CALCULATION OF DOSE, DOSE EQUIVALENT, AND
RELATIVE BIOLOGICAL EFFECTIVENESS FOR HIGH
CHARGE AND ENERGY ION BEAMS

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Abstract—The Green's function for the transport of ions of
high charge and energy is utilized with a nuclear fragmenta-
tion database to evaluate dose, dose equivalent, and RBE for
C3H10Tl/2 cell survival and neoplastic transformation as a
function of depth in soft tissue. Such evaluations are useful to
estimates of biological risk for high altitude aircraft, space
operations, accelerator operations, and biomedical applica-
tions.

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Key words: dose equivalent; relative biological effectiveness;
dose; tissue, body

INTRODUCTION

The biological response of living tissues depends (in part) on the temporal and spatial fluctuations of the
energy deposits of the ions within the tissue system. Such fluctuations depend not only on the specific environment
(or particle fluence) to which the person is exposed but also on how that environment (or particle fluence) is
modified by interaction with the human body in reaching specific tissues. Only by knowledge of the specific
radiation types and their physical properties at the tissue site can a basis for estimating risk be found. Even if the
environment to which the individual is exposed is known precisely, the energy deposits within specific tissues deep
in the body are largely known through theoretical esti-
mates and are therefore limited by the uncertainty in the
calculational models. Although methods for exposure evaluations in high energy nucleonic fields are relatively
well developed (Alsmiller et al. 1970; Nabelssi and
Hertel 1993), there are several practical applications in
which exposures with high charge and energy (HZE)
ions are of concern. Among these are the natural cosmic
ray environment consisting of relativistic nuclei of all
elements for which HZE ions contribute 80% of the free
space dose equivalent (Wilson et al. 1991). Even for
modest depths of shield penetration (30 g cm⁻²), the
HZE dose equivalent is on the order of 45% of the total
and is of considerable importance to protection of astro-
nauts and possible future high speed aircraft (≈ Mach 2.5
or greater) on high latitude routes (for example, New
Computational models are required for treatment planning
in which cell killing for HZE tumor therapy is the critical
parameter. With the further development of HZE acceler-
tors, there is a growing concern among policy setting bodies
on evaluation of exposures for the scattering of HZE
particles from the beamline (L. Brackenbush 1994) for
which the present calculations were undertaken. Clearly, an
accurate conversion of the environment to estimates of
exposure fields at specific tissue sites is an important issue
in HZE radiation protection problems.

In our previous paper on the closed form solution for
the HZE Green’s function, we reviewed the computa-
tional procedures for space and laboratory HZE transport
of various groups (Wilson et al. 1990). It is noted in that
review that numerical solution methods for the Boltz-
mann equation are best suited to space radiations where
energy spectra are smooth over large energy intervals and
less suited to the simulation of laboratory beams which
exhibit large spectral variation over a very limited energy
domain and a large energy derivative resulting in large
truncation errors (Wilson et al. 1990, 1991). In the
present paper, we apply these analytical methods (Wilson
et al. 1990) to evaluate dosimetric quantities as a function
of penetration depth in tissue equivalent materials to
estimate specific organ exposures.

TRANSPORT EQUATION

The Green’s function is introduced as a solution of the following transport equation:

\[
\frac{\partial}{\partial x} \frac{\partial}{\partial E} \sum_j G_{jm}(x, E, E') = \sum_k \sigma_{jk} G_{km}(x, E, E'),
\]

where \( G_{jm}(x, E, E') \) is the flux of ion type \( j \) at depth \( x \) with energy \( E \) (in MeV/amu) resulting from a unit flux of...
type \( m \) ions of energy \( E' \) at the boundary, \( \bar{S}_j(E) \) is the change in \( E \) per unit distance, \( \sigma_k \) is the total macroscopic reaction cross section, and \( \sigma_{mk} \) the macroscopic cross section for collision of ion type \( k \) to produce an ion of type \( j \). The boundary condition is given by

\[
G_{jm}(0, E, E') = \delta_{jm}\delta(E - E'),
\]

where \( \delta_{jm} \) is Kronecker’s delta and \( \delta(E - E') \) is Dirac’s delta. The dose and dose equivalent for monoenergetic ions are given by

\[
D_m(x, E') = \sum_j A_j \bar{S}_j(E) G_{jm}(x, E, E') \, dE;
\]

and

\[
H_m(x, E') = \sum_j A_j \bar{S}_j(E) \mathcal{Q}(L_j) G_{jm}(x, E, E') \, dE.
\]

If \( G_{jm}(x, E, E') \) is known as a closed form solution (Wilson et al. 1990), then eqns (3) and (4) can be evaluated by simple integration techniques and the associated errors in numerically solving eqn (1) are avoided (Wilson et al. 1991).

