Least Squares Best Fit Method for the Three Parameter Weibull Distribution: Analysis of Tensile and Bend Specimens With Volume or Surface Flaw Failure

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Summary

Material characterization parameters obtained from naturally flawed specimens are necessary for reliability evaluation of nondeterministic advanced ceramic structural components. The least squares best fit method is applied to the three parameter uniaxial Weibull model to obtain the material parameters from experimental tests on volume or surface flawed specimens subjected to pure tension, pure bending, four point or three point loading. Several illustrative example problems are provided.

Introduction

The objective of this report is to apply the least squares best fit (LSBF) method to evaluate the parameters used in the uniaxial Weibull three parameter model. These parameters, scale factor $G_0$, Weibull modulus $m$, and threshold (location) parameter $G_u$, are material dependent. Weibull two or three parameter models are used to specify a probabilistic distribution for monolithic ceramic materials. The success in the use of the two parameter model rather than the three parameter model depends on the importance of ignoring the threshold (location) parameter. Disregarding the threshold parameter is conservative and simplifies matters. This simplification can be justified only by comparing the predicted behavior of a component with its observed performance.

Equations are developed to obtain the three material parameters from inert volume or surface flawed data. Inert data imply fast fracture (no subcritical crack growth). The inert data are obtained from experimental tests on specimens subjected to either pure tension, pure bending, and four or three point loading (fig. 1). Ideally the data are obtained under conditions representative of the service environment.

Several applications are presented in the section entitled EXPERIMENTAL APPLICATIONS. Experimental data are analyzed for volume flaw failure of silicon nitride (SNW-1000) specimens tested in four point bending (ref. 1). In addition, analysis is made of surface flaw failure data of silicon carbide specimens, annealed in both the longitudinal and transverse direction and tested in three point bending (Private communication from Sung Choi and Jonathan Salem, NASA Lewis Research Center). The four point bend volume flaw data are also used for a four point bend surface flaw analysis to illustrate the application of the developed equations. It is realized that these data are not representative of the physical problem.

Symbols

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<thead>
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<td>$L$</td>
<td>beam length</td>
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<td>$m$</td>
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<td>volume in tension</td>
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<td>$W$</td>
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<td>$x,y,z$</td>
<td>Cartesian coordinates</td>
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<tr>
<td>$\delta_1, \delta_2, \delta_3$</td>
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<td>stress distribution in the specimen j at fracture</td>
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<td>$\sigma_{f_{\text{max}}}$</td>
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<td>computed maximum principal stress in specimen j based on $P_f$ and assumed material parameters</td>
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<td>$G_0$</td>
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<td>$G_u$</td>
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<td>$\sigma_0$</td>
<td>characteristic strength</td>
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Subscripts

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<th>Assumed</th>
<th>Computed</th>
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<tr>
<td>assumed</td>
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Analysis Based on Three Parameter Uniaxial Weibull Model

The three parameter uniaxial Weibull model is used to describe the material inert strength probabilistic distribution. For both volume and surface flawed specimens, least squares best fit (LSBF) methods are developed to obtain the three material parameters from experimental tests on pure tension, pure bending, and four or three point loaded specimens. The necessary and sufficient condition for a solution is satisfied when the three computed parameters produce the lowest value of the sum of the residuals squared, that is, when
\[
\sum_{j=1}^{n} \left( \sigma_{\text{comp}}(x,y,z) - \sigma_{\text{max}} \right)^2 = \text{minimum},
\]
where \(n\) is the number of specimens tested. The \(P_j, \sigma_{\text{max}}\) data points are obtained from the experimental tests where \(P_j = (j - 0.3)(n + 0.4)\), \(P_j\) and \(\sigma_{\text{max}}\) are, respectively, the probability of failure and maximum principal tensile stress in the \(j\)th specimen at failure. \(\sigma_{\text{comp}}\) is the computed maximum failure stress based on the value of \(P_j\) and the computed inert strength material parameters.

Pure Tension (Fig. 1(a)), Volume Flaws

\[
P_j = 1 - \exp \left[ - \int V_j \left( \frac{\sigma_{\text{max}}(x,y,z) - \sigma_{\text{uv}}}{\sigma_{\text{ov}}} \right)^{m_v} \right] (1)
\]

\[
\sigma_{\text{max}}(x,y,z) \geq \sigma_{\text{uv}}
\]

where \(V_j\) is the volume in tension of the \(j\)th specimen with a stress distribution throughout the volume denoted by \(\sigma_{\text{max}}(x,y,z)\), \(\sigma_{\text{uv}}\) is the threshold stress, \(\sigma_{\text{ov}}\) is the scale factor, and \(m_v\) is the Weibull modulus. For this case \(\sigma_{\text{max}}(x,y,z) = \sigma_{\text{max}}\) and the tensile gage volume of specimen \(j\) is \(V_j = \frac{L \cdot b \cdot W_j}{2}\). Hence

\[
\ln \left[ \frac{\ln(1 - P_j)}{V_j} \right] = m_v \ln(\sigma_{\text{max}} - \sigma_{\text{uv}}) - m_v \ln(\sigma_{\text{ov}}) (2)
\]

The following system of \(n\) linear equations is solved in a LSBF sense:

\[
\begin{bmatrix}
\ln(1 - P_{j1})^{-1} \\
\ln(1 - P_{j2})^{-1} \\
\vdots \\
\ln(1 - P_{jn})^{-1}
\end{bmatrix} =
\begin{bmatrix}
\ln(\sigma_{\text{max}} - \sigma_{\text{uv}}) \\
\ln(\sigma_{\text{max}} - \sigma_{\text{uv}}) \\
\vdots \\
\ln(\sigma_{\text{max}} - \sigma_{\text{uv}})
\end{bmatrix}
\begin{bmatrix}
m_v \\
m_v \\
\vdots \\
-m_v \ln(\sigma_{\text{ov}})
\end{bmatrix}
\]

In matrix notation \(\{Y\} = [A] \{X\}\), where the \(j\)th term in the column vector \(\{Y\}\) is \(y_j = \ln \left[ \frac{\ln(1 - P_j)}{V_j} \right]\) and vector \(\{X\} = \left\{ m_v, -m_v \ln(\sigma_{\text{ov}}) \right\}\). The equation that must be satisfied to obtain the LSBF solution is

\[
\{X\} = [A^T A]^{-1} [A^T] \{Y\} (4)
\]

where superscript \(T\) defines the transpose.

