The Kirchhoff Formulas for Moving Surfaces in Aeroacoustics - The Subsonic and Supersonic Cases

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ABSTRACT

One of the active areas of computational aeroacoustics is the application of the Kirchhoff formulas to the problems of the rotating machinery noise prediction. The original Kirchhoff formula was derived for a stationary surface. In 1988, Farassat and Myers derived a Kirchhoff Formula obtained originally by Morgans using modern mathematics. These authors gave a formula particularly useful for applications in aeroacoustics. This formula is for a surface moving at subsonic speed. Later in 1995 these authors derived the Kirchhoff formula for a supersonically moving surface. This technical memorandum presents the viewgraphs of a day long workshop by the author on the derivation of the Kirchhoff formulas. All necessary background mathematics such as differential geometry and multidimensional generalized function theory are discussed in these viewgraphs. Abstraction is kept at minimum level here. These viewgraphs are also suitable for understanding the derivation and obtaining the solutions of the Ffowcs Williams-Hawkings equation. In the first part of this memorandum, some introductory remarks are made on generalized functions, the derivation of the Kirchhoff formulas and the development and validation of Kirchhoff codes. Separate lists of references by Lyrintzis, Long, Strawn and their co-workers are given in this memorandum. This publication is aimed at graduate students, physicists and engineers who are in need of the understanding and applications of the Kirchhoff formulas in acoustics and electromagnetics.

INTRODUCTION

When Ffowcs Williams and Hawkings published their now famous paper on the noise from moving surfaces in 1969 [1], they used a level of mathematical sophistication unfamiliar to engineers who would later be the main users of this work. Advanced generalized function theory and differential geometry were employed by these authors to derive the Ffowcs Williams-Hawkings (FW-H) equation and to obtain some important qualitative results in this paper. The subject of generalized functions is very abstract, particularly as described in books written by mathematicians. The level of differential
geometry needed in acoustics is, however, basic and at the level essentially fully
developed by the end of the nineteenth century. Both of these subjects are not
emphasized in engineering education. It is possible to teach advanced generalized
function theory to engineers if some of the abstractions are left out initially. One needs
to learn how to work with multidimensional Dirac delta functions and their derivatives
concentrated on moving surfaces, i.e. with support on moving surfaces. This goal can
be achieved.

This technical memorandum is on the derivation of the Kirchhoff formulas for moving
surfaces. The main part of this memorandum is the copies of the viewgraphs based on
lectures delivered by the author in the Workshop on Kirchhoff Formulas for Moving
Surfaces at NASA Langley Research Center on February 15, 1995 (see Appendix).
Attempt was made to present all the mathematical machinery needed in the derivation
of Kirchhoff formulas. One of the publications of the author [2], NASA TP-3428 (May
1994), should also be consulted, if needed, to fill in some details. The author and M. K.
Myers have published two papers on the derivation of Kirchhoff formula for moving
surfaces [3, 4] which should be easily comprehended by the readers reading the
material in the Appendix.

Below we briefly introduce the concept of Generalized Functions. Then we discuss
the derivation of the subsonic and supersonic Kirchhoff formulas. Finally we make
some remarks on the development and validation of codes based on the Kirchhoff
formulas.

GENERALIZED FUNCTIONS

Our main reference for this section is NASA TP-3428 [2]. To derive the Kirchhoff
formulas for moving surfaces, we need to learn how to manipulate multidimensional
Dirac delta functions and their derivatives. Some knowledge of differential geometry
and tensor analysis is also essential. In addition to [2], we give some other useful
references on generalized functions as well as on differential geometry and tensor
analysis in this paper [5-13]. To learn about generalized functions, we need a change of
paradigm in the way we look at ordinary functions. Ordinary functions are locally
(Lebesgue) integrable functions, i.e., functions that have a finite integral over any finite
interval. This change of paradigm is actually very familiar in mathematics. For
example, learning about fractions, negative numbers and complex numbers involves a change of paradigm although we are not told that the change is occurring.

How do we think of an ordinary function \( f(x) \)? We think of this function as a table of ordered pairs \((x, f(x))\). A graph of a function is a plot of this table. In generalized function theory, we need to work with mathematical objects such as the Dirac delta "function" \( \delta(x) \) with the sifting property

\[
\int_{-\infty}^{\infty} \phi(x) \delta(x) dx = \phi(0)
\]  

(1)

It can be shown that no ordinary function has this property. The Dirac delta function is an example of a generalized function which is not an ordinary function. To include \( \delta(x) \) and other such useful but strange objects in mathematics, we change our method of thinking about functions as follows. Suppose we take a space of functions \( D \) which will be called test function space. We will be more specific about \( D \) below. Now given an ordinary function \( f(x) \), let us define the functional

\[
F[\phi] = \int_{-\infty}^{\infty} f \phi dx, \quad \phi \in D.
\]  

(2)

If we take the space \( D \) large enough, then there is a possibility that the table of functional values \( F[\phi] \) where \( \phi \in D \) can identify \( f(x) \). This is actually true if we take the space \( D \) as the space of all \( c^\infty \) functions which are identically zero beyond a bounded interval, i.e., with compact support. Therefore, the new paradigm of viewing a function is: *think of the function \( f(x) \) in terms of the table \( \{F[\phi], \phi \in D\} \).* We can show that this table includes an uncountable number of elements.

Next, one shows that the functional \( F[\phi] \) given by eq. (2) is *linear and continuous* for an ordinary function \( f(x) \) [2, 7-9]. We ask whether all continuous linear functionals are produced by ordinary functions from eq. (2). The answer is no. For example, the functional

\[
\delta[\phi] = \phi(0) \quad \phi \in D
\]  

(3)
is linear and continuous. Therefore, the class of linear and continuous functionals is larger than the class generated by ordinary functions through eq. (2). Now, using our new paradigm of thinking of a function as a table generated by the functional rule we say:

a generalized function is identified by the table produced using a continuous linear functional on space $D$.

By an abuse of terminology, we say that:

generalized functions are continuous linear functionals on space $D$.

By this definition the functional in eq. (3) is (represents) the Dirac delta function! Note that each continuous linear functional on space $D$ produces (represents, identifies, gives) one generalized function. Ordinary functions then become a subset of generalized functions called regular generalized functions. Other functions are called singular generalized functions.

Next the operations on ordinary functions are extended to all generalized functions in such a way that they are equivalent to the old definitions when applied to ordinary functions. To do this, one should write the operation in the language of functionals on space $D$. For example, the derivative of generalized function $F[\phi]$ is defined by

$$F'[\phi] = -F[\phi']$$

In this way, many operations on ordinary functions can be extended to generalized functions [2, 5-9].

Finally, we mention here that the space of generalized functions on $D$ is called $D'$. For any singular generalized function $F[\phi]$, we use eq. (2) with a symbolic function $f(x)$ under the integral sign. Here the integral does not represent an ordinary integral but stands for the rule specified by $F[\phi]$. For example, $\delta(x)$ is a symbolic function which is interpreted as follows. Interpret $\int \delta(x)\phi(x)\,dx$ as $\delta[\phi] = \phi(0)$ for all $\phi \in D$, i.e., in our new way of looking at functions as a table of functional values on space $D$. 

4
\[ \delta(x) \equiv \{ \phi(0), \phi \in D \} \quad (5) \]

Of utmost importance to us are delta functions and their derivatives with support on a surface \( f = 0 \). Here \( f = f(\vec{x}) \) or \( f = f(\vec{x}, t) \). We give the following two results [2] assuming that \( |\nabla f| = 1 \) on \( f = 0 \), which is always possible:

\[ \int \phi(\vec{x}) \delta(f) \, d\vec{x} = \int_{f=0} \phi \, dS \quad (6) \]

\[ \int \phi(\vec{x}) \delta'(f) \, d\vec{x} = \int_{f=0} \left[ -\frac{\partial \phi}{\partial n} + 2H_f \phi \right] dS \quad (7) \]

where \( H_f \) is the local mean curvature of the surface \( f = 0 \) with \( dS \) the element of the surface area. Also if the function \( f(\vec{x}) \) has a discontinuity across a surface \( g(\vec{x}) = 0 \) with the jump defined as

\[ \Delta f = f(g = 0^+) - f(g = 0^-) \quad (8) \]

then

\[ \nabla f = \nabla f + \Delta f \nabla g \, \delta(g) \quad (9) \]

where \( \nabla f \) is the generalized gradient of \( f(\vec{x}) \) (see [2]). Finally, we mention here that the Green's function method is valid for finding solutions of differential equations with discontinuities (weak solutions) provided that all derivatives in the differential equation are viewed as generalized derivatives.

THE KIRCHHOFF FORMULAS FOR MOVING SURFACES

Assume that \( f(\vec{x}, t) = 0 \) is the moving Kirchhoff surface defined such that \( |\nabla f| = 1 \) on this surface. Let \( \phi \) satisfy the wave equation in the exterior \( \overline{\Omega} \) of \( f = 0 \), i.e.,

\[ \Box^2 \phi = 0 \quad \vec{x} \in \overline{\Omega} \quad (10) \]

Extend \( \phi \) to the entire unbounded space as follows, calling the extended function \( \bar{\phi} \)
The governing equation for deriving the Kirchhoff formula for moving surfaces is then found by applying the generalized wave operator (D'Alembertian) to \( \bar{\phi} \) to get [2-4]:

\[
\square^2 \bar{\phi} = -\left( \phi_n + \frac{1}{c} M_n \phi_t \right) \delta(f) - \frac{1}{c} \frac{\partial}{\partial t} [M_n \phi \delta(f)] - \nabla \cdot \left[ \phi \bar{n} \delta(f) \right]
\] (12)

where \( M_n = v_n / c \) is the local normal Mach number on \( f = 0 \), \( \phi_n = \partial \phi / \partial n \) and \( \phi_t = \partial \phi / \partial t \).

We can now apply the Green's function method for the wave operator in the unbounded space to eq. (11) to find the Kirchhoff formula for subsonically moving surfaces [3]. The formula involves a Doppler singularity making it inappropriate for a supersonically moving surface. For supersonic surfaces, we derive the Kirchhoff formula for an open surface (e.g. a panel). The reason is that the Kirchhoff surface is usually divided into panels and the formula is applied individually to each panel. The subsonic formula, applies to both open and closed surfaces. However, the supersonic formula differs for open and closed surfaces. If the formula for an open surface is known, obtaining the formula for a closed surface is trivial.

The governing equation for deriving the supersonic Kirchhoff formula for a panel is

\[
\square^2 \bar{\phi} = -\left( \phi_n + \frac{1}{c} M_n \phi_n \right) H(\tilde{f}) \delta(f) - \frac{1}{c} \frac{\partial}{\partial t} [M_n \phi H(\tilde{f}) \delta(f)] - \nabla \cdot \left[ \phi \bar{n} H(\tilde{f}) \delta(f) \right]
\] (13)

where \( H(\tilde{f}) \) is the Heaviside function, \( \tilde{f} \) is a function such that \( \tilde{f} > 0 \) on the panel and \( f = \tilde{f} = 0 \) defines the edge of the panel. The derivatives on the right side of eq. (13) are brought inside to get three source terms involving \( H(\tilde{f}) \delta(f) \), \( H(\tilde{f}) \delta'(f) \) and \( \delta(\tilde{f}) \delta(f) \) [4]. The solutions of the wave equation with these kinds of sources are given by the author [2]. The Kirchhoff formula for a supersonically moving surfaces using the above method was derived and presented by Farassat and Myers [4]. It is a
particularly simple and straightforward result and easy to apply. This formula requires
the mean curvature $H_F$ of the surface $\Sigma: F(\tilde{y}; \tilde{x}, t) = [f(\tilde{y}, \tau)]_{\text{ret.}}$. We give the formula
for calculation of $H_F$ in the Appendix in terms of the geometric and kinematic
parameters of the Kirchhoff surface $f = 0$.

**SOME REMARKS ON DEVELOPMENT AND VALIDATION OF KIRCHHOFF CODES**

The development of a Kirchhoff code requires a good subroutine for retarded time
calculation if the Kirchhoff surface is rotating. The possibility of multiple emission times
for a supersonic panel complicates retarded time calculation, particularly for two nearly
equal emission times. If the Kirchhoff surface is not selected properly for the supersonic
formula, there is the possibility of a singularity [4]. This singularity can be avoided as
suggested by Farassat and Myers [4] or by using two different Kirchhoff surfaces for
different intervals of the observer time. There is a fool-proof test of the Kirchhoff code
that must not be ignored by code developers. Both of the Kirchhoff formula for moving
surfaces, as well as that for a stationary surface, are written such that $\tilde{\phi} = 0$ inside a
closed surface. Therefore, to test a Kirchhoff code, use a point source inside the closed
surface and specify $\phi$, $\dot{\phi}$ and $\phi_n$ analytically on the Kirchhoff surface $f = 0$. If the
observer is now put anywhere inside $f = 0$ and $\tilde{\phi} \neq 0$, then there is a bug in the code.
One must rule out conceptual misunderstanding of the parameters in the formulation
first. It is recommended that one should be familiar with the complete details of the
derivation of the Kirchhoff formulas to avoid conceptual misunderstanding.

