Learning Impasses in Problem Solving

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Abstract
Problem Solving systems customarily use backtracking to deal with obstacles that they encounter in the course of trying to solve a problem. This paper outlines an approach in which the possible obstacles are investigated prior to the search for a solution. This provides a solution strategy that avoids backtracking.

Introduction
Many weak methods of problem solving are based upon the idea that a problem can be solved by choosing a sequence of goals and satisfying them in some order. GPS (Newell and Simon 1972) was amongst the first to set out this approach. Since then the work of Ernst and Goldstein (Ernst and Goldstein 1982), Korf (Korf 1985), and Guvernir (Guvernir 1987) has built upon this idea. The culmination of this kind of approach is, in some ways, the Soar system, which through the creation of a large production system with learning capabilities is able to incorporate many of the weak problem solving systems into a single system.

If one compares Soar and Korf's system they take quite different approaches to the problem of what should be learned and when it should be learned. Korf's system is able to specify in advance exactly what macros it needs to learn. This will yield benefits in the system's ability to determine which macro to use at a given point in the solution, at the price of requiring long searches for some of the more complex macros. Soar on the other hand learns only the solutions to the difficulties that actually arise. This conservative attitude toward learning means that the system can encounter problems in matching expensive chunks that do not arise in Korf's situation.

This paper looks for a half way house between these two strategies. We would like to obtain the benefits of easier pattern matching afforded by Korf's system without having to pay the price of the large amount of search that his system needs. Our approach is to show that for a substantial number of problems one can anticipate the impasses that will be encountered by a problem solver. These can then be modeled and solved in small pieces of the larger problem, thus avoiding the deep searches required in Korf.

Problems, Strategies and Impasses
We review briefly the definitions that we will need. A consensus as to the appropriate definitions seems to be emerging (Banerji 1983), (Benjamin et al. 1989), and (Nilsuma and Kitahashi 1985). Our definitions follow this trend. Some of them have appeared previously in (Hodgson 1989).

Problems and Subproblems
Our definition of a problem is based upon the idea of an action.

Definition 1 A free problem $P$ is a triple $(S, \Omega, a)$ where $a$ is a partial map

$$a : S \times \Omega \to S$$

The set $S$ is called the state space of the problem and the set $\Omega$ is the move set of the problem. The map $a$ represents the effect of the moves on the state space. The effect of a move $\omega$ on the state $s$ is to give the state $a(s, \omega)$. The element $a(s, \omega)$ fails to exist precisely when $(s, \omega)$ is not in the domain of $a$; that is, when $\omega$ cannot be applied to the state $s$. A sequence $\Sigma = (\omega_1, \ldots, \omega_k)$ of moves on $P$ is called admissible at $s$ if the composition

$$a(s, \Sigma) = a(a(\ldots(a(s, \omega_1), \omega_2), \ldots, \omega_k)\ldots)$$

exists.

We need a notion of maps between problems.

Definition 2 Given two problems $P_1 = (S_1, \Omega_1, a_1)$ and $P_2 = (S_2, \Omega_2, a_2)$, with a pair of maps $f : S_1 \to S_2$ and $g : \Omega_1 \to \Omega_2$ The pair $(f, g)$ defines a strict homomorphism $F : P_1 \to P_2$ provided that

1. Given two points $s_1$ and $s_2$ such that $f(s_1) = f(s_2)$, then if the move $\omega$ applies to $s_1$ it also applies to $s_2$ and

2. The equation

$$f(a_1(s, \omega)) = a_2(f(s), g(\omega))$$

is satisfied in the sense that whenever the right-hand side exists so does the left-hand side.
A strict homomorphism $F$ is a monomorphism if the underlying maps $f$ and $g$ are one to one.

We now turn to the notion of (strict) isomorphism.

**Definition 3** Two problems $P_1$ and $P_2$ are strictly isomorphic if there exists a pair of mutually inverse strict homomorphisms $F : P_1 \Rightarrow P_2$ and $H : P_2 \Rightarrow P_1$ between them.

We can use monomorphisms in a natural way to define subproblems.

**Definition 4** Let $P = (S, \Omega, a)$ be a problem. A problem $P_0 = (S_0, \Omega_0, a_0)$ is a (strict) subproblem of $P$ if there exists a problem monomorphism $(f, g) = F : P_0 \Rightarrow P$.

