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Relativistic Navigation: A Theoretical Foundation

Slava G. Turyshev

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Space Administration

Jet Propulsion Laboratory
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Abstract

We present a theoretical foundation for relativistic astronomical measurements in curved space-time. In particular, we discuss a new iterative approach for describing the dynamics of an isolated astronomical N-body system in metric theories of gravity. To do this, we generalize the Fock–Chandrasekhar method of the weak-field and slow-motion approximation (WFSMA) and develop a theory of relativistic reference frames (RFs) for a gravitationally bounded many-extended-body problem. In any proper RF constructed in the immediate vicinity of an arbitrary body, the N-body solutions of the gravitational field equations are formally presented as a sum of the Riemann-flat inertial space-time, the gravitational field generated by the body itself, the unperturbed solutions for each body in the system transformed to the coordinates of this proper RF, and the gravitational interaction term. We develop the basic concept of a general WFSMA theory of the celestial RFs applicable to a wide class of metric theories of gravity and an arbitrary model of matter distribution.

We apply the proposed method to general relativity. Celestial bodies are described using a perfect fluid model; as such, they possess any number of internal mass and current multipole moments that explicitly characterize their internal structures. The obtained relativistic corrections to the geodetic equations of motion arise because of a coupling of the bodies' multiple moments to the surrounding gravitational field. The resulting relativistic transformations between the different RFs extend the Poincaré group to the motion of deformable self-gravitating bodies. Within the present accuracy of astronomical measurements we discuss the properties of the Fermi-normal-like proper RF that is defined in the immediate vicinity of the extended compact bodies. We further generalize the proposed approximation method and include two Eddington parameters (γ, β). This generalized approach was used to derive the relativistic equations of satellite motion in the vicinity of the extended bodies. Anticipating improvements in radio and laser tracking technologies over the next few decades, we apply this method to spacecraft orbit determination. We emphasize the number of feasible relativistic gravity tests that may be performed within the context of the parameterized WFSMA. Based on the planeto-centric equations of motion of a spacecraft around the planet, we suggested a new null test of the Strong Equivalence Principle (SEP). The experiment to measure the corresponding SEP violation effect could be performed with the future *Mercury Orbiter* mission. We discuss other relativistic effects, including the perihelion advance and the redshift and geodetic precession of the orbiter's orbital plane about Mercury, as well as the possible future implementation of the proposed formalism in software codes developed for solar-system orbit determination. All the important calculations are completely documented, and the references contain an extensive list of cited literature.

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0 Notations and Definitions.

In this report, the notations are the same as in (Landau & Lifshitz, 1988). In particular, the small latin letters $n, m, k...$ run from 0 to 3 and the Greek letters $\alpha, \beta, \gamma, ...$ run from 1 to 3; the italic capitals A, B, C number the bodies and run from 1 to N; the comma denotes a standard partial derivative and the semicolon denotes a covariant derivative; repeated indices imply an Einstein rule of summation; round brackets surrounding indices denote symmetrization and square brackets denote anti-symmetrization. The geometrical units $c = G = 1$ are used throughout the report, where G is the universal gravitational constant and c is the speed of light. We designate $\epsilon_{\alpha\beta\delta}$ as the fully anti-symmetric Levi–Civita symbol ($\epsilon_{123} = 1$); the metric convention is accepted to be $(+ - - -)$; $\gamma_{mn} = diag(1, -1, -1, -1)$ is the Minkowski metric in Cartesian coordinates of the inertial RF; $\gamma^A_{mn}(z^p_A)$ is the Minkowski metric in the coordinates (z^p_A) of the RF$_A$ that is constructed in the immediate vicinity of an arbitrary body (A); g_{mn} denotes the effective Riemann metric of the curved space-time; and $g = det(g_{mn})$. To enable one to deal conveniently with sequences of many spatial indices, we shall use an abbreviated notation for 'multi-indices' where an upper-case letter in curly brackets denotes a multi-index, while the corresponding lower-case letter denotes its number of indices, for example: $\{P\} := \mu_1\mu_2 \ldots \mu_p$, $S_{\{J\}} := S_{\mu_1\mu_2\ldots\mu_j}$. When needed, we also use $\{L-1\}$: $\mu_1\mu_2 \ldots \mu_{l-1}$, so that the tensor $T_{a\{L-1\}} = T_{a\mu_1\mu_2\ldots\mu_{l-1}}$ has l indices. We also denote $z^{\{L\}} = z^{\mu_1}z^{\mu_2}\ldots z^{\mu_l}$ and $\partial^{\{L\}}/\partial z^{\{L\}} = \partial^l/\partial z^{\mu_1}\partial z^{\mu_2}\ldots\partial z^{\mu_l}$. The explicit expression for the symmetric and trace-free (STF) part of the tensor $T_{\{P\}}$ is given in (Thorne, 1980; Blanchet and Damour, 1986, 1989). For any positive integer l we shall denote $l! = l(l-1)\cdot\ldots\cdot 2\cdot 1; l!! = l(l-2)\cdot\ldots\cdot(2\,\text{or}\,1)$ as usual. A dot over any function means a differentiation with respect to time.

1 Introduction and Overview.

1.1 The Motivation and the Structure of the Report.

The principal objective of spacecraft navigation is to determine the present and future trajectory of a craft. This is usually done by measuring the spacecraft's coordinates and then by correcting (fitting and adjusting) the predicted spacecraft trajectory using those measurements. There are three different types of measurements that are used in spacecraft navigation: radiometric (range and Doppler), very long baseline interferometry (VLBI), and optical (Standish, 1995). As well as serving navigational needs, high precision Doppler and laser and radio range measurements of the velocity of and the distance to celestial bodies and spacecraft are presently the best ways to collect important information about relativistic gravity within the solar system. Combined with the technique of ground- and space-based VLBI, these methods provide us with a unique opportunity to explore the physical phenomena in our universe with very high precision. Most remarkable is the increase in accuracy of the modern VLBI observations, especially in applications to problems of modern geodesy (Soffel *et al.*, 1991; Herring, 1995). Thus, the delay residuals are presently of the order of 30–50 picoseconds (ps), which corresponds to an uncertainty in length of $\sim$ 1 cm. In the navigation of interplanetary spacecraft, the short arcs of spacecraft range and Doppler measurements, reduced with Earth orientation information referred to the International Earth Rotation Service's (IERS) celestial system, lead to a position determination in the extragalactic RF with an accuracy on the order of $\sim$ 20 milliarcseconds (mas). At the same time, the VLBI observations of the spacecraft with respect to an extragalactic radio source enable one to measure directly one component of the spacecraft position in this extragalactic RF to an accuracy of about $\sim$ 5 mas (Border *et al.*, 1982; Folkner *et al.*, 1994). As a result, the

use of such precise methods enables one to study the dynamics of celestial bodies and spacecraft with an unprecedented accuracy.

In addition to these line-of-sight methods, the computer revolution of the 1990's has revived interest in the classical approach for determining the gravity field based on the spherical harmonics representation. It is now believed that the use of spherical harmonics to high degree and order, where only high frequency noise is present in the raw Doppler residuals, is one of the best reduction approaches because it allows a fully three-dimensional analysis. Thus, the gravitational spherical harmonics of the Earth gravity field are currently known up to the 70th degree and order for the solutions based upon the spacecraft tracking data only, and to the 360th degree and order with surface measurements included (Rapp *et al.*, 1991; Nerem *et al.*, 1995). Let us mention that currently there exists the possibility of determining the Venus gravity field to 120th degree and order (Konopliv *et al.*, 1995). It should be noted that the determination of the multipolar structure of the Newtonian gravitational field of the Earth and planets with such high resolution and accuracy enables one to take into account the relativistic corrections to the gravitational field of these bodies. Then, by using modern techniques of data reduction, one may generate highly precise solutions that have applications beyond that of serving the geodetic needs (Hellings, 1986; Herring, 1996). For example, these very important results are widely in use as the necessary foundation for studies of many modern scientific problems, such as

(i). The problem of developing a more precise definition of the masses and the multipole structure of the Sun, Earth, other planets, their satellites, and asteroids (Standish, 1992; Standish, 1994; Schubert *et al.*, 1994; Konopliv *et al.*, 1995).

(ii). The establishment of better values for gravitational and other astronomical constants, as well as the testing of the hypothesis of their dependence on time that was predicted by a number of modern theories of gravity (Dirac, 1937; Anderson *et al.*, 1986; Will, 1993).

(iii). The study of the dynamics and the evolution of the solar system, aimed at a better understanding of its metrological characteristics. This will help to solve some cosmogonical problems, such as determining whether or not there is a second asteroid belt (the Kuiper belt) behind the Saturnian orbit (Anderson *et al.*, 1986), giving better numerical estimates of the quantity of dark matter in the solar system (Braginsky, 1994; Anderson, *et al.*, 1995), and determining whether or not our Sun has a companion star.

(iv). Experimental tests of modern gravitational theories in the WFSMA (Damour, 1983; Will, 1993; Lebach, *et al.*, 1995; Anderson *et al.*, 1996), including the establishment of upper limits on the amplitude and energy density of gravitational radiation (Anderson *et al.*, 1986). Also, the search for gravitational waves, their detection, and studies of mechanisms of wave generation, as well as their propagation and interaction with matter. These studies will increase our knowledge of the early age of the universe, its cosmological evolution, and the behavior of stellar systems, as well as further confirming the hypothesis of the existence of unseen matter in the universe (Anderson, *et al.*, 1995).

The modern approach to conducting these different scientific studies should be based upon the use of a well established common relativistic framework for both collecting and interpreting astronomical observations. Until recently, this task had been done by taking into account only the post-Newtonian corrections to the solar static spherically symmetric (Schwarzschild) gravitational field. The basic relativistic effects, such as Mercury's perihelion advance, gravitational light deflection, redshift, and time delay (the Shapiro effect), have been calculated with

post-Newtonian accuracy by a number of authors, and the corresponding results are well known (Brumberg, 1972; Misner *et al.*, 1973; Will, 1993). It should be noted that during the last 10 years the precision of theoretical predictions of satellite motion has increased considerably. This has happened because some of the leading static-field post-Newtonian perturbations in the dynamics of the planets, the Moon and artificial satellites have been included in the equations of motion (eq.m.) and time and position transformation (Moyer, 1971; Moyer 1981; Dickey *et al.*, 1989; Huang *et al.*, 1990; Dickey *et al.*, 1994; Habib *et al.*, 1994; Williams *et al.*, 1996). However, due to enormous progress in the accuracy of astronomical observations at the present time, we must now take into account the much smaller relativistic effects caused by the post-post-Newtonian corrections to the solar gravitational field as well as the post-Newtonian contributions from the lunar and planets' gravities. Moreover, it is also well understood that the effects due to the non-stationary behavior of the solar system gravitational field as well as its deviation from spherical symmetry should also be considered (Kopejkin, 1988). The successful solution of these problems requires a detailed critical review of modern observational methods and the development of a consistent and physically well-founded theory of relativistic celestial mechanics and relativistic RFs. This theory should provide one with reliable physical grounds for theoretical studies of the new relativistic gravitational phenomena as well as meet the needs of practical astronomy.

It has long been considered that such a theory already exists in the form of the parameterized post-Newtonian formalism (PPN) (Nordevedt, 1968a,b; Will, 1971; Will & Nordtvedt, 1972; Will, 1993). However, based on the present understanding of the problem, this point of view is not correct. Indeed, the foundation of the PPN formalism is based upon the existence of an exclusive set of inertial RFs. Usually, the origin of such a frame either coincides with the solar system barycenter or it may be transformed to one by the post-Galilean coordinate transformations (Chandrasekhar & Contopulos, 1967; Kopejkin, 1988; Will, 1993). The resultant barycentric inertial RF is perfectly suited for analyzing both light ray propagation in the proximity of the Sun and the motion of the planets around the Sun. However, it does not address some very practical needs of modern astronomy, such as providing a description of a satellite's motion around the Earth (or other planet), studying properties of the Earth's rotation, or collecting and interpreting data from satellite laser ranging (SLR), lunar laser ranging (LLR), or ground- or space-based VLBI. These difficulties are caused by the fact that the planet's center of mass, in general, does not move along the geodesic line. The corresponding deviations are very small (Misner *et al.*, 1973; Brumberg, 1972; Will, 1993) and a product of the coupling of the planet's internal multipole moments to the external gravitational field. It is well-known that geodesic motion in the general theory of relativity, for example, can be is viewed as free fall. Moreover, in the immediate vicinity of the free-falling body, one may introduce a *local* quasi-inertial RF. In this RF, an external gravitational field should manifest itself in the form of tidal forces only (Synge, 1960; Bertotti & Grishchuk, 1990). However, the PPN coordinate system, with its origin at the center of mass of the planet, does not satisfy this last condition, and therefore it may not be treated as a quasi-inertial RF (Kopejkin, 1988). However, from the practical point of view of collecting and interpreting experimental data, one needs to use a set of RFs with well-defined geometrical and physical properties. Thus, it has been shown that a poor choice of coordinate transformations for defining the proper RF may lead to unnecessary complications in the equations of motion. These equations may appear to contain non-physical (or fictitious) forces acting on the bodies in the system. Although these forces are simply a result of a 'bad' choice of RF, their appearance in the equations of motion may make the scientific interpretation of the collected results much more difficult. For example, the term with an amplitude of about one meter in the relativistic theory of motion of the moon (Brumberg, 1958; Baierlein, 1967) has no

real physical meaning when built on the basis of the proper coordinates. The appearance of this term is an artifact of the choice of coordinates and, therefore, the one-meter term is not observable (Soffel *et al.*, 1986; Kopejkin, 1988). This example suggests that a clear understanding of the dynamic properties of a chosen RF will help make the separation between physically measurable quantities and coordinate-induced ones and, hence, will simplify the analysis of the data obtained.

From this standpoint, the detailed construction of a relativistic theory of astronomical RFs is greatly needed. It is especially important because at present almost all the astronomical observations (such as optical, radio, Doppler, laser, etc.) are performed and/or processed by experimental equipment placed on the Earth's surface. Moreover, there is great demand for reliable relativistic navigation in outer space for near-future space missions, such as space-based gravitational-wave astronomy. Let us also note that there are near-term plans for launching several drag-free satellites with GPS receivers onboard: Gravity Probe B (GP-B) (Bardas *et al.*, 1989), LAGEOS III, a satellite test of the equivalence principle (STEP), and the *Mercury Orbiter* mission, which has been proposed by the European Space Agency as a cornerstone mission under the Horizon 2000 Plus program (Anderson, Turyshev *et al.*, 1996). There exist plans to include the post-post-Newtonian contributions to the light propagation effects coming from the solar gravitational field and the post-Newtonian gravitational perturbations by the planets of the solar system (Klioner & Kopejkin, 1992). In particular, one of the most promising projects is the deployment into Earth orbit of a precision optical interferometer (POINTS). This satellite will be designed to be able to measure the arcs between the pairs of stars separated on the sky by the right angle with an anticipated accuracy on the order of a few microarcseconds (μas) (Chandler & Reasenberg, 1990). These plans encourage the development of orbit determination algorithms that would enable one to process the data with the required relativistic accuracy. This alone will require substantial work to be done in development of a number of theoretical and practical questions, such as

(i). The construction of a dynamic inertial RF and a more precise definition of the orbital elements of the Sun, Earth, moon, planets, and their satellites (Standish *et al.*; 1992, Chandler *et al.*, 1994; Dickey *et al.*, 1994; Williams *et al.*, 1996; Standish, 1995).

(ii). The construction of a kinematic inertial RF, based on the observations of stars and quasars from spaceborne astronomical observatories (Fukushima, 1991a; Standish *et al.*, 1992).

(iii). The construction of a precise ephemeris for the motion of bodies in the solar system to support reliable navigation in the solar system (Denisov *et al.*, 1989; Standish *et al.*, 1995; Standish, 1995). The construction of precise radio-star catalogs for spacecraft astro-orientation and navigation in outer space beyond the solar system.

(iv). The comparison of dynamic and kinematic inertial RFs, based on the observations of spacecraft on the background of quasars, pulsars, and radio stars, as well as the verification of the zero points of the coordinates in the inertial RF (Jacobs *et al.*, 1993; Folkner *et al.*, 1994; Fukushima, 1995).