There have been many derivations of the Kirchhoff formula for uniform rectilinear
motion of the Kirchhoff surface [14, 15]. These formulas do not have the generality of
Morgans formula derived and rewritten in a new form using modern mathematics by
Farassat and Myers [3]. Myers and Hausmann [16] were among the first to use the new
Kirchhoff formula in aeroacoustics. Other researchers include Lyrintzis, Long, Strawn
and Di Francescantonio [17]. We give separately the publications of Lyrintzis, Long,
Strawn and their co-workers.

**CONCLUDING REMARKS**

The availability of high resolution aerodynamics and turbulence simulation make the
Kirchhoff formulas discussed here attractive in aeroacoustics. The mathematics for
derivation of these formulas have been under development in the last decade and are
well within the reach of modern engineers. The final form of the formulas are simple and relatively easy to apply. The present paper is written as a guide to understanding the mathematical derivation as well as application of these results.

The viewgraphs in the Appendix give all the necessary mathematical background for the derivation of the Kirchhoff formulas. Note that the mathematical part of the Appendix is also suitable for understanding the derivation and the solutions of the Ffowcs Williams-Hawkings equation. This publication is aimed at graduate students, physicists and engineers who are in need of the understanding and applications of the Kirchhoff formulas in acoustics and electromagnetism.

REFERENCES


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**Journal/Refereed Papers**


Conference Papers


PUBLICATIONS OF L. N. LONG AND CO-WORKERS

Journal/Refereed Papers


Conference Papers


PUBLICATIONS OF R. C. STRAWN AND CO-WORKERS


APPENDIX

Workshop Viewgraphs
The Kirchhoff Formulas for Moving Surfaces in Aeroacoustics—The Subsonic and Supersonic Cases

by

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Based on Lectures Delivered in Workshop on Kirchhoff Formulas for Moving Surfaces, NASA Langley Research Center, February 15, 1995
Available Methods of Noise Prediction in Aeroacoustics

Today we have three methods available. These are:

1. **The Acoustic Analogy** introduced into aeroacoustics by Lighthill (1952). Applications to rotating blades are based on Ffowcs Williams-Hawking (FW-H) equation (1969). It is the most developed method and is widely in use in the aircraft industry.

2. **The Kirchhoff Formula** based method. Originally suggested by Hawking in aeroacoustics (1979), this method is currently under development. Availability of high resolution aerodynamics and powerful computers may make this approach very popular in the future.

3. **The CFD Based CAA** (Computational Aeroacoustics). This method is under development and is the least mature of the three methods. It may be appropriate for some problems. Computational Techniques developed here will also help the above two methods.
Classical Kirchhoff Formula (1882)

\[4\pi \phi(\hat{x}, t) = \int_{f=0} \frac{[-\phi_n + c^{-1}\phi \cos \theta]_{\text{ret}}}{r} dS + \int_{f=0} \frac{[\phi \cos \theta]_{\text{ret}}}{r^2} dS\]

ret: retarded time, \(\phi_n = \frac{\partial \phi}{\partial n}\), \(r = |\hat{x} - \hat{y}|\)

- Gives \(\phi\) in terms of values of \(\phi\), \(\dot{\phi}\) and \(\phi_n\) on the Kirchhoff surface. It is Green's Identity for the wave equation. Compare with the following identity for Laplace equation:

\[4\pi \phi(\hat{x}) = -\int_{f=0} \frac{\phi_n}{r} dS + \int_{f=0} \frac{\phi \cos \theta}{r^2} dS\]

- We derive both above results by the same method using generalized function theory.
Classical Kirchhoff Formula (Cont’d)

- Derived in 1882 by G. Kirchhoff


- Applications in optics, electromagnetism and acoustics are very extensive. Until recently the classical Kirchhoff formula has been used either as approximation or for qualitative understanding of fields governed by the wave equation. The availability of high speed digital computers has changed this picture. Simulation of the wave field is possible and rewarding! Extension to moving surfaces has opened new applications.

Why are Kirchhoff Formulas Important in Acoustics

- Accurate prediction of the noise of helicopter rotors, propellers and ducted fans, particularly at design stage, is needed to reduce the passenger and public annoyance and to meet noise standards.

- Low noise aircraft and propulsion systems sell better in the international market. Therefore, noise prediction tools to meet U.S. aircraft and engine industry needs must be developed.

- Kirchhoff formulas for moving surfaces coupled to advanced CFD codes supply an efficient and powerful tool for noise prediction. See box above.
What is this Workshop About?

- Our **primary purpose** in this workshop is the derivation of two Kirchhoff formulas for subsonic and supersonic surfaces.

- When working with inhomogeneous wave equations for moving sources using classical methods, we notice that the algebraic manipulations quickly become complicated. We lose track of cancellations and simplifications. We need special tools from mathematics which give us simple and direct method of derivation.

- The **secondary purpose** of this workshop is to give all the necessary tools from generalized function (GF) theory, P.D.E.'s and differential geometry.
Method of Deriving Kirchhoff Formulas

- We reduce the derivation of the three Kirchhoff formulas (stationary, subsonic and supersonically moving surface) here to the solution of wave equation \( \Box^2 \phi = Q \) where \( Q \) is a generalized function (such as \( q \delta(f) \)). This is the most direct approach to deriving Kirchhoff formulas. One must, therefore, learn some generalized function theory. The source distributions are on moving surfaces and invariably the geometry of these surfaces enters the derivation. Without the knowledge of differential geometry of surfaces, we cannot identify surface curvature terms and other geometric quantities resulting in a large number of meaningless terms in the Kirchhoff formula. A formula in this form is not very useful in applications.

- **Note:** In applications, the Kirchhoff surface is divided into panels and the contributions of individual panels are added together. The stationary and subsonic Kirchhoff formulas remain unchanged for open or closed surfaces. We derive the supersonic formula for an open surface only. The extension to a closed surface is trivial.
Elements of Generalized Function Theory
Models of Functions

Old (Conventional) Model: We think of a function as a table of ordered pairs \((x, f(x))\) where for each \(x\), \(f(x)\) is unique. This table can be graphed as shown and usually has an uncountable number of ordered pairs.

![Graph showing \((x_0, f(x_0))\)]

New Model: We think of a function \(f\) by its action (functional values) on a given space of ordinary functions called test function space. This action for ordinary functions is defined by

\[
F[\phi] = \int f(x) \phi(x) \, dx
\]

The function \(f\) is now defined (identified, thought of) by the new table \(\{F[\phi], \phi\text{ is in the test function space}\}\). This view of ordinary functions now allows us to incorporate \(\delta(x)\) into mathematics rigorously.
A Familiar Example of Thinking About Functions by New Model

Consider space of periodic functions with period $2\pi$. Take the *test function space* to be the space formed by functions $\phi_n = \exp(inx)$, $n = 0, \pm 1, \pm 2, \ldots$. Let $f$ be periodic with period $2\pi$. The Fourier coefficients of $f$ can be viewed as functionals on test function space by the relation

$$F[\phi_n] = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{inx} \, dx$$

From the theory of Fourier analysis, we know that the following table of Fourier coefficients (i.e., *functional values* of $f$ on test function space) contains the same information as $f(x)$:

$$\{F[\phi_n], n = 0, \pm 1, \pm 2, \ldots \}$$

Note that if $f(x) \neq g(x)$, where $g(x)$ is another periodic function with period $2\pi$, then

$$F[\phi_n] \neq G[\phi_n] = \frac{1}{2\pi} \int_0^{2\pi} g(x)e^{inx} \, dx$$

for some $n$, i.e., the new table uniquely defines functions.
Elementary Generalized Function Theory

The main reason to develop the generalized function theory is to include mathematical objects such as the Dirac delta "function" \( \delta(x) \). This function has the sifting property

\[
\int_{-a}^{a} \phi(x) \delta(x) \, dx = \phi(0)
\]

To include these objects in mathematics, we need to change our thinking about functions. The reason we must change our thinking about functions is that no ordinary function can have the sifting property. We must therefore enlarge the space of functions by a process familiar in mathematics: define (look at, view) functions in a way which includes all ordinary functions as well as objects like the Dirac delta function. This is a change of paradigm familiar to us when we learned fractions, negative numbers and complex numbers.
Definition of Generalized Functions

- A functional on a space of functions $\Omega$ is a mapping (a rule) of $\Omega$ into scalars (real or complex numbers).

**Examples:** Take $\Omega$ as space of differentiable functions. The following are functionals on $\Omega$, $\phi \in \Omega$

i) $F[\phi] = \phi'(0) + 2\phi(1)$  

ii) $F[\phi] = \int_0^1 \phi^2(x)dx$

iii) $F[\phi] = \sin[\phi(0)]$  

iv) $F[\phi] = 2\phi(1) + \int_{-1}^1 \phi(x)dx$

- In the theory, the functionals act on various test function spaces depending on the problem. We define generalized functions on the following test function space:

**Space $D$ of Test Functions:** infinitely differentiable functions with bounded support.

- The **support** of a function $\phi$ is the closure of the set on which $\phi \neq 0$. We use $\text{supp } \phi$ for support of $\phi$. 

Definition of Generalized Functions (Cont’d)

• Example of functions in D:

i) Let \( \phi(x; a) = \begin{cases} 
\exp\left[\frac{a^2}{x^2 - a^2}\right] & |x| < a \\
0 & |x| \geq a
\end{cases} \)

\[ \Rightarrow \phi(x; a) \in \mathcal{D} \]

ii) Let \( g(x) \) be any continuous function, then

\[ \psi(x) = \int_{b}^{c} g(y)\phi(x - y; a)dy \]

where \([b, c]\) is a finite interval, belongs to \( \mathcal{D} \). We can show that \( \text{supp} \psi(x) = [b - a, c + a] \).

• Example (ii), above, shows that space \( \mathcal{D} \) is populated with an uncountably infinite number of functions. This means that the table of functional values on \( \mathcal{D} \) in our new model of functions has an uncountably infinite number of members.
Definition of Generalized Functions (Cont’d)

- By an ordinary function we mean a locally (Lebesgue) integrable function.
- A Reminder: In our new model of thinking about functions, we identify an ordinary function \( f(x) \) by table \( \{ F[\phi] = \int f\phi dx, \phi \in \mathcal{D} \} \).

The functional \( F[\phi] = \int f\phi dx \) is linear and continuous. We define linearity and continuity below.

- A functional on \( \mathcal{D} \) is linear if \( F[\alpha \phi_1 + \beta \phi_2] = \alpha F[\phi_1] + \beta F[\phi_2] \) for all \( \phi_1 \) and \( \phi_2 \) in \( \mathcal{D} \).

- Examples: \( \phi \in \mathcal{D} \)

  i) \( F[\phi] = \phi(0) \) is linear

  ii) \( F[\phi] = 2\phi'(1) - \int f\phi dx \), \( f \) an ordinary function, is linear

  iii) \( F[\phi] = \phi^2(0) \) is nonlinear
Definition of Generalized Functions (Cont'd)

- A sequence of functions \( \{\phi_n\} \) in \( D \) converges to zero in \( D \), written as \( \phi_n \xrightarrow{D} 0 \), if \( \phi_n \) and all its derivatives converge uniformly to zero and \( \text{supp } \phi_n \subset I \) for all \( n \) where \( I \) is a fixed bounded interval.

- A functional on \( D \) is continuous if \( F[\phi_n] \xrightarrow{n \to \infty} 0 \) if \( \phi_n \xrightarrow{D} 0 \).

This definition seems very strange but gives generalized functions some of their nicest properties.

Examples:

i) Let \( \phi_n = \frac{1}{n} \phi(x; a) \), where \( \phi(x; a) \) was defined earlier, \( \frac{1}{n} \phi_n \xrightarrow{n \to \infty} 0 \) because \( \text{supp } \phi_n = [-na, na] \) becomes unbounded as \( n \to \infty \).

ii) \( \phi_n = \frac{1}{n} \phi(x; a) \), \( \phi_n \xrightarrow{n \to \infty} \phi \), \( \phi \) is continuous.

iii) Linear functionals in the examples on previous vugraph are continuous. (It is also linear.)

iv) \( \delta[\phi] = \phi(0), \phi \in D \), is continuous.
Definition of Generalized Functions (Cont’d)

- Any ordinary function \( f \) defines a continuous linear functional on \( D \) by the relation \( F[\phi] = \int f\phi \, dx, \ \phi \in D \). But ordinary functions do not exhaust all continuous linear functionals on \( D \).