The answer is obtained in the following manner: Assume a value for \(\sigma_{\text{uv}}\), and solve for \(m_v\) and \(\sigma_{\text{ov}}\). With these values, compute the model failure stresses \(\sigma_{\text{comp}}\) at all of the \(n\) \(P_j\) data points, where

\[
\sigma_{\text{comp}}(x,y,z) = \sigma_{\text{ov}} \left[ \frac{\ln(1 - P_j)}{V_j} \right]^{m_v} + \sigma_{\text{uv}} (5)
\]

Evaluate the sum of the squares of the residuals, where

\[
\text{Sum} = \sum_{j=1}^{n} \left( \sigma_{\text{comp}}(x,y,z) - \sigma_{\text{max}} \right)^2 (6)
\]

Repeat the process for another value of \(\sigma_{\text{uv}}\). Compute the new sum of the squares of the residuals (eq. (6)). Continue until the parameters \((m_v, \sigma_{\text{ov}}, \sigma_{\text{uv}})\) produce the minimum value of the sum of the residuals squared.
Pure Tension (Fig. 1(a)), Surface Flaws

\[ P_f = 1 - \exp \left[ - \int_{A_{Tj}} \left( \frac{\sigma_f(x,y,z) - \sigma_{us}}{\sigma_{os}} \right)^{m_s} \, dA \right] \]

(7)

\[ \sigma_f(x,y,z) \geq \sigma_{us} \]

For pure tension, \( \sigma_f(x,y,z) = \sigma_{f_{\text{max}}}, \) and the area in tension is \( A_{Tj} = 2 L_2(W_j + b_j) \) where \( L_2 \) is the gage length, \( W \) is the width, and \( b \) is the thickness. Hence,

\[ \ln \left[ \frac{\ln(1 - P_f)}{A_{Tj}} \right]^{1/m_s} \left( \frac{\ln(1 - P_f)}{A_{Tj}} \right)^{-1/m_s} = \ln(\sigma_{f_{\text{max}}} - \sigma_{us}) - \ln(\sigma_{os}) \]

(8)

Equation (8) is the basis of a LSBF evaluation of the Weibull parameters. From the inert data, a set of \( n \) linear equations is obtained. In matrix notation \( \{Y\} = [A] \{X\} \) where the \( j \)th term of the column vector \( \{Y\} \) is \( \ln \left[ \frac{\ln(1 - P_f)}{V_{Tj}} \right]^{1/m_v} \left( \frac{\ln(1 - P_f)}{A_{Tj}} \right)^{-1/m_v} = \ln(\sigma_{f_{\text{max}}} - \sigma_{uv}) - \ln(\sigma_{ov}) \)

(9)

The \( j \)th term of the column vector \( \{Y\} \) is \( \ln \left[ \frac{\ln(1 - P_f)}{V_{Tj}} \right]^{1/m_v} \left( \frac{\ln(1 - P_f)}{A_{Tj}} \right)^{-1/m_v} \) and the vector \( \{X\} \) is \( \left[ \begin{array}{c} m_s \ln(\sigma_{f_{\text{max}}} - \sigma_{us}) - m_s \ln(\sigma_{os}) \\ -m_v \ln(\sigma_{ov}) \end{array} \right] \). The matrix \([A]\) is the same as in equation (3). Assume \( \sigma_{ov}, \) and solve for \( m_s \) and \( \sigma_{us} \). With these values, compute the failure stresses, \( \sigma_{f_{\text{max}},\text{comp}} \) for all \( n \) specimens. The computed failure stress for the \( j \)th specimen is

\[ \sigma_{f_{\text{max},\text{comp}}} = \sigma_{os} \left[ \frac{\ln(1 - P_f)}{A_{Tj}} \right]^{1/m_s} + \sigma_{us} \]

(10)

Evaluate equation (6), the sum of the residuals squared. Continue the process for another value of \( \sigma_{us} \). Compute the sum of the residuals squared. Continue until the parameters \( (m_s, \sigma_{us}, \sigma_{ov}) \) produce the minimum value of the sum by equation (6).

Pure Bending (Fig. 1(b)), Volume Flaws

Substituting the expressions \( dV = L_2b_j \, dy \) and \( \sigma_f(x,y,z) = \sigma_{f_{\text{max}}} \) into equation (1) results in

\[ (1 - P_f)^{-1} = \exp \left[ \frac{1}{\sigma_{ov}} \right]^{m_v} \frac{b_j L_2 \int 2 \sigma_{f_{\text{max}}} - y}{\delta_{ij} W_j} dy \]

\[ = \exp \left[ \frac{1}{\sigma_{ov}} \right]^{m_v} \frac{V_{Tj}}{1 + m_v} \left( \frac{\sigma_{f_{\text{max}}} - \sigma_{uv}}{\sigma_{f_{\text{max}}}} \right)^{1/m_v} \]

(11)

The next assumed value is \( \sigma_{f_{\text{max}},\text{assumed}} = 0.5 \left[ \sigma_{f_{\text{max}}} + \sigma_{f_{\text{max,previous assumed}}} \right] \). Repeat this process until \( \sigma_{f_{\text{max},\text{comp}}} \) is within some specified tolerance of \( \sigma_{f_{\text{max}},\text{assumed}} \). Then compute the sum of the residuals squared by means of equation (6). Repeat all of the previous steps until the minimum value of the sum of the residuals squared is obtained. The parameters \( (m_v, \sigma_{ov}, \sigma_{uv}) \) associated with this minimum are the solution.

Pure Bending (Fig. 1(b)), Surface Flaws

From equation (1)

\[ (1 - P_f)^{-1} = \exp \left[ \frac{1}{\sigma_{os}} \right] \int_{A_{Tj}} \left( \frac{\sigma_f(x,y,z) - \sigma_{us}}{\sigma_{f_{\text{max}}}} \right)^{m_v} \, dA_j \]

(12)
where $A_{Tj}$ is the tensile surface area of specimen $j$. Therefore, considering both the side and bottom surfaces of the specimen yields

\[
(1 - P_{fj})^{-1} = \exp \left( \frac{1}{\sigma_{os}} \int_{0}^{\frac{L_2}{2}} \left( \sigma_{fj}(x,y,z) - \sigma_{us} \right) \, dx \right) \times dy + \frac{b_j L_2 \left( \sigma_{f_{j\max}} - \sigma_{us} \right) \, dx}{2} \left( 1 + m_v \right)
\]

