Multilevel atomic coherent states and atomic holomorphic representation

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Abstract
The notion of atomic coherent states is extended to the case of multilevel atom collective. Based on atomic coherent states, a holomorphic representation for atom collective states and operators is defined. An example is given to illustrate its application.

1 Introduction

Atomic coherent states have been introduced in quantum optics for more than twenty years[1−3]. They are the analogue of boson coherent states and exhibit approximate classical behavior when the mean atom numbers on the relevant levels are large. But until now such states are only available for collective of two-level atoms. We have extended this formulation to multilevel case[4]. Since many optical processes involve atoms of three or more levels, it is expected that this extension will play a role in the theory of such processes as cascade superfluorescence and superradiant lasing. In addition, like the photonic counterparts[5], the atomic coherent states may also be used to define a holomorphic representation. In some cases, it is convenient to use this representation to treat the collective interaction of atoms with the light field.

We shall give a brief introduction of our work in the following.

2 Multilevel atomic coherent states

For concreteness let us consider the fully symmetrical states of N three-level atoms. In Fock representation, such states are denoted by $|n_3, n_2, n_1\rangle$, where n_3, n_2, n_1 are numbers of atoms in the upper, middle and lower level respectively. The observables of l-th atom may be expressed by the generators $\hat{s}_{jk}^{(l)}$ of the group $SU(3)$, where $\hat{s}_{jk}$ are level-change operators for $j \neq k$, and are population operators for $j = k$. The collective atomic operators $\hat{S}_{jk}$ is defined as

$$\hat{S}_{jk} = \sum_{l=1}^{N} \hat{s}_{jk}^{(l)} \tag{1}$$

Each atomic coherent state in this subspace is characterized by two complex parameters α and β:

$$|\alpha,\beta\rangle = \frac{1}{[1+|\alpha|^2(1+|\beta|^2)]^{N/2}} e^{\beta\hat{S}_{12}} e^{\alpha\hat{S}_{23}} |N,0,0\rangle. \tag{2}$$

We might as well define atomic coherent state by $e^{\alpha\hat{S}_{21}}$ and subsequently by $e^{\alpha\hat{S}_{32}}$ operating on $|0,0,N\rangle$. This latter definition is somewhat more close to that of photon coherent state.

The meaning of parameters α and β can be seen from the expectation values of $\langle n_1\rangle$, $\langle n_2\rangle$, $\langle n_3\rangle$ and $\langle S_{jk}\rangle$ for $j \neq k$:

$$\langle n_3\rangle : \langle n_2\rangle : \langle n_1\rangle = 1 : |\alpha|^2 : |\alpha\beta|^2, \tag{3}$$

and the phase of α and β are just those of $\langle S_{32}\rangle$ and $\langle S_{21}\rangle$ respectively.

If both $|\alpha|$ and $|\beta|$ are of order 1, namely all $\langle n_3\rangle$, $\langle n_2\rangle$ and $\langle n_1\rangle$ are of order N. then all $\frac{\delta n_j}{\langle n_j\rangle} and \frac{\sqrt{\langle\hat{S}_{ij}\hat{S}_{ji}\rangle - |\langle\hat{S}_{ij}\rangle|^2}}{|\langle\hat{S}_{ij}\rangle|}$ are of order $\frac{1}{\sqrt{N}}$. This results confirm that atomic coherent states tend to display classical behavior when $\langle n_j\rangle$'s grow large.

The states with different α, β are not orthogonal to each other. But they form an overcomplete set in the discussed subspace. We have found the weight function

$$\xi(|\alpha|,|\beta|) = \frac{(N+1)(N+2)}{\pi^2} \frac{|\alpha|^2}{[1+|\alpha|^2(1+|\beta|^2)]}, \tag{4}$$

such that

$$\int d^2\alpha d^2\beta \; \xi(|\alpha|,|\beta|)|\alpha,\beta\rangle\langle\alpha\beta| = 1. \tag{5}$$

The above dicussion can be extended to other subspaces. It is also easy to be extended to atoms with more levels[4].

3 Atomic holomorphic representation

Like bosonic case, one can define an atomic holomorphic representation based on atomic coherent states. We shall illustrate this interesting concept in the simplest case, a collective of two level atoms, and consider the fully symmetrical subspace.

A holomorphic representation employs a holomorphic function $f(\alpha^*)$ which is analytic in the whole α^* plane to represent every state in the discussed subspace.

Actually, $f(\alpha^*)$ is a polynomial of order N, a rather well behaved function. Accordingly, each atomic operator will be represented by a holomorphic function of two complex variables α^* and β. We shall do some explanation in the following.

The completness of the two-level atomic coherent states in the discussed subspace is expressed by

$$1 = \frac{N+1}{\pi}\int \frac{d^2\alpha}{(1+|\alpha|^2)^2}|\alpha\rangle\langle\alpha|. \tag{6}$$

By this, we get

$$\begin{aligned} |f\rangle &= \frac{N+1}{\pi}\int \frac{d^2\alpha}{(1+|\alpha|^2)^2}|\alpha\rangle\langle\alpha|f\rangle \\ &= \frac{N+1}{\pi}\int \frac{d^2\alpha}{(1+|\alpha|^2)^{N/2+2}} f(\alpha^*)|\alpha\rangle, \end{aligned} \tag{7}$$

in which

$$f(\alpha^*) = (1+|\alpha|^2)^{N/2}\langle\alpha|f\rangle = (1+|\alpha|^2)^{N/2}\sum_{n=0}^{N}\langle\alpha|N-n,n\rangle\langle N-n,n|f\rangle, \tag{8}$$

where $|N-n,n\rangle$ denotes the Fock state in the fully symmetric subspace.