- **Definition of Generalized Functions:** A continuous linear functional on space \( D \) defines a *generalized function*. The space of all generalized functions is denoted \( D' \)

**Examples:** \( \phi \in D \)

i) \( \delta[\phi] = \phi(0) \) defines a generalized function. We can show that there is no ordinary function \( f(x) \) such that \( \int f(x)\phi(x) \, dx = \phi(0) \). This means that \( D' \) is larger than the space of ordinary functions.

ii) \( G[\phi] = 2\phi'(1) + 3\int f\phi \, dx, \ f \) ordinary function, defines a generalized function
Definition of Generalized Functions (Cont'd)

- It is inconvenient to work with functional notation in mathematical manipulations. For this reason, we introduce the notation of symbolic functions for those generalized functions which are not ordinary functions. Ordinary functions are called regular generalized functions. Other generalized functions are called singular generalized functions. For singular generalized function $F[\phi]$, we define the symbolic function $f(x)$ so that

$$\int f(x)\phi(x)\,dx = F[\phi]$$

for $\phi \in D$. It is important to recognize that the integral on the left is just a symbol standing for $F[\phi]$ and one should not treat it as an ordinary integral.

This is the picture of the space of generalized functions $D'$ we should have in mind.
Some Operations on Generalized Functions

Note: All test functions are in space $D$ ($c^\infty$ fns with compact supp.)

i) *Equality* of two generalized functions on an open interval $I$: $F[\phi] = G[\phi]$ on $I$ if for all $\phi$ in $D$ such that $\text{supp } \phi \subset I$, we have $F[\phi] = G[\phi]$ (symbolically $f(x) = g(x)$).

**Example:** $\delta(x) = 0$ on $(0, \infty)$ since $\delta[\phi] = \phi(0) = 0$ for all $\phi$ such that $\text{supp } \phi \subset (0, \infty)$. This means that a singular generalized function can be equal to an ordinary function (here $f = 0$) on an open interval.

ii) *Multiplication* of a generalized functions $F[\phi]$ with a $c^\infty$ function $a(x)$: $aF[\phi] = F[a\phi]$ (left side is defined by right side).

**Example:** $a\delta[\phi] = \delta[a\phi] = a(0)\phi(0)$ or symbolically $a(x)\delta(x) = a(0)\delta(x)$, an important result!

**Note:** Multiplication of two singular generalized functions or a regular and a singular generalized functions may not be defined.
Some Operations on Generalized Functions (Cont'd)

iii) *Addition* of generalized functions: \((F + G)[\phi] = F[\phi] + G[\phi]\) or symbolically \((f + g)(x) = f(x) + g(x)\)

iv) *Shift Operation*: \(E_h F[\phi] = F[E_{-h} \phi]\) where \(E_{-h} \phi = \phi(x - h)\)

*Example*: \(E_h \delta[\phi] = \delta[E_{-h} \phi] = \phi(-h)\) or symbolically
\[
\int E_h \delta(x) \phi(x) dx = \int \delta(x + h) \phi(x) dx = \phi(-h)
\]

**Note**: Generalized functions are not defined at a point but on open intervals. In practice, this does not cause problems.

- We can define other operations such a *dilation*:
\[
\int \delta(\alpha x) \phi(x) dx = \frac{1}{|\alpha|} \phi(0) \Rightarrow \delta(\alpha x) = \frac{1}{|\alpha|} \delta(x), \text{ and Fourier transform (F.T.)}\]

\(\hat{F}[\phi] = F[\hat{\phi}], \hat{\phi} = \text{F.T.}(\phi)\) where \(\phi\) now belongs to space of rapidly decreasing test functions \(S\). For our purpose, the most important operation on generalized functions is *differentiation*.  

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Differentiation of Generalized Functions

All test functions are in D.

- \( f(x) \) ordinary function, differentiable, \( F[\phi] = \int f \phi \, dx \), we must identify \( F'[\phi] \) with \( \int f' \phi \, dx \). But \( F'[\phi] = \int f' \phi \, dx = -\int f \phi' \, dx = -F[\phi'] \) since \( \phi' \in D \).

Therefore, we use the relation:

\[
F'[\phi] = -F[\phi']
\]

as the definition of derivative of any generalized function \( F[\phi] \). Similarly \( F^{(n)}[\phi] = (-1)^n F[\phi^{(n)}] \), i.e., generalized functions have derivatives of all orders.

Examples:

i) \( \delta'[\phi] = -\delta[\phi'] = -\phi'(0) \) or \( \int \delta'(x) \phi(x) \, dx = -\phi'(0) \)

ii) \( \delta''[\phi] = (-1)^2 \delta[\phi''] = \phi''(0) \) or \( \int \delta''(x) \phi(x) \, dx = \phi''(0) \)

Note: If an ordinary function is differentiable on real line, then \( f'_{\text{gen.}} = f' \). However, generalized derivative of an ordinary function can be a singular generalized function.
Differentiation of Generalized Functions (Cont’d)

Notation: For ordinary (regular G.F.’s) functions, we use $\bar{f}'(x)$ or $\frac{df}{dx}$ for $f'_\text{gen.}$ to distinguish generalized from ordinary derivative.

Example: Generalized derivative of an ordinary function with a jump.

$F[\phi] = \int f\phi dx, \phi \in D$

$F'[\phi] = -F[\phi'] = -\int f \phi' dx$

\[
= -\left( \int_{-\infty}^{x_0^{-}} + \int_{x_0^{+}}^{\infty} \right) f \phi' dx = \int f'\phi dx + \Delta f \phi(x_0)
\]

or symbolically

$$
\bar{f}'(x) = f'(x) + \Delta f \delta(x - x_0)
$$
Differentiation of Generalized Functions (Cont’d)

Example: Generalized derivative of Heaviside function

\[ h(x) = \begin{cases} 
1 & x > 0 \\ 
0 & x < 0 
\end{cases} \]

\[ \tilde{h}'(x) = \delta(x) \text{ since } h'(x) = 0 \text{ on } (0, \infty) \cup (-\infty, 0). \]

• Note: Even at this level of exposition, we can do a lot we could not do by using ordinary functions. We can discuss Green’s function of an O.D.E., for example.
Some Important Results of Generalized Function Theory

- **Structure Theorem of D’:** Generalized functions in D’ are generalized derivatives of finite order of continuous functions.

- **Sequences of Generalized Functions:** A sequence \( \{F_n[\phi]\} \) of generalized functions is *convergent* if for all \( \phi \in D \), the sequence of numbers \( \{F_n[\phi]\} \) is convergent.

- **Theorem:** The space D’ is complete.

This theorem implies that a convergent sequence of generalized functions gives (i.e., converges to) a generalized function. This theorem is the basis of the sequential approach to generalized function theory (see books by Lighthill and Jones).
Some Important Results of Generalized Function Theory (Cont’d)

• Exchange of Limit Processes: We can exchange limit processes when we are dealing with generalized functions. This result is very important in applications.

Examples:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}; \quad \sum_i \int_{\Omega} \ldots = \int \sum_i \ldots; \quad \frac{\partial}{\partial x_i} \int_{\Omega} \ldots = \int \frac{\partial}{\partial x_i} \ldots;$$

$$\lim_{n \to \infty} \sum_m \ldots = \sum_m \lim_{n \to \infty} \ldots; \quad \frac{d}{dx} \lim_{m \to \infty} \ldots = \lim_{m \to \infty} \frac{d}{dx} \ldots.$$  

Note: In the rule of exchanging the order of differentiation and integration, we assume that $\Omega$ is independent of $x$. 
Green's Function of a 2nd Order Linear O.D.E.

Given
\[ \begin{align*}
lu &= f(x) \quad x \in [0, 1] \quad \text{Linear 2nd order O.D.E.} \\
BC_1[u] &= a_1u(0) + b_1u'(0) + c_1u(1) + d_1u'(1) = 0 \\
BC_2[u] &= a_2u(0) + b_2u'(0) + c_2u(1) + d_2u'(1) = 0 \\
\end{align*} \]

Linear Homogeneous BCs

Assume there is a function \( g(x, y) \) (Green's function) such that

\[ u(x) = \int_0^1 f(y)g(x, y)dy \quad (1) \]

We are interested in solutions where \( u \in C^1 \) and \( u \) is twice differentiable so that \( \bar{u}'' = u'' \) and \( \bar{u}' = u' \) and, therefore, \( \bar{lu} = lu \). Here \( \bar{lu} \) stands for the differential equation where ordinary derivatives are replaced by generalized derivatives. From eq. (1), we have

\[ \bar{lu}(x) = \bar{l} \int_0^1 f(y)g(x, y)dy = \int_0^1 f(y)\bar{l}g(x, y)dy \quad \text{(exchange of limit process)} \]

\[ = f(x) \quad \text{(by the O.D.E.)} \]

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Green's Function of a 2nd Order Linear O.D.E.
(Cont'd)

Therefore

\[ \overline{I}_x g(x, y) = \delta(x - y) \]

We will interpret this equation later. Note that since the boundary conditions are linear:

\[ BC_1[u] = BC_1, x \int_0^1 f(y) g(x, y) dy \]

\[ = \int_0^1 f(y) BC_1, x [g(x, y)] dy = 0 \]

A similar result also holds for \( BC_2[u] \).

\[ \therefore \quad BC_1, x [g(x, y)] = 0, \quad BC_2, x [g(x, y)] = 0, \]

i.e., \( g(x, y) \) in variable \( x \) satisfies both \( BC \)'s.
Green's Function of a 2nd Order Linear O.D.E.
(Cont'd)

- What is the interpretation of $\overline{l} g(x, y) = \delta(x - y)$?

Let $l = A(x) \frac{d^2}{dx^2} + B(x) \frac{d}{dx} + C(x)$, then $g(x, y)$ and $\frac{\partial g}{\partial x}(x, y)$ must have some kind of discontinuity at $x = y$.

Let $g(x, y) = \begin{cases} 
g_1(x, y) & x < y \\
g_2(x, y) & x > y 
\end{cases}$
Green's Function of a 2nd Order Linear O.D.E.  
(Cont'd)

Then \[ \frac{\partial g}{\partial x} = \frac{\partial g}{\partial x} + \Delta g \delta(x - y) \]

\[ \frac{\partial^2 g}{\partial x^2} = \frac{\partial^2 g}{\partial x^2} + \Delta \left( \frac{\partial g}{\partial x} \right) \delta(x - y) + \Delta g \delta'(x - y) \]

\[ I_x g(x, y) = I_x g(x, y) + \left[ A(y) \Delta \left( \frac{\partial g}{\partial x} \right) + B(y) \Delta g \right] \delta(x - y) + A(x) \Delta g \delta'(x - y) \]

= \delta(x - y) \text{ (by the result of previous page)}

\[ \therefore \Delta g = 0 \text{ at } x = y \text{ and } \Delta \left( \frac{\partial g}{\partial x} \right) = \frac{1}{A(y)} \text{ at } x = y \]

This means \( I_x g_1(x, y) = I_x g_2(x, y) = 0 \), \( g(x, y) \) is continuous at \( x = y \) and \( \frac{\partial g}{\partial x} \) has a jump equal to \( 1/A(y) \) at \( x = y \).
Generalized Functions in Multidimensions

- **Space D in Multidimensions**: This space is formed by $c^\infty$ functions with bounded support. Define

$$\phi(\hat{x}; a) = \begin{cases} \exp\left[\frac{a^2}{a^2 - |\hat{x}|^2}\right] & |\hat{x}| < a, \\ 0 & |\hat{x}| \geq a \end{cases}$$

$$|\hat{x}| = \left[ \sum_{i=1}^{n} x_i^2 \right]^{1/2}$$

This belongs to $D$ in $n$ dimensions. Given any continuous function $g(\hat{x})$ and any bounded region $\Omega$

$$\psi(\hat{x}) = \int_{\Omega} g(\hat{y})\phi(\hat{x} - \hat{y}; a) \, d\hat{y}$$

$$\Rightarrow \psi(\hat{x}) \in D$$

- As in the case of space $D$ in one dimension, the space $D$ in $n$ dimension is populated by an uncountably infinite number of functions.
Generalized Functions in Multidimensions (Cont’d)

- **Generalized functions** in $n$ dimensions are continuous linear functionals on $n$ dimensional test function space $D$.