(14)

where

\[
\sigma_{fj}(x,y,z) = \frac{2y \sigma_{f_{j\max}}}{W_j}
\]

and

\[
\delta_{ij} = \frac{\sigma_{us} W_j}{2 \sigma_{f_{j\max}}}
\]

Thus

\[
\ln \left( \frac{1}{L_2 W_j} \right) - \ln \left( \frac{1 - \sigma_{us}}{\sigma_{f_{j\max}}} \right) + \frac{b_j L_2 (1 + m_v)}{W_j}
\]

\[
= m_v \ln(\sigma_{f_{j\max}} - \sigma_{us}) - \ln \left[ \sigma_{os}^m (1 + m_v) \right]
\]

(15)

Solve in a LSBF sense the set of linear equations obtained by means of equation (15) and denoted in matrix form by $\{Y\} = [A] \{X\}$. The $j$th term of the column vector $\{Y\}$ is

\[
\ln \left( \frac{1}{L_2 W_j} \right) - \ln \left( \frac{1 - \sigma_{us}}{\sigma_{f_{j\max}}} \right) + \frac{b_j L_2 (1 + m_v)}{W_j}
\]

and the vector $\{X\} = \left[ - \ln \left[ \sigma_{os}^m (1 + m_v) \right] \right]$. The LSBF parameters are obtained in the following way: Assume $\sigma_{us}$, and keep this value fixed. To evaluate the vector $\{Y\}$, assume a value for $m_v$ ($m_v$ assumed) based on the two parameter solution. Evaluate the column vector $\{Y\}$ and the matrix $[A]$. Solve for the vector $\{X\}$ by equation (4). Compare the computed value of the Weibull modulus $m_{s,\text{comp}}$ with $m_{s,\text{assumed}}$. Repeat this process until both values, $m_{s,\text{comp}}$ and $m_{s,\text{assumed}}$, are within some specified tolerance. With these parameters ($m_v$, $\sigma_{f_{j\max}}$, $\sigma_{us}$) and $P_{fj}$, compute the $n$ values of $\sigma_{f_{j\max,\text{comp}}}$ where

\[
\sigma_{f_{j\max,\text{comp}}} = \sigma_{os} \frac{(1 + m_v)(1 - P_{fj})^{-1}}{L_2 W_j \left( 1 - \frac{\sigma_{us}}{\sigma_{f_{j\max,\text{assumed}}} \right) + b_j L_2}
\]

(16)

Assume a value for $\sigma_{f_{j\max,\text{assumed}}}$, and iterate until $\sigma_{f_{j\max,\text{comp}}}$ is within some specified tolerance of $\sigma_{f_{j\max,\text{assumed}}}$ Obtain the sum of the residuals squared by equation (6). Repeat the process. The parameters that produce the minimum value of the sum of the residuals squared are the solution.

**Four Point Bend Specimen Fig. 1(c), Volume Flaws**

Substituting the inner span and outer span stress distributions $\sigma_{fj}(x,y,z) = 2 \sigma_{f_{j\max}} y/W_j$ and $\sigma_{f_{j\max}}(x,y,z) = 4 \sigma_{f_{j\max}} Xy/(L_1 W_j)$ into equation (1) results in

\[
P_{fj} = 1 - \exp \left( - \frac{L_1 b_j W_j}{2(1 + m_v)} \left( \frac{\sigma_{f_{j\max}}}{\sigma_{os}} \right)^{m_v} \right) \frac{1}{\delta_{ij}} \left( \frac{2y - \sigma_{uv}}{\sigma_{f_{j\max}}} \right)^{1+m_v} dy
\]

(17)

where $\delta_{ij} = \sigma_{uv} W_j/(2 \sigma_{f_{j\max}})$. From equation (17) we obtain

\[
\ln \left( \frac{1}{L_2 W_j} \right) - \ln \left( \frac{1 - \sigma_{us}}{\sigma_{f_{j\max}}} \right) + \frac{b_j L_2 (1 + m_v)}{W_j}
\]

\[
+ \frac{L_2}{L_1} \left( 1 - \frac{1}{\sigma_{f_{j\max}}} \right)^{1+m_v} = m_v \ln(\sigma_{f_{j\max}} - \ln[\sigma_{os}^m (1 + m_v)]
\]

(18)

Solve the set of linear equations obtained by equation (18), denoted in matrix form by $\{Y\} = [A] \{X\}$. For constant values $L_1$ and $L_2$, the $j$th term of column vector $\{Y\}$ is

\[
y_j = \ln \left( \frac{1}{L_2 W_j} \right) - \ln \left( \frac{1 - \sigma_{us}}{\sigma_{f_{j\max}}} \right) + \frac{b_j L_2 (1 + m_v)}{W_j}
\]

(19)
The $j$th row of matrix $[A]$ is $\begin{bmatrix} \ln(\sigma_{f_{\text{max}}}) & 1.0 \end{bmatrix}$ and $\{X\} = \begin{bmatrix} m_v \\ -\ln(\sigma_{f_{\text{max}}}(1 + m_v)) \end{bmatrix}$. Assume a value for the threshold stress $\sigma_{\text{uv}}$ and an appropriate value for the Weibull modulus $m_v$, assumed (based on the two parameter solution). Evaluate column vector $\{Y\}$ and matrix $[A]$. Solve for solution vector $\{X\}$ by equation (4). With $\sigma_{\text{uv}}$ fixed, solve for $m_v$,comp. Iterate until $m_v$,comp is within a given tolerance of $m_v$,assumed.

To compute the sum of the squares of the residuals by equation (6), we obtain

$$\sum (Y - \{X\})^2$$

from the following equation:

$$\sigma_{f_{\text{max}},\text{comp}} = \frac{\sigma_{\text{ov}}}{\sqrt{V_{Tj}}} \left[ \frac{(1 + m_v)\ln(1 - P_{ij})^{-1}} {\int_{\delta_{ij}}^{\delta_{ij} + \delta_2} \frac{2y}{W_j - \sigma_{f_{\text{max}},\text{assumed}}} dy + \frac{L_2}{L_1} \left( \frac{1 - \sigma_{\text{uv}}}{\sigma_{f_{\text{max}},\text{assumed}}} \right)^{1 + m_v}} \right]^{1/m_v}$$

Assume a value of $\sigma_{f_{\text{max}},\text{assumed}}$, and iterate until $\sigma_{f_{\text{max}},\text{comp}}$ is within some specified tolerance of $\sigma_{f_{\text{max}},\text{assumed}}$. Evaluate the sum of the residuals squared by equation (6). Repeat the process. The parameters that produce the minimum value of the sum of the residuals squared are the solution.