Therefore, the motivation for this research is quite natural: In order to propose the necessary recommendations for corrections to existing software codes, we will re-examine the basic concepts of high-precision navigation in the solar system. The principal goal of this report is to provide one with a solid theoretical foundation for the relativistic astronomical measurements in the curved space-time. To reach this goal we, by using the methods of the WFSMA, will develop

a new approach to the relativistic treatment of the satellite orbit determination problem. This approach will be based upon a new theory of coordinate transformations (*i.e.*, the theory of relativistic RFs) and measurement models in relativistic celestial mechanics. The outline of the present report is as follows:

The next subsection contains a brief historical introduction to the problem of motion of N weakly interacting self-gravitating extended bodies. To specify our theoretical studies, we will present a qualitative description of the astronomical N-body systems of interest. In order to provide a solid motivation for this research, we will analyze the different methods used to approach this problem and will present their advantages and the encountered difficulties.

In Section 2, we discuss the conventional PPN barycentric approach, which is based on the solution to the gravitational one-body problem. Recognizing that the generalization of the obtained results into a general case of motion of an arbitrary N-body system is not straightforward, we analyze the conditions necessary to derive the restricted solution for the motion of the general N-body problem. We also discuss ways to obtain the complete multipolar solution to the problem in the general case.

Section 3 is devoted to a general description of the new method proposed to overcome the above mentioned problems. We discuss a new iterative approach to describing the dynamics of an isolated astronomical N-extended-body system in the metric theories of gravity. The N-body solution of the gravitational field equations in the proper RF_A originated in an arbitrary body (A) is formally presented as a sum of the four following terms: (i) γ^A_{kl}, which is the Riemann-flat inertial space-time: $R^A_{mnst}(\gamma^A_{kl}) = 0$; (ii) $h^{(0)A}_{mn}$, which is the gravitational field generated by the body (A) itself; (iii) $h^{(0)B}_{mn}$, the perturbations caused by other bodies in the system ($B \neq A$); and, finally, (iv) the gravitational interaction term h^{int}_{mn}. This method is presented in its most general form and, hence, it is valid for a number of metric theories of gravity. We discuss the general properties of the post-Newtonian non-rotating coordinate transformations and present the straight, inverse, and mutual coordinate transformations. As a possible way of generalizing the results obtained, we discuss the use of the rotational coordinate transformations. In addition, we discuss the necessary conditions for constructing a proper RF with well-defined dynamical properties. Physically, these conditions should provide one with an additional inertial force acting on the body in its proper RF such that the body will be in a state of equilibrium. Mathematically, these conditions required that the total dipole moment of the system of the fields produced by matter, the field of inertia, and the gravitational field taken jointly will vanish for all times.

In Section 4, we apply the proposed formalism to the case of general relativity. The celestial bodies are assumed to consist of a perfect fluid and possess any number of the internal mass and current multiple moments that characterize the internal structure of such bodies. We present the physical and mathematical definitions of the proper RF in the WFSMA. We find the explicit solution for the interaction term. This enables us to construct all the necessary expressions for the metric tensor in both the barycentric inertial and the arbitrarily parameterized proper quasi-inertial RFs.

In Section 5, we present the general solution for the *global* and *local* problems, as well as show the general solution for the functions of the coordinate transformation in the case of bodies with a weak external gravitational field. In particular, within the present accuracy of radio

measurements, we discuss the generalized Fermi-normal-like proper RF, which is defined in the immediate vicinity of such extended bodies.

In Section 6, we generalized the results obtained on the case of a system of N arbitrarily shaped and deformable extended bodies. To do this, we study the existence of the conservation laws in the proper RF. It turns out that the existence of these laws in the WFSMA may be shown explicitly in the case of well separated celestial bodies. This allows us to evaluate the surface integrals on the boundaries of the domains occupied by the celestial bodies and present the explicit coordinate transformations between the different RFs in the WFSMA of the general relativity. These results are the extension of the post-Galilean transformations obtained by Chandrasekhar and Contopulos (1967) on the case of a system of interacting celestial extended bodies. We discuss the properties of the corresponding quasi-group of motion and its application to the study of the dynamics of an arbitrary N-body gravitational problem.

Section 7 is devoted to future relativity missions in the solar system. In order to provide the framework to study relativistic gravity for a number of gravitational theories, our previous derivations will be generalized on the case of the tensor-scalar theories. As a result, we include in the analysis the two Eddington parameters (γ, β), which allows us to develop a parameterized theory of astronomical RFs. By analyzing the equations of motion in the two-parameter Fermi-normal-like RF, we have obtained an interesting result: that although some terms in the planetocentric eq.m. of the spacecraft around the planet are zero for the case of general relativity, they may produce an observable effect in scalar-tensor theories. This allows us to propose a new null test of the SEP. Also in this section, we discuss the other relativistic gravitational experiments possible with the future *Mercury Orbiter* mission, which has been proposed by the European Space Agency as a cornerstone mission under the Horizon 2000 Plus program. The motivation for this research is to determine what scientific information may be obtained during this mission, how accurate these measurements can be, and what will be the significance of the knowledge obtained. We present there both quantitative and qualitative analyses of measurable effects such as Mercury's perihelion advance, the redshift experiment, and the precession phenomena of the Hermean orbital plane.

In Section 8, we present the hierarchy of the celestial RFs, including the four frames that are widely in use for the practical needs of modern relativistic astronomy. Thus, in a compact and explicit form, we show the coordinate transformations between the barycentric and the geocentric RFs, between the geocentric and the satellite RFs, and between the geocentric and the topocentric RFs. This presentation contains the two Eddington parameters, (γ, β), which makes the obtained results valid for a wide class of metric theories of gravity. In the discussion, we present a number of possible areas for immediate practical application of the theory of astronomical RFs developed in this report. We present our conclusions and recommendations for future research on relativistic gravity in the solar system and beyond.

In order to avoid cumbersome calculations and to simplify the presentation of the main results in the text, some expressions and intermediate relations will be presented in appendices. In Appendix A, we present the generalized gravitational potentials. Appendix B is devoted to a discussion of the structure of the post-Newtonian power expansion of general geometrical quantities such as the metric tensor, g_{mn}; the Christoffel symbols; and the Riemann tensor, R_{mnkl}, in coordinates of an arbitrary RF with respect to small parameters. Appendix C contains the general theory of relativistic coordinate transformations. We discuss there the transformation of the base vectors for different coordinate transitions. In Appendix D, we present the features

of the transformations of different equations and quantities, such as the covariant gauge conditions, the Ricci tensor, the gravitational field solutions, and the energy-momentum tensor. The transformation rules for the generalized gravitational potentials under the post-Newtonian coordinate transformation are presented in Appendix E. The Christoffel symbols in the proper RF are calculated in Appendix F. The calculation of the form of the inertial part of the metric tensor in the proper RF and the form of the interaction term, as well as the components of the Riemann tensor in this frame, are presented in Appendix G. In Appendix H, we present some useful identities that are used in Section 6 to study the existence of the conservation laws in an arbitrary proper RF. And, finally, in Appendix I, we have presented the astrophysical parameters used for estimations of the magnitudes of the gravitational effects in Section 7.

1.2 The Problem of Relativistic Astronomical Measurements.

Classical Newtonian mechanics is based upon the principles of Euclidean geometry. The physical experiments, within the accuracy available at that time, had confirmed the two basic postulates of this geometry: that time is absolute and homogeneous and that space is also absolute and, not only homogeneous, but also isentropic. These properties of time and space were discovered because, for the then-known physical forces,[1] the corresponding eq.m. of Newton's mechanics preserved their form under the Galilean group of motion. These properties may be written for two different RFs moving relative to each other with constant speed $\vec{v}$ as

$$t' = t + a, \qquad \vec{r}\,' = \vec{r} - \vec{b} - \vec{v}t, \tag{1.1}$$

where parameters a and $\vec{b}$ are the constant time shift and the displacement of the origin of the coordinate system, respectively. This form-invariancy suggested that, independent of the state of motion of these RFs (they may be either at rest or uniformly moving along a straight line relative to each other), all the mechanical phenomena will behave exactly the same way in any such RF. This principle has become known as the principle of relativity (Poincaré, 1904). Note that transformations (1.1) are given in Cartesian coordinates. One may choose another coordinate system (CS) in the same RF without changing its state of motion (say $\vec{x}$) by simply rotating the coordinate axes: $r^\alpha = \mathcal{R}^\alpha_\beta x^\beta$, where $\mathcal{R}^\alpha_\beta$ is a constant orthogonal rotation matrix.

Thus, Newton's mechanics had introduced into physics notions both of an absolute distance between two points in three-dimensional space and of absolute time. In other words, he asserted that time and coordinates are directly measurable quantities. Because of this, the theory of gravitational measurements in celestial mechanics long was based upon the three laws of Newton's mechanics and coordinate transformations, (1.1). From the practical point of view, there were two astronomical RFs of primary importance: the barycentric frame (BRF), which is related to the barycenter of the solar system, and the geocentric frame (GRF), whose origin coincides with the Earth's center of mass. Because of the recent progress in the relativistic treatment of an isolated N-body system, there is now clear and unambiguous agreement on an asymptotical BRF (which is valid even through the post-Newtonian level of the WFSMA). By assuming that the solar system as a whole is completely isolated, one may put its barycentric RF to be non-accelerated (or to say 'at rest') and absolutely non-rotating. The latter condition implies (i) the absence of the centripetal and Coriolis forces (dynamical inertiality) and (ii) that the coordinate directions to the remote light sources (such as quasars) must be constant (kinematic inertiality). In addition, the absence of any external sources of gravity enables one to consider only the proper

[1]There were only two known natural forces at this time: gravity and elasticity. The first one was described by Newton's gravitational law and the second by Hooke's law.

(or 'inertial') gravitational field of the solar system. As a result, such an RF was used for a long time as the basic tool for solving almost all the problems in practical astronomy (even relativistic ones).

As far as the GRF is concerned, the situation turns out to be more complicated. If one attempts to describe the *local* gravitational environment of some extended body from an N-body system (for example, the Earth in the solar system), first of all, based on the results of a study of the existence of the energy-momentum conservation laws, one generally defines the barycentric inertial RF: $(t', \vec{r}\,')$. Then, one may introduce a non-rotational accelerated GRF $(t, \vec{r})$, which is defined at the center of mass of the extended body under study by a coordinate transformation similar to that of (1.1):

$$t' = t, \qquad \vec{r}\,' = \vec{r} + \vec{r}_0(t), \tag{1.2}$$

where $\vec{r}_0(t)$ is the Newtonian barycentric radius vector of the body.

To analyze the gravitational environment of the body under consideration, one presents the effective potential in the body's vicinity in the form

$$\overline{U}(\vec{r}) = U_0(\vec{r}) + U^{\text{tid}}(\vec{r}), \tag{1.3a}$$

where U_0 is the body's own gravitational potential. The influence of the external bodies in the chosen frame manifests itself in the form of gravitational tidal forces only. The corresponding tidal gravitational potential, U^{tid}, may be given by

$$U^{\text{tid}}(\vec{r}) = U^{\text{ext}}(\vec{r}_0 + \vec{r}) - U^{\text{ext}}(\vec{r}_0) - \left(\vec{r} \cdot \vec{\nabla} U^{\text{ext}}(\vec{r}_0)\right). \tag{1.3b}$$

This potential is searched for as the solution of the usual Poisson equation in the form

$$\triangle U^{\text{tid}} = -4\pi\rho_0^{\text{ext}} \tag{1.4a}$$

with the boundary conditions

$$\vec{\nabla} U^{\text{tid}}(\vec{r}_0) = 0, \qquad U^{\text{tid}}(\vec{r}_0) = 0, \tag{1.4b}$$

where ρ_0^{ext} is the mass density of the external gravity in the vicinity of the body under study. As a result, the theory of astronomical observations becomes inseparable from the problem of determining the motion of celestial bodies, because the Newtonian eq.m. for the body's center of mass is determined as follows:

$$\ddot{\vec{r}}_0 = -\vec{\nabla} U^{\text{ext}}(\vec{r}_0). \tag{1.5}$$

One may also verify that, in the proper RF for an extended body constructed this way, the body's own center of mass will be at rest during the time of the experiment. Indeed, by integrating the *local* eq.m. of the Newtonian hydrodynamics (Fock, 1955),

$$\rho_0 \frac{d\vec{v}}{dt} = -\rho_0 \vec{\nabla}\overline{U} + \vec{\nabla} p, \tag{1.6}$$

over the body's compact volume, one obtains the desired result: $\ddot{m}_0^\alpha = 0$, where m_0^α is the body's first (dipole) mass moment. In the body's vicinity, the external gravity produces negligibly small tidal perturbations of the *local* motion, which are presently well known (Standish *et al.*, 1992). This leads to so-called 'quasi-inertial' properties of GRF. The kinematic advantage of these *local*

coordinates, $(t, \vec{r})$, is that the RF, when obtained this way, moves with the considered body. Their dynamic usefulness comes from the fact that coordinate transformations (1.2) allow one (to some extent) to decouple the motion of the studied body from the *global* dynamics of the system as a whole (Pars, 1965; Brumberg, 1972; Damour, Soffel & Xu (here and after, DSX), 1991). These are the reasons why this proper RF (or GRF) has become very useful for studying *local* physics in a body's vicinity.

The situation changed drastically when, by generalizing Faraday's thoughts on electric and magnetic phenomena, Maxwell discovered a set of equations describing electromagnetic fields. These equations successfully described the two then-'new' forces corresponding to electromagnetic and optical phenomena. However, it turned out that the famous Maxwell–Lorentz equations of electromagnetism were not form-invariant under the Galilean transformations, (1.1). This was an indicator that either the laws of Newtonian mechanics were incomplete or these transformations were wrong. From the other side, recall that transformations (1.1) were a simple consequence of the laws of Newton's mechanics. It became clear that even if some other set of equations were substituted for these laws, transformations (1.1) may not provide a form-invariancy for this new set. Thus, it became obvious that the principle of relativity must have a more fundamental character. In the Poincaré interpretation this principle was reformulated so that the physical laws should be the same for two particular observers, one being at rest and the other one being in the state of steady straight-line motion so that there is no means to find out whether or not the second observer is moving. The significance of this principle was that it stated that there are no such things as absolute space or time and, moreover, it implied the impossibility of an absolute motion in the general law of nature.

As we know now, the understanding of this theory sparked a revolutionary change in the course of theoretical physics in the beginning of the 20th century. The answer to this problem was given in a series of works by Poincaré (1904) and Minkowski (1908) (see also Lorentz *et al.*, 1923): that space and time must be united together to form a four-dimensional pseudo-Euclidean geometry. The coordinates of two points in this four-dimensional manifold are denoted as $(ct, \vec{r}) \rightarrow x^n \equiv (x^0, x^\alpha)$, where $n = 0, 1, 2, 3$ and c is the speed of light. The square of the geodesic distance ds^2 between the two infinitely close points of this space-time (interval) is given by the four-dimensional analog of the Pythagorean theorem: $ds^2 = \gamma_{mn}(x) dx^m dx^n$. The function $\gamma_{mn}(x)$ is the metric tensor, which has become the main object for defining the structure of studied space-time (Eisenhart, 1926). These metric coefficients only (as referred to a particular coordinate system), together with the coordinate differentials, will provide one with physically measurable quantities. In Cartesian coordinates of the Galilean (inertial) RF, for all the points of the pseudo-Euclidean space-time, this metric function may be chosen in the form of the Minkowski metric: $\overset{(0)}{\gamma}_{mn} = \mathrm{diag}(1, -1, -1, -1)$. As a result of such a change, the coordinates lost their absolute meaning and could not be used for direct physical observations. Even the differentials do not have a physical sense, because they are not directly connected with either the distance between two points in three-dimensional space or with the temporal evolution of the physical processes.

By analyzing the Maxwell–Lorentz equations of the electromagnetic field and the interval in the form $ds^2 = \overset{(0)}{\gamma}_{mn} dx^m dx^n$, Poincaré was the first to point out that the set of these field

equations and the quantity ds^2 are form-invariant under Lorentz' transformations, which form the Poincaré group of motion:

$$t' = \gamma\Big(t - \frac{(\vec{v}\cdot\vec{r})}{c^2}\Big), \qquad \gamma = \Big(1 - \frac{v^2}{c^2}\Big)^{-\frac{1}{2}}, \tag{1.7a}$$

$$\vec{r}\,' = \vec{r} + (\gamma - 1)\frac{\vec{v}(\vec{v}\cdot\vec{r})}{v^2} - \gamma\vec{v}t, \tag{1.7b}$$

where $\vec{v}$ is the constant relative speed between the two RFs. Thus, the study of electromagnetic phenomena led to the discovery of a new theory of the structure of space-time.

