfy \(2\phi - \theta_1 - \theta_2 = 0\), from eq. (25) it follows that

\[
< (\Delta \hat{E}_2)^N > = (N - 1)!\Omega^{N/2}[\sec^2 |k|L + \tan^2 |k|L + 2 \sqrt{1 - \frac{\epsilon^2}{\Omega^2} \sec |k|L \tan |k|L}]^{N/2}. \quad (29)
\]

when \(0 < |k|L < \pi\), the right-hand side is greater than \((N - 1)!\Omega^{N/2}\).

From eqs. (26) and (29), we obtain

\[
< (\Delta \hat{E}_1)^N > \cdot < (\Delta \hat{E}_2)^N > = [(N - 1)!\Omega^N[1 + 4 \frac{\epsilon^2}{\Omega^2} \sec^2 |k|L \tan^2 |k|L]^{N/2}. \quad (30)
\]

Eq. (30) shows that \(< (\Delta \hat{E}_1)^N >\) and \(< (\Delta \hat{E}_2)^N >\) can not be made arbitrarily small simultaneously. We call eq. (30) the generalized uncertainty relations in NDFWM, and the right-hand side is dependent on \(\epsilon, \Omega, N,\) and \(|k|L\).

In the degenerate case \(\omega_1 = \omega_2 = \Omega, \epsilon = 0\) from eqs. (26), (28) and (30) we obtain

\[
< (\Delta \hat{E}_1)^N > = (N - 1)!\Omega^{N/2}[\sec |k|L - \tan |k|L]^N, \quad (31)
\]

\[
q_N = [\sec |k|L - \tan |k|L]^N - 1, \quad (32)
\]

\[
< (\Delta \hat{E}_1)^N > \cdot < (\Delta \hat{E}_2)^N > = [(N - 1)!\Omega^N. \quad (33)
\]

When \(N = 2\),

\[
< (\Delta \hat{E}_1)^2 > = \Omega[\sec |k|L - \tan |k|L]^2, \quad (34)
\]

\[
q_2 = [\sec |k|L - \tan |k|L]^2 - 1, \quad (35)
\]
$\langle (\Delta \hat{E}_1)^2 \rangle \cdot \langle (\Delta \hat{E}_2)^2 \rangle = \Omega^2$  \hspace{1cm} (36)

These results are in agreement with the conclusions in the previous relevant references\textsuperscript{[8],[5]}.

6 Acknowledgements

This research was supported by the National Natural Science Foundation of China, and Tianjin Natural Science Foundation.

References