It is worth noting that the requirement that the underlying state map be a monomorphism has the effect that even the weaker definitions of isomorphism for underlying maps such as the weak homomorphism of Niitsuma and Kitahashi (Niitsuma and Kitahashi 1985) lead to the same subproblems.

As an example of the concepts developed here we can take the sliding tile puzzles. In particular we might take the fifteen puzzle. The state space is then the set of all legal arrangements of the fifteen tiles and the blank in the 4 x 4 array. The move set can be given by the set $\Omega = \{U, D, L, R\}$ where the letter indicates the direction in which a tile is moved. A typical subproblem can be obtained by restricting one's attention to the tiles in the top half, (assuming that the blank lies in the top half). Moves on this subproblem are restricted to be those in which the blank remains in the top half of the array.

**Strategies**

So far we have not recognised that problems are supposed to represent things that are to be solved. To do this we define a problem instance for a problem $P$ as a triple $(P, \sigma, G)$ where $\sigma$ is a state of $P$ called the start state, and $G$ is a subset of the state space called the goal set. A solution to the problem instance is a sequence $\Sigma$ of moves which is admissible at $\sigma$ and such that $a(\sigma, \Sigma) \in G$.

Informally a strategy is a sequence of intermediate subproblem instances between the initial state and the goal. We can distinguish two classes of strategies. In the first the successive state spaces overlap; we call these ample strategies. In the other the successive state spaces are disjoint; we call these abutment strategies.

**Definition 5** An ample strategy for a problem instance $(S, \Omega, a, G)$ is a sequence $\{P_0, \ldots, P_k\}$ of subproblems of $P = (S, \Omega, a)$ such that the state spaces of successive subproblems have non-trivial intersection, that is $S_{i-1} \cap S_i \neq \emptyset \forall i \in 1, \ldots, k$. Furthermore $\sigma \in S_0$ and $G \subseteq S_k$.

An abutment strategy for a problem instance is a sequence $\{P_0, \ldots, P_k\}$ of subproblems of $P$ such that

1. $\sigma \in S_0$,
2. $G \subseteq S_k$,
3. $S_{i-1} \cap S_i = \emptyset \forall i \in 1, \ldots, k$,
4. for each $i \in 1, \ldots, k$ there exists at least one pair $(x_{i-1}, x_i)$ of points of $S$ such that there is a move $\omega \in \Omega$ for which $a(x_{i-1}, \omega) = x_i$.

A solution is based upon a strategy if it is obtained by concatenating a sequence of solutions to the intermediate subproblems.

We illustrate the two kinds of strategies with examples. For our example of an ample strategy we consider the case of Fool's disk. This problem has been discussed by Ernst and Goldstein (Ernst and Goldstein 1982). It consists of four concentric rings each of which is free to rotate about the common center. Each ring has eight numbers on it, appearing at 45 degree intervals around the ring. The goal of the problem is to rotate the rings so that the sum of each radius is 12. The standard strategy is as follows:

- By using only rotations through 45 degrees, make the sum on each pair of perpendicular diameters 48. $P_0$ thus has the same state space as $P$ but a smaller move set.
- By using only rotations through 90 degrees, make the sum along each diameter 24. $P_1$ has as state space a set of states in which the sum on each pair of perpendicular diameters is 48. The move set is again a subset of the original one.
- By using only rotations through 180 degrees make the sum along each radius 12. $P_2$ has as state space a set of states in which the sum along each diameter is 24 and once again the move set is a subset of the original.

This strategy, when successful (about which more later), reduces the amount of search from $8^3$ moves (the center ring can be kept fixed) to $8 \times 3$ moves.

Our second example is an elegant solution of the five puzzle that has been presented by Banerji (Banerji 1990). He observes that there is a way to represent the states of the five puzzle by points on the faces of a dodecahedron. The sequence of moves that circulates the blank around the circumference of the the puzzle moves through all the states on one face. Passage from one face to another is effected by the moves that slide the blank up and down in the centre column. The strategy in this case consists of choosing the sequence of faces (each of which is a subproblem) through which one must pass from the start to the goal.

