1 Introduction

In mankind's enduring pursuit to go faster and further with greater economy and safety in its diverse variety of vehicles that travel across land and sea or through the air and space, we are taxing our materials to their utmost capabilities. Consequently, the need for accurate material models to describe the various physical properties of a given material is much more critical in the design and development of these vehicles than it has ever been, and this need can only be expected to continue to grow.

The analysis of metallic response for high temperature applications requires mathematical models capable of predicting accurately the short-term plastic strains, the long-term creep strains, and interactions between them. Viscoplastic models attempt to do that. Multiaxial, cyclic and nonisothermal histories are normal service conditions, not exceptional ones, all of which challenge the predictive capabilities of such models.

Prior to the advent of the computer, viscoplasticity was a theory in its infancy; however, over the past two decades substantial advancements have been made to the theory. Because of viscoplasticity's innate nature, which leads to systems of first-order, ordinary, differential equations that are nonlinear, coupled, and mathematically stiff, a unique mathematical structure (like that of elasticity) is not to be expected. Nevertheless, these past two decades have given the community a vast wealth of experience with a variety of evolution equations—what works, what does not, and in many cases, some physical insight as to why. Using this experience base, we have set out to develop a viscoplastic model whose predictive capabilities are in reasonable agreement with experiments. A viscoplastic model is constructed within this theoretical framework by defining material functions that have close ties to the physics of inelasticity. As a consequence, this model is easily characterized—only steady-state creep data, monotonic stress-strain curves, and saturated stress-strain hysteresis loops are required. The model is applied to the copper alloy NARlloy Z.

2 Elasticity

The stress, $\sigma_{ij}$, is taken to be related to the infinitesimal strain, $\varepsilon_{ij}$, through the constitutive equations of an isotropic Hookean material, viz.

$$S_{ij} = 2\mu(E_{ij} - \varepsilon_{ij}^0) \quad \text{where} \quad \delta_{kk} = 0,$$

and

$$\sigma_{kk} = 3\kappa(\varepsilon_{kk} - \alpha(T - T_0)\delta_{kk}),$$

which are characterized by the shear, $\mu$, and bulk, $\kappa$, elastic moduli, and where

$$S_{ij} = \sigma_{ij} - 1/3\sigma_{kk}\delta_{ij} \quad \text{and} \quad E_{ij} = \varepsilon_{ij} - 1/3\varepsilon_{kk}\delta_{ij}$$

denote the deviatoric stress and strain, respectively. The mean coefficient of thermal expansion, $\alpha$, acts on the difference between the current temperature, $T$, and some reference temperature, $T_0$. The Kronecker delta, $\delta_{ij}$, has the value 1 if $i=j$, otherwise it is 0. Repeated Latin indices are summed from 1 to 3.

The Zener parameter is a temperature normalized measure of and
where $C = 0$ and $C > 0$ are the material constants. For
stress states below power-law breakdown, i.e., when $\mathbf{ISI} > C$, the
steady-state Zener parameter of Garofalo reduces to the
power-law relationship

$$Z_{ss} = A \left| \mathbf{ISI} \right|^n,$$  
(10)

thereby designating dislocation climb as the rate-controlling
mechanism. (Note: $A$, $C$, and $n$ are independent in Eq. (9) but
not in Eq. (10).) Similarly, when the stress exceeds power-law
breakdown, i.e., when $\mathbf{ISI} > C$, Garofalo's Zener parameter
reduces to the exponential relationship

$$Z_{ss} = A' \exp \left[ \frac{\mathbf{ISI}}{C} \right],$$  
(11)

where $A' = A/2^n$ and $C' = C/n$, thereby designating dislocation
slide as the rate-controlling mechanism. The ability of Eqs. (7)-(9) to correlate the stationary creep-rate data of NAR-
loy Z is demonstrated in Fig. 1. The material constants obtained
from this correlation are given in Table 2. Because none of
these data lie within the power-law domain, the exponential
creep equation, Eq. (11), was used to determine values for $A'$
and $C'$ leading to the straight line fit shown in the log/linear
plot of Fig. 1(a), where $A' = 5 \times 10^7$ and $C' = 3.5$ MPa
for the predefined values of $Q = 450,000$ J/mole (Lewis, 1970)
and $T = 400^\circ C$ (assumed). Taking $n = 4$ (assumed), the values

$\text{Table 1 Elastic constants for NARloy Z (Anonymous, 1986)}$

<table>
<thead>
<tr>
<th>Constants</th>
<th>Units</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>K$^{-1}$</td>
<td>$16.5 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>MPa</td>
<td>52,000</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>MPa/K</td>
<td>$-14$</td>
</tr>
<tr>
<td>$\nu$</td>
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</tr>
</tbody>
</table>

$\mu = \mu_0 + \mu_1 T$, $T$ is in K.

to 3 in the usual manner. Equation (1) characterizes the
deviatoric stress response, while Eq. (2) characterizes the
hydrostatic stress response. The plastic strain, $\varepsilon_p$, and thermal strain, $\alpha(T - T_0)\delta\nu$, are, in essence, eigenstrains that represent devi-
ations from deviatoric and hydrostatic elastic behaviors, re-
spectively.