If all failures occur within the inner span and the tensile stress distribution outside the inner span is neglected ($L_1 = 0.0$), equation (17) becomes the pure bend solution, that is,

$$P_{ij} = 1 - \exp \left[ -\frac{L_2 b_j W_j}{2(1 + m_v)} \frac{\sigma_{f_{\text{max}}}}{\sigma_{\text{ov}}} m_v (1 - \frac{\sigma_{\text{uv}}}{\sigma_{f_{\text{max}}}})^{1 + m_v} \right]$$

Therefore,

$$\ln \left( \frac{\ln(1 - P_{ij})^{-1}} {L_2 b_j W_j} \right) + \ln(1 + m_v) \ln(\sigma_{f_{\text{max}}} - \sigma_{\text{uv}})$$

and the vector is $\{X\} = \begin{bmatrix} m_v \\ -\ln(1 + m_v)\sigma_{\text{ov}}^{m_v} \end{bmatrix}$. To obtain an initial value of $m_v$, let $\sigma_{\text{uv}} = 0$ and solve for the uniaxial Weibull two parameter distribution satisfying equation (4). Starting with this computed value of $m_v$ as $m_v$,assumed and a fixed value of $\sigma_{\text{uv}}$, evaluate the integral (eq. (22)) in column vector $\{Y\}$. The integral is evaluated numerically between
the lower limit $\delta_{ij}$ and upper limit $W_j/2$. Obtain from solution vector $\{X\}$, $m_{v,\text{comp}}$. A solution is obtained when $m_{v,\text{comp}}$ is within some specified tolerance of $m_{v,\text{assumed}}$. When this does not occur, the next choice for $m_{v,\text{assumed}}$ is 0.5 ($m_{v,\text{comp}} + m_{v,\text{previous assumed}}$). Iterate until $m_{v,\text{assumed}}$ is within some specified tolerance of $m_{v,\text{comp}}$. To compute the sum of the residuals squared by equation (6), we evaluate $\sigma_{ij,max,\text{comp}}$ in the following way: For the $n$ data values $(P_j, \sigma_{ij,max})$ where $j = 1, n$, assume $\sigma_{ij,max,\text{assumed}} = \sigma_{ij,max}$. The lower integration limit is $\delta_{ij} = \sigma_{uv}W_j/(2\sigma_{ij,max,\text{assumed}})$. With this limit, solve for $\sigma_{ij,max,\text{comp}}$:

$$\sigma_{ij,max,\text{comp}} = \sigma_{uv} V_{ij} m_{v} \frac{1}{W_{j}} \left(1 + m_{v}\right) \ln \left(1 - P_{ij}\right)^{-1} \int_{\delta_{ij}}^{1} \left(2y/W_{j} - \sigma_{uv}/\sigma_{ij,max,\text{assumed}}\right)^{1+m_{v}} dy$$

(23)

Assume a new value for $\sigma_{ij,max,\text{assumed}} = 0.5 \left(\sigma_{ij,max,\text{previous assumed}} + \sigma_{ij,max,\text{comp}}\right)$. Repeat the process, integrating over the new limit $\delta_{ij}$ until the previous assumed value is within some specified tolerance of the computed new value ($\sigma_{ij,max,\text{comp}} = \sigma_{ij,max,\text{assumed}}$). Compute the sum of the residuals squared by equation (6). Repeat the process assuming a new value for $\sigma_{uv}$, and continue until the minimum sum of the residuals squared by equation (6) is obtained. When this occurs, the values of $m_{v}$, $\sigma_{uv}$, and $\sigma_{uv}$ are the three material parameters.

Four Point Bend Specimen (Fig. 1(c)), Surface Flaws

Substitute into equation (7) the inner span side surface uniaxial tensile stress distribution $\sigma_{ij}(x,y,z) = 2\sigma_{ij,max}y/W_{j}$, the bottom surface tensile stress distribution $\sigma_{ij}(x,y,z) = \sigma_{ij,max}$, and the outer span uniaxial tensile surface stress distributions $\sigma_{ij}(x,y,z) = 4\sigma_{ij,max}x/(L_1W_j)$ and $\sigma_{ij}(x,y,z) = 2\sigma_{ij,max}x/L_1$. Normalizing the area with respect to $L_1W_j$ results in the following equation:

$$P_j = 1 - \exp \left[-\left(\frac{\sigma_{ij,max}}{\sigma_{uv}}\right)^{m_{v}} \frac{W_{j}}{L_1W_j} \left(\frac{2y}{W_{j}} - \frac{\sigma_{uv}}{\sigma_{ij,max}}\right)^{1+m_{v}} \int_{\delta_{ij}}^{1} \left(1 - \frac{\sigma_{uv}}{\sigma_{ij,max}}\right)^{1+m_{v}} dy \right]$$

$$\times + \frac{L_2W_j + L_1b_{j}}{L_1W_j} \left(1 - \frac{\sigma_{uv}}{\sigma_{ij,max}}\right)^{1+m_{v}} + \frac{(1 + m_{s})L_2b_{j}}{L_1W_j}$$

$$\times \left(1 - \frac{\sigma_{uv}}{\sigma_{ij,max}}\right)^{1+m_{s}}$$

(24)

where $\delta_{ij} = W_j\sigma_{uv}/(2\sigma_{ij,max})$. Thus

$$\ln \frac{\ln \left(1 - P_{ij}\right)^{-1}}{L_1W_j} = \ln \int_{\delta_{ij}}^{1} \frac{W_{j}}{\sigma_{ij,max}} \left(1 - \frac{\sigma_{uv}}{\sigma_{ij,max}}\right)^{1+m_{v}} dy$$

$$+ \frac{L_2W_j + L_1b_{j}}{L_1W_j} \left(1 - \frac{\sigma_{uv}}{\sigma_{ij,max}}\right)^{1+m_{v}} + \frac{(1 + m_{s})L_2b_{j}}{L_1W_j}$$

$$\times \left(1 - \frac{\sigma_{uv}}{\sigma_{ij,max}}\right)^{1+m_{s}}$$

(25)