The above equations can be simplified (Wilson 1977) by transforming the energy into the residual range of the ions as

\[
r_j = \int_0^E \frac{de}{\bar{S}_j(e)},
\]

and defining new field functions as

\[
\tilde{g}_{jm}(x, r_j, r_{j'},) = \bar{S}_j(E) G_{jm}(x, E, E').
\]

As a result, eqn (1) becomes

\[
\left( \frac{\partial}{\partial x} - \frac{\partial}{\partial r_j} + \sigma_j \right) \tilde{g}_{jm}(x, r_j, r_{j'}) = \sum_k \nu_k \sigma_{jk} \tilde{g}_{km}(x, r_k, r_{k'}),
\]

with the boundary condition

\[
\tilde{g}_{jm}(0, r_j, r_{j'}) = \delta_{jm} \delta(r_j - r_{j'}). \tag{8}
\]

Note that \( \nu_j \) is the range scale factor such that \( \nu_j r_j = v_m r_m \) and can be expressed as

\[
\nu_j = Z_j^2 / A_j. \tag{9}
\]

The solution to eqn (1) can be written as a perturbation series as

\[
\tilde{g}_{jm}(x, r_j, r_{j'}) = \sum_i g_{jm}^{(i)}(x, r_j, r_{j'}), 
\]

where

\[
g_{jm}^{(0)}(x, r_j, r_{j'}) = \exp(-\sigma_j x) \delta_{jm} \delta(x + r_j - r_{j'}), \tag{11}
\]

and the first collision term is

\[
g_{jm}^{(1)}(x, r_j, r_{j'}) = \nu_j \sigma_{jm} \exp(-\sigma_j x) - \sigma_m v_m, \tag{12}
\]

where \( g_{jm}^{(1)}(x, r_j, r_{j'}) \) takes on non-zero values for

\[
\frac{\nu_m}{\nu_j} (r_{j'} - x) \leq r_j \leq \frac{\nu_m}{\nu_j} (r_{j'} - x), \tag{13}
\]

and

\[
x_j = (\nu_m x + \nu_j r_j - \nu_m r_{j'}) / (\nu_m - \nu_j); \tag{14}
\]

\[
x_m = (\nu_m r_m - \nu_j x - \nu_j r_j) / (\nu_m - \nu_j). \tag{15}
\]

Note that in eqns (12)–(15), \( x_j + x_m = x \) for all \( r_j \) and \( r_{j'} \).

The significance of \( x_m \) is that it is the distance ion \( m \) traveled from the boundary to the collision site at which the ion \( j \) was produced and must now travel distance \( x_j \) before reaching \( x \). Higher order terms in the perturbation series are given in the Appendix.

To this point of development, the perturbative Green’s function (eqn 10) can be approximated by summing the terms given by eqns (11), (A7), and (A9).

The problem in explicitly summing eqn (10) is that the higher order terms of eqn (A9) contain thousands of terms, each of which is composed of sums of tens of thousands of terms for the second order correction \( i = 2 \) and millions of terms for the third order correction. Since the terms given by eqns (A7) and (A9) are dependent only on spatial coordinates, we seek an alternative means of summing the series (10) implicitly.

**NONPERTURBATIVE GREEN’S FUNCTION**

We now introduce nonperturbation terms for the summation in eqn (10). First we recall (Wilson et al. 1990) that the \( g \)-function of \( n \) arguments is generated by the perturbation solution of the transport equation neglecting ionization energy loss given by

\[
\mathcal{D}_j \mathcal{G}_m(x) = \sum_k \sigma_{jk} \mathcal{G}_k(x), \tag{16}
\]

subject to the boundary condition

\[
\mathcal{G}_m(0) = \delta_m. \tag{17}
\]

The solution is

\[
g_m(x) = \delta_m g(m) + \sigma_m g(j, m) + \sum_k \sigma_{mk} \mathcal{G}_k(j, k, m) + \ldots \tag{18}
\]

