HIGHER-ORDER SQUEEZING OF QUANTUM FIELD
AND THE GENERALIZED UNCERTAINTY RELATIONS
IN NON-DEGENERATE FOUR-WAVE MIXING

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Abstract

It is found that the field of the combined mode of the probe wave and the phase-conjugate
wave in the process of non-degenerate four-wave mixing exhibits higher-order squeezing to
all even orders. And the generalized uncertainty relations in this process are also presented.

With the development of techniques for making higher-order correlation measurement in quantum optics, the new concept of higher-order squeezing of the single-mode quantum electromagnetic field was first introduced and applied to several processes by Hong and Mandel in 1985. Lately Xi-zeng Li and Ying Shan have calculated the higher-order squeezing in the process of degenerate four-wave mixing and presented the higher-order uncertainty relations of the fields in single-mode squeezed states. As a natural generalization of Hong and Mandel's work, we introduced the theory of higher-order squeezing of the quantum fields in two-mode squeezed states in 1993. In this paper we study for the first time the higher-order squeezing of the quantum field and the generalized uncertainty relations in non-degenerate four-wave mixing (NDFWM) by means of the above theory.

1 Definition of higher-order squeezing of two mode quantum fields

The real two mode output field $\hat{E}$ can be decomposed into two quadrature components $\hat{E}_1$ and $\hat{E}_2$, which are canonical conjugates

$$\hat{E} = \hat{E}_1 \cos(\Omega t - \phi) + \hat{E}_2 \sin(\Omega t - \phi),$$

$$[\hat{E}_1, \hat{E}_2] = 2iG_0.$$  \hspace{1cm} (1)

Then the field is squeezed to the $N$th-order in $\hat{E}_1 (N = 1, 2, 3, \cdots)$ if there exists a phase angle $\phi$ such that $<(\Delta \hat{E}_1)^N>$ is smaller than its value in a completely two-mode coherent state of the field, viz.,

$$<(\Delta \hat{E}_1)^N> < <(\Delta \hat{E}_1)^N>_{\text{two-mode coh.}}.$$  \hspace{1cm} (2)

This is the definition of higher-order squeezing of two mode quantum fields.
2 Scheme for generation of higher-order squeezing via NDFWM

The scheme is shown in the following figure:

FIG. 1. Schematic for generation of higher-order squeezing via NDFWM. \( M_1, M_2, M_3 \) are mirrors, BS is the 50\%–50\% beam splitter

Where two strong, classical pump waves of complex amplitude \( v_1 = |v_1|e^{i\phi_1} \) and \( v_2 = |v_2|e^{i\phi_2} \) with the same frequency \( \Omega \) are incident on a nonlinear crystal possessing a third-order \((\chi^{(3)})\) nonlinearity. The length of the medium is \( L \). \( \hat{a}_4 \) is the annihilation operator of the transmitted–probe wave with frequency \( \omega_4 \), \( \hat{a}_s \) is the annihilation operator of the phase-conjugate wave with frequency \( \omega_s \), and

\[
\Omega = \frac{\omega_s + \omega_4}{2}
\]  

(4)

The effective Hamiltonian of this interaction system has the form of

\[
\hat{H} = \hbar \omega_4 \hat{a}_s^+ \hat{a}_s + \hbar \omega_4 \hat{a}_4^+ \hat{a}_4 + \hbar g_0 (v_1 v_2 \hat{a}_s^+ \hat{a}_4^+ e^{-2\Omega t} + H.C)
\]  

(5)

where \( g_0 \) is the coupling constant, \( t \) is the time propagation of light in NL crystal.

By solving the Heisenberg Equation of motion we get the output mode

\[
\hat{a}_s(t) = [\mu \hat{a}_s(L) + \nu \hat{a}_4^+(0)]e^{-i\omega_4 t}, \quad (z = L - ct \text{ for } \hat{a}_s)
\]  

(6)

\[
\hat{a}_4(t) = [\mu \hat{a}_4(0) + \nu \hat{a}_s^+(L)]e^{-i\omega_4 t}, \quad (z = ct \text{ for } \hat{a}_4)
\]  

(7)

where

\[
\begin{align*}
\mu &= \sec|k|L, \\
\nu &= -ie^{i(\phi_1 + \phi_2)}\tan|k|L, \\
|k| &= \frac{\eta |v_1| |v_2|}{c}.
\end{align*}
\]  

(8)
3 Combined mode and its quadrature components

It can be verified that the field of either $\hat{a}_3(0)$ or $\hat{a}_4(L)$ mode does not exhibit higher-order squeezing.