The form-invariancy of the metric tensor under transformations (1.7) has suggested a more general physical property, namely: for all the possible coordinate transformations between the two arbitrary RFs, which preserve the form of the metric tensor γ_{mn}, the physical phenomena in both obtained frames will behave in exactly the same way. As a result, the principle of relativity becomes simply a consequence of the latter property. The next logical step was to generalize the equations of Newton's mechanics based on this new four-dimensional relativistic treatment. The resultant set of equations of motion has become known as the relativistic mechanics of Poincaré (Sard, 1970). This theory was formulated in a covariant form which allows one to study the physical processes in any physical RF. Note that, independently of Minkowski and Poincaré, Einstein had also formulated a new theory of space-time—the special theory of relativity (Lorentz *et al.*, 1923; Landau & Lifshitz, 1988). However, this theory was formulated based only on the Poincaré group of motion and was constrained to the class of inertial RFs only.

The discovery of the pseudo-Euclidean geometry had finally undermined any absolute meaning of finite time or finite distance and had substituted instead a purely relative one. Now the interval ds^2—the square of the infinitesimal distance in four-dimensional manifold—had become the only absolute quantity. For example, based on the Lorentz transformations, the time in two different RFs was no longer the same, but rather depended on the relative speed between the frames:

$$\Delta t' = \int_{t_0}^{t_1} dt\Big(1 - \frac{\vec{v}^2(t)}{c^2}\Big)^{\frac{1}{2}}. \tag{1.8a}$$

Moreover, the length of an object in two RFs was also no longer invariant. Thus, a rod, which has a length dl_0 in a rest frame, will experience the length contraction in an inertially moving frame in the direction $\vec{n} = \vec{v}/v$, parallel to the speed of motion $\vec{v}$:

$$d\vec{l} = \vec{n}dl_0\Big(1 - \frac{\vec{v}^2}{c^2}\Big)^{\frac{1}{2}}. \tag{1.8b}$$

It should be stressed that formulas (1.8) are simply the consequence of the properties of pseudo-Euclidean geometry. It should be noted that, together with the properties of this geometry, the language of the 'microscopic' (or field) description has appeared in theoretical physics as the necessary tool for theoretical studies of physical processes. This 'field' terminology deals with the densities of physical quantities in a relativistic coordinate-independent way, rather than providing a coordinate-dependent (or RF-dependent) regular 'macroscopic' treatment, and it has become a very powerful substitution for the latter. As a result, for the special relativistic treatment of gravitational observations, contrary to Newtonian mechanics, one should always appeal to the notion of the 'proper' quasi-inertial RF of a body in order to correctly define the body's mass, its barycenter, and the intrinsic multipole moments.

For a long time, it was thought that the special theory of relativity, and hence the relativistic mechanics of Poincaré, were theories that described the physical processes solely in different inertial RFs (which may be linked to each other by the Lorentzian transformations, (1.7)). From the other side, real astronomical phenomena unavoidably involve descriptions based on non-inertial RFs, which, by a misunderstanding (partially based on the Equivalence Principle), were considered as a prerogative of the general theory of relativity only. However, this is not true. Based on the discovery of the pseudo-Euclidean space-time made by Poincaré and Minkowski, one may use an infinite class of admissible RFs, both inertial and non-inertial, in order to describe the physical phenomena in the real world. Indeed, the Riemann curvature tensor, which defines the intrinsic geometry of space-time, is zero in any of these frames. However, observing any physical process enables one to confidently distinguish the situations when an experiment is performed in an inertial or in a non-inertial frame. This means that the following generalized principle of relativity (Logunov, 1987) is valid: Independent of the state of motion of the RF chosen for the experiment (either inertial or non-inertial), one may define an infinite set of other RFs for which the physical phenomena will behave in exactly the same way. Moreover, one may not establish, by any means, in which RF from this equivalent set the experiment is performed. As a result, by defining the admissible coordinate transformations that leave the metric tensor in the chosen RF form-invariant, one defines the entire infinite set of physically equivalent RFs. Thus, from Poincaré's equations of relativistic mechanics and the requirement of the form-invariancy of the metric tensor, one may find another fundamental group of motion in the pseudo-Euclidean space-time, namely, the relativistic group of uniformly accelerated motion of a monopole particle. Indeed, for a particle with mass m_0 moving under the influence of a constant force $\vec{f} = (f, 0, 0)$, the law of motion is given by

$$t' = t, \qquad x' = x + \frac{c^2}{a}\Big[\Big(1 + \frac{a^2t^2}{c^2}\Big)^{\frac{1}{2}} - 1\Big], \tag{1.9}$$

where $a = f/m_0$ is the corresponding constant acceleration. The interval of the two-dimensional space-time in the co-moving RF takes the form

$$ds^2 = \frac{c^2dt^2}{1 + a^2t^2/c^2} - \frac{2at\ dtdx}{\Big(1 + a^2t^2/c^2\Big)^{\frac{1}{2}}} - dx^2. \tag{1.10}$$

From this it is easy to show that the corresponding two-parametric group of motion for the uniformly accelerated RFs may be presented as follows:

$$t' = \gamma\Big(t + t_0 + \frac{vx}{c^2} + \frac{v}{a}\Big[\Big(1 + \frac{a^2t^2}{c^2}\Big)^{\frac{1}{2}} - 1\Big]\Big), \qquad \gamma = \Big(1 - \frac{v^2}{c^2}\Big)^{-\frac{1}{2}},$$

$$x' = \gamma\Big(x + vt + \frac{c^2}{a}\Big[\Big(1 + \frac{a^2t^2}{c^2}\Big)^{\frac{1}{2}} - 1\Big]\Big) -$$

$$-\frac{c^2}{a}\left(\left[1 + \frac{a^2}{c^2 - v^2}\Big(t + t_0 + \frac{vx}{c^2} + \frac{v}{a}\Big[\Big(1 + \frac{a^2t^2}{c^2}\Big)^{\frac{1}{2}} - 1\Big]\Big)^2\right]^{\frac{1}{2}} - 1\right) + x_0, \tag{1.11}$$

where t_0 and x_0 are the group constants.

One can see that, in order to preserve the form-invariancy of the metric tensor for the time translation (given by a parameter t_0) contrary to the Poincaré group, (1.7), this nonlinear group of motion requires the transformation of spatial coordinates as well. Thus, the non-inertiality of

the RF makes the physical analysis more difficult than in the case of inertial RFs. The situation becomes even more complicated if one decides to describe the motion of an extended object. This is because the bodies in this case, besides the 'usual' Lorentzian relativistic contractions, will experience other dynamic effects generated by the properties of the RF chosen for the analysis. In practice, one is usually faced with the problem of extracting the RF-induced effects. As the properties of the pseudo-Euclidean space-time are well established, this problem may be solved in a satisfactory manner by constructing a quasi-inertial RF in the vicinity of the body under consideration. The vanishing of a Riemann curvature leads to a maximum possible number of the Killing vectors in this geometry (N=10), which enables us to separate physically observable and coordinate-induced quantities in a satisfactory manner. Note that the corresponding theoretical methods of the classical mechanics of Poincaré are presently well tested in different experimental situations, and they are used extensively in many areas of modern relativistic physics, such as high-energy physics, theoretical astrophysics, and solid-state physics. Astronomers, however, previously had not fully accepted these methods into real astronomical practice, as there was little observational data with relativistic accuracy.

This situation has changed dramatically during the last two decades, and now that the accuracy of astronomical observations enables us to perform studies of the physical processes in the universe with much higher precision, the problem of relativistic gravitational measurements has become very important. This has led to numerous experiments testing different hypotheses that have laid the foundations for a number of recent theories of gravity (Will, 1993). Gravity, however, remains the last yet unexplored frontier of modern theoretical physics (Hawking & Israel, 1987; Damour & Schäfer, 1991; Damour & Taylor, 1992). This is mainly because the weakness of the gravitational interaction in the solar system presents great difficulties when planning and performing gravitational experiments. The other reason is that the discovery of the field equations of the general theory of relativity has changed our physical conceptions once again. According to this theory, not only are space and time united together by forming a four-dimensional Riemann manifold with the general metric tensor g_{mn}, but also it is matter that is responsible for generating the properties of this space-time. In other words, space-time tells matter how to move and matter tells space-time how to curve (Misner *et al.*, 1973). There are many other gravitational theories currently under consideration, but the metric theories of gravity have taken a special position among all the possible theoretical models. The reason is that, independent of the many different principles at their foundations, the gravitational field in these theories affects matter directly through the metric tensor of Riemann space-time g_{mn}, which is determined from the field equations of a particular theory of gravity. In contrast to Newtonian gravity, this tensor contains the properties of a particular gravitational theory and also carries the information about the gravitational field of the bodies themselves. This property of the metric tensor enables one to analyze the motion of matter in one or another metric theory of gravity based only upon the underlying principles of modern theoretical physics.

The situation with relativistic measurements has become even more complicated. Because it is well known that in the Riemann space-time one cannot have an explicit mathematical definition for the proper RF, it is permissible to introduce any coordinate system. As a rule, before solving these equations, four restrictions (coordinate or gauge conditions) must be imposed on the components of the g_{mn}. These conditions extract a particular subset from an infinite set of space-time coordinates. Inside this subset, the coordinates are linked by smooth differentiable transformations that do not change the coordinate conditions being chosen. In general relativity, for example, there exists no absolute time or Euclidean space. Besides, one may not, in the

general case, introduce some 'privileged' RF in space-time. Contrary to the Newtonian theory of gravity, coordinates in curved space-time have no physical meaning and cannot be measured directly by astronomical observations.

Nevertheless, there are some special cases in which one may speak about privileged coordinates in general relativity. One such case is space-time having a weak gravitational field and slowly moving matter. The density of the total non-linear Riemann metric tensor g^{mn} of such space-time may be linearized and presented as a sum of the density of the pseudo-Euclidean background metric γ^{mn} plus the small perturbations caused by the physical gravitational field h^{mn}: $\sqrt{-g}g^{mn} = \sqrt{-\gamma}\gamma^{mn} + h^{mn}$. Then, in the Galilean inertial RF, such a space-time may be covered by coordinates that differ only slightly from the absolute time and Cartesian space coordinates of the Newtonian theory of gravity. We shall call these space-time coordinates quasi-Cartesian. These quasi-Cartesian coordinates are the most convenient coordinate system for developing a relativistic theory of astronomical RFs inside the solar system. They are also used in the case of an isolated astronomical system that consists of N well-separated and extended bodies possessing a weak gravitational field and moving with slow orbital and rotational velocities (such as our solar system).

The solution of the field equations of general relativity in the WFSMA for an isolated distribution of matter is presently well known (Will, 1993). There have been a number of attempts to describe the motion of different gravitationally bounded astronomical systems. This problem of describing the motion of a system consisting of N massive monopole particles was first considered by Einstein *et al.* (1938); the rigid uniform rotation of the bodies was included by Papapetrou (1948, 1951), Fock (1955), etc. It was shown that the post-Newtonian equations of Einstein, Infeld, and Hoffmann (EIH) governing the motions of N mass points allow the same ten classical integrals as the equations of Newtonian gravity, namely, those expressing conservation of energy, linear momentum, and the uniform motion of the center mass of the body. Moreover, Chandrasekhar & Contopulos (1967) had shown there exists a way to introduce the notion of the 'center of mass' of such a system, which enables one to construct the barycentric inertial RF_0. Thus, by studying the problem of the form-invariancy of the metric tensor and the corresponding post-Newtonian EIH eq.m., they had shown that both of these expressions are invariant under the following 'post-Galilean' coordinate transformations that establish a correspondence between frames with uniform relative motion:

$$t' = \Big(1 + \frac{v^2}{2c^2} + \frac{3v^4}{8c^4}\Big)t - \frac{v_\mu x^\mu}{c^2}\Big(1 + \frac{v^2}{2c^2}\Big) + \frac{1}{2c^4}\sum_B m_B n_\mu x^\mu + \mathcal{O}(c^{-6}), \tag{1.12a}$$

$$x'^\alpha = x^\alpha + \Big(1 + \frac{v^2}{2c^2}\Big)v^\alpha t - \frac{v_\mu x^\mu}{2c^2}v^\alpha + \frac{\sigma}{c^2}\epsilon^\alpha_{\mu\beta}x^\mu v^\beta + \mathcal{O}(c^{-4}), \tag{1.12b}$$

where v^α is the constant velocity of the uniform motion, m_B is the post-Newtonian rest mass of the distribution of matter under study, and σ is some arbitrary constant. One can see that both of the equations in (1.12) contain additional terms beyond those obtained by expanding the Lorentz transformation, (1.7). The last term in eq. (1.12a) is the contribution that is unique to general relativity, and it is this term that gives transformation (1.12) its non-Lorentzian character. The other additional term in eq. (1.12b) represents an arbitrary infinitesimal rotation that may be satisfactorily explained in terms of the Poincaré group. As a result, the obtained post-Galilean transformations are generalizations of the Lorentzian transformations, (1.7), in the gravitational case.

These post-Galilean transformations, (1.12), are of little use for astronomical observations as they were obtained in order to demonstrate the existence of the barycentric inertial RF_0 and they are not suited for the construction of an astronomical RF for even massive monopole bodies. This is simply because such proper RFs generally will not be inertial, but rather quasi-inertial. Moreover, expressions (1.12) do not account for the multipolar structure of the extended bodies. However, we need some transformation that will work, since, in order to present all the necessary expressions for the metric tensor and the equations of motion with the same post-Newtonian accuracy, one must have a physically grounded definition of the transformation rules between the RFs. To find this transformation, one must expand the Newtonian contributions in terms of the intrinsic mass and current multipole moments of the bodies (Damour, 1983, 1986). The greater the required accuracy, the larger the number of these terms that must be taken into account. It is known that the fully relativistic definition of these moments may be given in the proper quasi-inertial RF only. Such a definition replaces that which was given in the rest frame of the one-body problem.[2] In presenting these transformations, one should also take into account that, due to the non-linear character of the gravitational interaction, these moments are expected to interact with external gravity, changing the state of motion of the body itself. Fock (1955) was the first to notice that in order to find the solution of the *global* problem (the motion of the N-body system as a whole), the solution for the *local* gravitational problem (in the body's vicinity) is required. In addition, one must establish their correspondence by presenting the coordinate transformation by which the physical characteristics of motion and rotation are transformed from the coordinates of one RF to another. Thus, one must find the solutions to the three following problems (Damour, 1987; DSX, 1991):

(1). The *global* problem:

- (i). We must construct the asymptotically inertial RF.
- (ii). We must find the barycentric inertial RF_0 for the system under study. This is primarily a problem of describing the *global* translational motion of the bodies constituting the N-extended-body system (*i.e.*, finding the geodesic structure of the space-time occupied by the whole system).

(2). The *local* problem:

- (i). We must establish the properties of the gravitational environment in the proximity of each body in the system (*i.e.*, finding the geodesic structure of the *local* region of the space-time in the body's gravitational domain).
- (ii). We must construct the *local* effective rest frame of each body.
- (iii). We must study the internal motion of matter inside the bodies as well as establish their explicit multipolar structure and rotational motion.

(3). The theory of the RFs:

- (i). We must find a way to describe the mutual physical cross-interpretation of the results obtained for the above two problems (*i.e.*, the fine mapping of the space-time).

[2]Note that, due to the breaking of the symmetry of the total Riemannian space-time by realizing the $3+1$ split (Thorne *et al.*, 1988), these moments will not form tensor quantities with respect to general four-dimensional coordinate transformations in the WFSMA. Instead, these quantities will behave as tensors under the sub-group of this total group of motion only, namely, the three-dimensional rotation. This is similar to the situation in classical electrodynamics, where electric $\vec{E}$ and magnetic $\vec{H}$ fields are not true vectors, but rather components of the 4×4 tensor of the electromagnetic field $F_{mn} = (\vec{E} \otimes \vec{H})$ (Landau & Lifshitz, 1988).

Because the solutions to the first two problems will not be complete without presenting the rules of the coordinate transformations between the *global* and the *local* (or planeto-centric) RFs chosen for such an analysis, the theory of astronomical RFs becomes inseparable from the problem of determining the motion of the celestial bodies. From the other side, if one attempts to describe the *global* dynamics of the system of N arbitrarily shaped extended bodies, one will discover that, even in the WFSMA, this solution will not be possible without appropriate description of the gravitational environment in the immediate vicinity of the bodies.