Solve the set of linear equations obtained from equation (25) (denoted by $\{Y\} = [A] \{X\}$, in matrix form) by the LSBF method (eq. (4)). The $j$th value of column vector $\{Y\}$ is

$$y_{j} = \ln \frac{\ln \left(1 - P_{ij}\right)^{-1}}{L_1W_j} - \ln \int_{\delta_{ij}}^{1} \frac{W_{j}}{\sigma_{ij,max}} \left(1 - \frac{\sigma_{uv}}{\sigma_{ij,max}}\right)^{1+m_{v}} dy$$

$$+ \frac{L_2W_j + L_1b_{j}}{L_1W_j} \left(1 - \frac{\sigma_{uv}}{\sigma_{ij,max}}\right)^{1+m_{v}} + \frac{(1 + m_{s})L_2b_{j}}{L_1W_j}$$

$$\times \left(1 - \frac{\sigma_{uv}}{\sigma_{ij,max}}\right)^{1+m_{s}}$$

(26)
and vector \( \{X\} = \left\{ -\ln\left[ m_s \right] \right\} \). Row \( j \) of matrix \([A]\) is \([\sigma_{f_{\text{max}}} \ 1.0] \). Assume a value for the threshold stress \( \sigma_{\text{us}} \) and an appropriate value for the Weibull modulus, \( m_s, \text{assumed} \) (based on the two parameter solution). Evaluate column vector \( \{Y\} \) and matrix \([A]\). The vector \( \{X\} \) is evaluated by equation (4). With \( \sigma_{\text{us}} \) fixed, solve for \( m_s, \text{comp} \). When \( m_s, \text{comp} \) is within a given tolerance of \( m_s, \text{assumed} \), a solution results. To compute the sum of the squares of the residuals by equation (6), the values of \( \sigma_{f_{\text{max}}, \text{comp}} \) are obtained from the following equation:

\[
\sigma_{f_{\text{max}}, \text{comp}} = \frac{\sigma_{\text{os}}}{(L_j W_j)^{m_s}}
\]

\[
\times \left[ \frac{\sigma_{\text{us}}}{\sigma_{f_{\text{max}}, \text{assumed}}} \right]^{1+m_s} \left( 1 + m_s \right) \ln \left( 1 - P_{f_j} \right)^{-1}
\]

To determine \( \sigma_{f_{\text{max}}, \text{comp}} \), the process is the same as that outlined for the four point bend, volume flawed specimen. Likewise, the evaluation of the material parameters is the same as that outlined for the four point bend, volume flawed specimen.

**Three Point Bend Specimen (Fig. 1(d)), Surface Flaws**

From equation (1)

\[
\ln \left( 1 - P_{f_j} \right)^{-1} = \left( \frac{1}{\sigma_{\text{os}}} \right)^{m_s} \left[ \frac{W_j L_j}{4} \int \int_{\delta_{i_j} \delta_{2,j}} \left( \frac{4 \sigma_{f_{\text{max}}, \text{assumed}} \sigma_{\text{us}}}{L_j W_j} - \sigma_{\text{us}} \right) \right] \ dx \ dy
\]

\[
+ \frac{L_j}{2} b_j \int \left( \frac{2 \sigma_{f_{\text{max}}, \text{assumed}} \sigma_{\text{us}}}{L_j} - \sigma_{\text{us}} \right) \ dx
\]

(27)

with \( \delta_{i_j} = \sigma_{\text{us}} W_j / (2 \sigma_{f_{\text{max}}}) \). \( \delta_{2,j} = \sigma_{\text{us}} L_j W_j / (4 \sigma_{f_{\text{max}}}) \) and \( \delta_{3,j} = \sigma_{\text{us}} L_j y / 2 \sigma_{f_{\text{max}}} \). Thus

\[
\ln \left[ \frac{\ln \left( 1 - P_{f_j} \right)}{L_j W_j} \right] - \ln \left[ \frac{w_j}{2} \int \int_{\delta_{i_j}} \left( \frac{2 y}{W_j} - \frac{\sigma_{\text{us}}}{\sigma_{f_{\text{max}}}} \right)^{1+m_s} \ dy \right] + b_j \left( 1 - \frac{\sigma_{\text{us}}}{\sigma_{f_{\text{max}}}} \right)^{1+m_s}
\]

\[
\times \left( 1 - \frac{\sigma_{\text{us}}}{\sigma_{f_{\text{max}}}} \right)^{1+m_s}
\]

\[
= m_s \ln \sigma_{f_{\text{max}}} - \ln \left[ (1 + m_s) \sigma_{\text{os}} \right]
\]

(28)

In matrix form \( \{Y\} = [A] \{X\} \) and the \( j \)th term of column vector \( \{Y\} \) is

\[
y_j = \ln \left[ \frac{\ln \left( 1 - P_{f_j} \right)}{L_j W_j} \right] - \ln \left[ \frac{w_j}{2} \int \int_{\delta_{i_j}} \left( \frac{2 y}{W_j} - \frac{\sigma_{\text{us}}}{\sigma_{f_{\text{max}}}} \right)^{1+m_s} \ dy \right]
\]

\[
+ b_j \left( 1 - \frac{\sigma_{\text{us}}}{\sigma_{f_{\text{max}}}} \right)^{1+m_s}
\]

and \( \{X\} = \left\{ \frac{m_s}{\sigma_{\text{os}} (1 + m_s)} \right\} \). The solution to this set of linear equations denoted symbolically by \( \{Y\} = [A] \{X\} \) is solved by equation (4). Starting with \( \sigma_{\text{us}} = 0.0 \), solve for \( m_s \) and the scale factor \( \sigma_{\text{os}} \). Next, assume a value for \( \sigma_{\text{os}} \). The integrand is a function of \( m_s \). Starting with an assumed value of \( m_s \), iterate until \( m_s, \text{assumed} \) is within some specified limit of
Then evaluate the scale factor. To find the value of \( \sigma_{ij_{\max,\text{comp}}} \) associated with the probability of failure \( P_{ij} \), and computed material parameters \( (m_s, \alpha_u, \alpha_v) \), satisfy the following condition:

\[
\sigma_{ij_{\max,\text{comp}}} = \left[ \frac{\int \left( 2y \left( \frac{\sigma_{us}}{\sigma_{ij_{\max,\text{comp}}}} \right)^{l+m_s} dy \right)^{1+m_s}}{b_j \left( 1 - \frac{\sigma_{us}}{\sigma_{ij_{\max,\text{comp}}}} \right)^{l+m_s} \ln(1 - P_{ij})^{-1} \left[ \frac{1+m_s}{L_j W_j} \right] \sigma_{us}^{m_s} m_s} \right]^{1/m_s} \tag{29}
\]

Assume values for \( \sigma_{ij_{\max,\text{comp}}} \) and iterate until the left side is equal to the right side constant. Compute the sum of the residuals squared via equation (6). Repeat the process assuming a new value for \( \sigma_{us} \), and continue until the minimum value of the sum of the residuals squared is obtained. When this occurs, the parameters \( (m_s, \alpha_u, \alpha_v) \) are the solution.