- **Examples:**
  
  i) $\int \delta(\hat{x})\phi(\hat{x})d\hat{x} = \phi(0)$
  
  ii) $\int \left[ \frac{\partial}{\partial x_i} \delta(\hat{x}) \right] \phi(\hat{x})d\hat{x} = -\frac{\partial \phi}{\partial x_i}(0)$

- From our point of view, the most important generalized functions are delta functions whose supports are on open or closed surfaces, e.g., $\delta(f)$. We need to interpret integrals of the form

  $$I_1 = \int \delta(f)\phi(\hat{x})d\hat{x} \quad \text{and} \quad I_2 = \int \delta'(f)\phi(\hat{x})d\hat{x}.$$
How Does $\delta(f)$ Appear in Applications?

Assume $g(\hat{x})$ is discontinuous across the surface $f(\hat{x}) = 0$ with the jump

$$\Delta g = g(f = 0^+) - g(f = 0^-)$$

Set up coordinate system $(u^1, u^2)$ on $f = 0$ and extend these coordinates to the vicinity of $f = 0$ along local normals. Take $u^3 = f$ as third local variable. Then (assuming $g$ is continuous in $u^1, u^2$)

$$\frac{\partial g}{\partial u^i} = \frac{\partial g}{\partial u^i}, \quad i = 1, 2 \quad \text{and} \quad \frac{\partial g}{\partial u^3} = \frac{\partial g}{\partial u^3} + \Delta g \, \delta(u^3)$$

$$\frac{\partial g}{\partial x_j} = \frac{\partial g}{\partial u^i} \frac{\partial u^i}{\partial x_j} = \frac{\partial g}{\partial u^i} \frac{\partial u^i}{\partial x_j} + \Delta g \frac{\partial u^3}{\partial x_j} \delta(u^3) = \frac{\partial g}{\partial x_j} + \Delta g \frac{\partial u^3}{\partial x_j} \delta(u^3)$$

Since $u^3 = f$, we have $\nabla g = \nabla g + \Delta g \nabla f \, \delta(f)$.

Similarly

$$\nabla \cdot \hat{g} = \nabla \cdot \hat{g} + \Delta \hat{g} \cdot \nabla f \, \delta(f)$$

$$\nabla \times \hat{g} = \nabla \times \hat{g} + \Delta \hat{g} \times \nabla f \, \delta(f)$$
How Does $\delta(f)$ Appear in Applications? (Cont’d)

- In our work the discontinuities in functions are either real (e.g., shock waves) or artificial (e.g., across blade surface in derivation of FW-H eq.).

- **Example:** *Shock surface sources* in Lighthill jet noise theory. Let the shock surfaces be defined by $f(\hat{x}, t) = 0$. We can show that Lighthill’s equation is valid in presence of shocks if we interpret the derivatives of the source term as generalized derivatives:

  \[
  \Box^2 p' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \\
  = \frac{\partial}{\partial x_i} \left[ \frac{\partial T_{ij}}{\partial x_j} \right] + \Delta T_{ij} \frac{\partial f}{\partial x_j} \delta(f) \\
  = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} + \Delta \left( \frac{\partial T_{ij}}{\partial x_j} \right) \frac{\partial f}{\partial x_i} \delta(f) + \frac{\partial}{\partial x_i} \left[ \Delta T_{ij} \frac{\partial f}{\partial x_j} \delta(f) \right]
  \]

  \[\text{Turbulence Source \hspace{1cm} Shock Surface Sources}\]

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Elements of Differential Geometry
Some Results From Differential Geometry

- Introduce the local surface variables \((u^1, u^2)\) on a surface. Define local tangent vectors \(\hat{r}_1 = \partial \hat{r} / \partial u^1\) and \(\hat{r}_2 = \partial \hat{r} / \partial u^2\). In general, these are not of unit length. Let \(g_{ij} = \hat{r}_i \cdot \hat{r}_j\), the first fundamental form is

\[
\begin{align*}
dl^2 &= g_{11}(du^1)^2 + 2g_{12}du^1du^2 + g_{22}(du^2)^2, \\
g_{12} &= g_{21}.
\end{align*}
\]

This gives the element of length of a curve on the surface. In this relation \(g_{ij}\)'s are known as coefficients of the first fundamental form. We define \(g_{(2)}\) as the determinant of coeff. of 1st fundamental form

\[
g_{(2)} = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = g_{11}g_{22} - g_{12}^2.
\]
Some Results From Differential Geometry (Cont'd)

- We can show that the element of surface area $dS$ is $dS = |\hat{r}_1 \times \hat{r}_2| du^1 du^2$.

Since $g_{(2)} = |\hat{r}_1 \times \hat{r}_2|^2$, we have $dS = \sqrt{g_{(2)}} du^1 du^2$.

Note: We use summation convention on repeated index below.

- Define $g^{ij}$ as elements of the inverse of the matrix $G = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$,

i.e., $G^{-1} = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} \Rightarrow g^{11} = \frac{g_{22}}{g_{(2)}}, g^{22} = \frac{g_{11}}{g_{(2)}}, g^{12} = g^{21} = \frac{-g_{12}}{g_{(2)}}$.

We have $g^{ij} g_{jk} = \delta^i_k$, where $\delta^i_k$ is the Kronecker delta.
Some Results From Differential Geometry (Cont'd)

- Define $b_{ij} = \hat{r}_{ij} \cdot \hat{n}$ where $\hat{r}_{ij} = \frac{\partial^2 \hat{r}}{\partial u^i \partial u^j}$. The second fundamental form is $\Pi = b_{11}(du^1)^2 + 2b_{12}du^1du^2 + b_{22}(du^2)^2$. Note that $b_{12} = b_{21}$ and $\hat{n}$ is the local unit normal. In this relation $b_{ij}$'s are known as coefficients of 2nd fundamental form.

- $b = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = b_{11}b_{22} - b_{12}^2$

The quantity $b$ is the determinant of coefficient of 2nd fundamental form.

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Some Results From Differential Geometry (Cont'd)

- What is the geometrical meaning of $\Pi$?
  
  \[ d = \hat{n} \cdot d\tilde{r} \]
  
  \[ = \hat{n} \cdot \left[ \dot{r}_1 du^1 + \dot{r}_2 du^2 + \frac{1}{2}[\dot{r}_{11}(du^1)^2 \right. \]
  
  \[ + 2\dot{r}_{12}du^1 du^2 + \dot{r}_{22}(du^2)^2 \left. \right] + \ldots \]
  
  \[ = \frac{1}{2}\Pi + O(du^i)^3 \quad \therefore \Pi \approx 2d \]

- Another relation for $b_{ij}$:
  \[ b_{ij} = -\dot{r}_i \cdot \hat{n}_j, \quad \hat{n}_j = \frac{\partial \hat{n}}{\partial u^j} \]

- Weingarten Formula:
  \[ \hat{n}_i = -b_i^j \cdot \dot{r}_j \]

where $b_{ij} = g^{jk}b_{ki}$. We are using summation convention here.
Some Results From Differential Geometry (Cont’d)

- **Gauss Formula:** \( \hat{\Gamma}_{ij} = \Gamma^k_{ij} \hat{n} + b_{ij} \hat{n} \) where \( \Gamma^k_{ij} \) is Christoffel symbol of 2nd kind.

- **Christoffel Symbols:** First kind \( \Gamma_{ijk} \), second kind \( \Gamma^k_{ij} \)

\[
\Gamma_{ijk} = \frac{1}{2} \left[ \frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} \right]
\]

\( \Gamma^k_{ij} = g^{kl} \Gamma_{ijkl} \) and \( \Gamma^l_{ij} = \Gamma^l_{ji} \)

**Note:** Christoffel symbols are not tensors while \( g_{ij}, g^{ij}, b_{ij}, b^i \) are.

- **A useful result:**

\[
\frac{\partial \sqrt{g(2)}}{\partial u^i} = \Gamma^k_{ik} \sqrt{g(2)}
\]
Some Results From Differential Geometry (Cont’d)

- Gauss Formula:

\[
b = \partial^2_{12} g_{12} - \frac{1}{2} (\partial^2_{22} g_{11} + \partial^2_{11} g_{22}) - (\Gamma^i_{11} \Gamma^j_{22} - \Gamma^i_{12} \Gamma^j_{12}) g_{ij}
\]

\[\partial^2_{ij} = \partial^2 / \partial u^i \partial u^j.\] See theorema egregium of Gauss. A very significant result!

- Let us parametrize a curve in space by length parameter \( s \).

The unit tangent \( \hat{t} \) to the curve is \( \hat{t} = \frac{d\hat{r}}{ds} \) and the local curvature \( k \) is given by

\[
k\hat{N} = \frac{d\hat{t}}{ds} = \hat{k}, \ k > 0 \quad \therefore \quad k = \left| \frac{d\hat{t}}{ds} \right| = \left| \frac{d^2\hat{r}}{ds^2} \right|.
\]

Note that \( \hat{N} \) always points to the center of curvature, i.e. \( \hat{N} \) is parallel to \( d\hat{t} / ds \).
Some Results From Differential Geometry (Cont’d)

- For a curve on a surface $\frac{d^2 \vec{r}}{ds^2}$ has components along tangent and normal to the surface.

\[
\frac{d\vec{r}}{ds} = \hat{r}_i \frac{du^i}{ds}
\]

\[
\hat{k} = \frac{d^2 \vec{r}}{ds^2} = \hat{r}_{ij} \frac{du^i}{ds} \frac{du^j}{ds} + \hat{r}_i \frac{d^2 u^i}{ds^2}
\]

\[
= \left( \frac{d^2 u^k}{ds^2} + \Gamma^k_{ij} \frac{du^i}{ds} \frac{du^j}{ds} \right) \hat{r}_k + b_{ij} \frac{du^i}{ds} \frac{du^j}{ds} \hat{n}
\]

\[
\hat{k}_g: \text{Geodesic curvature vector} \quad \hat{k}_n: \text{Normal curvature vector}
\]

Geodesic curvature is an *intrinsice* while normal curvature is an *extrinsic* quantity.
Some Results From Differential Geometry (Cont'd)

- The normal curvature is a signed quantity. If $k_n > 0$, then the center of curvature of the curve obtained by the intersection of a plane containing $\hat{n}$ and the surface, is on the side of $\hat{n}$ points to. Note that $i = \frac{du}{ds}$ the components of unit tangent to this curve and $k_n = b_{ij} i_j$.

- There are two directions at a point on a surface orthogonal to each other where $k_n$ achieves its maximum and minimum values. These are known as the principal directions with principal curvatures $k_1$ and $k_2$ (normal curvatures).

Euler's Formula

\[ k_n(\alpha) = k_1 \cos^2 \alpha + k_2 \sin^2 \alpha \]
Some Results From Differential Geometry (Cont’d)

- Mean Curvature: \( H = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}b^i = \frac{1}{2}(b^1 + b^2) \), \( b^i = g^{ik}b_{kj} \)

- Gaussian Curvature: \( K = k_1 k_2 = b^1 b^2 - b^2 b^1 \)

- Theorem Egregium of Gauss: The Gaussian curvature \( K \) depends on \( g_{ij} \) and their first and second derivatives.

  We have \( K = \frac{b}{g} \), and \( b \) was given in terms of \( g_{ij} \) and their first and second derivatives. Thus the Gaussian curvature is an intrinsic quantity.

- \( \hat{n}_1 \times \hat{n}_2 = K \hat{r}_1 \times \hat{r}_2 \)

  \( \hat{r}_i = \frac{\partial \hat{r}}{\partial u^i}, \quad \hat{n}_i = \frac{\partial \hat{n}}{\partial u^i} \)

- By Euler’s formula

\[
H = \frac{1}{2}\left[ k_n(\alpha) + k_n(\alpha + \frac{\pi}{2}) \right]
\]
Some Results From Differential Geometry (Cont'd)

- The *principal directions* can be found from solving the quadratic equation:

\[
\begin{vmatrix}
(du^2)^2 & -du^1 du^2 & (du^1)^2 \\
 b_{11} & b_{12} & b_{22} \\
 g_{11} & g_{12} & g_{22}
\end{vmatrix} = 0
\]

Remember \((du^1, du^2)\) defines the direction \(r_1 du^1 + r_2 du^2\).

- \(H^2 \geq K\), \(H^2 = K\) if and only if the two fundamental forms are proportional.

- Let us displace surface \(S\) given by \(\hat{r}(u^1, u^2)\) by distance \(a = \text{const.}\) along local normal to get \(S'\) given by \(\hat{R}(u^1, u^2, a) = \hat{r}(u^1, u^2) + a\hat{n}(u^1, u^2)\).