Concerning the problem of astronomical data reduction, first of all, one must find the connection between the coordinate quantities and the physically observable ones. Until quite recently, relativistic reduction of astrometric observations was based on the use of the barycentric RF and covariant definitions of observables (Zel'manov, 1956; Synge, 1960; Misner *et al.*, 1973; Ivanitskaja, 1979; Soffel, 1989; Brumberg, 1991a,b; Nordtvedt, 1995). Thus, interval ds^2 in terms of observable coordinates $dX^{\overline{n}} \equiv (cd\tau, dr^{\alpha})$ is taken to be diagonal, and it is usually presented in the form of pseudo-Euclidean Minkowski space-time in the Galilean RF as follows:

$$ds^2 = g_{mn}(x)dx^m dx^n = \gamma_{\overline{m}\overline{n}} dX^{\overline{m}} dX^{\overline{n}} \equiv c^2 d\tau^2 - dr^2, \tag{1.13}$$

where the physical time $d\tau$ and the three-dimensional physically measurable distance dr^2 are given by

$$d\tau = \sqrt{g_{00}}dt + \frac{g_{0\alpha}dx^{\alpha}}{c\sqrt{g_{00}}}, \qquad dr^2 = \Big(- g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}\Big) dx^{\alpha}dx^{\beta}. \tag{1.14}$$

In this method, the directly measurable quantities by definition are the tetrad components (or base vectors) σ^n of the null wave vector of a photon projected onto a space-like hypersurface being orthogonal to the four-velocity of an observer u^n: $\sigma^n = P^n_l k^l$, where $P^m_n = \delta^m_n + u^m u_n$ is the projection operator that satisfies the conditions $P^{\alpha}_{\alpha} = 2$ and $P^m_k P^k_n = P^m_n$. By definition, the physically observable components of the vector σ^n in the locally orthonormalized tetrad basis of an observer has only the spatial components $\sigma^{\overline{\alpha}}$, while the temporal one, $\sigma^{\overline{0}}$, is equal to zero. Contraction of the components σ^n with the basis vectors $\lambda^{\overline{p}}_n$, *i.e.*, $\sigma^{\overline{p}} = \lambda^{\overline{p}}_n \sigma^n$, is a covariant quantity that is independent of the choice of RF. This gives the procedure of relaying the coordinate quantities dx^p to the observable ones $dX^{\overline{p}} \equiv (cd\tau, dr^{\alpha})$ as follows: $dX^{\overline{p}} = \lambda^{\overline{p}}_n dx^n$. From equation (1.13), we can find the following relation:

$$ds^2 = \gamma_{\overline{p}\overline{n}} dX^{\overline{p}} dX^{\overline{n}} = \lambda^{\overline{s}}_m \lambda^{\overline{p}}_n \gamma_{\overline{s}\overline{p}} dx^m dx^n = g_{mn} dx^m dx^n, \tag{1.15}$$

which provides one with the necessary equation for finding the components of the tetrad:

$$g_{mn} = \lambda^{\overline{s}}_m \lambda^{\overline{p}}_n \gamma_{\overline{s}\overline{p}}. \tag{1.16}$$

From this equation and with the help of relations (1.14), one, in principle, may find all the necessary basis vectors $\lambda^{\overline{p}}_n$ (Logunov, 1987; Soffel, 1989). Using this technique as well as the special methods of the Riemann geometry, one may establish the relationships between the basis vectors and transform the measurable components $\sigma^{\overline{n}} = (0, \sigma^{\overline{\alpha}})$ to the coordinates of the barycentric RF. However, the reduction formula obtained this way has been proven to contain a non-observable coordinate-induced contribution in the relativistic terms (Klioner & Kopejkin, 1992). For example, the barycentric velocity of the astrometric spacecraft orbiting the Earth is not directly observable and cannot be derived with the requisite accuracy with this barycentric method. To solve this and some other problems unavoidably arising in the solely barycentric approach, a consistent relativistic theory of astronomical RFs is needed.

As we have mentioned earlier, a well-defined proper RF must be linked with the inertial RF_0 by relativistic coordinate transformations that introduce no spurious terms into the metric or the equations of motion of the relativistic *local* problem. However, the precise definition of the quasi-inertial proper RF in a curved space-time (even in the WFSMA) is not quite straightforward. We know that in freely falling inertial frames, the external gravitational field appears only in the form of tidal interactions. Up to these tidal corrections, freely falling bodies behave as if external gravity were absent (Synge, 1960; Bertotti & Grishchuk 1990). The general theoretical consideration in this case is usually based on the geodesic equation

$$\frac{du^n}{ds} = -\Gamma^n_{kl} u^k u^l. \tag{1.17}$$

This equation may be interpreted as if on the left side we have the four-acceleration of the particle, while on the right side is the force acting upon the particle. By careful choice of the coordinates, one may make the Christoffel symbols Γ^n_{kl} vanish in the immediate vicinity of the body's world line, which will put this force equal to zero (Fermi, 1922a,b; Landau & Lifshitż, 1988). This allows one to use the analogy of inertial motion and, as a result, the four-velocity may be parameterized by the natural parameter s along the geodesic: $u^n = a^n s + b^n$, with a^n and b^n being the arbitrary constant parameters. The analysis shows that in the vicinity of the world line of the origin of this well-defined RF_A, the coordinate transformation from the inertial RF_0 $(x^n) \equiv (x^0, x^\mu)$ to the physically justified RF_A $(y^n_A) \equiv (y^0_A, y^\mu_A)$ must have the structure of a Taylor expansion with respect to the powers of a spatial coordinate y^μ_A (Manasse & Misner, 1963; Manasse, 1963; Misner *et al.*, 1973):

$$x^n = x^n_{A_0}(y^0_A) + e_{A\alpha}{}^n(y^0_A) \cdot y^\alpha_A + \frac{1}{2}\Gamma_A{}^n_{\alpha\beta}(y^0_A) \cdot y^\alpha_A y^\beta_A + \mathcal{O}(y^3_A), \tag{1.18}$$

where the function $x^n_{A_0}(y^0_A)$ represents the world line's description of the origin of the coordinates (y^p_A), and the functions $e_{A\alpha}{}^n$ and $\Gamma_A{}^n_{\alpha\beta}$ are coefficients of expansion. This relativistic transformation should replace the post-Galilean transformations (1.12) as well as the special relativistic group of motion of the uniformly accelerated RFs, (1.7), allowing them both to be generalized in the case of a system of N arbitrary extended self-gravitating bodies.

It should be noted that the use of the approach described above was based upon the geodesic equation (1.17), but, as we know, extended bodies do not move along the geodesic lines. Instead, the interaction of their intrinsic mulipole moments with external gravity causes deviation of their motion from the geodesic. This means that this geodesic method is valid only for the case of monopole structureless test particles. In order to provide the dynamic definition for the proper RF, one should obtain the eq.m. of the extended bodies and require that the acceleration of the body will vanish in its proper RF. One way to do this is to generalize the Fock–Chandrasekhar approach in derivation of the eq.m. for the extended bodies, which is based upon the equation of the conservation of the density of the energy-momentum tensor $\hat{T}^{mn}$ in the form $\nabla_m \hat{T}^{mn} = 0$. One may expect that the correct transformations will modify the structure of expressions (1.18) in the higher-order terms of the spatial coordinates: $\sim y^{\{K\}}_A$, where $(k \geq 3)$.

We should mention here that in the scientific literature, in addition to the expression 'reference frame,' the notion of a 'coordinate system' (CS) has recently come into use (Kopejkin, 1988; Brumberg & Kopejkin, 1988a,b; DSX, 1991-1994). This confusion in terminology partially came from a misunderstanding of the basic principles of the theory of relativistic observables in the curved space-time developed by Zel'manov (1956). In accord with his chronogeometric

classification, one should distinguish between these two physically different concepts. Thus, the RF is an arbitrary set of four coordinates chosen to define the position of the body under study. As we know, in order to properly describe the motion of the N-body system, one should have at least N+1 of these RFs (DSX, 1991). The CS is the coordinates one may choose to describe the physical processes in the vicinity of the body in its proper RF. A coordinate system is a particular code for labeling points in an RF by some numbers. However, once the RF has been chosen, one may not make the choice of the CS arbitrarily. In order to introduce the CS one must fulfill the chronogeometric requirements, which basically state or say that, while introducing the CS, one shouldn't change the state of motion of the RF already chosen for solution. In other words, the choice of the CS should provide one with a new RF that should be physically equivalent to the old one. In practice, one usually may introduce an infinite class of CSs without violating this equivalency (Zel'manov, 1956; Logunov, 1987; Denisov & Turyshev, 1989). From the other side, it is well known that in curved space-time there are no inertial RFs even in the WFSMA; instead one may introduce only quasi-inertial ones. Moreover, a non-optimal choice of the CS may change the dynamic properties of the RF and may significantly complicate the eq.m. of the bodies, leading to the wrong conclusions (Kopejkin, 1988). This means that a clear physical definition for the RF is very important. Such a definition should enable one to study the form-invariancy of the corresponding metric tensor. As a result, one may reconstruct the group of motion, which leaves this metric tensor form-invariant, and which will provide one with the class of admissible physically equivalent coordinate transformations in the RF of interest. We will keep this relativistic terminology, and in our further discussion, we will distinguish between the CS and the RF.

As we noted before, the properties of the proper RF should be based primarily upon the structure of the metric tensor and the equations of motion of the *local* problem. For practical reasons, in order to establish the physical characteristics of the proper RF_A constructed for a particular body (A) from the system, it is best to use the well-known properties of the freely falling RFs as a first approximation when examining the interaction between the bodies. Thus, the expected properties of a physically well-defined proper RF_A may be expressed as follows:

(i). The gravitational field solutions for both relativistic *global* and *local* problems should be obtained with the same covariant gauge conditions. At least up to the terms describing the motion of the mass monopoles, the metric tensor and the eq.m. of the *local* problem must not depend on the 'absolute' velocity of the motion of the origin of the proper RF_A relative to the inertial RF_0. Both the tensor and the eq.m. in this case may admit the dependence on the relative velocities of the bodies only (Fock, 1955; Kopejkin, 1988). The body's own translational motion in its proper RF should vanish.

(ii). This field in the *local* region must be made up of four physically different contributions, namely, the proper and external gravitational fields, the field of inertia, and the gravitational interaction term. The proper gravitational field outside the body should be describable by the set of mass and current intrinsic multipole moments including the monopole, the dipole, etc. (Thorne & Hartle, 1985; Kopejkin, 1988). The gravitational field of the external bodies must be presented in the proper RF_A solely in the form of tidal terms generated by mass and current multipole moments of these bodies (Fermi, 1922a,b; Synge, 1960). The field of inertia is due to the specific properties of the coordinate transformations chosen for the construction of this RF. The interaction term describes the mutual coupling of the three above-named terms.

(iii). Coordinate transformations between the different RFs should be homogeneous functions omitting the infinite number of non-singular partial derivatives. These functions should not violate the gauge conditions chosen for the problem and must be completely defined by means of the *local* gravitational field at the origin of the coordinates of a particular quasi-inertial proper RF.

For a long time it was thought that the physically adequate *local* RF must physically resemble a frame that falls freely in the background field created only by external bodies (Kopejkin, 1988). However, this is not true. This effect is due to the presence of the gravitational interaction term, which reflects the non-linear nature of gravity. When describing the motion of a monopole particle, one may use this analogy and describe the motion of the body as if the external gravity were absent, but, in the general case of the extended self-gravitating body, one must take into account the coupling of the body's intrinsic multipoles to the external field. The existence of this coupling should be reflected in the form of the transformation functions. As a result, one should not think that the 'good' proper RF may be realized as a locally inertial RF for a massless test body (Manasse & Misner, 1963; Misner *et al.*, 1973; Ni, 1977; Ni & Zimmerman, 1978). Physically, we are looking for an RF where one may effectively separate the *local* physics from the external gravitational environment. This is why we would like to apply such an elegant and simple Newtonian tidal approach to the post-Newtonian physics of the WFSMA. From the mathematical standpoint, we are looking for a solution to the *local* problem for which the resultant space-time in the proper RF_A will be tangent to the total effective space-time generated by all the bodies in the system, including the body (A). It was shown that the solution with these properties could be found only at the immediate vicinity of the body and that the smaller the Riemann curvature of the effective space-time, the further out would be the boundary of validity of this solution (Brumberg & Kopejkin, 1988). Note that the existence of a well-defined proper RF has been more or less explicitly assumed by many authors (see, for example, Misner *et al.*, 1973; Li & Ni, 1978, 1979a,b; Will, 1993; Nordtvedt, 1995).

1.3 The Qualitative Description of the Astronomical Systems of Interest.

In order to provide a quantitative description of the relativistic motion of an astronomical N-body system, let us first qualitatively define the small parameters involved in the description of such a system. It is known that there are several major methods for studying the dynamics of such systems (Damour, 1983, 1986), depending on the relationships between the astrophysical parameters characterizing the orbital motion; rotation; gravitational field inside and outside the bodies; their sizes, shapes, and internal structures; and the distance between the bodies. We shall investigate a structure of space-time for the case of a gravitationally bounded and isolated distribution of matter. We will restrict our attention to only N-body systems, such as our solar system, which have slowly moving matter and weak gravitational fields both outside and inside the bodies. Let us assume that non-gravitational forces are absent and that the bodies are well separated. Our assumptions then are that the velocities of the orbital motion of the bodies, v_B, are non-relativistical ones (*i.e.*, considerably smaller than the speed of light c, $v_B \ll c$) and that any two arbitrary bodies in the system are at distances r_{BA_0} that are considerably greater than their radii, L_A and L_B: $r_{BA_0} > L_A, L_B$. Note that the motion of the bodies at distances $r_{BA_0} \sim r_{g_A}, r_{g_B}$, where r_g is the gravitational radius of the body, has a highly unpredictable character and will require very different mathematical techniques (Shapiro & Teukolsky, 1986a,b; Thorne, 1989). Furthermore, let us denote the following quantities for each body in the system: m_B is the mass of the body (B); r_{B_0} is the Newtonian barycentric radius vector of

this body; L_B is its mean radius; D_B is the minimal distance between the body under question and its nearest companion in the system; $\vec{u}_B$ is the internal velocity (rotational $\vec{v}_{rot}$ and plus oscillatory $\vec{v}_{osc}$) of the element of the body's matter in the proper RF$_B$; ω_B is the frequency of its rotation in this RF$_B$; $I_B^{\{K\}}$ and $S_B^{\{K\}}$ are its internal mass and current moments of the kth order, respectively; and, finally, M_0 and L_0 denote the mass and maximal diameter of the entire system.

Then, making use of the definitions above, we will concentrate our attention on a solution of the problem of motion of such a gravitationally bounded astronomical system of N extended bodies in the WFSMA. This approximation may be used successfully if the system of interest admits the existence of the following four groups of small parameters induced by the *local* and *global* of the bodies in the system (denoted with the (l) and (g) subscripts, respectively):

(1). *The shape- and size-induced parameters.* We presume that for each body in the system the following parameters of a pure geometrical nature may be introduced:

(i). $\delta_g \sim sup[\delta_g^B = L_B/D_B] \ll 1$, which describes the quasi-point structure of each body in the system;

(ii). $\delta_l \sim sup[\delta_l^B = I_B^{\{K\}}/m_B L_B^k] \ll 1$, which characterizes a dimensionless measure of the deviation of the distribution of the body's matter from a spherically symmetric distribution.

(2). *The special relativistic parameters.* The orbital and rotational motions of the bodies in the system generate the following dimensionless parameters:

(i). $\epsilon_g \sim sup[\epsilon_g^B = v_B/c] \ll 1$, characterizing the speed of the orbital motion of the bodies;

(ii). $\epsilon_l \sim sup[\epsilon_l^B = u_B/c] \sim S_B^{\{1\}}/m_B L_B c \sim \omega_B L_B/c \ll 1$, describing the slowness of the rotational motion of the bodies.

(3). *The general relativistic parameters.* The gravitational field produced by the bodies in the system may be characterized as follows:

(i). $\eta_g \sim sup[\eta_g^B = c^{-2}Gm_B/D_B] = r_{g_B}/D_B \ll 1$, which describes the weakness of the gravitational field outside the bodies;

(ii). $\eta_l \sim sup[\eta_l^B = c^{-2}Gm_B/L_B] = r_{g_B}/L_B \ll 1$, which describes the weakness of the gravitational field inside the bodies.

(4). *The background-induced parameters.* For an isolated system, the absence of initial inhomogeneity of space-time caused by in-fallen radiation, external gravitational sources, or cosmological evolution may be characterized by the parameters

(i). $h \sim ||g_{mn}^{<0>} - \gamma_{mn}||/(M_0/L_0) \ll 1$, which describes the smallness of the maximal deviation of the background metric $g_{mn}^{<0>}$ from the Minkowskian metric γ_{mn} everywhere in the system.

(ii). $\sigma \sim \dot{h}/\omega_B \ll 1$, which describes the quasi-stationary behavior of the background metric.