A term by term comparison of the series (18) with the series (10) shows that in the spectral average approximation (see Appendix) we may write

\[
g_{jm}^{(1)}(x, r_j, r_{j'}) = \exp(-\sigma_j x) \delta_{jm} \delta(x + r_j - r_{j'}) \tag{19}
\]

\[
+ \nu_j [g_m(x) - \exp(-\sigma_j x) \delta_m] \frac{x(v_m - \nu_j)}{v_m - \nu_j}.
\]
The advantage of eqn (19) is that \( g_{jm}(x) \) satisfies the convolution product relation (Wilson et al. 1993a) as

\[
g_{jm}(x) = \sum_k g_{jk}(x - y)g_{km}(y) \tag{20}
\]

for any positive values of \( x \) and \( y \). A solution for small \( x \) is easily obtained from the first few terms of eqn (18). Eqn (20) may then be used to propagate the function \( g_{jm}(x) \) over the entire solution domain taking arbitrarily large steps. Although the nonperturbation Green’s function in the form of eqns (19) and (20) provides a rapid computational method, the spectral terms are replaced by averages over the spectral domains of higher order terms. These energy averaged spectra can be corrected with the spectral distributions of perturbation theory. The first collision Green’s function is given as

\[
g^{(1)}_{jm}(x, r_j, r_m') = \frac{\nu_j}{(\nu_m - \nu_j)} \exp(-\sigma_{m} x_{m}) \tag{21}
\]

We rewrite \( g^{(1)}_{jm} \) in terms of its average value (given by eqn A7) as

\[
g^{(1)}_{jm}(x, r_j, r_m') = \frac{\nu_j}{(\nu_m - \nu_j)} g^{(1)}_{jm}(x) + b_{jm}(x)(r_j - \bar{r}_j), \tag{22}
\]

where \( g^{(1)}_{jm}(x) = \sigma_{jm} \delta(j, m) \) is given by eqn (A7) and

\[
\bar{r}_j = \frac{r_{j'} + r_{j''}}{2}
\]

is the midpoint \( \bar{r}_j \) between its limits given by eqn (A8). The \( b_{jm} \) term of eqn (22) has the property that

\[
\int_{r_j}^{r_m} b_{jm}(x)(r - \bar{r}_j) \, dr = 0 \tag{23}
\]

to insure that the first term of eqn (22) is indeed the average spectrum as required. The spectral slope parameter is

\[
b_{jm}(x) = \frac{\nu_j \sigma_{jm} \left[ \exp(-\sigma_{m} x) - \exp(-\sigma_{n} x) \right]}{x(\nu_m - \nu_j)(\nu_m - \nu_j)}. \tag{24}
\]

Note that the spectral slope results from the difference in mean free paths of the projectile and secondary ion. The ion \( j \) at \( x \) produced near \( x_m = 0 \) is less attenuated than those produced locally \( (x_m = x) \). Since those produced locally are generally of lower energy than those produced at \( x = 0 \) we obtain the spectral slope.

**DEPTH-DOSE RELATIONS**

We consider the evaluation of dose equivalent as given by eqn (4) using the present formalism. The quality factor chosen is the latest recommended values of the ICRP (1990) given by

\[
Q(L) = \begin{cases} 
1 & \text{L} < 10 \text{ keV/\mu m} \\
0.32L - 2.2 & 10 \leq \text{L} \leq 100 \text{ keV/\mu m} \\
300/L & \text{L} > 100 \text{ keV/\mu m}
\end{cases} \tag{25}
\]

It is clear from the above formalism that

\[
H_{m}(x, E') = Q[L_{m}(E_0)]A_{m}E_{m}(E_0)\exp(-\sigma_{m} x) + \sum_j A_{j} g_{jm}(x) \delta_{m}\exp(-\sigma_{m} x)
\times \int_{E_{j}}^{E_{m}} Q[L_{j}(E)] \, dE
\]

\[
+ \sum_j A_{j} b_{jm}(x) \int_{E_{j}}^{E_{m}} Q[L_{j}(E)] \left[ R_{j}(E) - \bar{r}_j \right] \, dE
\]

where \( E_{jm} \) and \( E_{m} \) are obtained from the limits in eqn (13). \( D_{m}(x, E') \) is similarly obtained from eqn (26) by setting \( Q(L) \Rightarrow 1. \)

