QUANTIZATION OF ELECTROMAGNETIC_FIELDS IN CAVITIES

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Abstract

A quantization procedure for the electromagnetic field in a rectangular cavity with perfect conductor walls is presented, where a decomposition formula of the field plays an essential role. All vector mode functions are obtained by using the decomposition. After expanding the field in terms of the vector mode functions, we get the quantized electromagnetic Hamiltonian.

1 Introduction

Recently we have carried out the field quantization in several rectangular cavities using the vector mode functions [1]. The vector mode functions have been obtained with the help of an orthogonal matrix. However, the procedure developed there has not been applicable to other cavities in a straightforward manner.

To overcome the above difficulty, we have presented another quantization scheme for the field in a circular cylindrical cavity [2]. All vector mode functions have been obtained by using a decomposition formula derived from Maxwell's equations directly. This method is more general than before, because it is also applicable to rectangular and spherical cavities.

In this paper, we applied the above method to a rectangular cavity with perfect conducting walls. Then spontaneous emission of an atom inside the cavity is calculated.

2 Decomposition of Electromagnetic Fields

In this section, we derive the decomposition formula for the field in the Cartesian coordinates. We shall use this result in performing the field quantization in a rectangular cavity in Sec. 3.

Maxwell's equations for the electric field $E$ and the magnetic field $B$ in free space are given by

\[ \nabla \cdot E = 0, \quad \nabla \cdot B = 0, \quad \nabla \times E = \frac{1}{c^2} \partial_t B, \quad \nabla \times B = 0, \quad (1) \]

where $c$ is the velocity of light in free space and $\partial_t = \partial/\partial t$.

The electromagnetic field $E$ and $B$ can be written in the Cartesian coordinates $(x, y, z)$ as

$E = E_x \hat{e}_x + E_y \hat{e}_y + E_z \hat{e}_z$, $B = B_x \hat{e}_x + B_y \hat{e}_y + B_z \hat{e}_z$, where $E_x \hat{e}_x$, $E_y \hat{e}_y$ is the transverse component of the field.
and $\mathbf{E}_z = e_z E'_z$. Here $e_x$, $e_y$, and $e_z$ are the unit vectors in the $x$, $y$, and $z$ directions, respectively. For simplicity, the derivative with respect to, for example, $x$ is described as $\partial/\partial x = \partial_z$.

Since $\nabla_x \times \mathbf{E}_z = 0$, the first equation in (1) gives

$$\partial_t \mathbf{B}_z = -\nabla_x \times \mathbf{E}_T, \quad \partial_t \mathbf{B}_T = -\nabla_x \times \mathbf{E}_z - \nabla_z \times \mathbf{E}_T. \quad (2)$$

Similarly, the second equation in (1) leads to

$$\frac{1}{c^2} \partial_t \mathbf{E}_z = \nabla_x \times \mathbf{B}_T, \quad \frac{1}{c^2} \partial_t \mathbf{E}_T = \nabla_x \times \mathbf{B}_z + \nabla_z \times \mathbf{B}_T. \quad (3)$$

Equations (2) and (3) give

$$\Delta_T \mathbf{E} = -\nabla \times \nabla \times \mathbf{E}_z + \nabla \times \partial_z \mathbf{B}_z, \quad \Delta_T \mathbf{B} = -\frac{1}{c^2} \nabla \times \partial_t \mathbf{E}_z - \nabla \times \nabla \times \mathbf{B}_z. \quad (4)$$