### Three Parameter Specimen Uniaxial Weibull Model

This report deals with material properties that are independent of the component geometry. However, a simple model is often used to obtain the inert strength probabilistic distribution of a given component (refs. 2 and 3). The characteristic strength parameter \( \sigma_0 \) in this model is component dependent and is not a material property. For completeness, this model is briefly mentioned. This model equation for volume flaws is formulated as

\[
P_{ij} = 1 - \exp \left[ -\left( \frac{\sigma_{ij_{\max}} - \sigma_{uv}}{\sigma_{0v}} \right)^{m_s} \right] \tag{30}
\]

For surface flaws, subscript \( v \) is replaced by subscript \( s \). Since the characteristic strength is not a material property, this model has its limitations. It is commonly used and is mentioned for completeness.

### Experimental Applications

The examples in this section make use of some of the developed equations. Inert failure data are analyzed from the following Modulus of Rupture (MOR) bar data:

1. Four point bend room temperature failure data of sintered silicon nitride, table I (ref. 1). All failures were due to volume flaws and occurred within the inner span. These data were also used for four point bend surface flaw analysis to illustrate the application of the developed equations.

2. Three point bend transverse annealed silicon carbide data at 1300 °C, table II (Private communication from Sung Choi and Jonathan Salem, NASA Lewis Research Center). All failures were caused by surface flaws.

3. Three point bend longitudinal annealed silicon carbide data at 1300 °C, table III (Private communication from Sung Choi and Jonathan Salem, NASA Lewis Research Center). All failures were caused by surface flaws.

To develop confidence in the method developed in this report, comparisons were made with the pure bend results from reference 1. The equations for the four point bend and three point bend specimens were then developed and programmed.

### Sintered Silicon Nitride (Pure Bend Analysis, Volume Flaws, Table I)

Monolithic silicon nitride data (SNW-1000, GTE Wesco Division, table I) obtained from reference 1 are used to compare the results of various LSBF techniques. All of the data in table I contain failures that occurred within the inner span. In reference 1, pure bend loading (fig. 1(b)) was therefore assumed applicable to these data. The three material parameters were computed using Cooper’s method (ref. 4), a modified LSBF approach (ref. 5), and the method developed herein. Table IV summarizes the results of the three techniques used in the analysis of these data and the results obtained herein of the two parameter model (\( \sigma_{uv} = 0 \)). Figure 2 is a plot of the data points and the cumulative Weibull two parameter distribution curve. Figure 3 is a plot of the data points and the cumulative three parameter Weibull distribution curve.

### Silicon Carbide (Three Point Bend Surface Flaws, Tables II and III)

Table V summarizes the results obtained for the two and three parameter uniaxial Weibull models. The cumulative distribution curves for three point bend data (Private communication from Sung Choi and Jonathan Salem, NASA Lewis Research Center) for transverse and longitudinal annealed silicon carbide and the data points are plotted in
figures 4 to 7. The two parameter distribution curves are plotted in figures 4 and 6. The three parameter distribution curves are plotted in figures 5 and 7.

Sintered Silicon Nitride (Four Point Bend Analysis, Volume Flaws, Table I)

Table VI summarizes the results for the two and three parameter uniaxial Weibull models. Figure 8 is a plot of the two parameter cumulative distribution curve and data points. Figure 9 is a plot of the three parameter cumulative distribution curve and data points.

Sintered Silicon Nitride (Four Point Bend Analysis, Surface Flaws, Table I)

The four point bend volume flaw data are also used for a four point bend surface flaw analysis to illustrate the application of the developed equations. It is realized that these data are not representative of the physical problem. Table VII summarizes the results for the two and three parameter uniaxial Weibull models applied to these data. Figure 10 is a plot of the two parameter cumulative distribution curve and data points. Figure 11 is a plot of the three parameter cumulative distribution curve and data points.

Discussion and Conclusions

Solutions are obtained from inert failure data based on the minimizing of the sum of the residuals squared as a necessary and sufficient condition. There are programs to evaluate the three parameters fitted to the specimen uniaxial Weibull model (eq. (30), refs. 2 and 3). The characteristic strength, $\sigma_{0v}$, a parameter in this model, is not a material property but component dependent. The results obtained using this model are only applicable to that specific component made from the same test material.

In this report, material property parameter estimation methods are developed based on the uniaxial Weibull model (eq. (1)). The parameters so obtained are applicable to any component made from the same test data material. For the sintered silicon nitride four point bend inert volume flaw failure data (table I), Cooper’s method (ref. 4), a modified LSBF method (refs. 1 and 3), and the approach developed herein were used to minimize the sum of the squares of the residuals based on the pure bend solution (all failures occurred within the inner span). Comparing the results reveals that the largest variation of the sum of the squares of the residuals (table IV) was less than 3 percent. Further comparison of the results of the three methods indicated the approach used herein was slightly less conservative in the low probability of failure regions ($P_f < 0.05$) and slightly more conservative in the upper region ($P_f > 0.25$). Figures 2 and 3 are plots of the data points and the computed cumulative distribution curves for the two and three parameter uniaxial Weibull models based on the pure bend solution.

The results for the silicon carbide three point bend surface flaw data (tables II and III) from longitudinal and transverse annealed specimens are summarized in table V. Comparing the two and three parameter models reveals that there are large differences in the Weibull moduli and scale factors. The cumulative distribution curves are plotted in figures 4 to 7. Superimposing the two and three parameter curves reveals small but significant differences because most designs are based on the very low probability of failure region.

The four point bend solutions to the two and three parameter uniaxial Weibull models applied to the data in table I are summarized in table VI. The cumulative distribution curves are plotted in figures 8 and 9. Figures 2 and 3 are the cumulative distribution curves for the pure bend solution. A comparison of figure 2 with figure 8 and figure 3 with figure 9 indicates the four point bend results for both cases are slightly more conservative in the lower probability of failure region and slightly less conservative in the higher probability of failure region.