We shall assume that any processes in the system may be considered to be adiabatic (~ 1 yr) in comparison to the characteristic time scale of the cosmological evolution of this background space-time ($\sim 10^{10}$ yr) (as described by the Robertson–Walker solution). Moreover, asymptotic regions of the isolated N-body system are presumed to be in a state of free fall. This means that the influence of the rest of the matter in the universe on the *local* dynamics is of the order 10^{-24}, while the relativistic gravitational perturbations in the system are expected to be in the range of $10^{-5} - 10^{-21}$ (Will, 1993). With these expected accuracies, the influence of the rest of the matter in the universe on the *local* dynamics of the bodies in the system may safely be neglected. Let us denote this background space-time as γ_{mn}. Although in the general case this background metric may have arbitrary properties, for the case of an isolated system of astronomical bodies and for the WFSMA, one may take this metric in the form of space-time with a constant curvature or introduce flat Minkowski space-time in the vicinity of the system under consideration. These assumptions are necessary in order to justify the existence of a barycentric asymptotically inertial RF.

With these assumptions and consequences, the dependence on the background-induced parameters h and σ in the corresponding eq.m. of the extended bodies may be neglected. The equations in this case may be schematically presented as follows (Damour, 1987):

$$\frac{d^2 x_B}{dt^2} = \mathcal{F}^B[\delta_g, \delta_l; \epsilon_g, \epsilon_l; \eta_g, \eta_l]. \tag{1.19}$$

This expression may be formally expanded with respect to powers of the remaining small parameters, which may be given by

$$\frac{d^2 x_B}{dt^2} = \sum_{k,l,m,n,p,q \geq 0} \mathcal{F}^B_{klmnpq} \cdot \delta_g^k \delta_l^l \epsilon_g^m \epsilon_l^n \eta_g^p \eta_l^q. \tag{1.20}$$

Depending on the relations between the parameters in any particular problem, there exist several basic approximation methods. Our approach uses an assumption of a weak gravitational field inside and outside the bodies as well as an assumption about the slowness of the dynamic processes in the system. For this case, some of the parameters introduced above are linked by equalities or inequalities. Thus, the first relation may be written as $\eta_g^B = \delta_g^B \eta_l^B$, which automatically gives $\eta_g^B \leq \delta_g^B \ll 1$ or, for the entire system, $\eta_g \leq \delta_g \ll 1$. Since we are considering a gravitationally bounded N-body system in the WFSMA, there should exist relations linked by the virial theorem $v_B^2/c^2 \sim r_{g_B}/D_B$ and $v_{osc}^2/c^2 \sim r_{g_B}/L_B$ (Fock, 1955; Chandrasekhar, 1965), such that the parameters ϵ_g and η_g are equivalent and connected by the following relation: $\epsilon_g^2 \sim \eta_g$. The parameters ϵ_l^B and η_l^B are different and vary from body to body in the system. One may also limit the behavior of matter forming the bodies such that 'arbitrary bodies' must have slowly changing internal mulipole moments: $\dot{I}_B^{\{K\}}/I_B^{\{K\}} \sim \epsilon_l^B \cdot k\omega_B$, $\dot{S}_B^{\{K\}} \sim \eta_l^B I_B^{\{K\}}$. By assuming this, we exclude from this analysis such systems where the bodies are rapidly changing their multipole structure with time. Fortunately, all the celestial bodies in our solar system satisfy these conditions.

Moreover, each body studied in this report will be supposed to be isolated, *i.e.*, the immediate vicinity of the body is devoid of matter and non-gravitational fields, and the distance, D_B (the scale of homogeneity of the space-time), is large compared with the body's size, L_B. For such an isolated body, one may split space-time up into three regions as measured in the body's 'instantaneous' proper RF_B (Misner *et al.*, 1973; Thorne & Hartle, 1985; Kopejkin, 1988): the *local region*, which contains a world-tube surrounding the body and extending out to some radius

$r_l > L_B$; the *buffer region*, extending from radius r_l to some large radius $r_0 < D_B$; and the *external region*, located outside the distance r_0. In the *local* region, the body's own gravitational field dominates, but in the external region, gravitational fields of other bodies become important. The buffer region is placed in the vicinity of the distance $r^* \sim D_B(m_B/M_0)^{1/3}$ from the body, which is defined from the condition that the body's gravitational influence is approximately equal to the gravitational influence of the external masses. The buffer region plays the role of an asymptotically flat space-time region for the gravitational field of the body in question. In other words, the total three-dimensional volume V_N, which is occupied by the N-body system under study, may be split into N non-intersecting domains defined around each body in the system plus the buffer domain d_0. The situation is similar to that in the problem of the study of stellar stability of the solar system (Gladman & Duncan, 1990; Holman & Wisdom, 1993). Within each domain d_B where the gravitational influence of a particular body (B) is dominant over external gravity, the orbits of massless test particles will be stable and remain well inside this domain. In the buffer domain, the trajectories of particles are unstable. As a result, the set of small parameters defined above, in the case of the *local* problem, should be supplemented by another parameter, namely the parameter of geodesic separation, $\lambda_B = |y_B|/D_B < 1$, where $L_B \leq y_B \leq r^*$ is the distance from the world line of the body (B) to the current point of interest inside the domain d_B. This interpretation enables us to evaluate the surface integrals at the boundaries of these interacting domains as well as to define the boundary of validity of the expansions with respect to the small parameter λ_B.

1.4 Different Methods of Constructing the Proper RF.

The metric approach in the theories of gravity permits one to choose any RF to describe the gravitational environment around the body under question. As we know, a poor choice of the new coordinates may cause unreasonable complications in the physical interpretations of the data obtained (see the related discussion in Kopejkin, 1988; Soffel & Brumberg, 1991). Recently, several different attempts were made to remove these complications and consequently improve the present solution to the N-body problem in the WFSMA (see, for example, Ashby & Bertotti, 1984, 1986; Brumberg & Kopejkin, 1988a,b; Kopejkin, 1988; Klioner, 1993; DSX, 1991-94). Although these methods represent a significant improvement in our understanding of the general problem, not one of them gives a complete 'recipe' to overcome the difficulties stated above.

The methods differ in their physical and mathematical treatment of the three problems, which constitute the general problem of motion of a gravitationally bounded astronomical N-body system (the *global* and the *local* problems and the theory of the RFs). One such method was proposed by Bertotti (1954) and has been further developed in a number of publications by Ashby and Bertotti (1984, 1986), Bertotti (1986), Ashby and Shahid-Saless (1990), Shahid-Saless, Hellings, and Ashby (1991), and Shahid-Saless (1992). An equivalent method was proposed and developed to the extent of practical applications by Fukushima (1988, 1991a,b, 1995a,b). In these works, the 'good' proper RF is constructed within the first post-Newtonian approximation (1PNA) of general relativity for a specific form of the EIH metric (Einstein *et al.*, 1938). The EIH metric was obtained in the inertial RF_0 and describes the gravitational field only outside the bodies, which may be regarded as massive point particles or spherically symmetric and non-rotating extended bodies (Fock, 1955).

In the Bertotti–Fukushima method, the construction of the *local* RF is based upon finding the background external metric for the body under consideration. The external metric is obtained from the complete EIH metric by dropping all of the divergent or undefined terms on the body's

center of inertia world line. Then, a *local* Fermi-normal-like frame (Fermi, 1922; Manasse & Misner, 1963; Misner *et al.*, 1973) is defined in the body's vicinity using the background metric with respect to which the body moves along the geodesic. After that, the coordinate transformation between the Fermi frame and background metric is obtained. The transformation is applied to the complete EIH metric and, thus, the 'good' proper RF is obtained. The body's gravitational field in this proper RF is spherically symmetric (Schwarzchild) and the gravitational field of distant bodies appears only through the curvature tensor of the background metric, *i.e.*, through the tidal effects.

The Bertotti–Fukushima method is conceptually simple. It confirms our expectations that the physically adequate proper RF exists and gives an insight into the structure of transformations (1.18). However, this method of construction of the Fermi normal coordinates for massive bodies has some drawbacks (Kopejkin, 1988), namely:

(i). The background external metric was not derived by solving the gravitational field equations.

(ii). There are physical and mathematical ambiguities in the way of constructing the external metric. These ambiguities are caused by the terms describing the back action of the gravitational field of the body under consideration on the external gravity produced by other bodies (Thorne & Hartle, 1985).

(iii). The method under review cannot be used for derivations of the eq.m. of bodies, *i.e.*, their world lines. A choice of the body's center of inertia world line as a geodesic is justified only *a posteriori* and with the help of quite a different technique (EIH, 1938; Papapetrou, 1948, 1951; Brumberg, 1972; Damour, 1983; Thorne & Hartle, 1985; Kopejkin, 1985, 1987).

(iv). The method has been elaborated only for the special case of spherically symmetric and non-rotating bodies. It is completely unclear how one might construct the Fermi normal coordinates in real astronomical situations that are considerably more complicated. This method is inapplicable even to the Earth itself, which has oblateness and rotation that may not be ignored (Kopejkin, 1988).

(v). The proposed coordinate transformations between the RFs are incomplete, which significantly limits the applicability of the results obtained in real astronomical practice.

An important method of construction of the 'good' proper RF was proposed by Thorne and Hartle (1985) (see also Fujimoto & Grafarend (1986)) and developed to some extent by Zhang (1985, 1986) and Suen (1986). The Thorne–Hartle method is conceptually elegant and has produced the largest corrections to the geodesic law of motion and the Fermi–Walker law of transport (Misner *et al.*, 1973). The method consists of determining the metric tensor from the Hilbert–Einstein equations under the condition that one satisfies the properties of the well-defined proper RF that were mentioned above. Thus, the metric in this method is derived entirely in the 'good' proper RF. The solutions of the gravitational field equations are searched for in a vacuum region of space-time under de Donder (harmonic) gauge conditions in the body's neighborhood where the gravitational field is weak. The metric tensor is represented in the form of an expansion in powers of the small parameters $m_B/r, r/R$, etc., where m_B is the body's mass, r is a distance from the body, and R is an inhomogeneity scale (distance between the bodies). The coefficients of the expansion are the internal and external multipole moments of the gravitational fields created both by the body under consideration and the external gravity, respectively. In this method, the information about the properties of the chosen RF is completely contained in the set of these multipole moments.

Although the Thorne–Hartle method represents an important progression in our understanding of the motion of unisolated bodies and their interaction with the external universe and provides an important insight into the physical structure of a multipole expansion of the metric tensor in different RFs, it cannot be used immediately in ephemeris astronomy. The main reasons for this are as follows:

(i). The finding of the solutions of the Hilbert–Einstein field equations and the matching of the asymptotic expansions were done formally. Since the goal of the paper was to find the largest corrections to the laws of motion and precession only, the method does not provide a complete multipole treatment of extended bodies. As a result, the internal multipole moments are not presented as integrals over the volumes of the sources and therefore have no clear physical meaning (Kopejkin, 1988).

(ii). The authors have not presented the coordinate transformation between the RFs used for the analysis. They have constructed only the 'instantaneous' proper RF, which coincides with the body's center of inertia at a particular moment of time. As time goes on, the origin of the 'instantaneous' proper RF propagates along a geodesic, but, in the general case, the body's center of inertia world line does not. The deviation from the geodesic is caused by the interaction of the body's own intrinsic multipole moments with the external gravity. This leads to a drifting of the 'instantaneous' proper RF from the body's center of inertia, which is not acceptable for astronomical practice (Soffel & Brumberg, 1991; Williams *et al.*, 1991).

Another method of constructing of the 'good' proper RF was proposed by D'Eath (1975a,b) (see also papers by Kates (1980a,b) and Damour (1983)). These papers are devoted to the derivation of the eq.m. of compact, strongly gravitating astrophysical objects such as black holes and neutron stars. The authors have applied an interesting mathematical method of matched asymptotic expansions, which was not developed to be used in practical astronomical applications for the more common case of weakly gravitating bodies. There have been many works in which construction of the 'good' proper RF has been accomplished with the help of infinitesimal transformations (Fukushima *et al.*, 1986; Hellings, 1986; Vincent, 1986). Unfortunately, the methods used in these works may not be considered to be satisfactory since they are based upon heuristic principles rather than exact theory (Kopejkin, 1988).

The critical breakthrough in construction of a relativistic theory of RFs appropriate for astronomical practice was achieved by Brumberg and Kopejkin (for a detailed description see Kopejkin, 1985, 1987, 1988; Brumberg & Kopejkin, 1988a,b; Voinov, 1990; Brumberg, 1991a,b, 1992; Klioner & Kopejkin, 1992; Brumberg *et al.*, 1993; Klioner, 1993; Klioner & Voinov, 1993). The relativistic theory developed by Brumberg and Kopejkin combined the basic ideas of Fock (1955) on the post-Newtonian approximation scheme; Thorne (1980) and Thorne & Hartle (1985) on multipole formalism; and D'Eath (1975a,b), Kates (1980a,b), Kates & Madonna (1982), and D'Eath & Payne (1992) on matched asymptotic expansions.

The Brumberg–Kopejkin method was the first to develop the three sub-problems of the gravitationally bounded astronomical N-body system. The authors identify the metric tensor of the relativistic *global* problem with the solution of an isolated distribution of matter in the inertial RF obtained in the 1PNA of general relativity (Fock, 1955; Brumberg, 1972; Will, 1993). The solution of the *local* problem is formally presented as an isolated one-body solution corrected by electric-type and magnetic-type external multipole moments (Thorne, 1980). The form of

these moments reflects the properties of the proper RF chosen for the analysis of the gravitational environment of the body under study. The structure of these moments as well as the post-Newtonian coordinate transformations between the inertial and the quasi-inertial RFs are derived by matching both solutions in the body's neighborhood.

This method demonstrates a notable progression in the theory of astronomical relativistic RFs developed to describe the motion of a system of N extended bodies in the WFSMA. However, this method also has some drawbacks:

(i). The authors have made *ad hoc* assumptions about the various multipole expansions of the metric tensor and coordinate transformations that are only partially justified by some later consistency checks (DSX, 1991).

(ii). The method to derive the solution to the Hilbert–Einstein gravitational field equations of the general theory of relativity based on the Anderson–DeCanio approach (Anderson & DeCanio, 1975; Anderson, J. L. *et al.*, 1982) is not covariant. In particular, based only on this method, it is not possible to derive the explicit solution to these field equations in an accelerated proper RF linked to the body's center of inertia. As a result, the introduced 'external' multipole moments do not have a clear physical meaning.

(iii). The obtained relativistic coordinate transformation between the different RFs is incomplete as it contains only contributions from the leading intrinsic multipoles of the body (the mass monopole and dipole and the current dipole). The contributions from the other intrinsic multipoles are hopelessly mixed with the external moments in the structure metric tensor of the *local* problem. Thus, the transformation does not take into account the non-linear coupling of the body's own gravitational field to external gravity even at the Newtonian level. As a result, the origin of the proper RF coincides with the center of inertia of the body at a particular moment in time only, and, as time goes on, they will drift apart.

(iv). The method under review does not provide us with the necessary microscopic description of relativistic phenomena in terms of densities of the gravitational fields. Thus, the mass of the bodies, the momentum, and the angular momentum were never explicitly defined. The parameters introduced to substitute these quantities were never checked as to whether or not they correspond to the integral conservation laws in the proper RFs of the bodies. In addition, the mass density of the gravitational field in the *local* region at the Newtonian level is given solely by the body's own mass density. But the *local* gravitational field contains tidal terms due to the external bodies. As a result, the theory does not admit a special relativistic treatment of the N-body problem in the sense of the mechanics of Poincaré.

Recently, a very powerful approach to this problem has been elaborated by Damour, Soffel, and Xu (DSX, 1991-1994), Blanchet *et al.* (1995), and Damour & Vokrouhlický (1995). It combines an elegant ('Maxwell-like') treatise of the space-time metric in both the *global* and *local* RFs with the Blanchet–Damour multipole formalism (Blanchet & Damour, 1986). This approach allows one to relate the multipole expansions of the gravitational field to the structure of the source of gravitation. This method, though very promising and attractive, still requires extensive development to make it useful for practical astronomical applications. Besides this, the method under review has some problems that should be worked out in a more physically grounded way. These include the following:

(i). The Blanchet–Damour 'external' multipole moments were defined in the rest frame of an idealized isolated distribution of matter, so they must be modified in order to take into account the non-inertiality of the proper RF as well as the interaction of the body's proper gravitational field with external gravity.

(ii). The proposed relativistic coordinate transformation between the different RFs is incomplete because it does not take into account the terms due to interaction of the body's own gravitational field with external gravity. Moreover, the suggested coordinate transformation completely neglects the precession term and does not include the terms due to interaction of the body's intrinsic multipoles with the external gravity. This means that the proper RF constructed with these transformations in the case of monopole structureless particles does not end up with an RF defined on a geodesic line, which is guaranteed by the Principle of Equivalence. It should be noted that the origin of the proper RF, in the general case of extended bodies, coincides with the center of inertia of the *local* field in the initial moment of time only, and it drifts away as time progresses. This leaves the quantities, calculated with respect to such a proper RF, physically ill defined (Damour & Vokrouhlický, 1995).