To rewrite Eq. (4), we must decompose the components $E_z$ and $B_z$ into two parts. Suppose that the field is in a finite region. Let us expand $E_z$ and $B_z$ in terms of a certain complete system of functions with mode $s$:

$$E_z(r,t) = \sum_s (E_{zs}(r,t) + \text{c.c.}), \quad B_z(r,t) = \sum_s (B_{zs}(r,t) + \text{c.c.}), \quad (5)$$

where $E_{zs}(r,t) = \tilde{E}_{zs}(r)e^{-i\omega st}$ and $B_{zs}(r,t) = \tilde{B}_{zs}(r)e^{-i\omega st}$. Here $\omega_{s\sigma}$ ($\omega_{s\sigma} \geq 0, \sigma = 1, 2$) is determined by using given boundary conditions. Since $E_z$ and $B_z$ satisfy the wave equation, the components $E_{zs}$ and $B_{zs}$ satisfy the Helmholtz equations:

$$\Delta E_{zs} = -k^2 E_{zs}, \quad \Delta B_{zs} = -k^2 B_{zs}(r), \quad (6)$$

where $k^2 = \omega^2/c^2$. We assume that the components satisfy

$$\partial_z^2 E_{zs} = -h^2 E_{zs}, \quad \partial_z^2 B_{zs} = -h^2 B_{zs}, \quad (7)$$

where $h_s$ is determined by the boundary conditions. Then we have two dimensional Helmholtz equations:

$$\Delta_T E_{zs}(r) = -g_{s1}^2 E_{zs}(r), \quad \Delta_T B_{zs}(r) = -g_{s2}^2 B_{zs}(r), \quad (8)$$

where $g_{s\sigma} = k^2 - h_s^2$.

Here we define two functions $F_\sigma$ from $E_{zs}$ and $B_{zs}$ with $g_{s\sigma}^2 \neq 0$ as

$$F_\sigma(r,t) = \sum_{g_{s\sigma}^2 \neq 0} [F_{s\sigma}(r,t) + \text{c.c.}] = \sum_{g_{s\sigma}^2 \neq 0} [\tilde{F}_{s\sigma}(r)e^{-i\omega_{s\sigma} t} + \text{c.c.}], \quad (9)$$

where

$$F_{s1} = E_{zs}/g_{s1}^2, \quad F_{s2} = B_{zs}/g_{s2}^2, \quad (g_{s\sigma}^2 \neq 0). \quad (10)$$

The functions $F_\sigma$ and their components $F_{s\sigma}$ satisfy the same equations as $E_z$ and $E_{zs}$, respectively. The component $F_\sigma$ is a solution of the Poisson equation. On the other hand, if there is a component $E_{zs}$ or $B_{zs}$ with $g_{s\sigma}^2 = 0$, Eq. (8) reduces to two dimensional Laplace equation.
The functions $\triangle_T F_\sigma$ satisfy
\[ -\triangle_T F_1 = \sum_{g_{s1} \neq 0} g_{s1}^2 F_{s1} + \sum_{g_{s2} \neq 0} E_{zs}, \quad -\triangle_T F_2 = \sum_{g_{s2} \neq 0} g_{s2}^2 F_{s2} = \sum B_{zs}. \] (11)

Here define $E_{0z}$ and $B_{0z}$ as
\[ E_{0z} = E_z + \Delta_T F_1 = \sum E_{zs}, \quad B_{0z} = B_z + \Delta_T F_2 = \sum B_{zs}, \] (12)
which satisfy $\triangle_T E_{0z} = 0$ and $\triangle_T B_{0z} = 0$. Then we have useful formulas for $E_z$ and $B_z$:
\[ E_z = -\Delta_T F_1 + E_{0z}, \quad B_z = -\Delta_T F_2 + B_{0z}. \] (13)

Using Eq. (13) and defining $F_\sigma$ as $F_\sigma = e_z F_\sigma$, we can rewrite Eq. (4) as
\[ \triangle_T \left( E - \nabla \times \nabla \times F_1 + \nabla \times \partial_t F_2 \right) = -\nabla \times \nabla \times E_{0z} + \nabla \times \partial_t B_{0z}, \] (14)
\[ \triangle_T \left( B - \frac{1}{c^2} \nabla \times \partial_t F_1 - \nabla \times \nabla \times F_2 \right) = -\frac{1}{c^2} \nabla \times \partial_t E_{0z} - \nabla \times \nabla \times B_{0z}, \] (15)
where $E_{0z} = e_z E_{0z}$ and $B_{0z} = e_z B_{0z}$. Define $E_0$ and $B_0$ as the quantities in the parentheses at the left hand side in Eqs. (14) and (15), respectively. The results of this section is summarized in the following theorem.