The ion flux within a water column is represented by the eighty most important isotopes lighter than \( ^{29}\text{Ni} \). The Green’s function contains 6,400 terms as related by 3,160 fragmentation cross sections. The ion flux within the column, due to a monoenergetic incident ion flux, is evaluated using the NUCFRG database (Wilson et al. 1987; Townsend et al. 1993; Wilson et al. 1994) for the oxygen constituents and the Silberberg-Tsao (Silberberg et al. 1976) model for the hydrogen constituents. The isobaric nuclear fragmentation cross sections are shown in Fig. 1. The nuclear absorption cross sections are shown in Fig. 2 for selected ions. In the following calculations, the nuclear cross sections were evaluated at the ion beam energy at the entrance to the water column. The results of the computations using eqn (26) with \( Q \Rightarrow 1 \) for \( D_{m}(x, E') \) are shown in Fig. 3 for \( ^{20}\text{Ne} \) beams and \( ^{40}\text{Ar} \) beams in a water column with the experimental measurements of Schimmerling et al. (1989) and Lyman...
et al. (1977). Reasonable agreement is obtained in each case although there are important differences. These differences are on the order of 20% or less and are consistent with the comparison of LET spectral measurements of specific fragments (Shavers et al. 1993) measured in the $^{20}$Ne experiments. Improvement must come through more accurate nuclear models, addition of energy dependence in the nuclear cross sections, and contributions from target breakup.

The average quality factors $H_m(x, E')/D_m(x, E')$ for fixed range for beams of $^{20}$Ne, $^{40}$Ar, and $^{56}$Fe are shown in Fig. 4. The quality factor is relatively constant over the range of the primary beam except in the Bragg peak region. Beyond the Bragg peak the average quality factor for $^{20}$Ne beams drops suddenly as expected and declines slowly to larger depths since only the lighter ions penetrate to large depths. The depression of the average quality factor at the Bragg peak of the $^{40}$Ar and $^{56}$Fe beams is due to the decline of the primary ion quality factor ($Q \approx 300/V_L$) and the rapid rise in the primary ion dose at the Bragg peak.

There is experimental evidence that relative biological effectiveness depends not only on the rate at which an ion gives up energy to the tissue medium (numerically equal to LET) but depends as well on the lateral extent of the energy deposit (track structure effects). For this reason we consider a track structure cellular repair model for survival and neoplastic transformation (Wilson et al. 1991) as applied to the C3H10T1/2 mouse embryo cell culture for which there are extensive data using HZE ion beams (Yang et al. 1985, 1989). The cellular repair model is derived from the following assumptions. Injury from gamma rays follows Poisson statistics with a characteristic dose $D_o$ and enzymatic repair is less efficient with increasing number of hits. Such a model is consistent with the concept of a high efficiency and fast but saturable repair enzyme pool in competition with a slower less efficient repair process (Tubiana et al. 1990). Injury within a particle track is mediated by the secondary electrons (delta rays) and is related to the gamma-ray

injury through $D_o$. An 'inactivation' cross section associated with the core of the ion track is approximately given by

$$\sigma = \sigma_0(1 - e^{-Z^2/\kappa \beta^2})^m,$$

where $\kappa$ is a nondimensional size parameter associated with the sensitive structure within the cell, $Z$ is the ion effective charge, and $\beta$ is the ion velocity in units of the velocity of light (Katz et al. 1971). In the track periphery, the injury level is below saturation and multihit repair kinetics as in the case of gamma-ray exposures are assumed and the probability of direct inactivation is given as $p = \sigma/\sigma_0$, where $\sigma_0$ is the Katz 'saturation' cross section. The probability of injury with multihit repair kinetics is $(1 - P)$. In the low dose rate limit (Wilson et al. 1993b), the nonsurviving fraction of exposed cells is

$$n_{\alpha}(t) \approx \frac{n_{\alpha}(t)}{\alpha_1 6^{1/3} (1 - P) D_o \sigma + L D}$$

Fig. 2. Nuclear absorption cross sections of selected projectiles in water.