(iii). The solutions of the Hilbert–Einstein equations in the different RFs were obtained using non-covariant gauge conditions. This does not provide one with a clear understanding of what part of the solution of the *local* problem is due to the gravitational field, what is caused by the contribution of the inertial sector of the space-time, and how these two interact with each other.

(iv). At this time, the method under review may not be extended for analysis of the WFSMA of other metric theories of gravity.

In light of this, the principle purpose of the present report is to develop a classic field approach to the problems of astronomical measurements in the WFSMA of a number of modern metric theories of gravity. This approach will combine the well-established methods of the relativistic mechanics of Poincaré with the Fock–Chandrasekhar treatment of the relativistic many-extended-body gravitational problem (Fock, 1955, 1957; Chandrasekhar, 1965). One of the main goals of this research was to develop a foundation for extending the applicability of the PPN formalism, which has become a very useful framework for testing the metric theories of gravity.

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2 Parametrized Post-Newtonian Metric Gravity.

In this section, we will discuss the status of the problem of constructing a solution to the gravitational field equations for a gravitationally bound astronomical N-body system. Within the accuracy of modern experimental techniques, the WFSMA provides a useful starting point for testing the predictions of different metric theories of gravity in the solar system. Following Fock (1955, 1957), the perfect fluid is used most frequently as the model of matter distribution when describing the gravitational behavior of celestial bodies in this approximation. The density of the corresponding energy-momentum tensor $\widehat{T}^{mn}$ is as follows:

$$\widehat{T}^{mn} = \sqrt{-g}\Big(\Big[\rho_0(1+\Pi)+p\Big]u^m u^n - pg^{mn}\Big), \tag{2.1}$$

where ρ_0 is the mass density of the ideal fluid in coordinates of the co-moving RF, $u^k = dz^k/ds$ are the components of invariant four-velocity of a fluid element, and $p(\rho)$ is the isentropic pressure connected with ρ by an equation of state. The quantity $\rho\Pi$ is the density of internal energy of an ideal fluid. The definition of Π is given by the equation based on the first law of thermodynamics (Fock, 1955; Chandrasekhar, 1965; Brumberg, 1972; Will, 1993):

$$u^n\Big(\Pi_{;n} + p\Big(\frac{1}{\widehat{\rho}}\Big)_{;n}\Big) = 0, \tag{2.2}$$

where $\widehat{\rho} = \sqrt{-g}\rho_0 u^0$ is the conserved mass density. Given the energy-momentum tensor, one may proceed to find the solutions of the gravitational field equations for a particular relativistic theory of gravity. The solution for an astronomical N-body problem is the one of most practical interest. In the following subsections, we will discuss the properties of the solution of an isolated one-body problem as well as the features of construction of the general solution for the N-body problem in both barycentric and planeto-centric RFs.

2.1 An Isolated One-Body Problem.

The solution for the isolated one-body problem in the WFSMA may be obtained from the linearized gravitational field equations of a particular theory under study. As we mentioned above, a perturbative gravitational field $h^{mn}_{(0)}$ in this case is characterized by the deviation of the density of the general Riemmanian metric tensor $\sqrt{-g}g^{mn}$ from the background pseudo-Euclidian space-time γ_{mn}, which is considered to be a zeroeth order[3] approximation for the series of successive iterations: $\sqrt{-g}g^{mn} - \sqrt{-\gamma}\gamma^{mn} = h^{mn}_{(0)}$, or equivalently,

$$g_{mn} = \gamma_{mn} + h^{(0)}_{mn}. \tag{2.3}$$

The search for the solution of the field equations is performed within a barycentric inertial RF_0 (x^p) that is singled out by the Fock–Sommerfeld boundary conditions imposed on the $h^{(0)}_{mn}(z^p)$ and $\partial_k h^{(0)}_{mn}(z^p)$ (Fock, 1955; Damour, 1987; Will, 1993):

$$\lim_{r\to\infty}\Big[h^{(0)}_{mn}(x^p);\ r\Big(\frac{\partial}{\partial x^0}h^{(0)}_{mn}(x^p) + \frac{\partial}{\partial r}h^{(0)}_{mn}(x^p)\Big)\Big] \ \to\ 0, \tag{2.4}$$

$$x^0 + r = \mathrm{const}, \qquad r^2 = -\gamma^{(0)}_{\mu\nu}x^\mu x^\nu.$$

[3]For most non-radiative problems in solar system dynamics, this tensor usually is taken to be a Minkowski metric (Damour, 1983, 1987; Will, 1993).

In order to accumulate the features of many modern metric theories of gravity in one theoretical scheme, to create a versatile mechanism to plan gravitational experiments, and to analyze the data obtained, Nordtvedt and Will have proposed a parameterized post-Newtonian (PPN) formalism (Nordtvedt, 1968a,b; Will, 1971; Will & Nordtvedt, 1972). This formalism allows one to describe the motion of celestial bodies for a wide class of metric theories of gravity within a common framework. The gravitational field in the PPN formalism is presumed to be generated by some isolated distribution of matter that is taken to be an ideal fluid, (2.1). This field is represented by the sum of gravitational potentials with arbitrary coefficients: the PPN parameters. The two-parameter form of this tensor in four dimensions may be written as follows:

$$h_{00}^{(0)} = -2U + 2(\beta - \tau)U^2 + 2\Psi + 2\tau(\Phi_2 - \Phi_w) + (1 - 2\nu)\chi_{,00} + \mathcal{O}(c^{-6}), \tag{2.5a}$$

$$h_{0\alpha}^{(0)} = (2\gamma + 2 - \nu - \tau)V_\alpha + (\nu + \tau)W_\alpha + \mathcal{O}(c^{-5}), \tag{2.5b}$$

$$h_{\alpha\beta}^{(0)} = 2\gamma_{\alpha\beta}(\gamma - \tau)U - 2\tau U_{\alpha\beta} + \mathcal{O}(c^{-4}), \tag{2.5c}$$

where γ_{mn} is the Minkowski metric.[4] The generalized gravitational potentials are given in Appendix A.

Besides the two Eddington parameters (γ, β), eq.(2.5) contains two other parameters, ν and τ. The parameter ν reflects the specific choice of gauge conditions. For the standard PPN gauge, it is given as $\nu = \frac{1}{2}$, but for harmonic gauge conditions, one should choose $\nu = 0$. The parameter τ describes a possible pre-existing anisotropy of space-time and corresponds to different spatial coordinates that may be chosen for modelling the experimental situation. For example, the case of $\tau = 0$ corresponds to harmonic coordinates, while $\tau = 1$ corresponds to the standard (Schwarzschild) coordinates. A particular metric theory of gravity in this framework with a specific coordinate gauge (ν, τ) may then be characterized by means of two of the above-mentioned PPN parameters (γ, β), which are uniquely prescribed for each particular theory under study. In the standard PPN gauge (i.e., in the case when $\nu = \frac{1}{2}, \tau = 0$), these parameters have clear physical meaning. The parameter γ represents the measure of the curvature of the space-time created by the unit rest mass; the parameter β is the measure of the non-linearity of the law of superposition of the gravitational fields in the theory of gravity (or the measure of the metricity). Note that general relativity, when analyzed in standard PPN gauge, gives $\gamma = \beta = 1$, whereas, for the Brans–Dicke theory, one has $\beta = 1, \gamma = \frac{1+\omega}{2+\omega}$, where ω is an unspecified dimensionless parameter of the theory.

The properties of an isolated one-body solution are well known. It has been shown (Lee *et al.*, 1974; Ni & Zimmerman, 1978; Will, 1993) that for an isolated distribution of matter in the WFSMA there exist a set of inertial RFs and ten integrals of motion corresponding to ten conservation laws. Therefore, it is possible to consistently define the multipole moments characterizing the body under study. For practical purposes, one chooses the inertial RF located in the center of mass of an isolated distribution of matter. By performing a power expansion of the potentials in terms of spherical harmonics, one may obtain the post-Newtonian set of 'canonical'

[4]Do not mix the post-Newtonian parameter γ and the Minkowski metric tensor γ_{mn}. As necessary, we will distinguish the determinant $det||\gamma_{mn}||$ with the special symbol.

parameters (such as unperturbed irreducible mass $I^{\{L\}}_{A(0)}$ and current $S^{\{L\}}_{A(0)}$ multipole moments) generated by the inertially moving extended self-gravitating body (A) under consideration:

$$I^{\{L\}}_{A(0)} = \Big[\int_A d^3z'_A \hat{t}^{00}_A(z'^p_A) z'^{\{L\}}_A\Big]^{STF}, \qquad S^{\{L\}}_{A(0)} = \Big[\epsilon^{\mu_1}{}_{\beta\sigma}\int_A d^3z'_A z'^{\beta}_A \hat{t}^{0\sigma}_A(z'^p_A) z'^{\mu_2}_A ... z'^{\mu_l}_A\Big]^{STF}, \quad (2.6a)$$

where $\hat{t}^{mn}_A$ is the components of the symmetric density of the energy-momentum tensor of matter and gravitational field taken jointly. As a result, the corresponding gravitational field $h^{mn}_{(0)}$ may be uniquely represented in the external domain as a functional of the set of these moments. Schematically this may be expressed as

$$h^{mn}_{(0)A} = \mathcal{F}^{mn}[I^{\{L\}}_{A(0)}, S^{\{L\}}_{A(0)}], \quad (2.6b)$$

where the functional dependence, in general, includes a non-local time dependence on the 'past' history[5] of the moments (Blanchet *et al.*, 1995). However, by assuming that the internal processes in the body are adiabatic, one may neglect this non-local evolution. As a result, an external observer may uniquely establish the gravitational field of this body through determination of these multipole moments, for example, by studying the geodesic motion of the test particles in orbit around this distribution of matter (Misner *et al.*, 1973).

2.2 The Limitations of the Standard PPN Formalism.

It turns out that the generalization of the results obtained for the one-body problem into a solution of the problem of motion of an arbitrary N-body system is not quite straightforward. Thus, the studies of post-Newtonian motion of extended bodies in PPN formalism begin by expanding the generalized gravitational potentials in the metric tensor and the corresponding eq.m of these bodies with respect to deviation from Newtonian motion. As a final result, one needs to have the generalization of expression (2.6*b*) for the case of the N-body problem. However, this generalization is usually done by using Galilean coordinate transformations similar to those of (1.2) from the Newtonian mechanics (Fock, 1955; Will, 1993):

$$x^0 = y^0_B + \mathcal{O}(c^{-2}), \qquad x^\alpha = y^\alpha_{B_0}(y^0_B) + y^\alpha_B + \mathcal{O}(c^{-2}), \quad (2.7)$$

where $y^\alpha_{B_0}$ is the Newtonian barycentric radius vector of the body (B) under study. It was noted that this accuracy is enough for the post-Newtonian terms in these eq.m. (Brumberg, 1972), but it is insufficient to account for the necessary special relativistic and gravitational corrections. Thus, as we know, if the body is spherically symmetric in the proper RF, in the other frame it will experience both the Lorentzian contraction (linked to the relative velocity between these frames) in the direction of velocity between these RFs and gravitational compression (or 'Einsteinian' contraction, which is linked with the external gravity) (Kopejkin, 1987). However, transformations (2.7) ignored completely these Lorentzian and gravitational contractions, as well as the relativistic geodetic precession and effects of the curvature of space-time. All these kinematic and dynamic effects appear in the expressions for the metric tensor and eq.m. of the *local* problem, where they are shown as terms depending on both (i) the 'absolute' velocity of the body's center of inertia with respect to the barycentric inertial RF_0 and (ii) the absolute value and first spatial derivative of the external gravitational potential U^{ext}. As a result, the relativistic eq.m. of the *local* problem differ essentially from the Newtonian eq.m., which do not depend on the 'absolute' velocity and contain only the second spatial derivative of U^{ext}, *i.e.*, the tidal terms.

[5]Gravitational radiation problems are not within the scope of the present report and, hence, the set of multipole moments, (2.6*a*), are used for both tensor and scalar-tensor theories.

The correct way to describe these phenomena is to use the appropriate coordinate transformations between the different RFs in the WFSMA. These transformations should generalize the expressions of the Poincaré group of motion, (1.7), for the problem of motion of the gravitationally bounded N-extended-body system. However, the standard PPN formalism was formulated once in the inertial RF and there is no way to construct such a transformation for the quasi-inertial proper frames of the bodies. This lack of transformation between the different RFs is a major limitation of this otherwise very useful method.

Nevertheless, by putting some additional restrictions on the shape and internal structure of the bodies, one may generalize the results presented above in the case of an N-body system. The assumption that the bodies possess only the lowest multipole mass moments considerably simplifies the problem. It has been shown (Fock, 1955; Lee, Lightmann & Ni, 1974; Ni & Zimmerman, 1978) that for an isolated distribution of matter in the WFSMA it is possible to consistently define the lowest conserved multipole moments, such as the total rest mass of the system, M_0; its center of mass, z_0^α; the momentum, p_0^α; and the total angular momentum, $S_0^{\alpha\beta}$, of the system. The definitions for the mass M_0 and coordinates of the center of mass of the body z_0^α in any inertial RF are given by the following formulae (for a more detailed analysis see Damour (1983) and Will (1993) and references therein):

$$M_0 = \int d^3x' \, \hat{t}^{00}(x'^p), \qquad z_0^\alpha(t) = \frac{1}{M_0} \int d^3x' \, \hat{t}^{00}(x'^p) x'^\alpha, \tag{2.8a}$$

where the energy density $\hat{t}^{00}(x'^p)$ of the matter and the gravitational field is given by

$$\hat{t}^{00}(x^p) = \hat{\rho}\Big[1 + c^{-2}\Big(\Pi - \frac{1}{2}U - \frac{1}{2}v_\mu v^\mu\Big) + \mathcal{O}(c^{-4})\Big], \tag{2.8b}$$

with $\hat{\rho}$ being the conserved mass density. In particular, the center of mass z_0^α moves in space with a constant velocity along a straight line: $z_0^\alpha(t) = p_0^\alpha \cdot t + k^\alpha$, where the constants $p_0^\alpha = dz_0^\alpha/dt$ and k_A^α are the body's momentum and center of inertia, respectively. Moreover, it was shown by Chandrasekhar & Contopulos (1967) that, in the case of point-like massive particles, the form of metric tensor (2.5) and the corresponding EIH eq.m. are invariant under coordinate transformations (1.12). This form invariancy justifies the word 'inertial' for harmonic RFs constructed under the Fock–Sommerfeld boundary conditions (2.3). One may choose from the set of inertial RFs the barycentric inertial RF_0 for such a system. In this frame, the functions z_0^α must equal zero for any moment of time. This condition may be satisfied by applying the post-Galilean transformations (1.12) to the metric (2.5), where the constant velocity and displacement of the origin should be selected in a such a way that p_0^α and k^α equal zero (for details, see Kopejkin, 1988; Will, 1993). The solar system barycentric RF_0, constructed using general relativity for the system of point-like massive particles, is widely in use in modern astronomical practice, for example, in the construction of planet ephemerides (Moyer, 1971; Lestrade and Chapront-Touzé, 1982; Newhall *et al.*, 1983; Akim *et al.*, 1986; Standish, 1995). Moreover, the coordinate time of the solar system barycentric (harmonic) RF_0 must be considered as the TDB time scale, which is extensively used in modern astronomical practice (Fukushima, 1995a).

2.2.1 The Simplified Lagrangian Function of an Isolated N-Body System.

In order to extract the information about the gravitational field of an N-body system, one should study the motion of light rays and test bodies in this gravitational environment. However, the standard methods of the PPN formalism (Will, 1993) do not enable us to develop the correct

theoretical model of the astrophysical measurements with the accuracy necessary to identify the multipolar structure of the gravitational fields of the bodies. In particular, it was noted that taking into account the presence of any non-vanishing internal multipole moments of an extended body significantly changes its equations of motion due to the coupling of these intrinsic multipole moments of the body to the surrounding gravitational field. For example, for a neutral monopole test particle, the external gravitational field completely defines the feducial geodesic world line that this test body follows (Fock, 1955; Will, 1993). On the other hand, the equations of motion for spinning bodies contain additional terms due to the coupling of the body's spin to the external gravity through the Riemann curvature tensor (Papapertou, 1948, 1951; Barker & O'Connel, 1975).