There is an important difference between these two examples. In the second example once the strategy is chosen no backtracking over the solutions to the intermediate problems is necessary but in the case of the fool's disk it may be necessary to backtrack since it is possible that the first arrangement in which the sum on all the diameters is 24 may not lead to a solution and another arrangement is needed. Niitsuma and Ki-
Proposition 6 Suppose that for each subproblem occurring in a strategy it is the case that any instance of the subproblem can be solved then no backtracking will be needed to construct a solution to the original problem following the strategy.

It may seem that the restriction on the state spaces of the intermediate problems is unduly restrictive. Yet it is exactly this that is needed to avoid backtracking. Thus one aim of our approach is to find strategies for which this hypothesis is true.

Impasses

At any stage in the execution of the strategy one has a subproblem instance \((P_1, \Omega_1, a_1, G_1)\) where in the case of a Soar strategy \(G_1 = S_i \cap S_{i+1}\) or in the case of an abutment strategy \(G_1 = \text{set of points} S_i\) from which a move to \(S_{i+1}\) is possible. We have seen that the strategy proceeds smoothly as long as these intermediate problems can be solved.

Definition 7 An intermediate problem for which there is no solution is called an impasse for the strategy.

This terminology follows one case of the use of the term in the Soar system, in so doing we are also following the usage of Ruby in (Ruby and Kibler 1989).

It is important to note that our definition of an impasse in a problem is dependent upon the strategy chosen to solve the problem. Thus in the Banerji solution to the five puzzle there are no impasses since each intermediate goal is attainable. By contrast in the more usual method in which the tiles are brought into position in a prearranged order there are impasses.

Learning the Impasses

Our approach to finding impasseless strategies is to improve an existing strategy by modifying the subproblems so that they do not contain any impasses. As an example we show that in the case of Fool's disk we can do this by enlarging the intermediate problems. This need not always be the case as we shall see in some of the examples that we discuss.

For the Fool's disk case we consider the intermediate problems defined as follows:

- By using only rotations through 45 degrees, make the sum on each pair of perpendicular diameters 48. \(P_0\) thus has the same state space as \(P\) but a smaller move set.
- Let \(P_1\) have as state space the set of all states in which the sum on each pair of perpendicular diameters is 48. The move set is again a subset of the original one. It may contain some moves through 45 degrees.
- Let \(P_2\) have as state space the set of all states in which the sum along each diameter is 24. The move set may contain moves through 90 or even 45 degrees.

It is clear that for these problems no backtracking into earlier problems is necessary.

Finding the Impasses

Problem solvers such as Soar (Laird et al. 1986) and the stepping stone method (Ruby 1989) find the impasses in the course of attempting to satisfy the current goal. A search procedure is then invoked to resolve the impasse and the resolution of the impasse becomes part of the problem solver's knowledge about the problem. This is an accurate representation of much human problem solving, but it does not tell the whole story. Often faced with a problem a human will actively consider the difficulties that may arise in the course of the resolution of the problem to see if they can be solved. One advantage of such an approach offers is that it allows one to take advantage of efficient storage techniques once one has determined that a small group of chunks will be adequate to solve the problem. This addresses some of the problems of expensive chunks (Tambe et al. 1990).

We give here a recognition criterion that forms the basis for an algorithm that can be used to produce impasses in problems. The criterion will be stated for the cases where the strategy is based upon the idea of reducing a set of features to goal values. We begin by formalising this notion.

Given a problem \(P\) a feature on \(P\) is a map

\[ f : S \rightarrow \mathcal{T}(f) \]

between the state space of \(P\) and some finite set \(\mathcal{T}(f)\) called the target of \(f\). A set \(\mathcal{F}\) of features is called discriminating if given any two state \(s_0\) and \(s_1\) of \(P\) there is some feature \(f \in \mathcal{F}\) such that \(f(s_0) \neq f(s_1)\). The set is called adequate for a goal \(G\) if given any feature \(f\) which is not a goal state there is some feature \(f\) such that \(f(s)\) is not a member of \(f(G)\).

The strategy associated to an ordering \(\{f_1, \ldots, f_k\}\) of a set of adequate features is the sequence of subproblems \(P_i = (S_i, \Omega, \alpha, \sigma_i, S_{i+1})\) where \(S_i\) is the set of all states for which the features \(f_1, \ldots, f_{i-1}\) have goal values, \(\alpha_i\) is the restriction of \(\alpha\) to \(\alpha^{-1}S_i \cap (S_i \times \Omega)\). For these strategies we can give a recognition criterion for impasses.