If we associate prime quantities to \(S'\), we can show the following

\[
g'_{(2)} = (1 - 2Ha + Ka^2)^2 g_{(2)}
\]

\[
\sqrt{g'_{(2)}} = (1 - 2Ha + Ka^2) \sqrt{g_{(2)}}
\]

\[
H' = \frac{H - Ka}{1 - 2Ha + Ka^2}
\]

\[
K' = \frac{K}{1 - 2Ha + Ka^2}
\]
Some Results From Differential Geometry (Cont'd)

- If we now define \( u^3 = a \) be the distance along local normal to a surface \( S \), then the three dimensional space near \( S \) can be parametrized by \((u^1, u^2, u^3)\) and \( g_{(3)} \), the determinant of coefficient of first fundamental form in 3D is given by

\[
g_{(3)} = g'_{(2)}(u^1, u^2, u^3) = [1 - 2Hu^3 + K(u^3)^2]^2 g_2(u^1, u^2)
\]

From this we find

\[
\left( \frac{\partial \sqrt{g'_{(2)}}}{\partial u^3} \right)_{u^3 = 0} = \left( \frac{\partial \sqrt{g'_{(2)}}}{\partial n} \right)_S = -2H \sqrt{g_{(2)}}
\]

Here \( H \) is the local mean curvature of the surface \( S \).
Some Results From Differential Geometry (Cont’d)

- Let us now have a vector field $\mathbf{\hat{Q}}$ in the vicinity of surface $S$: $f = 0$. We want to write $\nabla \cdot \mathbf{\hat{Q}}$ in a new way. First parametrize the 3D space in the vicinity of $S$ as shown. Then, let $Q^i$ be the contravariant components of $\mathbf{\hat{Q}}$. We have

$$\nabla \cdot \mathbf{\hat{Q}} = \frac{1}{\sqrt{g(3)}} \frac{\partial}{\partial u^i} [\sqrt{g(3)} Q^i] .$$

Now using the result of previous page that $g(3) = g'(2)(u^1, u^2, u^3)$, we have, using $\alpha = 1, 2$

$$\nabla \cdot \mathbf{\hat{Q}}_S = \left[ \frac{1}{\sqrt{g'(2)}} \frac{\partial}{\partial u^\alpha} [\sqrt{g'(2)} Q^\alpha] + \frac{\partial Q^3}{\partial u^3} + \frac{Q^3}{\sqrt{g'(2)}} \frac{\partial}{\partial u^3} \right]_S$$
Some Results From Differential Geometry (Cont’d)

Since $Q^3 = Q_n$:

$$ (\nabla \cdot \mathbf{Q})_S = \nabla_2 \cdot \mathbf{Q}_T + \frac{\partial Q_n}{\partial n} - 2HQ_n $$

A very useful result

$\mathbf{Q}_T = \mathbf{Q} - Q_n$ surface component of $\mathbf{Q}$ on $S$.

$\nabla_2 \cdot \mathbf{Q}_T$ is the surface divergence of $\mathbf{Q}_T = Q^1 r_1 + Q^2 r_2$:

$$ \nabla_2 \cdot \mathbf{Q}_T = \frac{1}{\sqrt{g(2)}} \frac{\partial}{\partial u^\alpha} \left[ \sqrt{g(2)} Q^\alpha \right] \quad \alpha = 1, 2 $$

Example: $\nabla \cdot [p \hat{n} \delta(f)] = \frac{\partial}{\partial n} [p \delta(f)] - 2H_f \delta(f) = p \delta'(f) - 2H_f \delta(f)$

where $p$ is the restriction of $p$ to $f = 0$ (explained later), and $H_f$ is the mean curvature of $f = 0$. Note that $\partial p / \partial n = 0$. This identity is used in deriving the supersonic Kirchhoff formula.
Integration of Delta Functions
and
Solution of Wave Equation
The Integration of $\delta(f)$ and $\delta'(f)$

We assume $f(\hat{x})$ is defined such that $|\nabla f| = 1$ on the surface $f = 0$. This can always be done. This means $df = dn = du^3$

- Parametrize the space in vicinity of surface $f = 0$ by variables $(u^1, u^2, u^3)$ as shown. Then

$$I_1 = \int \phi(\hat{x}) \delta(f) d\hat{x}$$

$$d\hat{x} = \sqrt{g(3)} du^1 du^2 du^3$$

$$= \sqrt{g'_2(u^1, u^2, u^3)} du^1 du^2 du^3$$

$$I_1 = \int \phi(\hat{x}) \delta(u^3) \sqrt{g'_2(u^1, u^2, u^3)} du^1 du^2 du^3 = \int [\phi(\hat{x})]_{u^3 = 0} \sqrt{g'_2(u^1, u^2, u^3)} du^1 du^2$$

$$I_1 = \int \phi(\hat{x}) \delta(f) d\hat{x} = \int_{f = 0} \phi(\hat{x}) ds$$
The Integration of $\delta(f)$ and $\delta'(f)$ (Cont'd)

\[
I_2 = \int \phi(\dot{x}) \delta'(f) d\dot{x} = \int \phi(\dot{x}) \delta'(u^3) \sqrt{g''(2)} \, du^1 du^2 du^3
\]

\[
= -\int \frac{\partial}{\partial u^3} \left[ \phi(\dot{x}) \sqrt{g'(2)} \right] u^3 = 0 \, du^1 du^2
\]

\[
= \int \left[ -\frac{\partial \phi}{\partial u^3} + 2H_f \phi \right] \sqrt{g'(2)} \, du^1 du^2
\]

where we used $(\frac{\partial}{\partial u^3} \sqrt{g'(2)}) u^3 = 0 = -2H_f \sqrt{g'(2)}$, $H_f$ local mean curvature on $f = 0$.

\[
I_2 = \int \phi(\dot{x}) \delta'(f) d\dot{x} = \int_{f = 0} \left[ -\frac{\partial \phi}{\partial n} + 2H_f \phi \right] ds
\]

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Integration of Product of Delta Functions

Let \( f(\hat{x}) = 0 \) and \( g(\hat{x}) = 0 \) be two intersecting surfaces in 3D. We want to integrate

\[
I = \int \phi(\hat{x}) \delta(f) \delta(g) d\hat{x}
\]

Let the two surfaces intersect along the curve \( \Gamma \). On local plane normal to \( \Gamma \), parametrize space by \( u^1 = f \), \( u^2 = g \), and \( u^3 = \gamma \), where \( \gamma \) is the distance along \( \Gamma \). Extend \( u^1 \) and \( u^2 \) to the space in the vicinity of the plane along local normal to the plane.
Integration of Product of Delta Functions (Cont’d)

Then

\[
d\hat x = \frac{du^1 du^2 du^3}{\sin \theta}, \quad \sin \theta = |\hat n \times \hat n'|\n\]

\[
I = \int \frac{\phi(\hat x)}{\sin \theta} \delta(u^1) \delta(u^2) du^1 du^2 du^3 = \int \frac{\phi(\hat x)}{\sin \theta} du^3
\]

\[
f = g = 0
\]

Also if $|\nabla f| \neq 1$ or $|\nabla g| \neq 1$

\[
I = \int \frac{\phi(\hat x)}{|\nabla f| |\nabla g| \sin \theta} d\gamma
\]

\[
f = 0, \quad g = 0
\]
Illustration of Manipulation of Generalized Functions

Let \( \vec{Q} \) be a vector field which is zero outside \( \Omega \) and nonzero inside \( \Omega \).

\[
\Delta \dot{Q} = \dot{Q}(f = 0^+) - \dot{Q}(f = 0^-) = -\dot{Q}_s
\]

\[
\overrightarrow{\nabla} \cdot \vec{Q} = \nabla \cdot \dot{Q} + \Delta \dot{Q} \cdot \hat{n} \delta(f) = \nabla \cdot \dot{Q} - \dot{Q}_n \delta(f)
\]

Now integrate \( \overrightarrow{\nabla} \cdot \vec{Q} \) over the entire 3D space. \( \int \overrightarrow{\nabla} \cdot \dot{Q} \, d\Omega = 0 \) since

\[
\int \overrightarrow{\nabla} \cdot \dot{Q} \, dx_1 \, dx_2 \, dx_3 = \int_{x_1 = -\infty}^{x_1 = \infty} \int_{x_2 = -\infty}^{x_2 = \infty} \int_{x_3 = -\infty}^{x_3 = \infty} \nabla \cdot \dot{Q}_n \, dx_2 \, dx_3 = 0
\]

Similarly for \( \frac{\partial Q_2}{\partial x_2} \) and \( \frac{\partial Q_3}{\partial x_3} \). Now, we have

\[
\int \overrightarrow{\nabla} \cdot \dot{Q} \, d\Omega = \int \left[ \nabla \cdot \dot{Q}_1 - \dot{Q}_n \delta(f) \right] \, dx_1 \, dx_2 \, dx_3 = 0
\]
Illustration of Manipulation of Generalized Functions (Cont'd)

This is the divergence theorem. This result is valid if \( \hat{Q} \) is discontinuous across a surface \( g = 0 \) inside \( \Omega \).

\[
\int_{\Omega} \nabla \cdot \hat{Q} d\Omega = \int_{\Omega} [\nabla \cdot \hat{Q} + \Delta \hat{Q} \cdot \hat{n}' \delta(g)] d\Omega
\]

\[
= \int_{\Omega} \nabla \cdot \hat{Q} d\Omega + \int_{\partial \Omega} \Delta Q_{n'} dS = \int_{\partial \Omega} Q_{n} dS
\]

\[
\int \nabla \cdot \hat{Q} d\Omega = \int_{\partial \Omega} Q_{n} dS - \int_{\partial \Omega} \Delta Q_{n'} dS
\]

\[
\Delta Q_{n'} = \hat{n}' \cdot [\hat{Q}(g = 0+) - \hat{Q}(g = 0-)]
\]

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Illustration of Manipulation of Generalized Functions
(Cont’d)

In deriving conservation laws in differential form from finite volumes involving discontinuous functions, whenever the divergence theorem is used to convert surface integrals into volume integrals, one should use generalized derivative. Such conservation laws have the jump conditions incorporated in them.

Example: Shock Jump Conditions: Let the shock surface be given by

\[ f(\mathbf{x}, t) = 0, \nabla f = \mathbf{n}, \Rightarrow \frac{df}{dt} = -v_n \text{ local shock normal speed,} \]

Mass continuity eq.: \[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \]

\[ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) + \Delta \rho \frac{\partial f}{\partial t} \delta(f) + \Delta(\rho u_i) \frac{\partial f}{\partial x_i} \delta(f) \]
\[ = 0 \]
\[ [-v_n \Delta \rho + \Delta(\rho u_n)] \delta(f) = 0 \]

\[ \therefore \Delta[\rho(u_n - v_n)] = 0 \]

Similarly for momentum and energy equations.
Things to Know About Green's Function of Wave Equation

- The Green's function of the wave equation in the unbounded space is

\[
G(\hat{y}, \tau; \hat{x}, t) = \begin{cases} 
\frac{\delta(g)}{4\pi r} & \tau \leq t \\
0 & \tau > t
\end{cases}
\]

\[
g = \tau - t + \frac{r}{c} \text{ outgoing wave}
\]

(\(\hat{y}, \tau\)) source space-time variables

(\(\hat{x}, t\)) observer space-time variables
Things to Know About Green's Function of Wave Equation

- There are many methods to derive $G(\hat{y}, \tau; \hat{x}, t)$ rigorously. It is easy to show that $G$ depends on $\hat{x} - \hat{y}$ and $t - \tau$. Using $\hat{x} - \hat{y} = \hat{r}$, $\lambda = t - \tau$, take spatial Fourier transform of $\tilde{\square}^{2}_{(\hat{r}, \lambda)} G = \delta(\hat{r})\delta(\lambda)$ to get a simple problem involving finding the Green's function of an O.D.E. in $\lambda$. The inverse spatial Fourier transform of the Green's function of the O.D.E. gives Green's function of the wave equation for both the outgoing and incoming waves.

$r = |\hat{x} - \hat{y}|$, $\hat{r} = \frac{\hat{r}}{r}$, $\frac{\partial r}{\partial x_i} = \hat{r}_i$, $\frac{\partial r}{\partial y_i} = -\hat{r}_i$

Useful things to remember
Things to Know About Green's Function of Wave Equation

The support of $\delta(g)$ is on the surface $g = 0$. The surface $g = 0$ is $r = |\hat{x} - \hat{y}| = c(t - \tau)$. This is the characteristic cone of the wave equation with vertex at $(\hat{x}, t)$. Since $\Box^2$ is a differential equation with constant coefficients, $g = 0$ is also the characteristic conoid with vertex at $(\hat{x}, t)$. This gives us the picture on the right. Note that we have drawn the 3D space $\Omega$ as a plane in the figure. Therefore, this figure is a 3D illustration of what happens in 4D (3D space + time).