Young's modulus, $E$, and Poisson's ratio, $\nu$, are the two
elastic constants that are usually determined via experiment.
The expressions,

$$\mu = \frac{E}{2(1 + \nu)} \quad \text{and} \quad \kappa = \frac{E}{3(1 - 2\nu)},$$  
(4)

define their interdependence with the elastic moduli of Eqs.
(1) and (2). Only two elastic moduli are independent for elas-
tically isotropic materials. Values for the elastic constants of
NARloy Z (typical composition: Cu-3%Ag-0.5%Zr) are given in Table 1.

3 Creep

The evolution of plastic strain which describes the classical
theory of creep (Odqvist, 1974) is given by

$$\dot{e}_p = \frac{1}{2} \dot{\varepsilon}^p \mathbf{ISI},$$  
(5)

with the subscript "ss" implying steady-state, and where $\dot{\varepsilon}^p$ denotes the magnitude of plastic strain-rate, and $\mathbf{ISI}$
denotes the magnitude of deviatoric stress. The subscript sig-
nifying steady state is not attached to $S_j$ because stress is a
controllable external variable, whereas creep rate is a response
variable. This equation states that an increment in creep strain
accumulates in the current direction of the deviatoric stress.
A dot is placed over a variable to signify its time rate-of-change.

The norms, or magnitudes, pertaining to the deviatoric ten-
sors of this paper are defined by

$$\mathbf{II}_j = \sqrt{2J_3/J_{ss}} \quad \text{and} \quad \mathbf{II} = \sqrt{1/2J_3/J_{ss}},$$  
(6)

where $J_3$ is any deviatoric "strain-like" tensor, and $J_{ss}$ is any
deviatoric "stress-like" tensor. These are the norms of von
Mises (1913), where the coefficients under the radical signs
are independent in Eq. (9) but

In the theory of creep, $\dot{\varepsilon}^p$ is described by a kinetic equa-
tion, i.e., an equation of state. Zener and Hollomon (1944)
determined that such a kinetic equation can, to a good
approximation, be decomposed into a product of two functions;

In the physical description of the thermal function, $\vartheta$, there
is a parameter called the activation energy, $Q$, which—for creep
at low stresses and elevated temperatures—is associated with
self-diffusion where the rate-controlling mechanism for de-
formation is dislocation climb (Sherby and Weertman, 1979).
for $A$ and $C$ have been converted to those of $A$ and $C$ that are reported in Table 2. The result is the curved line presented in the log/log plot of Fig. 1(b). We note that the value of $C$ for NARloy Z, i.e., 14 MPa, obtained with this choice for $n$, i.e., 4, is in agreement with the value of $C$ for Cu, i.e., 13 MPa, reported in Freed and Walker (1993a).

This continuum representation for creep is well established. Our viscoplastic model reduces analytically to this creep model also appear in our viscoplasticity. The evolution of the back stress accounts for the more gradual transition from elastic to plastic behavior, while the evolutions of the drag strength and yield stress account for the more gradual transition from elastic to plastic behavior, while the evolutions of the back stress accounts for kinematic (flow-induced anisotropic) hardening effects.

### 4 Viscoplastic Flow

A general mathematical structure for viscoplasticity (Freed et al., 1991) may admit up to three kinds of internal state variables; they are: (i) the (scalar-valued) drag strength, $D > 0$; (ii) the (scalar-valued) yield stress, $Y \geq 0$; and (iii) the (deviatoric tensor-valued) back stress, $B_{ij}$. The drag strength and yield stress account for isotropic hardening effects, while the back stress accounts for kinematic (flow-induced anisotropic) hardening effects.

### 5 Creep == Viscoplasticity

In the process of going from creep theory to viscoplasticity, one must remove the steady-state constraint that is present in creep, and thereby extend the domain of admissible states to include transient behavior. In other words, viscoplasticity is capable of modeling both primary and secondary creep be-
havior. The modeling of transient behavior is done through the introduction of internal state variables. Although the purpose of viscoplasticity is to model rate-dependent transient behavior, it is not unreasonable to also require that it reduces to creep theory under steady-state conditions. An important objective in our development of a viscoplastic theory is that it reduces analytically to creep theory when at steady state. Not only is this a realistic requirement, but it also strengthens the physics of the theory, and it simplifies greatly the process of model characterization—about half of our viscoplastic material constants come from correlating stationary creep-rate data alone.