**Theorem 1:** If the components $E_{zs}$ and $B_{zs}$ satisfy Eq. (7), the field can be decomposed into three components as follows:
\[ \mathbf{E} = \nabla \times \nabla \times F_1 - \nabla \times \partial_t F_2 + E_0, \quad \mathbf{B} = \frac{1}{c^2} \nabla \times \partial_t F_1 + \nabla \times \nabla \times F_2 + B_0, \] (16)
where $E_0$ and $B_0$ satisfy
\[ \triangle_T E_0 = -\nabla \times \nabla \times E_{0z} + \nabla \times \partial_t B_{0z}, \quad \triangle_T B_0 = -\frac{1}{c^2} \nabla \times \partial_t E_{0z} - \nabla \times \nabla \times B_{0z}. \] (17)

Theorem 1 plays a central role in performing the field quantization in this paper. It is worth emphasizing that each term in Eq. (16) is a solution to Maxwell's equations.

### 3 Vector Mode Functions and Field Quantization

The cavity we treat here is enclosed by rectangular walls having sides $L_x, L_y,$ and $L_z$ in the $x, y,$ and $z$ directions, respectively: $0 < x < L_x, 0 < y < L_y,$ and $0 < z < L_z$. We assume that the cavity has perfectly conducting walls. The tangential component of the electric field $E_{\text{tan}}$ and the normal component of the magnetic field $B_{\text{norm}}$ must accordingly vanish at the boundaries of the cavity.
The above boundary condition reduces to that for the $z$ components

$$E_z = 0, \quad \partial_z B_z = 0, \quad (x = 0, \ L_z), \quad (x > 0), \quad (x < 0),$$

$$B_z = 0, \quad \partial_z E_z = 0, \quad (y = 0, \ L_y), \quad (y > 0), \quad (y < 0),$$

$$E_z = 0, \quad \partial_z E_z = 0, \quad (z = 0, \ L_z).$$

The solution to the Helmholtz equation (6) for the components $E_{za}$ and $B_{za}$ under the above boundary conditions is given by

$$E_{za}(r, t) = C_1(t) \sin(\ell \pi x / L_z) \sin(m \pi y / L_y) \cos(n \pi z / L_z),$$

$$B_{za}(r, t) = C_2(t) \cos(\ell \pi x / L_z) \cos(m \pi y / L_y) \sin(n \pi z / L_z),$$

where the mode index is $s = (\ell, m, n)$ ($\ell, m, n = 0, \pm 1, \pm 2, \cdots$).

From the solution (21) we have

$$g_{so}^2 = g_s^2 = (\ell \pi / L_z)^2 + (m \pi / L_y)^2,$$

$$k_{so}^2 = k_s^2 = (\ell \pi / L_z)^2 + (m \pi / L_y)^2 + (n \pi / L_z)^2.$$  

(22)

Consequently, we can use Theorem 1 in the preceding section. Although it follows from Eq. (22) that $g_{so}^2 \geq 0$, we can prove that $g_{so}^2 > 0$, which results from the following lemma. We omit its proof.

**Lemma:** $E_0, \ B_0 = 0$ and $g_{so}^2 > 0$. As a result, the term with $\ell = m = 0$ in Eq. (21) cannot be used.

Let us next obtain the functions $F_\sigma$, whose definitions are given in Eqs. (9) and (10). That is, the functions are given by

$$F_1(r, t) = \frac{E_{za}(r, t)}{g_s^2} = i \frac{\sqrt{\omega_o}}{2 \varepsilon_0} a_{s1}(t) \psi_{s1}(r),$$

$$F_2(r, t) = \frac{B_{za}(r, t)}{g_s^2} = i \frac{\sqrt{\omega_o}}{2 \varepsilon_0} a_{s2}(t) \psi_{s2}(r),$$