- **Note:** $g = 0$ is a cone because if the 4-vector $\hat{A} = (\hat{x} - \hat{y}, t - \tau)$ lies on $g = 0 \Rightarrow \alpha \hat{A} = [\alpha(\hat{x} - \hat{y}), \alpha(t - \tau)]$ also lies on $g = 0$. This is the property of a cone.

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Things to Know About Green's Function of Wave Equation

- Visualization of domain of dependence of ($\hat{x}$, $t$) in four dimensions.

Fix ($\hat{x}$, $t$) and $\tau \Rightarrow r = c(t - \tau)$ is a sphere with center at $\hat{x}$ and radius $c(t - \tau)$. Any source on this sphere at time $\tau$, contributes to $\hat{x}$ at time $t$. As $\tau$ increases, the radius shrinks, hence we have a collapsing sphere. Radius becomes zero at $\tau = t$. 

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The Collapsing Sphere Concept

Equation of collapsing sphere: \( r = c(t - \tau), (\hat{x}, t) \) fixed

The \( \Sigma \)-surface is the locus of \( \Gamma \)-curves in space. If the blade surface is described by \( f(y, \tau) = 0 \), the equation of the \( \Sigma \)-surface is:

\[
F(y, \hat{x}, t) = \left[ f(y, \tau) \right]_{ret} = f(y, t - r/c) = 0, (\hat{x}, t) \text{ fixed}
\]
Construction of $\Sigma$-Surface for a Helicopter Rotor Blade

In this construction, we have taken a rotor blade of zero thickness rotating with rotational Mach number 0.67 and forward Mach number 0.15. The observer is in the rotor plane. The circles are the intersection of the collapsing sphere with the plane containing the rotor. The circles are drawn at equal source time intervals. The observer time is $t = \tau + r/c$ where $r$ is the radius of the collapsing sphere at $\tau$. Note that $t$ is fixed for the above $\Sigma$-surface.

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Use of Green’s Functions for Discontinuous Solutions

Green’s function can be used to find discontinuous solutions if the derivatives in the differential equation are treated as generalized derivatives. This adds to usefulness of Green’s function.

**Example:** *Green’s Identity for Laplace Equation*

Let \( \tilde{\phi}(\tilde{x}) = \begin{cases} \phi(\tilde{x}) & \tilde{x} \in \Omega \\ 0 & \tilde{x} \notin \Omega \end{cases} \Rightarrow \nabla^2 \tilde{\phi} = 0 \text{ everywhere.} \)

\[
\nabla \tilde{\phi} = \nabla \tilde{\phi} + \Delta \tilde{\phi} \hat{n} \delta(f) = \nabla \tilde{\phi} - \phi \hat{n} \delta(f)
\]

\[
\nabla^2 \tilde{\phi} = \nabla^2 \tilde{\phi} - \nabla \phi \cdot \hat{n} \delta(f) - \nabla \cdot [\phi \hat{n} \delta(f)]
\]

\[= - \frac{\partial \phi}{\partial n} \delta(f) - \nabla \cdot [\phi \hat{n} \delta(f)]\]
Use of Green’s Functions for Discontinuous Solutions 
(Cont’d)

Since this equation is valid in the unbounded space, we can use the Green’s function $-\frac{1}{4\pi r}$ to get the Green’s identity

$$4\pi \tilde{\phi}(\hat{x}) = \int \frac{1}{r} \frac{\partial \phi}{\partial n} \delta(f)d\hat{y} + \nabla \hat{x} \cdot \int \frac{\phi \hat{n}}{r} \delta(f)d\hat{y}$$

$$= \int \frac{1}{r} \frac{\partial \phi}{\partial n} dS + \nabla \hat{x} \cdot \int \frac{\phi \hat{n}}{r} dS = \int \frac{\phi_n}{r} dS - \int \frac{\phi \cos \theta}{r^2} dS$$

This method tells us that when $\hat{x} \notin \Omega$, $\tilde{\phi} = 0$ which is not obvious from the classical derivation. The exterior problem is similar.

Note: $r = |\hat{x} - \hat{y}|$ is the only term in the integrands of the above integrals which is a function of $\hat{x}$. We assume that $\hat{x}$ is not located on $S$ and $S$ is piecewise smooth. The justification for the exchange of the divergence and integral operators follows from classical analysis.
The Two Forms of the Solution of Wave Equation
(Volume Sources)

We want to find the solution of $\Box^2 \phi = Q(\hat{x}, t)$

$$4\pi \phi(\hat{x}, t) = \int \frac{1}{r} Q(\hat{y}, \tau) \delta(g) d\hat{y} d\tau$$

All volume integrals are over unbounded 3 space and all time integrals are over $(-\infty, t)$.

i) Let $\tau \rightarrow g \Rightarrow \frac{\partial g}{\partial \tau} = 1$ and

$$4\pi \phi(\hat{x}, t) = \int \frac{1}{r} Q(\hat{y}, g + t - \frac{r}{c}) \delta(g) dg d\hat{y}$$

Integrate with respect to $g$ to get

$$4\pi \phi(\hat{x}, t) = \int \frac{1}{r} Q(\hat{y}, t - \frac{r}{c}) d\hat{y} = \int \frac{[Q]_{ret}}{r} d\hat{y}$$

Retarded Time Solution
The Two Forms of the Solution of Wave Equation (Volume Sources) (Cont'd)

ii) Let $y_3 \rightarrow g \Rightarrow \frac{\partial g}{\partial y_3} = -\frac{1}{c} \hat{r}_3$

$$4\pi \phi(\hat{x}, t) = \int \frac{cQ(\hat{y}, \tau)}{r} \delta(g) \, dg \frac{dy_1 dy_2}{|\hat{r}_3|} \, d\tau$$

Since in the inner integrals $(\hat{x}, t)$ and $\tau$ are fixed, then $\frac{dy_1 dy_2}{|\hat{r}_3|} = d\Omega$ element of surface area of sphere $r = c(t - \tau)$. Integrate with respect to $g$ to get:

$$4\pi \phi(\hat{x}, t) = \int_{-\infty}^{t} \frac{d\tau}{t - \tau} \int_{r = c(t - \tau)} Q(\hat{y}, \tau) \, d\Omega$$

Collapsing Sphere Solution
Derivation of the Stationary, Subsonic and Supersonic Kirchhoff Formulas
The Governing Wave Equation for Deriving Kirchhoff Formulas

We consider the exterior problem here.

$\Omega$: The exterior unbounded space

Let $\tilde{\phi}(\tilde{x}, t) = \begin{cases} \phi(\tilde{x}, t) & \tilde{x} \in \Omega \\ 0 & \tilde{x} \notin \Omega \end{cases} \Rightarrow \Box^2 \tilde{\phi} = 0$ everywhere

$$\frac{\partial \tilde{\phi}}{\partial t} = \frac{\partial \phi}{\partial t} + \phi \frac{\partial f}{\partial t} \delta(f) = \frac{\partial \phi}{\partial t} - v_n \phi \delta(f)$$

where $v_n = -\frac{\partial f}{\partial t}$ is the local normal velocity on $f = 0$
The Governing Wave Equation for Deriving Kirchhoff Formulas (Cont’d)

Next take the second time derivative of $\ddot{\phi}$:

$$\frac{\partial^2 \ddot{\phi}}{\partial t^2} = \frac{\partial^2 \ddot{\phi}}{\partial t^2} + \frac{\partial \phi}{\partial t} \frac{\partial f}{\partial t} \delta(f) - \frac{\partial}{\partial t} [\nu_n \phi \delta(f)] = \frac{\partial^2 \ddot{\phi}}{\partial t^2} - \nu_n \phi_t \delta(f) - \frac{\partial}{\partial t} [\nu_n \phi \delta(f)]$$

Similarly for the space derivatives we have:

$$\nabla \ddot{\phi} = \nabla \ddot{\phi} + \phi \ddot{\phi} \delta(f), \quad \nabla^2 \ddot{\phi} = \nabla^2 \ddot{\phi} + \phi_n \delta(f) + \nabla \cdot [\phi \ddot{\phi} \delta(f)]$$

The above results give:

$$\Box^2 \ddot{\phi} = \frac{1}{c^2} \frac{\partial^2 \ddot{\phi}}{\partial t^2} - \nabla^2 \ddot{\phi} = \Box^2 \ddot{\phi} - \left(\frac{\nu_n \phi_t}{c^2} + \phi_n \right) \delta(f)$$

$$-\frac{1}{c^2} \frac{\partial}{\partial t} [\nu_n \phi \delta(f)] - \nabla \cdot [\phi \ddot{\phi} \delta(f)]$$
The Governing Wave Equation for Deriving Kirchhoff Formulas (Cont’d)

Since $\square^2 \tilde{\phi} = 0$, and using $M_n = v_n/c$, we get

$$\square^2 \tilde{\phi} = - \left( \phi_n + \frac{1}{c} M_n \phi_t \right) \delta(f) - \frac{1}{c} \frac{\partial}{\partial t} [M_n \phi \delta(f)] - \nabla \cdot \left[ \phi \hat{n} \delta(f) \right]$$

We now solve this wave equation for stationary, subsonic and supersonic surfaces.


**Derivation of the Classical Kirchhoff Formula**

The Kirchhoff surface \( f(\hat{x}) \) is now stationary so that \( M_n = 0 \). The governing wave equation is

\[
\Box^2 \phi = -\phi_n \delta(f) - \nabla \cdot [\phi \hat{n} \delta(f)]
\]

\[
4\pi \tilde{\phi}(\hat{x}, t) = -\int \frac{\phi_n}{r} \delta(f) \delta(g) d\hat{y} d\tau - \nabla \cdot \int \frac{\phi \hat{n}}{r} \delta(f) \delta(g) d\hat{y} d\tau
\]

where \( \phi_n \) and \( \phi \) in the integrands are functions of \((\hat{y}, \tau)\). Now let \( \tau \to g, \frac{\partial g}{\partial \tau} = 1 \), and integrate with respect to \( g \), to get

\[
4\pi \tilde{\phi}(\hat{x}, t) = -\int \frac{\phi_n(\hat{y}, t-r/c)}{r} \delta(f) d\hat{y} - \nabla \cdot \int \frac{\phi(\hat{y}, t-r/c) \hat{n}}{r} \delta(f) d\hat{y}
\]

We have dealt with these integrals before. The integration of \( \delta(f) \) gives

\[
4\pi \tilde{\phi}(\hat{x}, t) = -\int_{f=0}^{1} \frac{1}{r} \phi_n(\hat{y}, t-r/c) dS - \nabla \cdot \int_{f=0}^{\hat{n}} \frac{\hat{n}}{r} \phi(\hat{y}, t-r/c) dS
\]
Derivation of the Classical Kirchhoff Formula (Cont'd)

Taking the divergence operator in and using subscript ret for retarded time, we get the classical Kirchhoff formula

\[
4\pi \tilde{\phi}(\hat{x}, t) = \int_{f = 0} \frac{[c^{-1}\phi \cos \theta - \phi_n]_{\text{ret}}}{r} dS + \int_{f = 0} \frac{\cos \theta \phi_{\text{ret}}}{r^2} dS
\]

In this equation \( \cos \theta = \hat{n} \cdot \hat{r} \). Again, our method tells that \( \tilde{\phi}(\hat{x}, t) = 0 \) in the interior of \( f = 0 \) which is not obvious from classical derivation.

**Note**: Only \( r \) is a function of \( \hat{x} \) in the integrands of the integrals in previous vugraph. We assume \( \hat{x} \) is not on \( S \) and \( S \) is piecewise smooth. The justification for bringing the divergence operator inside the integral follows from classical analysis.
Derivation of the Subsonic Kirchhoff Formula

We now assume a deformable surface moving at subsonic speed.