In order for a viscoplastic theory to reduce analytically to creep theory when at steady-state (i.e., when $\dot{B} = 0, D = 0$, and $Y = 0$ for $\nu \neq 0$) two conditions must be satisfied. First, the back stress must be coaxial with the stress at steady state so that the directions of plastic strain-rate defined by Eqs. (5) and (12) are also coaxial at steady state. And second, it is necessary that the kinetics of viscoplasticity, Eq. (13), reduce analytically to the kinetics of creep, Eq. (7), under steady-state conditions. The evolution law for back stress given in Eq. (28) satisfies this first constraint. To satisfy the second constraint, one must first hypothesize a relationship between the steady-state and transient Zener parameters, and then hypothesize another one between the internal and external variables, when at steady state (Freed and Walker, 1990). We therefore suppose that

$$Z = Z_{\text{ss}} \left( \frac{\|S - B\| - Y}{D} \right),$$

(14)
in support of Eq. (13). This relationship implies that the transient Zener parameter, $Z$, has the same functional form as the steady-state Zener parameter, $Z_{\text{ss}}$, but with a different argument; in particular, and in accordance with Eq. (9), we take

$$Z = A \sinh\left( \frac{\|S - B\| - Y}{D} \right),$$

(15)

which is similar in form to the kinetics of Miller's (1976) viscoplastic model, but with a yield stress and without a power acting on the Macauley bracket.

Furthermore, we shall suppose that

$$I_{B_{\text{ss}}} = f_{\text{ss}} \|S\| \|S\|, \quad D_{\text{ss}} = D_0 + \delta \|S\|,$$

and

$$Y_{\text{ss}} = (1 - f) f_{\text{ss}} \|S\| \|S\|,$$

(16)
in support of experimental evidence, where $f_{\text{ss}} > 0$ and $\delta > 0$ are the steady-state fractions of applied stress that are associated with the internal stress (i.e., the back and yield stresses) and the drag strength, respectively, such that $1/2 < f < 1$. The parameter $f$ partitions the internal stress between isotropic and kinematic contributions, such that $0 < f < 1$. The fact the drag strength is taken to be proportional to the saturation stress is a consequence of the fact that the drag strength represents the material's innate strength to resist plastic flow, i.e., $D$ is a strength parameter—not a stress parameter. We take the internal stress to be a nonlinear function of the applied stress at saturation because that is what the experimental data of Argon and Takeuchi (1981) and Čadek (1987) suggest. A similar hypothesis to that of Eq. (16) is given in Freed and Walker (1993a) for the case where the internal stress is composed of two back stresses with no yield stress.

Because the applied stress and the back stress must be coaxial at steady state, as discussed above, it follows that

$$\|S - B_{\text{ss}}\| = \|S\| - \|B_{\text{ss}}\|.$$

(17)

Therefore, upon equating the arguments of the Zener parameters in Eqs. (7) and (14), while utilizing Eqs. (16) and (17), one obtains the result

$$\|S\| - I_{B_{\text{ss}}} = \|S\| - I_{B_{\text{ss}}}.$$

(18)
Table 3 Additional viscoplastic constants for NARlroy Z

<table>
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<tr>
<th>Constants</th>
<th>Units</th>
<th>Value</th>
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<tr>
<td>δ</td>
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</tr>
<tr>
<td>f</td>
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<td>11</td>
</tr>
<tr>
<td>ε</td>
<td></td>
<td>6.4</td>
</tr>
</tbody>
</table>

\[ D_b = C/100 \]

To be physically meaningful, \(||B|| \geq 0, D > 0, \) and \(Y \geq 0 \). Furthermore, their steady-state values ought to increase monotonically with increasing stress (Freed and Walker, 1990). This is verified easily for our hypothesis, Eqs. (14), (16), and (17), as long as \(0 \leq ||B|| \leq ||B||_{\text{max}}, D_0 \leq D \leq D_{\text{max}} \) and \(0 \leq Y \leq Y_{\text{max}} \).