We consider the field of the combined mode of $\hat{a}_3(t)$ and $\hat{a}_4(t)$

$$\hat{E}(t) = \sqrt{\frac{\omega_3}{2}} \hat{a}_3(t) - i \sqrt{\frac{\omega_4}{2}} \hat{a}_4(t) + (H.C)$$

$$= \sqrt{\frac{\Omega}{2}} \hat{a}_3(t) - i \sqrt{\frac{\Omega}{2}} \lambda_4 \hat{a}_4(t) + (H.C)$$

where

$$\lambda_3 = \sqrt{\frac{\omega_3}{\Omega}}, \lambda_4 = \sqrt{\frac{\omega_4}{\Omega}}$$

and $-i$ denotes the phase delay. The units are chosen so that $h = c = 1$.

$\hat{E}(t)$ can be decomposed into two quadrature components $\hat{E}_1$ and $\hat{E}_2$, which are canonical conjugates

$$\hat{E}(t) = \hat{E}_1 \cos(\Omega t - \phi) + \hat{E}_2 \sin(\Omega t - \phi),$$

where

$$\Omega = \frac{\omega_3 + \omega_4}{2},$$

and $\phi$ is an arbitrary phase angle that may be chosen at will.

$\hat{E}_1$ can be expressed in term of initial modes $\hat{a}_3(L)$ and $\hat{a}_4(0)$,

$$\hat{E}_1 = g \hat{a}_3(L) + h \hat{a}_4(0) + g^* \hat{a}_4^+(L) + h^* \hat{a}_3^+(0),$$

where

$$g = \sqrt{\frac{\Omega}{2}} [\lambda_3 \mu e^{-i\phi} + \lambda_4 \nu^* e^{i(\phi + \pi/2)}] e^{i\epsilon t},$$

$$h = \sqrt{\frac{\Omega}{2}} [\lambda_3 \mu e^{-i(\phi + \pi/2)} + \lambda_4 \nu^* e^{i\phi}] e^{-i\epsilon t},$$

$$\epsilon = \Omega - \omega_3 = \omega_4 - \Omega,$$

$\epsilon$ is the modulation frequency.

Now we define

$$\hat{B} = g \hat{a}_3(L) + h \hat{a}_4(0),$$

$$\hat{B}^+ = g^* \hat{a}_3^+(L) + h^* \hat{a}_4^+(0),$$

then

$$\hat{E}_1 = \hat{B} + \hat{B}^+, \quad (19)$$

where $B^+$ is the adjoint of $\hat{B}$. 63
4 Higher-order noise moment $< (\Delta \hat{E}_1)^N >$ and higher-order squeezing

By using the Campbell–Baker–Hausdorff formula, we get the Nth-order moment of $\Delta \hat{E}_1$,

$$< (\Delta \hat{E}_1)^N > = <:: (\Delta \hat{E}_1)^N :> + \frac{N(1)}{1!} \frac{1}{2} C_0 <:: (\Delta \hat{E}_1)^{N-2} :> + \frac{N(2)}{2!} \frac{1}{2} C_0^2 <:: (\Delta \hat{E}_1)^{N-4} :> + \cdots + (N - 1)!! C_0^{N/2}. \quad (N \text{ is even}) \quad (20)$$

where

$$N^{(r)} = N(N - 1) \cdots (N - r + 1), \quad C_0 = \frac{1}{2i} [\hat{E}_1, \hat{E}_2] = [\hat{B}, \hat{B}^+], \quad (21)$$

and $:: :$ denotes normal ordering with respect to $\hat{B}$ and $\hat{B}^+$. We take the initial quantum state to be $|\alpha >_4 |0 >_S$, which is a product of the coherent state $|\alpha >_4$ for $\hat{a}_4(0)$ mode and the vacuum state for $\hat{a}_S(L)$ mode. Since $|\alpha >_4 |0 >_S$ is the eigenstate of $\hat{B}$, we get

$$<:: (\Delta \hat{E}_1)^N :> = <:: (\Delta \hat{B} + \Delta \hat{B}^+)^N :> = \sum_{\gamma = 0}^{N} \left[ \begin{array}{c} N \\ \gamma \end{array} \right] \gamma <0|\alpha : (\Delta \hat{B}^+)^\gamma (\Delta \hat{B})^{N-\gamma} : |\alpha >_4 |0 >_S = 0. \quad (22)$$

Then from (20),

$$< (\Delta \hat{E}_1)^N >= (N - 1)!! C_0^{N/2}, \quad (23)$$

$$C_0 = [\hat{B}, \hat{B}^+] = |\sigma|^2 + |\mu|^2 = \frac{\Omega}{2} \left( (\lambda_2^2 + \lambda_4^2) (|\mu|^2 + |\nu|^2) + 2\lambda_2 \lambda_4 \mu^* \nu e^{i(\theta^2 + \frac{\pi}{4})} + \mu \nu e^{-i(\theta^2 + \frac{\pi}{4})} \right). \quad (24)$$

where

$$\lambda_2^2 + \lambda_4^2 = 2, \quad \lambda_2 \lambda_4 = \sqrt{1 - \frac{\epsilon^2}{\Omega^2}}. \quad (25)$$