An 'absolute' limit of the PPN formalism takes into account the lowest multipole moments of the bodies only, such as the rest mass m_A of the body (A), its intrinsic spin moment $S_A^{\alpha\beta}$, and the quadrupole moment $I_A^{\alpha\beta}$. The general solution with such assumptions is also known (see Damour, 1986, 1987 and references therein, and Turyshev, 1990). In order to analyze the motion of bodies in the solar system barycentric RF_0, one may obtain the restricted Lagrangian function L_N describing the motion of N self-gravitating bodies, which may be presented as follows:

$$L_N = \sum_A^N \frac{m_A}{2} v_{A\mu} v_A{}^\mu \Big(1 - \frac{1}{4} v_{A\mu} v_A{}^\mu\Big) - \sum_A^N \sum_{B\neq A}^N \frac{m_A m_B}{r_{AB}} \Bigg(\frac{1}{2} + (3+\gamma-4\beta)E_A -$$

$$-(\gamma - \tau + \frac{1}{2}) v_{A\mu} v_A{}^\mu + (\gamma - \tau + \frac{3}{4}) v_{A\mu} v_B{}^\mu - (\frac{1}{4} + \tau) n_{AB\lambda} n_{AB\mu} v_A{}^\lambda v_B{}^\mu + \tau (n_{AB\mu} v_A{}^\mu)^2 +$$

$$+ \frac{n_{AB\lambda}}{r_{AB}} \Big[(\gamma + \frac{1}{2}) v_{A\mu} - (\gamma+1) v_{B\mu}\Big] S_A^{\mu\lambda} + n_{AB\lambda} n_{AB\mu} \frac{I_A^{\lambda\mu}}{r_{AB}^2}\Bigg) + (\beta - \tau - \frac{1}{2}) \sum_A^N m_A \Big(\sum_{B\neq A}^N \frac{m_B}{r_{AB}}\Big)^2 -$$

$$-\tau \sum_A^N \sum_{B\neq A}^N \sum_{C\neq A,B}^N m_A m_B m_C \Big[\frac{n_{AB\lambda}}{2 r_{AB}^2} (n_{BC}^\lambda + n_{CA}^\lambda) - \frac{1}{r_{AB} r_{AC}}\Big] + \sum_A^N m_A \mathcal{O}(c^{-6}), \tag{2.9}$$

where m_A is the isolated rest mass of a body (A), the vector r_A^α is the barycentric radius vector of this body, the vector $r_{AB}^\alpha = r_B^\alpha - r_A^\alpha$ is the vector directed from body (A) to body (B), and the vector $n_{AB}^\alpha = r_{AB}^\alpha / r_{AB}$ is the usual notation for the unit vector along this direction. It should be noted that expression (2.9) does not depend on parameter ν, which confirms that this parameter is the gauge parameter only. The tensor $I_A^{\mu\nu}$ is the STF (Thorne, 1980) tensor of the reduced quadrupole moment of body (A), defined as

$$I_A^{\mu\nu} = \frac{1}{2m_A} \int_A d^3 z'_A \widehat{\rho}_A(z'^p_A) \Big(3 z'^\mu_A z'^\nu_A - \gamma^{\mu\nu} z'_{A\beta} z'^\beta_A\Big). \tag{2.10}$$

The tensor $S_A^{\mu\nu}$ is the body's reduced intrinsic STF spin moment which is given as:

$$S_A^{\mu\nu} = \frac{1}{m_A} \int_A d^3 z'_A \widehat{\rho}_A(z'^p_A) [\underline{v}_A^\mu z'^\nu_A - \underline{v}_A^\nu z'^\mu_A], \tag{2.11}$$

where $\underline{v}_A^\mu$ is the velocity of the intrinsic motion of matter in the body (A). Finally, the quantity E_A is the body's gravitational binding energy:

$$E_A = \frac{1}{2m_A} \int\int_A d^3 z'_A d^3 z''_A \frac{\widehat{\rho}_A(z'^p_A) \widehat{\rho}_A(z''^p_A)}{|z'^\nu_A - z''^\nu_A|}. \tag{2.12}$$

Let us note that the Lagrangian function is obtained with the condition that, in the proper RF of each body (A) in the system, the body's dipole mass moments vanish:

$$I_A^{\{1\}} \equiv m_A^\alpha = \int d^3 z'_A \hat{t}_A^{00}(z_A'^p) z_A'^\alpha = 0, \tag{2.13a}$$

where $\hat{t}_A^{00}$ is defined by the following expression:

$$\hat{t}^{00}(z_A^p) = \hat{\rho}_A\Big[1 + c^{-2}\Big(\Pi - \frac{1}{2}U_A - \frac{1}{2}\underline{v}_\mu \underline{v}^\mu\Big) + \mathcal{O}(c^{-4})\Big]. \tag{2.13b}$$

Expression (2.13a), together with the condition $\dot{m}_A^\alpha = 0$, may be considered as an indirect post-Newtonian definition of the proper RF$_A$ in the PPN formalism.

2.2.2 The Simplified Barycentric Equations of Motion.

In this subsection, we will present the barycentric equations of motion that follow from the Lagrangian function (2.9). The assumption that bodies in the system possess the lowest intrinsic multipole moments enables us to obtain only the corresponding simplified equations of motion. Thus, with the help of the expressions (2.9), for an arbitrary body (A), these equations will read as follows:

$$\ddot{r}_A^\alpha = \sum_{B\neq A} \frac{\mathcal{M}_B}{r_{AB}^2}\hat{n}_{AB}^\alpha + \sum_{B\neq A}\frac{m_B}{r_{AB}^2}\Big[\mathcal{A}_{AB}^\alpha + \frac{\mathcal{B}_{AB}^\alpha}{r_{AB}} + \frac{\mathcal{C}_{AB}^\alpha}{r_{ab}^2} -$$

$$-\frac{n_{AB}^\alpha}{r_{AB}}\Big((2\beta + 2\gamma - 2\tau + 1)m_A + (2\beta + 2\gamma - 2\tau)m_B\Big)\Big] +$$

$$+ \sum_{B\neq A}\sum_{C\neq A,B} m_B m_C \mathcal{D}_{ABC}^\alpha + \mathcal{O}(c^{-6}), \tag{2.14}$$

where, in order to account for the influence of the gravitational binding energy E_B, we have introduced the passive gravitational rest mass $\mathcal{M}_B$ (Nordtvedt, 1968b; Will, 1993) as follows

$$\mathcal{M}_B = m_B\Big(1 + (3 + \gamma - 4\beta)E_B + \mathcal{O}(c^{-4})\Big). \tag{2.15}$$

The unit vector n_{AB} must also be corrected using the gravitational binding energy and the tensor of the quadrupole moment $I_A^{\alpha\beta}$ of the body (A) under question:

$$\hat{n}_{AB}^\alpha = n_{AB}^\alpha\Big(1 + (3 + \gamma - 4\beta)E_A + 5n_{AB\lambda}n_{AB\mu}\frac{I_A^{\lambda\mu}}{r_{AB}^2}\Big) + 2n_{AB\beta}\frac{I_A^{\alpha\beta}}{r_{AB}^2} + \mathcal{O}(c^{-4}). \tag{2.16}$$

The term $\mathcal{A}_{AB}^\alpha$ in expression (2.14) is the orbital term, which is given as follows:

$$\mathcal{A}_{AB}^\alpha = v_{AB}^\alpha n_{AB\lambda}\Big(v_A^\lambda - (2\gamma - 2\tau + 1)v_{AB}^\lambda\Big) +$$

$$+ n_{AB}^\alpha\Big(v_{A\lambda}v_A^\lambda - (\gamma + 1 + \tau)v_{AB\lambda}v_{AB}^\lambda - 3\tau(n_{AB\lambda}v_{AB}{}^\lambda)^2 - \frac{3}{2}(n_{AB\lambda}v_B^\lambda)^2\Big). \tag{2.17}$$

The spin-orbital term $\mathcal{B}_{AB}^\alpha$ has the form

$$\mathcal{B}^{\alpha}_{AB} = (\frac{3}{2} + 2\gamma)v_{AB\lambda}(S^{\alpha\lambda}_{A} + S^{\alpha\lambda}_{B}) + \frac{1}{2}v_{A\lambda}(S^{\alpha\lambda}_{A} - S^{\alpha\lambda}_{B}) +$$

$$+\frac{3}{2}(1+2\gamma)n_{AB\lambda}v_{AB\beta}\Big[n^{\beta}_{AB}(S^{\alpha\lambda}_{A} + S^{\alpha\lambda}_{B}) - n^{\alpha}_{AB}(S^{\beta\lambda}_{A} + S^{\beta\lambda}_{B})\Big] +$$

$$+\frac{3}{2}n_{AB\lambda}\Big[n^{\alpha}_{AB}(v_{A\beta}S^{\beta\lambda}_{B} - v_{B\beta}S^{\beta\lambda}_{A}) + n_{AB\beta}v_{AB}{}^{\beta}S^{\alpha\lambda}_{B}\Big]. \tag{2.18}$$

The term $\mathcal{C}^{\alpha}_{AB}$ is caused by the oblateness of the bodies in the system:

$$\mathcal{C}^{\alpha}_{AB} = 2n_{AB\beta}I^{\alpha\beta}_{B} + 5n^{\alpha}_{AB}n_{AB\lambda}n_{AB\mu}I^{\lambda\mu}_{B}. \tag{2.19}$$

And, finally, the contribution $\mathcal{D}^{\alpha}_{abc}$ to the equations of motion (2.14) of body (A) (caused by the interaction of the other planets (B≠A, C≠A,B) with each other) is presented as

$$\mathcal{D}^{\alpha}_{ABC} = \frac{n^{\alpha}_{AB}}{r^2_{AB}}\Big[(1-2\beta)\frac{1}{r_{BC}} - 2(\beta+\gamma)\frac{1}{r_{ac}}\Big] +$$

$$+\tau\frac{\mathcal{P}^{\alpha\lambda}_{AB}}{r^3_{AB}}(n_{bc\lambda} + n_{ca\lambda}) + \tau\frac{n_{AB\lambda}}{r^2_{AB}}\frac{\Lambda^{\alpha\lambda}_{AC}}{r_{AC}} + \frac{1}{2}(1+2\tau)\frac{n_{BC\lambda}}{r^2_{BC}}\frac{\Lambda^{\alpha\lambda}_{ac}}{r_{ac}} + 2(1+\gamma)\frac{n_{BC}{}^{\alpha}}{r^2_{BC}r_{AB}}, \tag{2.20}$$

where $\Lambda^{\mu\nu}_{AB} = \eta^{\mu\nu} + n^{\mu}_{AB}n^{\nu}_{AB}$ and $\mathcal{P}^{\mu\nu}_{AB} = \eta^{\mu\nu} + 3n^{\mu}_{AB}n^{\nu}_{AB}$ are the projecting and the polarizing operators, respectively.

The metric tensor (2.5), the Lagrangian function (2.9), and the equations of motion (2.14)–(2.20) define the behavior of the celestial bodies in the post-Newtonian approximation in the PPN formalism. These equations may be simplified considerably by taking into account that the leading contribution to these equations is the solar gravitational field. With such an approximation, they are used to produce the numerical codes in relativistic orbit determination formalisms for planets and satellites (Moyer, 1981; Huang *et al.*, 1990; Ries *et al.*, 1991; Standish *et al.*, 1992) as well as to analyze the gravitational experiments in the solar system (Will, 1993; Pitjeva, 1993; Anderson *et al.*, 1996). It should be noted here that in the present numerical algorithms for celestial mechanics problems (Moyer, 1971; Moyer, 1981; Brumberg, 1991; Standish *et al.*, 1992; Will, 1993), the bodies in the solar system are assumed to possess the lowest post-Newtonian mass moments only, namely, the rest masses and the quadrupole moments. The corresponding barycentric inertial RF_0 defined in the harmonic coordinates for general relativity ($\gamma = \beta = 1; \nu = \tau = 0$) has been adopted for the fundamental planetary and lunar ephemerides (Newhall *et al.*, 1983; Standish *et al.*, 1992).

However, if one attempts to describe the *global* dynamics of the system of N arbitrarily shaped extended bodies, one will discover that even in the WFSMA this solution will not be possible without an appropriate description of the gravitational environment in the immediate vicinity of the bodies (Kopejkin, 1988; DSX, 1991). Thus, one needs to present the post-Newtonian definition for the proper intrinsic multipole moments for the bodies in order to describe their interaction with the surrounding gravitational field as well as to obtain the corresponding corrections to the laws of motion and precession of the extended bodies in this system. This could be done correctly only by using the theory of the quasi-inertial proper RF with well-defined dynamic and kinematic properties. In the next section, we will discuss a new perturbative method for finding the solution for the relativistic N-extended body problem and will formulate the corresponding theory of relativistic astronomical RFs in curved space-time.

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3 WFSMA for an Isolated Astronomical N-Body System.

In this section, we will discuss the principles of a new iterative method for generating the solutions to an arbitrary N-body gravitational problem in the WFSMA. This formalism will be based upon the construction of proper RFs in the vicinities of each body in the system. Such frames are defined in the gravitational domain, d_B, occupied by a particular body (B). One may expect that, in the immediate vicinity of this body, its proper gravitational field will dominate, while the existence of the external gravity will manifest itself in the form of the tidal interaction only. Therefore, in the case of the WFSMA in the closest proximity to the body under study, this proper RF should resemble the properties of an inertial frame, and the solution for an isolated one-body problem $h_{mn}^{(0)B}$ should adequately represent the physical situation. However, if one decides to perform a physical experiment at some distance from the world tube of the body, one should consider the existence of the external gravity as well. This is true because external gravity plays a more significant role at large distances from the body, and this should be taken into account. As we noticed earlier in Section 1, the physically adequate description of this nature of gravity could be made in the well-justified proper RF only. Let us mention that the dynamical properties of the inertial frames presently are well justified and correctly modelled both physically and mathematically. In particular, the properties of the barycentric inertial RF_0 are based upon the properties of an N-body generalization of an unperturbed isolated one-body solution of the gravitational field equations in an inertial RF given by (2.5). These properties are well established and widely in use in modern astronomical practice (Moyer, 1971; Moyer, 1981; Brumberg, 1991; Will, 1993). However, as we discussed earlier, this N-body generalization is based on the assumption that the bodies in the system possess the lowest intrinsic mass and current multipoles only. In order to account for the influence of higher-order multipoles, the coordinate transformations to the proper RF are necessary. This proper RF should take into account both Lorentzian and Einsteinian features of the motion of extended bodies in the external gravitational field. In the next subsection, we will concentrate on formulating the basic principles of a new method for constructing such transformations for a wide class of metric theories of gravity.

3.1 The General Form of the N-Body Solution.

In order to construct a general solution for the N-body problem in a metric theory of gravity, let us make a few assumptions. First of all, let us assume that there exists a background space-time γ_{mn} with the dynamic and cosmological properties discussed in the Section 1. Note that these properties do not forbid the existence of incoming and outgoing gravitational radiation. We will discuss this case further. We shall assume that the solution of the gravitational field equations $h_{mn}^{(0)}$ for an isolated unperturbed distribution of matter is known and is given by relations (2.5). We further assume that for each body (B) in the system, one may establish a unique correspondence to each such solution: (B) $\Leftrightarrow$ $h_{mn}^{(0)B}$.

With these assumptions, we may construct the total solution of the *global* problem g_{mn} in an arbitrary RF as a formal tensorial sum of the background space-time metric γ_{mn}, the unperturbed solutions $h_{mn}^{(0)B}$ plus the gravitational interaction term h_{mn}^{int}. Thus, in the coordinates $x^p \equiv (x^0, x^\nu)$ of the barycentric inertial RF_0, one may search for the desired total solution in the following form:

$$g_{mn}(x^p) = \gamma_{mn}(x^p) + h_{mn}(x^p) =$$
$$= \gamma_{mn}(x^p) + \sum_{B=1}^{N} \frac{\partial y_B^k}{\partial x^m} \frac{\partial y_B^l}{\partial x^n} h_{kl}^{(0)B}(y_B^q(x^p)) + h_{mn}^{int}(x^p), \tag{3.1}$$

where the coordinate transformation functions $y_B^q = y_B^q(x^p)$ are yet to be determined. The interaction term h_{mn}^{int} will be discussed below.