6 The Model

A succinct description of our viscoplastic model is given below. The stress is acquired through the constitutive equations

\[ S_{ij} = 2\mu (E_{ij} - \delta_{ij}) \quad \text{and} \quad \sigma_{kk} = 3\kappa (\dot{\varepsilon}_{kk} - \alpha (T - T_0) \delta_{kk}). \quad (25) \]

The flow equation and kinetics that describe plastic straining are given by

\[ \dot{\varepsilon}_p = \frac{1}{2\mu} \frac{S_{ij}}{||S||} (S_{ij} - B_{ij}) \quad \text{and} \quad \dot{||S||} = \delta Z, \quad (26) \]

respectively, with the von Mises norm of effective stress being defined by

\[ ||S - B|| = \sqrt{1/2 (S_{ij} - B_{ij}) (S_{ij} - B_{ij})}. \quad (27) \]

The evolutions of back stress and drag strength are given by

\[ \dot{B}_{ij} = 2H \left( \frac{\dot{\varepsilon}_p - B_{ij}}{D} \right) \frac{||S||}{2}, \quad \text{and} \quad \dot{D} = h \left( \frac{||\dot{\varepsilon}_p||}{2} - \Lambda ||\dot{\varepsilon}_p|| - \varphi \right), \quad (28) \]

respectively, such that \(D_0 \leq D \leq D_{\text{max}}\), while the yield stress is related through the state function

\[ Y = (1 - f) \frac{(D - D_0) (C - D)}{\delta C}, \quad (29) \]

which is not an evolution equation. Associated with these relationships are the material functions:

\[ \vartheta = \begin{cases} \exp \left( -\frac{Q}{kT} \right) & \text{when } T \leq T \leq T_m, \\ \exp \left( \frac{Q}{kT} \left( \ln \left( \frac{T}{T_m} \right) + 1 \right) \right) & \text{when } 0 < T \leq T_m, \end{cases} \quad (30) \]

\[ Z = A \sinh^\alpha \left( \frac{||S - B|| - Y}{D} \right), \quad (31) \]

\[ H = (0.1 + 0.9\xi) \mu \quad \text{and} \quad L = f \frac{(D - D_0) (C - D)}{\delta C}, \quad (32) \]

\[ \Lambda = \xi \quad \text{and} \quad \rho = A \sinh^\alpha \left( \frac{D - D_0}{\delta C} \right), \quad (33) \]

with

\[ \xi = \sqrt{\frac{1}{8} (S_{ij} - B_{ij} - B_{ij}) L} \quad \text{and} \quad \frac{S_{ij} - B_{ij} - B_{ij}}{||S - B|| - L}. \quad (34) \]

Restricting the drag strength to be bound by the interval \(D_0 \leq D \leq D_{\text{max}}\), the following modifications apply:

\[ \sigma = E (\varepsilon - \dot{\varepsilon}_p - \alpha (T - T_0)), \quad \dot{\varepsilon}_p = \text{sgn}[\sigma - \beta] \frac{||\dot{\varepsilon}_p||}{\sqrt{3}} \]

and

\[ \dot{\beta} = 3H \left( \frac{\dot{\varepsilon}_p - \beta}{3L} \frac{||\dot{\varepsilon}_p||}{3} \right), \quad (35) \]

given that \(\sigma = \sigma_1 = 3/2S_{11}, \beta = \beta_{11} = 3/2B_{11}, \varepsilon = \varepsilon_{11}, \) and \(\varepsilon_p = \varepsilon_{p_1}.\)

The ability of the model to correlate (not predict) monotonic and cyclic material behavior is presented in Figs. 3 and 4, respectively, for NARlroy Z. Data for this material are sparse, thereby not permitting a more detailed assessment of the model's predictive capability. In part, this demonstrates a design objective in our development of this model—the capability to characterize the model from a sparse data set.

This model is not perfect, and certainly not ideal, but hopefully it represents another step in that direction. It is a sim-
plified continuum description of complex microscopic phenomena; nevertheless, its development has been guided by the physics of these phenomena. There are several known deficiencies associated with this model. They are: the predicted, transient, rate dependence, which is extrapolated from steady-state dependence, does not always match experimentally observed rate dependence, for example, the rate dependence exhibited during a stress relaxation experiment (Freed and Walker, 1993a); predicted transient behavior in the region of transition between the domains of power-law and exponential behaviors, which is also taken from steady-state behavior, does not always agree with experimental observations (Loh, 1993); and the well-known fact of excessive, predicted, ratchetting behavior, which is a consequence of our using the Armstrong and Frederick (1966) evolution equation for back stress (Freed and Walker, 1993c).

7 Closure

By designing the development of our viscoplastic model in such a manner that it reduces analytically to a creep model under steady-state conditions, we have incorporated essential physics into our model, and we have also simplified greatly the process that one must go through in order to completely characterize a material with this model. In this sense, we have developed a viscoplastic model with an eye towards its characterization. This has particular merit because parameter estimation of a viscoplastic model is, in general, a very complex process that all too often prohibits its use in applications. A model's relative ease of characterization without the need for exotic experiments is often considered by many industrial users of viscoplasticity to be of greater value than the model's ability to predict accurately a material's behavior (within reason). Our model was developed with this fact in mind, where we have sought to strike a balance between accuracy and ease of characterization using physics as our guidepost.

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References