Four point bend volume flaw data in table I are used for a four point bend, surface flaw analysis to illustrate the application of the developed equations. It is realized that these data are not representative of the physical problem. The results are plotted in figures 10 and 11 and summarized in table VII.

Justification for applying the three parameter model rather than the two parameter model will depend on which model better predicts the behavior of a component with its observed performance.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, March 1996
### References


### TABLE I.—FOUR POINT BEND SILICON NITRIDE VOLUME FLAW INERT FAILURE DATA (fig. 1(a))

<table>
<thead>
<tr>
<th>Specimen number</th>
<th>Failure strength, MPa</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>613.9</td>
</tr>
<tr>
<td>2</td>
<td>623.4</td>
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<tr>
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<td>4</td>
<td>642.1</td>
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<td>662.4</td>
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### TABLE II.—THREE POINT BEND SILICON CARBIDE SURFACE FLAW TRANSVERSE ANNEALED INERT FAILURE DATA (fig. 1(c))

<table>
<thead>
<tr>
<th>Specimen number</th>
<th>Thickness, $b_j$, mm</th>
<th>Depth, $W_j$, mm</th>
<th>Failure load, kg</th>
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TABLE IV.—WEIBULL PARAMETERS OBTAINED FROM SILICON NITRIDE (SNW-1000) FOUR POINT BEND VOLUME FLAW INERT FAILURE DATA (TABLE I)  
[Data in Table I are analyzed as a pure bend solution over the inner span.]

The probability of failure for a given value of $\sigma_{fj_{max}}$ is defined as $P_{ij} = 1.0 - \exp \left[ -\frac{1}{\sigma_{ov}} \left( \frac{V_{Tj}}{1 + m_v} \right)^{1 + m_v} \right]$.

<table>
<thead>
<tr>
<th>Pure bend solution (volume flaws, fig. 1(b))</th>
<th>Weibull modulus, $m_v$</th>
<th>Scale factor, $\sigma_{ov}$, MPa $\cdot m^{3/2}$</th>
<th>Threshold stress, $\sigma_{uv}$, MPa</th>
<th>Sum of residuals squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>Starlinger, et al. and Cooper - LSBF (ref. 1)</td>
<td>1.625</td>
<td>0.00258276</td>
<td>560.84</td>
<td>2684.4</td>
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<tr>
<td>Starlinger, et al. - Modified LSBF (ref. 1)</td>
<td>1.677</td>
<td>0.00370464</td>
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<td>2664.0</td>
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<td>LSBF*</td>
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<td>0.00218469</td>
<td>565.195</td>
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<tr>
<td>Two parameter LSBF method*</td>
<td>11.306</td>
<td>150.1733</td>
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</table>
| *LSBF applied to eq. (11).  
* $\sigma_{uv}$ set equal to zero in eq. (11). |

TABLE V.—WEIBULL PARAMETERS OBTAINED FROM THREE POINT BEND SILICON CARBIDE SURFACE FLAW INERT FAILURE DATA (TABLES II AND III)

The probability of failure for a given value of $\sigma_{fj_{max}}$ defined as

$P_{ij} = 1.0 - \exp \left[ \frac{\sigma_{fj_{max}}}{\sigma_{os}} \left( \frac{\delta_{ij} W_j}{(1 + m_s)} \right)^{1 + m_s} \right]$,

where $\delta_{ij} = \frac{\sigma_{us} W_j}{2 \sigma_{fj_{max}}}$.

<table>
<thead>
<tr>
<th>LSBF best method (surface flaws, fig. 1(d))</th>
<th>Weibull modulus, $m_s$</th>
<th>Scale factor, $\sigma_{os}$, MPa $\cdot m^{3/2}$</th>
<th>Threshold stress, $\sigma_{us}$, MPa</th>
<th>Sum of residuals squared</th>
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</thead>
<tbody>
<tr>
<td>Transverse-annealed two-parameter model</td>
<td>9.294</td>
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<td>Longitudinal-annealed two-parameter model</td>
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<td>Longitudinal-annealed three-parameter model</td>
<td>5.893</td>
<td>39.78</td>
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TABLE VI. — WEIBULL PARAMETERS OBTAINED FROM SILICON NITRIDE (SNW-1000) FOUR POINT BEND VOLUME FLAW INERT FAILURE DATA (TABLE I)

The probability of failure for a given value of \( \sigma_{f_{\text{max}}} \) is defined as

\[
P_{f_j} = 1 - \exp \left( - \left( \frac{\sigma_{f_{\text{max}}}}{\sigma_{0\nu}} \right)^{m_{\nu}} \frac{L_j b_j W_j}{2(1 + m_{\nu})} \left[ \frac{W_j^{2y} - \sigma_{0\nu}}{\sigma_{f_{\text{max}}}^{1 + m_{\nu}}} dy + \frac{L_j L_{1f_{\text{max}}}}{L_1 W_j} \left( 1 - \frac{\sigma_{0\nu}}{\sigma_{f_{\text{max}}}} \right)^{1 + m_{\nu}} \right] \right)
\]

where \( \delta_{ij} = \frac{\sigma_{0\nu} W_j}{2\sigma_{f_{\text{max}}}} \)

<table>
<thead>
<tr>
<th>LSBF method (volume flaws, fig. 1(c))</th>
<th>Weibull modulus, ( m_v )</th>
<th>Scale factor, ( \sigma_{0\nu} ), MPa - m^3/MPa</th>
<th>Threshold stress, ( \sigma_{0\nu} ), MPa</th>
<th>Sum of residuals squared,</th>
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<td>Weibull two parameter model</td>
<td>10.841</td>
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<td>Weibull three parameter model</td>
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TABLE VII. — WEIBULL PARAMETERS OBTAINED FROM SILICON NITRIDE (SNW-1000) FOUR POINT BEND VOLUME FLAW INERT FAILURE DATA

[For illustrative purposes the data in table I are analyzed as surface flaw inert failure data.]