Fig. 3. Bragg curves for (a) Ne and (b) Ar ion beam according to present calculations and experiments.
where \( D \) is the accumulated dose, \( L \) is the linear energy transfer, \( \alpha_{m1} \) is the misrepair rate for once hit cells, \( \alpha_1 \) is the total rate at which the enzyme \((Z)\) forms a repair complex \((C')\) and completes the repair through the reaction kinetics given by

\[
Z + DNA \rightarrow C' \rightarrow Z + DNA_m
\]

where subscript \( l \) denotes injured DNA and subscript \( m \) denotes misrepair. Note that \((1 - \alpha_{m1}/\alpha_1)\) and \(\alpha_{m1}/\alpha_1\) are the branching ratios of the reaction to perfectly repaired and misrepaired states. In eqn (28), the \( L^{1/3}(1-P)D/D_0 \) is the fraction of cells with injury in the track periphery and \( \alpha_{m1}/\alpha \) is the probability the cell is permanently harmed. The fraction of cells exposed within the ion core with no chance of recovery is \( \sigma D/L \). A similar result holds for the fraction of transformed cells in the low dose rate limit with appropriate parameters for transformation. The repair rates and efficiencies (which depend on the cell cycle status) are found from the experiments of Yang et al. (1985, 1989) and the low dose rate limit parameters for resting Go phase and exponential growth phase cells are given in Table 1. Unlike conventional dosimetric analysis wherein radiation quality is represented by LET dependent factors, the repair kinetics model is driven by track-structure dependent injury coefficients (eqn 27). The \( \text{RBE}_m \) is found by finding the gamma-ray dose and ion dose which leads to the same fraction of changed cells (note \( P = 0 \) for gamma rays) as

\[
\text{RBE}_m = 1 - P + \frac{\alpha_1 \sigma D_0}{\alpha_{m1} 3 \sqrt{6} L} \quad (29)
\]

The subscript \( m \) on RBE denotes the maximum value obtained in the low dose rate limit. The \( \text{RBE}_m \) values derived from a cell kinetics model (Wilson et al. 1993b) for C3H10T1/2 cells exposed and repaired in stationary phase at the low dose rate limit are substituted for \( Q(L) \) in eqn (27) for ion beam exposures. The \( \text{RBE}_m \) for cell survival (S) and cell neoplastic transformation (T) are shown in comparison to the average quality factors. The quality factor is intended to represent stochastic processes (mainly cancer induction) and should best correlate with the RBE for neoplastic transformed cells (a pre-cancerous cell). The RBE for cell survival is more indicative of deterministic effects for which large RBEs

Table 1. Cellular kinetic ratios and parameters.

<table>
<thead>
<tr>
<th>Cellular kinetic ratio ( \alpha_{m1}/\alpha_1 )</th>
<th>Go phase</th>
<th>Exponential phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>Survival</td>
<td>~0.03</td>
<td>0.30</td>
</tr>
<tr>
<td>Transformation</td>
<td>~0.002</td>
<td>0.01</td>
</tr>
</tbody>
</table>

Katz C3H10T1/2 cell parameters

<table>
<thead>
<tr>
<th></th>
<th>( \sigma_0 ) cm(^2)</th>
<th>( \kappa )</th>
<th>( m )</th>
<th>( D_m ) Gy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Survival</td>
<td>( 5 \times 10^{-7} )</td>
<td>750</td>
<td>3</td>
<td>2.8</td>
</tr>
<tr>
<td>Transformation</td>
<td>( 7 \times 10^{-11} )</td>
<td>475</td>
<td>3</td>
<td>116</td>
</tr>
</tbody>
</table>
have been measured for mouse embryo hemopoiesis (Jiang et al. 1994) irradiated with high LET alpha particles. Many of the qualitative features of the RBE are represented by the average quality factor but important track structure dependent differences can be observed. The quality factor increases from beam entrance to the position at which the primary ion reaches 100 keV/μm and declines to greater depth as seen for neon and argon ions. The iron ions enter with LET > 100 keV/μm and the quality factor declines throughout the medium. The quality factor approaches the stochastic RBE(T) for the lighter more penetrating fragments downstream from the Bragg peak. Near beam entrance, the width of the primary track is maximum and the RBE(T) lies substantially below the LET dependent quality factor. It is accidental that RBE(S) coincides with quality factor at beam entrance for neon and argon. The diverse behavior of RBE(S) and RBE(T) is a result of the specific track structure parameters associated with survival and transformation (Table 1) and are consistent with the data of Yang et al. (Wilson et al. 1993b). The quality factor is most appropriate for representing the cell transformation RBE for lighter ions such as 20Ne and appears to be conservative for heavier ions. A quality factor for deterministic effects is not defined but would be more indicative of RBE(S). Note that the present result neglects the secondaries produced from tissue constituents (target fragments).

