The relation of squeezing and $Q(\alpha)$ function is discussed in this paper. By means of $Q$ function, the squeezing of field with gaussian $Q(\alpha)$ function or negative $P(\alpha)$ function is also discussed in detail.

1 Introduction

In quantum optics, $P(\alpha), Q(\alpha)$ and $W(\alpha)$ are common quasiprobability distribution functions [1], but only $Q(\alpha)$ preserves a good function (positive and nonregular). Recently, by means of Fokker--Plank equation for $Q$ function, M. S. Kim et al. discussed the fourth-order squeezing [2]. In this paper, we consider the relation between $Q$ function and squeezing, and study the squeezing of field with a gaussian $Q$ function or negative $P(\alpha)$ function.

For any field density operator $\rho$, the $Q$ function is defined as

$$Q(\alpha) = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle$$

it satisfies the normalization condition

$$\int d\alpha^2 Q(\alpha) = 1$$

For antinormally ordered operator $f(a, a^+) = f^{(-)}(a, a^+)$, one can get following equation

$$\langle f(a, a^+) \rangle = \int d^2 \alpha Q(\alpha) f^{(-)}(\alpha, \alpha^*) = 1$$

where $a$ and $a^+$ are annihilation and creation operators respectively. Defining parameter

$$S = \langle (a + a^+)^2 \rangle - \langle a + a^+ \rangle^2$$

For squeezing, $S$ should be negative.

Now, we suppose that $Q$ function can be expanded as following form

$$Q(\alpha) = \frac{1}{\pi} e^{\beta |\alpha|^2} \sum C_{m,n} a^m a^{*n}, (C_{m,n} = C_{n,m}^*)$$

Using mathematical identity [3]
\[
\int \frac{d^2 \alpha}{\pi} e^{-\beta |\alpha|^2 + \rho \alpha^*} = \frac{1}{\beta} e^{\rho/\beta}, (\beta > 0)
\] (6)

one can have
\[
\int \frac{d^2 \alpha}{\pi} \alpha^m \alpha^* \alpha e^{-\beta |\alpha|^2} = \frac{n! \delta_{mn}}{\beta^{m+1}}
\] (7)

and the normalization condition is
\[
\sum_m C_{m,m} m! / \beta^{m+1} = 1
\] (8)

By means of equations (3) and (7), we have
\[
\langle a + a^+ \rangle = \sum_m \frac{2(m + 1) \Re C_{m,m+1}}{\beta^{m+2}}
\] (9)

\[
\langle a^2 + a^+ a^2 \rangle = \sum_m \frac{2(m + 2) \Re C_{m,m+2}}{\beta^{m+3}}
\] (10)

\[
\langle a^+ a \rangle = \sum_m \frac{(m + 1) - \beta}{\beta^{m+2}} m! C_{m,m}
\] (11)

and
\[
S = \sum_m \frac{2(m + 1) \Re C_{m,m+1}}{\beta^{m+2}} + 2 \sum_m \frac{m + 1 - \beta}{\beta^{m+2}} m! C_{m,m} - \left[ \sum_m \frac{2(m + 1) \Re C_{m,m+1}}{\beta^{m+2}} \right]^2
\] (12)

If the field exists squeezing, then
\[
\sum_m \left[ \frac{(m + 2) \Re C_{m,m+2}}{\beta^{m+3}} + \frac{(m + 1 - \beta)m! C_{m,m}}{\beta^{m+2}} \right] < \left[ \sum_m \frac{2(m + 1) \Re C_{m,m+1}}{\beta^{m+2}} \right]^2
\] (13)

2 Squddzing of field with gaussian Q function

We introduce the gaussian Q function as
\[
Q(\alpha) = \sqrt{t^2 - 4 |A|^2} \exp \left[-t (a^* - \omega^*)(\alpha - \omega) + A^*(a^* - \omega^*)^2 + A(\alpha - \omega)^2 \right]
\] (14)
where \( t > 2|A| \). Using integration formula [3]

\[
\int \frac{d^2z}{\pi} e^{-\mu|z|^2 + \epsilon z^* + rz + oz^*} = \frac{1}{\sqrt{\mu^2 - 4fg}} e^{\frac{\mu r + r' z + rz'}{\mu^2 - 4fg}}
\]

(15)

and equation (3), one can show

\[
\langle a + a^+ \rangle = \omega + \omega^* \tag{16}
\]

\[
\langle a^2 + a^{*+2} \rangle = \omega^{*2} + \omega^2 + \frac{2(A + A^*)}{t^2 - 4|A|^2} \tag{17}
\]

\[
\langle a^+ a \rangle = |\omega|^2 + \frac{t}{t^2 - 4|A|^2} - 1 \tag{18}
\]

and easily obtain

\[
S = \frac{2(A + A^* + 4|A|^2 + t - t^2)}{t^2 - 4|A|^2} \tag{19}
\]

Thus the condition for the existence of squeezing is

\[
A + A^* + 4|A|^2 < t^2 - t \tag{20}
\]

If \( A = 0 \), squeezing means \( t > 1 \), if \( t < 1 \) and \( A = 0 \), no squeezing exists in the field. It is worth to point out that the field with \( A = 0 \) and \( t > 1 \) has not been found up till now.

3 Squeezing of field with negative \( P(\alpha) \) function

The relation of \( P(\alpha) \) and \( Q(\alpha) \) is

\[
Q(\alpha) = \int \frac{d^2\beta}{\pi} e^{-|\beta - \alpha|^2} P(\beta) \tag{21}
\]

for nonclassical field, its \( P(\alpha) \) function has two situations [4]; i) \( P(\alpha) \) is negative, ii) \( P(\alpha) \) is more singular than \( \delta - \) function. We consider the nonclassical field with negative \( P(\alpha) \) function [5]

\[
\rho = \int d^2a P(\alpha) |\alpha\rangle \langle \alpha| \tag{22}
\]

Suppose \( P(\alpha) \) as

\[
P(\alpha) = \frac{1}{\pi} e^{-t|\alpha|^2} \sum_{i,j} P_{i,j} \alpha^i \alpha^{*j} \tag{23}
\]

Using equations (6) and (21), we obtain
\[ Q(\alpha) = \frac{1}{\pi} \sum_{i,j} P_{i,j} e^{-\frac{1}{2}|\alpha|^2} \sum_{l=0}^{\min(i,j)} \frac{i! j! \alpha_{i-l} \alpha_{j-l}}{l!(i-l)!(j-l)!(l+t)^{(i+j-l)}} \]  
(24)

Comparing with equation (6), one can have

\[ \beta = \frac{t}{1+t} \]  
(25)

\[ C_{m,n} = \sum_{l} P_{m+l,n+l} \frac{(m+l)!(n+l)!}{l! m! n! (1+t)^{m+n+l}} \]  
(26)

Obviously, the field with negative P function can exhibit squeezing for some situation, but if \( P(\alpha) \) is only the function of \(|\alpha|\), i.e., \( P(\alpha) \) is sphere symmetry in phase space, then

\[ P_{i,j} = 0 \quad (i \neq j) \]  
(27)

\[ C_{m,n} = 0 \quad (m \neq n) \]  
(28)

Form equation (12), one can get

\[ S > 0 \]  
(29)

In conclusion, it is clearly that no squeezing exists in the field with negative \( P(\alpha) \) function which is sphere symmetry in phase space.

References

W. H. Louisell, Quantum statistical properties of radiation (Wiley, New York, 1973)