The probability of failure for a given value of \( \sigma_{f_{\text{max}}} \) is defined as

\[
P_{f_j} = 1 - \exp \left( - \left( \frac{\sigma_{f_{\text{max}}}}{\sigma_{0\nu}} \right)^{m_{\nu}} \frac{L_j W_j}{2(1 + m_{\nu})} \left[ \frac{W_j^{2y} - \sigma_{0\nu}}{\sigma_{f_{\text{max}}}^{1 + m_{\nu}}} dy + \frac{L_j L_{1f_{\text{max}}}}{L_1 W_j} \left( 1 - \frac{\sigma_{0\nu}}{\sigma_{f_{\text{max}}}} \right)^{1 + m_{\nu}} \right] \right)
\]

where \( \delta_{ij} = \frac{\sigma_{0\nu} W_j}{2\sigma_{f_{\text{max}}}} \)

<table>
<thead>
<tr>
<th>LSBF method (surface flaws, fig. 1(c))</th>
<th>Weibull modulus, ( m_a )</th>
<th>Scale factor, ( \sigma_{0\nu} ), MPa - m^2/MPa</th>
<th>Threshold stress, ( \sigma_{0\nu} ), MPa</th>
<th>Sum of residuals squared,</th>
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</tbody>
</table>
Figure 1.—Specimen loading and geometry. (a) Pure tension. (b) Pure bend silicon nitride (SNW-1000) specimen; \(L_2 = 19.6\) mm, \(b = 4.0\) mm, \(W = 3.1\) mm. (c) Four point bend silicon nitride (SNW-1000) specimen; \(L_1 = 20.8\) mm, \(L_2 = 19.6\) mm, \(b = 4.0\) mm, \(W = 3.1\) mm (table I). (d) Three point bend silicon carbide specimen; \(L_1 = 19.936\) mm (tables II and III contain \(F_j\), \(b_j\), and \(W_j\) values).
Figure 2.—Distribution curve for two parameter Weibull model from volume flawed inert silicon nitride (SNW-1000) data via specimens subjected to pure bend analysis over inner span (Fig. 1(b), table I). Weibull modulus $m_v = 11.306$; scale factor $\sigma_{ov} = 150.173 \text{ MPa} \cdot m^{3/m_v}$; threshold stress $\sigma_{uv} = 0.0 \text{ MPa}$; sum of residuals squared = 13440.

Figure 3.—Distribution curve for three parameter Weibull model from volume flawed inert silicon nitride (SNW-1000) data via specimens subjected to pure bend analysis over inner span (Fig. 1(b), table I). Weibull modulus $m_v = 1.608$; scale factor $\sigma_{ov} = 0.0021847 \text{ MPa} \cdot m^{3/m_v}$; threshold stress $\sigma_{uv} = 565.2 \text{ MPa}$; sum of residuals squared = 2744.
Figure 4.—Distribution curve for two parameter Weibull model from surface flawed inert transverse annealed silicon carbide data via specimens subjected to three point bend analysis (Fig. 1(d), table II). Weibull modulus $m = 9.294$; scale factor $\sigma_{os} = 114.5 \text{ MPa}$; threshold stress $\sigma_{us} = 0.0 \text{ MPa}$; sum of residuals squared = 2665.

Figure 5.—Distribution curve for three parameter Weibull model from surface flawed inert transverse annealed silicon carbide data via specimens subjected to three point bend analysis (Fig. 1(d), table II). Weibull modulus $m = 4.024$; scale factor $\sigma_{os} = 11.708 \text{ MPa}$; threshold stress $\sigma_{us} = 190.0 \text{ MPa}$; sum of residuals squared = 1942.
Figure 6.—Distribution curve for two parameter Weibull model from surface flawed inert longitudinal annealed silicon carbide data via specimens subjected to three point bend analysis (Fig. 1(d), table III). Weibull modulus $m_b = 9.161$; scale factor $\sigma_{gs} = 114.14 \text{ MPa} \cdot \text{m}^{2/\text{ms}}$; threshold stress $\sigma_{us} = 0.0 \text{ MPa}$; sum of residuals squared = 1417.

Figure 7.—Distribution curve for three parameter Weibull model from surface flawed inert longitudinal annealed silicon carbide data via specimens subjected to three point bend analysis (Fig. 1(d), table III). Weibull modulus $m_b = 5.893$; scale factor $\sigma_{gs} = 39.78 \text{ MPa} \cdot \text{m}^{2/\text{ms}}$; threshold stress $\sigma_{us} = 120.0 \text{ MPa}$; sum of residuals squared = 1259.
Figure 8.—Distribution curve for two parameter Weibull model from volume flawed inert silicon nitride (SNW-1000) data via specimens subjected to four point bend analysis (Fig. 1(c), table I). Weibull modulus \( m_v = 10.84 \); scale factor \( \sigma_{uv} = 141.7 \text{ MPa} \cdot \text{m}^{-3/\nu} \); threshold stress \( \sigma_{uv} = 0.0 \text{ MPa} \); sum of residuals squared = 11713.

Figure 9.—Distribution curve for three parameter Weibull model from volume flawed inert silicon nitride (SNW-1000) data via specimens subjected to four point bend analysis (Fig. 1(c), table I). Weibull modulus \( m_v = 1.443 \); scale factor \( \sigma_{ov} = 0.000680 \text{ MPa} \cdot \text{m}^{-3/\nu} \); threshold stress \( \sigma_{uv} = 564.0 \text{ MPa} \); sum of residuals squared = 2039.
Figure 10.—Distribution curve for two parameter Weibull model from volume flawed inert silicon nitride (SNW-1000) data via specimens subjected to four point bend analysis. As an illustrative example these data were analyzed as if they came from surface flawed specimens (Fig. 1(c), table 1). Weibull modulus $m_s = 10.84$; scale factor $\sigma_{os} = 325.2$ MPa–m²/ms; threshold stress $\sigma_{us} = 0.0$ MPa; sum of residuals squared = 12067.

Figure 11.—Distribution curve for three parameter Weibull model from volume flawed inert silicon nitride (SNW-1000) data via specimens subjected to four point bend analysis. As an illustrative example these data were analyzed as if they came from surface flawed specimens (Fig. 1(c), table 1). Weibull modulus $m_s = 1.997$; scale factor $\sigma_{os} = 1.6895$ MPa–m²/ms; threshold stress $\sigma_{us} = 575.0$ MPa; sum of residuals squared = 1928.
**Title and Subtitle**: Least Squares Best Fit Method for the Three Parameter Weibull Distribution: Analysis of Tensile and Bend Specimens With Volume or Surface Flaw Failure

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**Abstract**: Material characterization parameters obtained from naturally flawed specimens are necessary for reliability evaluation of nondeterministic advanced ceramic structural components. The least squares best fit method is applied to the three parameter uniaxial Weibull model to obtain the material parameters from experimental tests on volume or surface flawed specimens subjected to pure tension, pure bending, four point or three point loading. Several illustrative example problems are provided.