**CONCLUSIONS**

At this point of development of HZE transport theory, the fluence spectra of the multiple charged ions can be reasonably calculated in one dimension assuming constant nuclear cross sections. Clearly, future efforts should concentrate on deriving important corrections associated with realistic cross section variation with energy. Even with this improvement several issues remain in comparing the results to the experimental data. The target breakup will make contributions at all depths but may be a source of error near the Bragg region and beyond and especially for the lighter ion beams. Beyond this, significant improvements are required in the nuclear models used in generating the nuclear database.

**REFERENCES**

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APPENDIX

The second collision term is similarly

\[ g_{jm}^{2}(x, r_{j}, r'_{m}) = \sum_{k} \frac{\nu_{k} \sigma_{jk} \sigma_{km}}{|\nu_{m} - \nu_{k}|} \]

\[ \times \int_{r_{j}}^{r_{m}} \exp(-\sigma_{jk} x_{j} - \sigma_{km} x_{m}) \, dx_{j} \]

where

\[ x_{j} + x_{k} + x_{m} = x \]

\[ \nu_{j} x_{j} + \nu_{k} x_{k} + \nu_{m} x_{m} = \nu_{m} r'_{m} - \nu_{k} r_{k} \]

(\text{A2})

(\text{A3})

and the integral limits in eqn (A1) depend on \( \nu_{m}, \nu_{k}, \nu_{j} \) and the physical requirements that the distances \( x_{j}, x_{k}, \) and \( x_{m} \) are all positive. We have used the rather weak dependence of the exponential argument of eqns (12) and (A1) on energy to simplify spectral quantities in the past (Wilson et al. 1984). Higher order terms are similarly derived (Wilson et al. 1990). Whenever \( r_{j} r'_{m} > x \) it can be shown that eqn (12) has the property that

\[ \int_{r_{j}}^{r_{m}} g_{jm}(x, r_{j}, r'_{m}) \, dr_{j} = \sigma_{jm} g(j, m) \]

(\text{A4})

where \( g(j, m) \) is the \( g \)-function of two arguments introduced in our prior paper (Wilson et al. 1990) and the integral limits are the interval limits defined by eqn (13). The \( g \)-functions are given by the following relations

\[ g(j) = \exp(-\sigma_{j} x) \]

(\text{A5})

and

(\text{A6})

The first collision Green’s function term may be approximated by the spectral averaged value as

\[ g_{jm}^{(1)}(x, r_{j}, r'_{m}) = \frac{1}{r_{m} - r_{j}} \int_{r_{j}}^{r_{m}} g_{jm}^{(1)}(x, r_{j}, r'_{m}) \, dr_{j} \]

(\text{A7})

and is functionally dependent only on depth of penetration \( x \) (Wilson and Badavi 1992). Its contribution applies only over the energy range

\[ \frac{\nu_{j}}{\nu_{m}} (r_{j} + x) \leq r'_{m} \leq \frac{\nu_{j}}{\nu_{m}} (r_{j} + x) \]

(\text{A8})

for \( \nu_{m} > \nu_{j} \). If \( \nu_{j} > \nu_{m} \), as can happen in neutron removal collisions, the negative of eqn (A7) is used and the inequalities of eqn (A8) are reversed (Wilson et al. 1991). The higher order terms are approximated (Wilson and Badavi 1992) by their spectral averages as

\[ g_{jm}^{(i)}(x, r_{j}, r'_{m}) \]

\[ = \sum_{k_{1}, k_{2}, \ldots, k_{i}} \frac{\nu_{j} \sigma_{jk_{1}} \sigma_{jk_{2}} \ldots \sigma_{jk_{i-1}} g(j, k_{1}, k_{2}, \ldots, k_{i-1}, m)}{x(\nu_{m} - \nu_{j})} \]

(\text{A9})

Note that \( g_{jm}^{(i)}(x, r_{j}, r'_{m}) \) is purely dependent on \( x \) for \( i > 0 \), and we represent the expression as \( g_{jm}^{(i)}(x) \).