(23)

where $\omega_{so} = \omega_s$ and we have introduced $a_{s0}(t) = a_{so}(0) e^{-\omega_{so} t}$ and $\psi_{s0}$ given by

$$\psi_{s1}(r, t) = c_{s1} \sin(\ell \pi x / L_z) \sin(m \pi y / L_y) \cos(n \pi z / L_z),$$

$$\psi_{s2}(r, t) = c_{s2} \cos(\ell \pi x / L_z) \cos(m \pi y / L_y) \sin(n \pi z / L_z),$$

(24)

(25)

with $c_{s1} = [8/(V k_{so}^2 g_s^2)]^{1/2}$ and $c_{s2} = [8/(V \omega_o^2 g_s^2)]^{1/2}$. The functions $\psi_{s\sigma}$ has the orthonormality property

$$\int_c dr \psi_{s\sigma}^*(r) \psi_{s'\sigma'}(r) = \frac{1}{8} |c_{s\sigma}|^2 V \delta_{s\sigma'},$$

(26)

where $\int_c dr = \int_{\text{cavity}} dx dy dz$ and $V$ is the cavity volume. Here the quantity $\cos x$ in Eqs. (24) and (25) must be changed to $1/\sqrt{2}$ when $x = 0$.

Substituting the functions $F_\sigma$ in Eq. (23) into $E$ in Eq. (16), we find

$$E(r, t) = i \sum_{s\sigma} \sqrt{\frac{\omega_o}{2 \varepsilon_0}} \left[ a_{so}(t) u_{s\sigma}(r) - a_{so}^*(t) u_{s\sigma}^*(r) \right].$$

(27)
where the vector mode functions \( u_{\sigma\sigma} \) are given by
\[
\begin{align*}
\mathbf{u}_{s1} &= \nabla \times \mathbf{e}_z \psi_{s1}, \\
\mathbf{u}_{s2} &= i \omega_s \mathbf{e}_z \psi_{s2}.
\end{align*}
\tag{28}
\]
The vector mode functions satisfy \( \nabla \cdot \mathbf{u}_{\sigma\sigma} = 0 \), \( (\Delta + k_{\sigma\sigma}^2)\mathbf{u}_{\sigma\sigma} = 0 \), and at the boundaries \( \mathbf{u}_{\sigma\sigma} |_{u_{\sigma\sigma}} = 0 \), \( \nabla \times \mathbf{u}_{\sigma\sigma} |_{\text{norm}} = 0 \). They also satisfy the orthonormality property needed for quantization:
\[
\int dr \mathbf{u}_{\sigma\sigma}^* (r) \cdot \mathbf{u}_{\sigma'\sigma'} (r) = \delta_{\sigma\sigma'} \delta_{\sigma'\sigma'}.
\tag{29}
\]
To get the quantized field, the functions \( a_{\sigma\sigma} (t) \) are regarded as annihilation operators satisfying the commutation relation \( [a_{\sigma\sigma} (t), a_{\sigma'\sigma'}^+ (t)] = \delta_{\sigma\sigma'} \delta_{\sigma'\sigma'} \). Then we get the following theorem.

**Theorem 2:** The quantized field and the Hamiltonian are given by
\[
\begin{align*}
\mathbf{E} (r, t) &= i \sum_{\sigma\sigma'} \sqrt{\frac{\hbar \omega_0}{2 \varepsilon_0}} \left[ a_{\sigma\sigma} (t) \mathbf{u}_{\sigma\sigma} (r) - a_{\sigma\sigma}^+ (t) \mathbf{u}_{\sigma\sigma}^* (r) \right],
\mathbf{B} (r, t) &= \sum_{\sigma\sigma'} \sqrt{\frac{\hbar}{2 \varepsilon_0 \omega_0}} \left[ a_{\sigma\sigma} (t) \nabla \times \mathbf{u}_{\sigma\sigma} (r) + a_{\sigma\sigma}^+ (t) \nabla \times \mathbf{u}_{\sigma\sigma}^* (r) \right],
H_R &= \sum_{\sigma\sigma'} \frac{1}{2} \hbar \omega_0 \left( a_{\sigma\sigma}^+ a_{\sigma\sigma} + a_{\sigma\sigma} a_{\sigma\sigma}^+ \right) = \sum_{\sigma} \hbar \omega_0 \left( a_{\sigma\sigma}^+ a_{\sigma\sigma} + \frac{1}{2} \right).
\end{align*}
\tag{30-32}
\]