Proposition 8 Let \(P_i = (S_i, \Omega, \alpha, \sigma_i, S_{i+1})\) be an intermediate problem for a strategy based upon an adequate set of features, then \(P_i\) is an impasse instance if either

- No move changing the value of \(f_i\) applies to \(\sigma_i\), or
- Any sequence of moves on \(P\) that reduces \(f_i\) from \(\sigma_i\) must change the value of at least one of \(f_1, \ldots, f_{i-1}\).
From this point forward the argument goes as follows. First, find an impasse. Second, produce a "smaller" example of the same impasse. Thirdly, expand the example to a subproblem in which the impasse can be resolved. Finally, show that the problem has a strategy based upon the new set of subproblems.

Examples of Impasses
To obtain an impasse of the first kind we can turn to Sussman's anomaly in the blocks world. The impasse can be succinctly described by the following figure.

![Sussman's Anomaly Diagram]

Although one can get "closer" to the goal by putting B on top of C in the position on the left hand side it will be necessary to undo this since the goal of putting A on B requires that the top of A be clear. Thus no move that will achieve the desired position for A is available.

To get an example of the second kind we consider the fifteen puzzle with the initial strategy of moving the tiles into position in the order 1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 10, 14, 11, 12, 15 (the ordering at the end is chosen to be a good one, we do not need to go this far though).

These are chart and atlas. The idea is that chart are pieces of a problem that are all modeled on some common subproblem. The important charts will be the ones that contain the resolutions of impasses.

Definition 9 Let P be a problem and s a state in P. Then a chart for P based upon a problem P₀ is a problem monomorphism P₀ → P whose image contains s.

An atlas for a problem P is a finite collection A of charts such that every point in the state space of P is in the image of some chart of A.

We define the images of two charts f₁ : P₁ → P and f₂ : P₂ → P to be incident if either

1. f₁(P₁) ∩ f₂(P₂) contains at least one move common to both subproblems, or
2. there exists a state s₁ ∈ f₁(P₁) and a state s₂ ∈ f₂(P₂) such that there is a reversible move ω with a(s₁, ω) = s₂.

The abstraction of a problem associated to an atlas is the graph whose vertices correspond to the embedded charts of the atlas with an edge between each pair of incident charts. A sequence of pairwise adjacent charts is called a chain.

We will want to distinguish between two types of abstraction. An abstraction in which the charts overlap will be called an ample atlas. One in which all the charts are incident but do not overlap will be called an absent atlas.

We give two examples of abstractions associated to an atlas. The first is based upon the earlier solution of the five puzzle. Here the charts consist of the images of the sub-problem of the five puzzle consisting of those states that are obtainable by moving the blank around the circumference of the puzzle. As Banerji remarks
(Banerji 1990) this represents the accessible states of the five puzzle on the faces of a dodecahedron. The faces of this are the points of the abstraction and the edges (which correspond to the move of the blank up or down in the middle column) correspond to the edges.

We can obtain an abstraction of the blocks world by "welding together" adjacent blocks so that we have only three big blocks to consider. Each big block is itself a blocks world and the three block world already contains the generating example of the impasse.

These two examples suggest that the correct choice of an atlas will allow one to give an impasse free strategy.

The Atlas Meta-strategy

Atlases serve as abstractions of a problem. Given a problem instance and an atlas on a problem we can define a problem instance on the atlas. The problem is to find a chain joining a chart containing the start position to a chart whose image intersects the goal.

Definition 10

Given an impasse \( I = (P_0, G_0, a, s_0, G_0) \) on a problem, a chart \( f : P_1 \rightarrow P \) is said to resolve the impasse if there is a monomorphism of \( P_0 \) into \( P_1 \) and if the instance \( I \) can be solved in \( P_1 \).

The main ideas of this paper can be summed up in the following.

Proposition 11 Let \( \{P_0, \ldots, P_k\} \) be a strategy for a problem \( P \) and let \( I \) denote the set of impasses for this strategy. Let \( \{Q_1, \ldots, Q_n\} \) be a set of charts of \( P \) such that each impasse is resolved in at least one of the charts \( Q_i \). There is an atlas \( A \) based upon the charts \( Q_i \) whose associated meta-strategy gives impasseless strategies for \( P \).