Governing equation:

\[
\ddot{\phi} = -\left(\phi_n + c^{-1} M_n \phi_t\right) \delta(f) - \frac{1}{c} \frac{\partial}{\partial t} [M_n \phi \delta(f)] - \nabla \cdot [\phi \dot{n} \delta(f)]
\]

\[
4\pi \tilde{\phi}(\hat{x}, t) = -\int \frac{1}{r} \left(\phi_n + c^{-1} M_n \phi_t\right) \delta(f) \delta(g) d\hat{y} d\tau - \frac{1}{c} \frac{\partial}{\partial t} \int \frac{1}{r} M_n \phi \delta(f) \delta(g) d\hat{y} d\tau - \nabla \cdot \int \frac{1}{r} \phi_n \delta(f) \delta(g) d\hat{y} d\tau
\]

Note that in the above equation \( \phi_t = \frac{\partial \phi(\hat{y}, \tau)}{\partial \tau} \).
Derivation of the Subsonic Kirchhoff Formula (Cont’d)

In the last integral, take divergence operator in. It only must operate on \( \frac{\delta(g)}{r} \) which depends on \( \hat{x} \). Now use the following result to write the last integral as two integrals:

\[
\nabla_{\hat{x}} \left[ \frac{\delta(g)}{r} \right] = -\frac{1}{c} \frac{\partial}{\partial t} \left[ \frac{\hat{r} \delta(g)}{r} \right] - \frac{\hat{r} \delta(g)}{r^2}, \quad \hat{\hat{r}} = \frac{\hat{r}}{r}
\]

\[
\nabla_{\hat{x}} \cdot \int \frac{1}{r} \phi \hat{n} \delta(f) \delta(g) d\hat{y} d\tau = \int \phi \delta(f) \hat{n} \cdot \nabla_{\hat{x}} \left[ \frac{\delta(g)}{r} \right] d\hat{y} d\tau
\]

\[
= -\frac{1}{c} \frac{\partial}{\partial t} \int \frac{1}{r} \phi \cos \theta \delta(f) \delta(g) d\hat{y} d\tau
\]

\[-\int \frac{1}{r^2} \phi \cos \theta \delta(f) \delta(g) d\hat{y} d\tau
\]

Substitute in equation for \( \tilde{\phi} \) above. We have used the rule for the exchange of limit processes for generalized functions here.
Derivation of the Subsonic Kirchhoff Formula (Cont'd)

\[ 4\pi \delta(t, t) = -\int \frac{1}{r} (\phi_n + c^{-1} M_n \phi \tau) \delta(f) \delta(g) d\gamma d\tau \]

\[ + \int \frac{1}{r^2} \phi \cos \theta \delta(f) \delta(g) d\gamma d\tau \]

\[ + \frac{1}{c} \frac{\partial}{\partial t} \int r \delta(f) \delta(g) d\gamma d\tau \]

We have two kinds of integrals in the above equation

\[ I_1 = \int Q_1(\gamma, \tau) \delta(f) \delta(g) d\gamma d\tau \]

\[ I_2 = \frac{1}{c} \frac{\partial}{\partial t} \int Q_2(\gamma, \tau) \delta(f) \delta(g) d\gamma d\tau \]
Derivation of the Subsonic Kirchhoff Formula (Cont’d)

- Let us parametrize \( S: f(\hat{y}, \tau) = 0 \) by surface coordinates \((u^1, u^2)\) with domain \( D(S) \). We assume \( D(S) \) is time independent. This is always possible. But \( g_{(2)} \), the det. of coef. of 1st fund. form is a function of time \( \tau \). Parametrize the space near \( f = 0 \) by taking \( u^3 = f \) and extend \((u^1, u^2)\) along local normal to \( f = 0 \). Now we have \( d\hat{y} = \sqrt{g_{(2)}} du^1 du^2 du^3 \) (strictly speaking, we should use \( g'_{(2)} \) but it makes no difference here).

We use \( Q_1(u^1, u^2, u^3, \tau) \) for \( Q_1[\hat{y}(u^1, u^2, u^3, \tau), \tau] \) in \( I_1 \)

\[
I_1 = \int Q_1(u^1, u^2, u^3, \tau) \delta(u^3) \delta(g) \sqrt{g_{(2)}} du^1 du^2 du^3 d\tau
\]

\[
= \int_{-\infty}^{t} \int_{D(S)} Q_1(u^1, u^2, 0, \tau) \delta(g) \sqrt{g_{(2)}} du^1 du^2 d\tau
\]
Derivation of the Subsonic Kirchhoff Formula (Cont’d)

Now let $\tau \rightarrow g$, $\frac{\partial g}{\partial \tau} = 1 - M_r$ because $g = \tau - t + |\dot{x} - \dot{y}(u^1, u^2, 0, \tau)|/c$,

$\vec{M} = \frac{\partial \dot{y}(u^1, u^2, 0, \tau)}{\partial \tau}$, \quad $M_r = \vec{M} \cdot \hat{r} \Rightarrow$

$$I_1 = \int_{D(S)} \left[ \frac{Q_1 \sqrt{g(2)}}{1 - M_r} \right]_{\tau^*} du^1 du^2$$

Here $\tau^*$ is the emission time of point $(u^1, u^2)$ on $f = 0$ for a fixed $(\dot{x}, t)$.

The emission time $\tau^*$ is the solution of:

$\tau^* - t + |\dot{x} - \dot{y}(u^1, u^2, 0, \tau^*)|/c = 0$
Derivation of the Subsonic Kirchhoff Formula (Cont’d)

Similar procedure for $I_2$ gives

$$I_2 = \frac{1}{c} \int_{D(S)} \frac{\partial}{\partial t} \left( \frac{Q_2 \sqrt{g(2)}}{1 - M_r} \right)_{\tau^*} du^1 du^2$$

$$I_2 = \frac{1}{c} \int_{D(S)} \left[ \frac{1}{1 - M_r} \frac{\partial}{\partial \tau} \left( \frac{Q_2 \sqrt{g(2)}}{1 - M_r} \right) \right]_{\tau^*} du^1 du^2$$

We note that $\tau^* = \tau^*(u^1, u^2; \dot{x}, t)$ so that $\frac{\partial}{\partial t} \bigg|_{\dot{x}} = \frac{\partial \tau^*}{\partial t} \frac{\partial}{\partial \tau^*}$. From the equation for emission time

$$\frac{\partial \tau^*}{\partial t} = \frac{1}{1 - M_r}.$$
Derivation of the Subsonic Kirchhoff Formula (Cont'd)

From the results for $I_1$ and $I_2$, we get

$$4\pi \hat{\phi}(\dot{x}, t) = -\int_{D(S)} \left[ \frac{(\phi_n + c^{-1}M_n \phi_r) \sqrt{g(2)}}{r(1-M_r)} \right]_{\tau^*} \, du^1 du^2$$

$$+ \int_{D(S)} \left[ \frac{\phi \sqrt{g(2)} \cos \theta}{r^2(1-M_r)} \right]_{\tau^*} \, du^1 du^2$$

$$+ \frac{1}{c} \int_{D(S)} \left[ \frac{1}{1-M_r} \frac{\partial}{\partial \tau} \left( \frac{(\cos \theta - M_n) \phi \sqrt{g(2)}}{r(1-M_r)} \right) \right]_{\tau^*} \, du^1 du^2$$
Derivation of the Subsonic Kirchhoff Formula (Cont'd)

- This result was originally derived by W. R. Morgans (Phil. Mag., vol. 9, 1930, 141–161). It was rederived by Farassat and Myers using the above method (JSV, vol. 123(3), 1988, 451–460). These authors have given a useful formula for applications in the following form

\[
4\pi \tilde{\phi}(\hat{x}, t) = \int_{D(S)} \left[ \frac{E_1 \sqrt{g(2)}}{r(1 - M_r)} \right]_{\tau^*} du^1 du^2 + \int_{D(S)} \left[ \frac{\phi E_2 \sqrt{g(2)}}{r^2(1 - M_r)} \right]_{\tau^*} du^1 du^2
\]

where \( E_1 \) and \( E_2 \) are long expressions given in the above reference. This formula was verified by using analytic input for rigid surfaces.

- **Note:** The above Kirchhoff formula has a Doppler singularity in the denominator for supersonic surfaces. This makes the above result unsuitable for a supersonic surface. However, the supersonic Kirchhoff formula is inherently less efficient on a computer. We prefer to use the above formula for any panel on the Kirchhoff surface without the Doppler singularity.
A Simple Trick in Preparation for Supersonic Kirchhoff Formula

To reduce algebraic manipulations and to obtain the simplest form of the supersonic Kirchhoff formula, we introduce the following trick. Note that in the governing wave equation for deriving Kirchhoff formula, we have terms involving time and space derivatives: \( \frac{\partial}{\partial t}[M_n \phi \delta(f)] \) and \( \nabla \cdot [\phi \hat{n} \delta(f)] \). We need to take these derivatives explicitly. We propose the following simplification of this process.

**Observation:** \( \phi(x) \delta(x) = \phi(0) \delta(x) \) take derivatives of both sides \( \phi'(x) \delta(x) + \phi(x) \delta'(x) = \phi(0) \delta'(x) \). It is obvious that the right side is simpler than the left side. What is so special about \( \phi(0) \delta(x) \)? Here \( \phi(x) \) is *restricted* to the support of the delta function, i.e., \( x = 0 \). Can restriction of \( \phi(x) \) to the support of \( \delta(f) \) in \( \phi(x) \delta(f) \) reduce manipulations when we take derivatives of \( \phi(x) \delta(f) \)? The answer is yes!
A Simple Trick in Preparation for Supersonic Kirchhoff Formula (Cont'd)

We use the notation \( \tilde{\phi}(\hat{x}) \) for restriction of \( \phi(\hat{x}) \) to the support of \( \delta(f) \).

Using the local parametrization of space near \( f = 0 \) ((\( u^1, u^2 \)) on \( f = 0 \), \( u^3 = \) distance from \( f = 0 \)), we have \( \tilde{\phi}(\hat{x}) = \phi(u^1, u^2, 0) \)

Similarly \( \tilde{\phi}(\hat{x}, t) = \phi(u^1, u^2, 0, t) \), note \( u^i = u^i(\hat{x}, t) \) we have

\[
\phi(\hat{x}, t)\delta(f) = \tilde{\phi}(\hat{x}, t)\delta(f)
\]

See NASA TP-3428 for some more explanation. In manipulation of term involving derivatives of the product of a delta function and an ordinary function, always restrict the ordinary function to the support of the delta function.
A Simple Trick in Preparation for Supersonic Kirchhoff Formula (Cont'd)

\[ \nabla[\phi(\dot{x}, t)\delta(f)] = \nabla\phi\delta(f) + \phi\nabla f\delta'(f) \]  
(A)

\[ \nabla[\phi(\dot{x}, t)\delta(f)] = \nabla_2\phi\delta(f) + \phi\nabla f\delta'(f) \]  
(B)

where \( \nabla_2\phi \) is the surface gradient of \( \phi \). As expected, the integration of the right side of (A) is algebraically somewhat more complicated than integration of the right side of (B).

- Note that

\[
\frac{\partial \phi}{\partial x_i} = \frac{\partial \phi}{\partial x_i} - n_i \frac{\partial \phi}{\partial n}, \quad \frac{\partial \phi}{\partial n} = 0, \quad \frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial t} + \nu \frac{\partial \phi}{n \partial n}
\]

Note: Wave propagation literature use \( \frac{\delta \phi}{\delta x_i} \) for \( \frac{\partial \phi}{\partial x_i} \) and \( \frac{\delta \phi}{\delta t} \) for \( \frac{\partial \phi}{\partial t} \).
Derivation of the Supersonic Kirchhoff Formula

We are now interested to develop the supersonic Kirchhoff formula for a panel on the Kirchhoff surface. This is only because of practical consideration. On the surface \( f(\tilde{x}, t) = 0 \), we define a panel by its edge curve \( \tilde{f} = 0 \) such that \( \tilde{f} > 0 \) on the panel and \( \nabla \tilde{f} = \hat{\nu} \) the local unit geodesic normal at the edge. \( f = \tilde{f} = 0 \). The geodesic normal is tangent to the panel and normal to the edge. Denoting Heaviside function by \( H(\tilde{f}) \), our governing differential equation for finding the Kirchhoff formula for the panel is

\[
\Box^2 \phi = -(\phi_n + c^{-1} M_n \phi_t) H(\tilde{f}) \delta(f) - \frac{1}{c} \frac{\partial}{\partial t} [M_n \phi H(\tilde{f}) \delta(f)] \\
- \nabla \cdot [\phi \hat{\nu} H(\tilde{f}) \delta(f)]
\]
Derivation of the Supersonic Kirchhoff Formula (Cont'd)

\[ \frac{1}{c} \frac{\partial}{\partial t} [ M_n \phi H(\tilde{f}) \delta(f) ] = \frac{1}{c} \frac{\partial}{\partial t} (M_n \phi) H(\tilde{f}) \delta(f) - M_n M_\nu \phi \delta(\tilde{f}) \delta(f) \]

\[ -M_n^2 \phi H(\tilde{f}) \delta'(f) \]

where \( M_\nu = \vec{M} \cdot \hat{v} \) is the local Mach number of the edge in the direction of \( \hat{v} \).