### 4 Spontaneous Emission

As an application, we consider the transition rates of an atom in the cavity, using the dipole approximation. The Hamiltonian is given by \( H = H_A + H_R + H_I \), where \( H_A \) is the free Hamiltonian for the atom, \( H_R \) for the field which is given in Eq. (32), and \( H_I = e \mathbf{D} \cdot \mathbf{E} (\mathbf{R}) \) (\( e \mathbf{D} \): the total electric dipole moment of the atom; \( \mathbf{R} = (X, Y, Z) \): the position of the atom).

At \( t = 0 \), the atom is in an energy state \( |i_0 \rangle \) (with energy \( E_{i_0} \)) and the field is in the vacuum. Then the probability per second of finding the atom in a state (with energy \( E_i \)) at sufficiently large time \( t \) is given by
\[
w = \frac{e^2 \pi}{\varepsilon_0 \hbar} \sum_{\sigma\sigma'} |\mathbf{u}_{\sigma\sigma} (\mathbf{R}) \cdot < i_0 | \mathbf{D} | i > |^2 \omega_0 \sin (\omega_0 - \omega_0) t \frac{\pi (\omega_0 - \omega_0)}{\pi (\omega_0 - \omega_0)},
\tag{33}
\]
where \( \hbar \omega_0 = E_{i_0} - E_i \).

Let us take the average of the coordinates \( Y \) and \( Z \) and take \( L_z \to \infty \). The transition rates \( w_y \) and \( w_z \) vanish (\( w_i \) indicates the rate where the dipole moment is along with the \( i \) direction). The rate \( w_x \) is given by
\[
w_x / w_0 = \frac{6}{\pi \xi_y} \sum_m (1 - m^2 / \xi_y^2)^{-1/2} \theta (1 - m^2 / \xi_y^2),
\tag{34}
\]
where \( \xi_i = \omega_0 L_i / c \pi, \xi_x = 1/2, w_0 = e^2 | < i_0 | \mathbf{D} | i > |^2 \omega_0^3 / (3 \pi \hbar \varepsilon_0 c^3), \theta (x) = 1 (x > 0), \) and
\[
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\]
\( \theta(x) \quad 0 \ (x < 0) \), which is shown in FIG. 1.

![Graph showing transition rate \( w_x \) for \( L_z \to \infty \), where the dipole moment is along with the \( x \) direction.](image)

FIG. 1. Transition rate \( w_x \) for \( L_z \to \infty \), where the dipole moment is along with the \( x \) direction.

Setting here \( \omega_0 = 10^{13} \) Hz, we have \( L_x = 4.7 \times 10^{-2} \) mm, so that the cavity is quite narrow. Also, FIG. 1 shows that the transition is forbidden when \( \xi_y < 1 \), i.e., \( L_y < 9.4 \times 10^{-2} \) mm., where the cavity is a thin tube in this case.

5 Conclusions

The quantization for the field in the cavity has been performed as follows: obtain the decomposition formula (16) in the Cartesian coordinates; solve the Helmholtz equations (6) for the components \( E_{za} \) and \( B_{za} \) under the boundary conditions; determine the functions \( F_\gamma \), substitute them into the decomposition formula (16), and obtain the vector mode functions satisfying the orthonormality property (29); then we arrive at the quantized field and Hamiltonian. In the whole process of quantization, the decomposition formula in Theorem 1 plays an important role.

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References