Substituting eqs. (8), (10), (24) into (23), we get the Nth-order moment of $\Delta \hat{E}_1$,

$$< (\Delta \hat{E}_1)^N > = (N - 1)!! \Omega^{N/2} |sec^2|k|L + tan^2|k|L - 2 \sqrt{1 - \frac{\epsilon^2}{\Omega^2} sec|k|L tan|k|L cos(2\phi - \theta_1 - \theta_2)|^{N/2}. \quad (26)$$

If $\phi$ is chosen to satisfy

$$2\phi - \theta_1 - \theta_2 = 0, \quad \text{or} \quad cos(2\phi - \theta_1 - \theta_2) = 1,$$

then the above eq. (25) leads to the result

$$< (\Delta \hat{E}_1)^N > = (N - 1)!! \Omega^{N/2} |sec^2|k|L + tan^2|k|L - 2 \sqrt{1 - \frac{\epsilon^2}{\Omega^2} sec|k|L tan|k|L|^{N/2}. \quad (26)$$

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When $0 < |k|L < \pi$, the right-hand side is less than $(N-1)!!\Omega^{N/2}$, which is the corresponding $N$th-order moment for two-mode coherent states. It follows that the field of the combined mode of the probe wave and the phase conjugate wave in NDFWM exhibits higher-order squeezing to all even orders.

The squeeze parameter $q_N$ for measuring the degree of $N$th-order squeezing is

$$q_N = \frac{< (\Delta \hat{E}_1)^N > - < (\Delta \hat{E}_1)^N >_{\text{two-mode coh.s}}}{< (\Delta \hat{E}_1)^N >_{\text{two-mode coh.s}}}
= [\sec^2|k|L + \tan^2|k|L - 2 - \frac{\epsilon^2}{\Omega^2} \sec|k|L \tan|k|L]^{N/2} - 1. \quad (28)$$

We find that $q_N$ is negative, and $q_N$ increases with $N$. This gives out the conclusion that the degree of higher-order squeezing is greater than that of the second order.

5 Generalized uncertainty relations in NDFWM

$\hat{E}_2$ can be regarded as a special case of $\hat{E}_1$ if $\phi$ is replaced by $\phi + \pi/2$. Then if $\phi$ is chosen to satisfy $2\phi - \theta_1 - \theta_2 = 0$, from eq. (25) it follows that

$$< (\Delta \hat{E}_2)^N > = (N-1)!!\Omega^{N/2}[\sec^2|k|L + \tan^2|k|L - 2 - \frac{\epsilon^2}{\Omega^2} \sec|k|L \tan|k|L]^{N/2}. \quad (29)$$

when $0 < |k|L < \pi$, the right-hand side is greater than $(N-1)!!\Omega^{N/2}$.

From eqs. (26) and (29), we obtain

$$< (\Delta \hat{E}_1)^N > \cdot < (\Delta \hat{E}_2)^N > = [(N-1)!!]^{2\Omega^N}[1 + 4 \frac{\epsilon^2}{\Omega^2} \sec^2|k|L \tan^2|k|L]^{N/2}. \quad (30)$$

Eq. (30) shows that $< (\Delta \hat{E}_1)^N >$ and $< (\Delta \hat{E}_2)^N >$ can not be made arbitrarily small simultaneously. We call eq. (30) the generalized uncertainty relations in NDFWM, and the right-hand side is dependent on $\epsilon, \Omega, N$, and $|k|L$.

In the degenerate case $\omega_1 = \omega_2 = \Omega, \epsilon = 0$ from eqs. (26), (28) and (30) we obtain

$$< (\Delta \hat{E}_1)^N > = (N-1)!!\Omega^{N/2}[\sec|k|L - \tan|k|L]^N, \quad (31)$$

$$q_N = [\sec|k|L - \tan|k|L]^N - 1, \quad (32)$$

$$< (\Delta \hat{E}_1)^N > \cdot < (\Delta \hat{E}_2)^N > = [(N-1)!!]^2 \cdot \Omega^N. \quad (33)$$

When $N = 2$,

$$< (\Delta \hat{E}_1)^2 > = \Omega[\sec|k|L - \tan|k|L]^2, \quad (34)$$

$$q_2 = [\sec|k|L - \tan|k|L]^2 - 1, \quad (35)$$
\[< (\Delta \hat{E}_1)^2 > \cdot < (\Delta \hat{E}_2)^2 > = \Omega^2 \]  \hspace{1cm} (36)

These results are in agreement with the conclusions in the previous relevant references\textsuperscript{[3][5]}. 

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\textbf{References}