The next section outlines a proof of this result and to a result on the length of the solutions that it produces.

Solutions and Their Length

The ideas required to construct the impasseless strategy are outlined below. The details have been worked out for the sliding tile puzzles, the Tower of Hanoi and the blocks world but in a manner that is somewhat problem dependent. Future work involves unifying the implementation so that it applies in a more problem independent way.

Resolving the Impasses

Let \( P = (S_1, G_1, a_S, s_1, S_{1+1}) \) be an impasse arising from the strategy based upon the set \( \{f_1, \ldots, f_k\} \) of features on a problem \( P \). The following sequence of steps is used to resolve the impasse.

SHRINK

The goal of this step is to remove from consideration those features that are not required to construct the impasse. In general given a set of features on a problem we can restrict to the moves that affect only these features. The required shrinking takes place by eliminating the features which are both fixed and whose value does not figure in the creation of the impasse.

ENLARGE

Moves that effect the remaining features are now added to produce a subproblem in which the impasse can be solved. At each step the move added should affect the smallest possible number of additional features.

An Example

We can illustrate this process with the example of the fifteen puzzle. We saw that an impasse can be reached when the first three tiles have been placed. The SHRINK process reduces this to an example equivalent to a three puzzle in which the tiles appear in the order 3, x, 4, blank, when read clockwise from the top left hand corner, (x denotes one of the possible tile values other than those already used.) We can then EXPAND to a five puzzle, which can be either horizontal or vertical in which the impasse is resolvable.

The next step is to determine whether there is an atlas for the problem whose charts are isomorphic to the set of subproblems obtained by resolving the impasses. If this is the case we then replace the original strategy by the following one. We suppose as before that we have a problem \( P \) with an adequate set of features \( \{f_1, \ldots, f_n\} \). In addition we assume that there is an atlas \( A \) whose charts are isomorphic to the impasse resolving subproblems obtained by the process outlined above.

Using the same ordering of features that was used for the original strategy that produced the impasses.

1. Set as the current subgoal the reduction of the next feature to its goal value.
2. As each feature comes up for reduction find a chain of minimal length joining the current state to a state in the current subgoal.
3. Extract the move sequence joining the current point to one in which the feature has been reduced.

Since the atlas contains a resolution of all impasses this method will solve the problem whenever there is in fact a solution.

The Length of a Solution

We can now give an estimate for the length of a solution found using this method. We need some preliminary definitions.

\( L \) will stand for the maximum chain length required to perform the reduction of a feature.

\( D \) will stand for the maximum distance between two states in a chart. When a particular chart \( C \) is referred to we will use \( D(C) \) for the distance on the chart. Note that this number can be infinite if the chart is an impasse chart.
will be the number of features on the problem.

\[ n \] will be the number of chains required to reduce all the features.

The first result is the following.

**Theorem 12** Let \( P \) be a problem with an ample atlas and features with values of \( L, D, n \) as above. Then the algorithm given above finds a solution of length at most \( L \times D \times n \).

Proof. For each feature the length of chain required to reduce it does not exceed \( L \), furthermore one each component of the chain the length of the move sequence required is less than \( D \).

The corresponding result for abutment atlases is the following. The proof is similar.

**Theorem 13** Let \( P \) be a problem with an abutment atlas and features with values of \( L, D, n \) as above. The algorithm above supplies a solution of length at most \( n(x \times D + L - 1) \).

Although these results are quite simple they give quite good estimates. For example in the case of the fifteen puzzle if we use the estimate of 22 as the maximum distance on the five puzzle (Banerji 1990) we get an estimate of \((22 \times 3 \times 15) + 3\) for the length of a solution. A more perspicuous version of the argument yields \((19 \times 22) + 3\).

### Summary and Conclusions

This paper has presented a method for solving problems that constructs the impasses associated to an initial strategy in order to be able to find a new strategy in which impasses will not arise.

The method can be applied to produce short solutions to the sliding tile puzzles as well as to the blocks world. Though the implementation is at this stage still very problem dependent. Future work will produce a version that is more general.

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### References