Using the divergence result derived earlier, we have

\[ \nabla \cdot [ \phi \hat{n} H(\tilde{f}) \delta(f) ] = -2H_f \phi H(\tilde{f}) \delta(f) + \phi H(\tilde{f}) \delta'(f) \]

where \( H_f \) is the local mean curvature of \( f = 0 \).
Derivation of the Supersonic Kirchhoff Formula (Cont’d)

The governing equation for deriving the supersonic Kirchhoff formula for a panel is

\[ \Box^2 \phi = - \left[ \phi_n + c^{-1} M_n \phi_t + c^{-1} (M_n \phi)_{t} - 2 H_f \phi \right] H(\tilde{f}) \delta(f) \]

\[ - (1 - M_n^2) \phi H(\tilde{f}) \delta'(f) + M_n M_v \phi \delta(\tilde{f}) \delta(f) \]

\[ \equiv q_1 H(\tilde{f}) \delta(f) + q_2 H(\tilde{f}) \delta'(f) + q_3 \delta(\tilde{f}) \delta(f) \]

We see that we have three kinds of sources which we will treat below.

- **Note:** The solution of wave equation with sources of the type in the above equation is given in detail in NASA Technical Paper 3428, May 1994, by F. Farassat.
Derivation of the Supersonic Kirchhoff Formula (Cont’d)

Let $\tilde{\phi} = \phi_1 + \phi_2 + \phi_3$ where $\phi_i$’s are solutions of wave equation with sources involving $q_i$ ($i = 1 - 3$).

$$\square^2 \phi_1 = q_1 H(\tilde{f}) \delta(f), \quad \square^2 \phi_2 = q_2 H(\tilde{f}) \delta'(f), \quad \square^2 \phi_3 = q_3 \delta(f) \delta(\tilde{f})$$

i) Solution of $\square^2 \phi_1 = q_1 H(\tilde{f}) \delta(f)$ [Eq. (4.23b), NASA TP-3428]

$$4\pi \phi_1(\hat{x}, t) = \int \frac{q_1(\hat{y}, \tau)}{r} H(\tilde{f}) \delta(f) \delta(g) d\hat{y} d\tau$$

Let $\tau \to g, \frac{\partial g}{\partial \tau} = 1$, integrate with respect to $g$

$$4\pi \phi_1(\hat{x}, t) = \int \frac{[q_1]_{ret}}{r} H(\tilde{F}) \delta(F) d\hat{y}$$
Derivation of the Supersonic Kirchhoff Formula (Cont'd)

Here $F(\hat{y}; \hat{x}, t) = [f(\hat{y}, \tau)]_{ret} = f(\hat{y}, t-r/c)$ and

$\tilde{F}(\hat{y}; \hat{x}, t) = [\tilde{f}(\hat{y}, \tau)]_{ret} = \tilde{f}(\hat{y}, t-r/c)$

We have treated integrals of this type before. We write the element of surface area of $F = 0$ by $d\Sigma$. The $\Sigma$-surface was explained earlier.

$$4\pi \phi_1(\hat{x}, t) = \int \frac{[q_1]_{ret}}{r\Lambda} d\Sigma$$

$$F = 0 \quad \tilde{F} > 0$$

$$\Lambda^2 = |\nabla F|^2 = 1 + M_n^2 - 2M_n \cos \theta$$

$$\cos \theta = \hat{n} \cdot \hat{r}$$
Derivation of the Supersonic Kirchhoff Formula
(Cont'd)

ii) Solution of \[ \Box^2 \phi_2 = q_2 H(\tilde{f}) \delta'(f) \] [Eq. (4.23e), NASA TP-3428]

\[ 4\pi \phi_2(\hat{x}, t) = \int \frac{q_2}{r} H(\tilde{f}) \delta'(f) \delta(g)d\hat{y}d\tau \]

Let \( \tau \rightarrow g, \frac{\partial g}{\partial \tau} = 1 \), integrate with respect to \( g \)

\[ 4\pi \phi_2(\hat{x}, t) = \int \frac{[q_2]_{ret}}{r} H(\tilde{F}) \delta'(\tilde{F})d\hat{y} \]

The interpretation of integrals involving derivatives of delta functions was given before. Note that we get a line integral on \( F = \tilde{F} = 0 \) because

\[ \frac{\partial}{\partial N} H(\tilde{F}) = \hat{N} \cdot \nabla \tilde{F} \delta(\tilde{F}) \]
Derivation of the Supersonic Kirchhoff Formula
(Cont'd)

\[ 4\pi \phi_2(\hat{x}, t) = \int_{F = 0}^{\tilde{F} = 0} \left\{ -\frac{1}{\Lambda} \frac{\partial}{\partial N} \frac{[q_2]_{\text{ret}}}{r \Lambda} + \frac{2H_F [q_2]_{\text{ret}}}{r \Lambda^2} \right\} d\Sigma \]

\[ -\int_{F = 0}^{\tilde{F} = 0} \frac{[q_2]_{\text{ret}} \cot \theta'}{r \Lambda^2} dL \]

\[ \tilde{N} = \frac{\nabla F}{|\nabla F|} = \frac{\hat{n} - M n \hat{r}}{\Lambda}, \quad \tilde{N} = \frac{\nabla \tilde{F}}{|\nabla \tilde{F}|} = \frac{\hat{\nu} - M \nu \hat{r}}{\tilde{\Lambda}} \]

\[ \tilde{\Lambda}^2 = |\nabla \tilde{F}|^2 = 1 + M^2 - 2M \nu \cos \tilde{\theta}, \quad \cos \tilde{\theta} = \hat{\nu} \cdot \hat{r} \]

\[ \cos \theta' = \tilde{N} \cdot \tilde{N} \quad H_F \text{ mean curvature of } \Sigma\text{-surface} \]
Derivation of the Supersonic Kirchhoff Formula (Cont'd)

iii) Solution of \( \Box^2 \phi_3 = q_3 \delta(\tilde{f})\delta(f) \) [Eq. (4.23f), NASA TP-3428]

\[
4\pi\phi_3(\hat{x}, t) = \int \frac{q_3}{r} \delta(\tilde{f})\delta(f)\delta(g)d\hat{y}d\tau
= \int \frac{1}{r} [q_3]_{ret} \delta(\tilde{F})\delta(F)d\hat{y}
\]

The interpretation of this integral was given before.

\[
4\pi\phi_3(\hat{x}, t) = \int \frac{1}{r} [q_3]_{ret} dL
\]

\[
F = 0 \quad \Lambda_0
\]

\[
\tilde{F} = 0
\]

where \( \Lambda_0 = |\nabla F \times \nabla \tilde{F}| = \Lambda \tilde{\Lambda} \sin \theta' \), \( \cos \theta' = \hat{N} \cdot \hat{\tilde{N}} \)
Derivation of the Supersonic Kirchhoff Formula (Cont’d)

Now putting the solutions for φ₁, φ₂, and φ₃ together in ̇ϕ, we get the supersonic Kirchhoff formula

\[
4\pi ̇\phi(x, t) = \int_{F=0}^{F>0} \frac{1}{r\Lambda} \left[ Q₁ + \frac{2H_F}{\Lambda} Q₂ + \frac{\hat{N} \cdot \nabla \Lambda}{\Lambda^2} Q₂ - \frac{\hat{N} \cdot \nabla Q₂}{\Lambda} \right] d\Sigma
\]

\[
+ \int_{F=0}^{F>0} \frac{\hat{N} \cdot \nabla r}{r^2 \Lambda^2} Q₂ d\Sigma + \int_{F=0}^{F=0} \frac{1}{r\Lambda_0} \left[ Q₃ - \frac{\hat{\Lambda} \cos \theta'}{\Lambda} Q₂ \right] dL
\]

where \( Q_i = [q_i]_{ret}, i = 1-3 \)

This equation was derived by Farassat and Myers in 1994. It was presented at ASME Int. Mech. Eng. Congress and Expo., Nov. 6–11, 1994, Chicago, Illinois. It was also published with improved derivation as a paper at the First Joint CEAS/AIAA Aeroacoustics Conference, June 12–15, 1995, Munich, Germany.
The Mean Curvature $H_F$ of $\Sigma$-Surface ($f = 0$ Rigid)

\[
H_F = \frac{-M_n}{r\Lambda} \left(1 - \sin^2\theta \right) + \frac{(1 - M_r)^2}{\Lambda^3} \left(\frac{1}{\Lambda} \left(\frac{\dot{r}}{\Lambda} \cdot \dot{\eta} + \kappa_1 \lambda \gamma^1 + \kappa_2 \lambda \gamma^2\right) + \sin^2\theta \left(\frac{1}{c} \dot{M}_n + M^2 \kappa_\gamma\right) + \frac{1}{\Lambda^3} \left(\ddot{\eta} \cdot \dot{\eta}\right)(\dot{\lambda} \cdot \dot{r} t)\right)
\]

\[
\Lambda = \left(1 + M_n^2 - 2M_n \cos\theta\right)^{1/2}, \quad \cos\theta = \hat{n} \cdot \hat{r}, \quad \hat{r} = \frac{\dot{x} - \dot{y}}{r}, \quad M = |\dot{M}|, \\
\hat{r}_t = \hat{r} - \hat{n} \cos\theta, \quad \dot{M}_t = \dot{M} - M_n \hat{n} \quad (\hat{r}_t \text{ and } \dot{M}_t \text{ are projections of } \hat{r} \text{ and } \dot{M} \text{ on local tangent plane to } f = 0), \quad \kappa_1 \text{ and } \kappa_2 \text{ local principal curvatures of } f = 0, \quad \kappa_\gamma \text{ local normal curvature of } f = 0 \text{ along } \dot{\eta}, \quad H_f \text{ local mean curvature of } f = 0,
\]

\[
\hat{\dot{\eta}} = \frac{1}{c} \hat{n} \times \vec{\omega}, \quad \vec{\omega} \text{ angular velocity, } \quad \hat{\dot{\eta}} = \hat{n} \times \hat{r}_t, \quad \hat{\lambda} = \hat{n} \times \dot{M}_t, \\
\dot{M}_n = \hat{n} \cdot \dot{M}, \quad (\lambda^1, \lambda^2) \text{ and } (\gamma^1, \gamma^2) \text{ components of } \hat{\lambda} \text{ and } \hat{\dot{\eta}} \text{ in principal directions with respect to unit basis vectors, respectively.}
\]
Selection of the Kirchhoff Surface for Supersonic Kirchhoff Formula

It can be shown that when the collapsing sphere leaves the Kirchhoff surface $f = 0$ tangentially at a point where $M_n = 1$, the supersonic Kirchhoff formula will develop a singularity. One can solve this problem by selecting a biconvex shape for Kirchhoff surface avoiding the above singularity condition. In rotor noise calculations, in-plane noise of high speed rotors is the most important. Reasonable shape of Kirchhoff surface is possible. Farassat and Myers have shown that the singularity from line integral in Kirchhoff formula is integrable. (See paper in *Theoretical and Computational Acoustics*, vol. 1, D. Lee et al. (eds), 1994, World Scientific Publishing.)

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References for Differential Geometry

Most of the results on differential geometry in previous pages were known by the end of the 19th century. The contribution of 20th century mathematicians has been development of general and powerful techniques to solve difficult problems but written in a supposedly rigorous style making differential geometry inaccessible to engineers. Here are a few useful books.


References for Generalized Functions


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**Abstract**

One of the active areas of computational aeroacoustics is the application of the Kirchhoff formulas to the problems of the rotating machinery noise predictions. The original Kirchhoff formula was derived for a stationary surface. In 1968, Farassat and Myers derived a Kirchhoff Formula obtained originally by Morgans using modern mathematics. These authors gave a formula particularly useful for applications in aeroacoustics. This formula is for a surface moving at subsonic speed. Later in 1995 these authors derived the Kirchhoff formula for a supersonically moving surface. This technical memorandum presents the viewgraphs of a day long workshop by the author on the derivation of the Kirchhoff formulas. All necessary background mathematics such as differential geometry and multidimensional generalized function theory are discussed in these viewgraphs. Abstraction is kept at minimum level here. These viewgraphs are also suitable for understanding the derivation and obtaining the solutions of the Farowcs Williams-Hawking equations. In the first part of this memorandum, some introductory remarks are made on generalized functions, the derivation of the Kirchhoff formulas and the development and validation of Kirchhoff codes. Separate lists of references by Lyrintzis, Long, Strawn and their co-workers are given in this memorandum. This publication is aimed at graduate students, physicists and engineers who are in need of the understanding and applications of the Kirchhoff formulas in acoustics and electromagnetics.