Invoking the well known value of $\langle\alpha|N-n,n\rangle$, we get

$$f(\alpha^*) = \sum_{n=0}^{N} f_n(\alpha^*)^n \tag{9}$$

with

$$f_n = \begin{pmatrix} N \\ n \end{pmatrix}^{\frac{1}{2}} \langle N-n,n|f\rangle. \tag{10}$$

Similarly, we can expand any atomic operator $\hat{T}$ by $|\alpha\rangle\langle\beta|$ as

$$\hat{T} = \frac{N+1}{\pi}\int \frac{d^2\alpha}{(1+|\alpha|^2)^{N/2+2}}\frac{d^2\beta}{(1+|\beta|^2)^{N/2+2}}T(\alpha^*,\beta)|\alpha\rangle\langle\beta|, \tag{11.1}$$

$$T(\alpha^*,\beta) = \sum_{m,n=1}^{N} T_{mn}(\alpha^*)^m\beta^n \tag{11.2}$$

where

$$T_{mn} = \langle N-m|\hat{T}|N-n,n\rangle \begin{pmatrix} N \\ m \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} N \\ n \end{pmatrix}^{\frac{1}{2}}. \tag{12}$$

The operation of $\hat{T}$ on a state $|f\rangle$ is described as follows. Let

$$|g\rangle = \hat{T}|f\rangle$$

then the holomorphic representation $g(\alpha^*)$ is given by

$$g(\alpha^*) = \frac{N+1}{\pi}\int \frac{d^2\beta}{(1+|\beta|^2)^{N+2}}T(\alpha^*,\beta)f(\beta^*). \tag{13}$$

Futhermore we have shown[4] that in holomorphic representation the collective atomic operators $\hat{S}_+$, $\hat{S}_-$, $\hat{S}_3$ may be also represented by differential operators as $\frac{\partial}{\partial\alpha^*}$, $(N\alpha^* - \alpha^{*2}\frac{\partial}{\partial\alpha^*})$, $(\frac{N}{2} - \alpha^*\frac{\partial}{\partial\alpha^*})$ when operating from left, and as $(N\beta - \beta^2\frac{\partial}{\partial\beta})$, $\frac{\partial}{\partial\beta}$, $(\frac{N}{2} - \beta\frac{\partial}{\partial\beta})$ when operating from right.

To our knowledge, there is only one problem which have been solved analytically for arbitrary value of N, namely a collective of two-level atoms in an external field[6],[7]. This problem can be solved in a comparatively easier way by our formulation. Let us see the simpler case of no detuning.

The atomic density operator $\hat{\rho}$ now obeys the master equation.

$$\frac{\partial\hat{\rho}}{\partial t} = -i\Omega[\hat{S}_+ + \hat{S}_-,\hat{\rho}] + \gamma[2\hat{S}_-\hat{\rho}\hat{S}_+ - \hat{\rho}\hat{S}_+\hat{S}_- - \hat{S}_+\hat{S}_-\hat{\rho}]. \tag{14}$$

In our holomorphic representation, this equation may be taken as

$$\begin{aligned}\frac{\partial\rho(\alpha^*,\beta,t)}{\partial t} &= (N\alpha^* - \alpha^{*2}\frac{\partial}{\partial\alpha^*} - \frac{\partial}{\partial\beta})[-i\Omega + \gamma(N\beta - \beta^2\frac{\partial}{\partial\beta})]\rho(\alpha^*,\beta,t) \\ &+ (c.c.\ with\ \alpha \leftrightarrow \beta).\end{aligned} \quad (15)$$

For the steady-state solution, we expand $\rho(\alpha^*,\beta)$ according to eq.(11.2):

$$\rho(\alpha^*,\beta) = \sum_{m,n=0}^{N} \rho_{mn}(\alpha^*)^m\beta^n. \quad (16)$$

Substituting this expansion into eq.(15) and setting the right-hand side be zero, we obtain a recurtion relation for ρ_{mn}, which can be solved analytically to give

$$\rho_{mn} = \frac{(N!)^2 i^{m-n}}{(N-n)!(N-m)!(\Omega/\gamma)^{m+n}}\rho_{00}. \quad (17)$$

The value of ρ_{00} is determined by normalization.

The density operator $\hat{\rho}$ is then given by

$$\hat{\rho} = \sum_{m,n}\rho_{mn}\begin{pmatrix} N \\ m \end{pmatrix}^{-\frac{1}{2}}\begin{pmatrix} N \\ n \end{pmatrix}^{-\frac{1}{2}}|m\rangle\langle n|. \quad (18)$$

The extension of holomorphic representation to three or more level atoms is obvious.

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