In order to describe the matter distribution, let us assume that the corresponding Lagrangian function L_M^{tot} may be given as

$$L_M^{tot} = \sum_B^N L_M^{(0)B} + L_M^{int},$$

where L_M^{int} is the Lagrangian describing interaction between the bodies. Then, the total energy-momentum tensor of matter in the system may be presented as follows:

$$T_{mn}(x^s) = 2\frac{\delta L_M^{tot}}{\delta g^{mn}} = \sum_B^N \frac{\partial y_B^k}{\partial x^m} \frac{\partial y_B^l}{\partial x^n} T_{kl}^B(y_B(x^s)) + 2\frac{\delta L_M^{int}}{\delta g^{mn}}. \tag{3.2}$$

For the case of compact and well separated bodies, we may take into account that the mutual gravitational interaction between the bodies affects their distribution of matter through the metric tensor only. Therefore we can neglect the second term in the expression above[6]. Then without any loss of accuracy, we obtain the total energy-momentum tensor of the matter distribution in the system in the following form:

$$T_{mn}(x^s) = \sum_{B=1}^{N} \frac{\partial y_B^k}{\partial x^m} \frac{\partial y_B^l}{\partial x^n} T_{kl}^B(y_B(x^s))(1 + \mathcal{O}(c^{-4})) = \sum_{B=1}^{N} \frac{\partial y_B^k}{\partial x^m} \frac{\partial y_B^l}{\partial x^n} T_{kl}^B(y_B(x^s)), \tag{3.3}$$

where T_{kl}^B is the energy-momentum tensor[7] of a body (B) as seen by a co-moving observer.

The unperturbed solution $h_{mn}^{(0)B}$ for the field equations in the WFSMA is presented in the form of the double power series with respect to two small scalar parameters: the gravitational coupling constant G and the orders of c^{-1}. It is clear that a similar set of small parameters may be used in order to construct an iterative N-body solution at least at the post-Newtonian level in the WFSMA. This means that all the functions and fields involved in the perturbation scheme (such as the interaction term h_{mn}^{int}, the coordinate transformation functions $y_B^q = y_B^q(x^p)$, the energy-momentum tensor T_{mn}^B, etc.) are also power expanded with respect to these small

[6]It is also true, if one recalls the result, that the interaction between the gravitational fields in the 1.5 post-Newtonian physics will appear in the g_{00} component of the metric tensor only and will have an $\mathcal{O}(c^{-4})$ order of magnitude.

[7]As a partial result of representation (3.3), one can see that the Newtonian mass density ρ_B of a particular body (B) is defined in the sense of a three-dimensional Dirac delta function. Thus, in the body's proper compact-support volume, one will have $\rho_B = m_B\delta(y_B^\nu)$, so that

$$\int_a d^3y'_A \rho_B(y'^p_A) = m_B\delta_{AB},$$

where δ_{AB} is the three-dimensional Kronekker symbol ($\delta_B^A = \delta_{AB}$; $\delta_{AB} = 1$ for $A = B$ and 0 for $A \neq B$). Then in any RF_A, the total density $\overline{\rho}$ of the whole N body system will be given by the expression $\overline{\rho}(y_A^p) = \sum_B^N \rho_B(y_A^p)$. This representation allows one to distinguish between the *local* and integral descriptions of the physical processes and, hence, provides correct relativistic treatment of the problem of motion of an astronomical N-body system.

scalar parameters. At this point, the actual form of the energy-momentum tensor T_{mn} is not of great importance. We prefer to keep this arbitrariness in our further calculations. The only restriction we will apply to the possible form of this tensor is based on the physical expectations; we will limit ourselves to such tensors which have the components of the following orders: $T_{00} \sim \mathcal{O}(1), T_{0\alpha} \sim \mathcal{O}(c^{-1}), T_{\alpha\beta} \sim \mathcal{O}(c^{-2})$.

One may establish the properties of solution (3.1) with respect to an arbitrary coordinate transformation simply by applying the basic rules of tensorial coordinate transformations. In particular, in the coordinates $y_A^p(x^q) \equiv (y_A^0, y_A^\nu)$ of an arbitrary proper RF$_A$, this tensor will take the following form:

$$g_{mn}^A(y_A^p) = \frac{\partial x^k}{\partial y_A^m}\frac{\partial x^l}{\partial y_A^n}\, g_{kl}(x^s(y_A^p)) = \gamma_{mn}^A(y_A^p) + h_{mn}^A(y_A^p) =$$

$$= \frac{\partial x^k}{\partial y_A^m}\frac{\partial x^l}{\partial y_A^n}\left(\gamma_{kl}(x^s(y_A^p)) + h_{kl}^{int}(x^s(y_A^p))\right) +$$

$$+ h_{mn}^{(0)A}(y_A^p) + \sum_{B\neq A}\frac{\partial y_B^k}{\partial y_A^m}\frac{\partial y_B^l}{\partial y_A^n}\, h_{kl}^{(0)B}(y_B^s(y_A^p)). \tag{3.4}$$

The expression for $T_{mn}(y_A^p)$ could be obtained analogously to that given by equations (3.3). To complete the formulation of the perturbative scheme, we need to introduce the procedure for constructing the solutions for the various unknown functions entering expressions (3.1)–(3.4), including the four functions of the coordinate transformations $y_B^q = y_B^q(y_A^p)$ and the interaction term h_{mn}^{int}.

We will construct the four functions of the coordinate transformations by applying the relativistic theory of celestial RFs in a curved space-time. To do this, we will use the most general form of the post-Newtonian non-rotating coordinate transformation between the barycentrical (inertial) coordinates (x^p) and the bodycentrical (quasi-inertial) coordinates (y_A^p):

$$x^0 = y_A^0 + c^{-2}K_A(y_A^0, y_A^\epsilon) + c^{-4}L_A(y_A^0, y_A^\epsilon) + O(c^{-6})y_A^0, \tag{3.5a}$$

$$x^\alpha = y_A^\alpha + y_{A_0}^\alpha(y_A^0) + c^{-2}Q_A^\alpha(y_A^0, y_A^\epsilon) + O(c^{-4})y_A^\alpha, \tag{3.5b}$$

where $y_{A_0}^\alpha(y_A^0)$ is the Newtonian radius vector of body (A). Transformations (3.5) should complement the post-Galilean coordinate transformations (1.12) in the case of the curved space-time generated by an arbitrary N-body system. Note that transformations (3.5) are presented as being parametrized by the set of three unknown functions, K_A, L_A, and Q_A^α. This is an example of that which will be referred to as the KLQ parameterization for the WFSMA. The functions K_A, L_A, and Q_A^α are expected to contain the information about the specific properties of the quasi-inertial RF$_A$ associated with the body (A). The form of these functions will be determined by the iterative procedure for constructing the quasi-inertial proper RF$_A$.

The way to construct the solution for the interaction term h_{mn}^{int} is quite straightforward: It is sufficient to require that the metric tensor in the form of eq.(3.1) or (3.4) will be the explicit solution of the gravitational field equations in the corresponding RF. Note that the second term in eq.(3.1) is linear with respect to the unperturbed solutions $h_{kl}^{(0)B}$ and that the transformation functions between the different RFs are determined by means of the external gravitational field in their origins. Only the interaction term should contain the information about the dynamic nonlinearity of the gravitational interaction. The form of this term should depend on the physical

features of the RFs chosen for the analysis. It should be noted that the search for the solution in the barycentric RF_0 is physically and mathematically more appropriate then in the bodycentric one. Moreover, to date no analysis has been made to propose a covariant boundary condition for the case of the non-inertial RF rather then the 'classical' Fock–Sommerfeld one. It is known that these conditions are applied asymptotically to the entire gravitational field from the system at the infinitive distance from the latter and are valid for the isolated distribution of matter. This means that making use of the Fock–Sommerfeld boundary conditions (Brumberg & Kopejkin, 1988a; DSX, 1991-1994) in a proper RF is mathematically weakly founded in order to find the general solution of the field equations in this frame. Based on this conclusion, we will perform the search for the h^{int}_{mn} in the coordinates of the barycentric inertial RF_0.

By taking into account that all the functions and fields in expressions (3.1)–(3.5) are presented in the form of a power expansion with respect to the set of small parameters, one may organize an iterative procedure in order to obtain the general solution for the problem. The two principle steps of this procedure are the supplementary conditions necessary for the solution of the gravitational field equations, which may be expressed by both the covariant gauge conditions and the boundary conditions.

In the proposed formalism, these conditions are taken to be as follows:

The covariant gauge conditions. The solutions of the field equations are assumed to satisfy the covariant harmonical de Donder gauge, which, for an arbitrary RF_B, may be written as follows:

$$\mathcal{D}^B_n\left(\sqrt{-g}_B g^{mn}_B(y^p_B)\right) = 0, \tag{3.6}$$

where $\mathcal{D}^B_n$ is the covariant derivative with respect to the metric $\gamma^B_{mn}(y^p_B)$ of the inertial Riemann-flat ($R^k_{nml}(\gamma^B_{mn}(y^p_B)) = 0$) space-time in these coordinates.[8] For most of the interesting practical problems in the WFSMA, this metric may be represented in quasi-Cartesian coordinates as the sum of two tensors, the Minkowski metric $\gamma^{(0)}_{mn}$ and the field of inertia ϕ_{mn}:

$$\gamma^B_{mn}(y^p_B) = \frac{\partial x^k}{\partial y^m_B}\frac{\partial x^l}{\partial y^n_B}\gamma_{kl}(x^s(y^p_B)) = \gamma^{(0)}_{mn} + \phi^B_{mn}(y^p_B). \tag{3.7}$$

Note that the term ϕ_{mn} appears to be parameterized by the coordinate transformation functions K_A, L_A, and Q^α_A defined in eqs.(3.5); thus, we have $\phi_{mn}(y^p_A) = \phi_{mn}[K_A, L_A, Q^\alpha_A]$, a formulation that will be referred to as the KLQ parameterization in the WFSMA.

The advantage of using these gauge conditions is that they allow us to construct the solutions to the field equations in a unique way without applying the technique of the, so-called, 'external multipole moments' (Brumberg & Kopejkin, 1988a; DSX, 1991). The conditions of eqs.(3.6) do not fix the harmonic RF in a unique way and, in definition of coordinates of this frame, some arbitrariness may still exist. Indeed, the coordinate transformation $y'^p_B = y^p_B + \zeta^p_B(y^q_B)$ with the function ζ^p_B satisfies the equation $g^{mn}(y^p_B)\mathcal{D}^B_m\mathcal{D}^B_n\,\zeta^p_B(y^q_B) = 0$ does not violate the chosen conditions (3.6). In all the particular cases, the remaining freedom of the harmonic RF might be

[8]In Cartesian coordinates of the inertial Galilean RF_0, the flat metric γ^B_{mn} can be chosen as $\gamma^{(0)}_{mn} = \mathrm{diag}(1,-1,-1,-1)$, so that the Christoffel symbols $\Gamma^{k(0)}_{mn} = 0$ all vanish and conditions (3.6) take the more familiar form of the harmonic conditions

$$\partial^B_n\left(\sqrt{-g}_B g^{mn}_B(y^p_B)\right) = 0,$$

which are equivalent to setting $\nu = \tau = 0$ in eqs.(2.5).

fixed by choosing the specific CS associated with the proper RFs for describing the dynamics of the N bodies in the system.[9]

The boundary conditions. The search for the general solution for $h^{int}_{mn}(x^p)$ is performed in a barycentric inertial RF_0, which is singled out by the Fock–Sommerfeld boundary conditions imposed on the h_{mn} and $\partial_k h_{mn}$:

$$\lim_{r\to\infty}\Big(h_{mn}(x^p);\ r\Big[\frac{\partial}{\partial x^0}h_{mn}(x^p)+\frac{\partial}{\partial r}h_{mn}(x^p)\Big]\Big)\ \to\ 0,$$

$$t+\frac{r}{c}=const, \tag{3.8a}$$

where $r^2=-\gamma^{(0)}_{\mu\nu}\,x^\mu x^\nu$. Note that conditions (3.8) must be satisfied along all past Minkowski light cones. Thus, these conditions define the asymptotically Minkowskian space-time in a weak sense, consistent with the absence of any flux of gravitational radiation falling on the system from an external universe (Damour, 1983, 1986). Moreover, one assumes that there exists such a quantity $h^{max}_{mn}=const$ (for the solar system, this constant is of the order of $\approx 10^{-5}$) for which the condition

$$h_{mn}(x^p)\ <\ h^{max}_{mn} \tag{3.8b}$$

should be satisfied for each point $(\vec{x})$ inside the system: $|\vec{x}|\le L_D$. Note that any distribution of matter is considered isolated if conditions (3.8) are fulfilled in any inertial RF (Damour, 1983; Kopejkin, 1987, 1988).

By making use of conditions (3.8), we have an opportunity to determine the interaction term $h^{int}_{mn}(x^p)$ in a unique way while solving the gravitational field equations of a metric theory of gravity.

3.2 The Post-Newtonian KLQ Parameterization.

It is well known that for practical description of the translational and rotational motions of the N-body system, one should introduce at least $(N+1)$ different RFs (Brumberg & Kopejkin, 1988; DSX, 1991). It is desirable that one of these frames be the inertial barycentric (RF_0) with coordinates denoted as $(x^p)\equiv(x^0,x^\mu)$. The origin of these coordinates is located at the center of the field of the entire N-body system. This particular RF will be used to describe the *global* dynamics of the whole system. The other N frames should be convenient for the description of the *local* gravitational environment in the immediate vicinity of the particular body (B) under consideration. The origins of corresponding coordinate grids, $(y^p_B)\equiv(y^0_B,y^\mu_B)$, should be associated with the centers of the *local* fields of the interacting bodies of interest.

In this subsection, we will establish the general relationships describing the straight, inverse, and mutual coordinate transformations between the different quasi-inertial RFs. We will show that, in the WFSMA, all these different types of coordinate transformations may be parametrized by the same set of functions, K_A, L_A, and Q^α_A. As a result, we will reconstruct in the general form of the post-Newtonian non-linear group of motion the background pseudo-Euclidean space-time for the WFSMA.

[9] Or equivalently, by choosing some specific form of g_{mn} (Thorne, 1980; Hellings, 1986; Fukushima, 1988) and the internal and 'external' moments in a vacuum power expansion of the metric tensor g_{mn} in a set of multipoles (Kopejkin, 1988; DSX, 1991).

3.2.1 The Properties of the Coordinate Transformations in the WFSMA.

As we mentioned above, in order to construct the relativistic theory of the RFs in celestial mechanics, one should not only solve the *global* and *local* problems, but also one should establish the rules of the coordinate transformations between these solutions that belong to the different RFs. To do this, let us discuss the expected physical and mathematical properties of the coordinate transformations given by expressions (3.5) in the form

$$x^0 = y_A^0 + c^{-2}K_A(y_A^0, y_A^\epsilon) + c^{-4}L_A(y_A^0, y_A^\epsilon) + \mathcal{O}(c^{-6})y_A^0,$$

$$x^\alpha = y_A^\alpha + y_{A_0}^\alpha(y_A^0) + c^{-2}Q_A^\alpha(y_A^0, y_A^\epsilon) + \mathcal{O}(c^{-4})y_A^\alpha.$$

These coordinates are expected to cover space-time in the immediate vicinity of the body under consideration. It is clear that such a mapping of the space-time may be performed by both the barycentric and bodycentric coordinates. This suggests that these coordinate transformations should be reversible. The functions K_A, L_A, and Q_A^α should contain the information about the specific physical properties of the RF chosen for analysis. It is generally believed that, in order to produce the transformations to the physically justified proper RF, the following properties of these functions should be satisfied:

(i). The functions K_A, L_A, and Q_A^α should be completely defined by means of the external gravitational field at the origin of the coordinate system of the proper RF$_A$ of body (A) for which the physically adequate proper RF is constructed. These functions should not contain any terms caused by the pure gravitational field of body (A) besides those with the coupling of the internal multipole moments of body (A) to the external gravitation.

(ii). In order to obtain reversible transformations, the transformation functions should be homogeneous and infinitely differentiable. Then, based on assumptions about the properties of a well-justified proper RF (given in the Section 1), the functions K_A, L_A, and Q_A^α should admit an additional Taylor expansion in power series of the spatial coordinate y_A^μ. For convenience, these series may originate on the world line of the center of the *local* field in the vicinity of body (A), so that these functions could be expressed as follows:

$$f_A(y_A^0, y_A^\mu) = \sum_l^\infty \frac{1}{l!} f_{A\{L\}}(y_A^0) \cdot y_A^{\{L\}}, \tag{3.9}$$

where function $f_A(y_A^0, y_A^\epsilon)$ is any function from K_A, L_A, or Q_A^α. As a result, the second derivatives taken from these functions will not depend on the order of the derivative's application, namely,

$$[\frac{\partial}{\partial y_A^m}, \frac{\partial}{\partial y_A^n}] f_A(y_A^0, y_A^\epsilon) = 0, \tag{3.10}$$

where the brackets are the usual notation for the commutator: $[a, b] = ab - ba$.

(iii). At the limit when gravitation is absent (G $\rightarrow$ 0), the theory becomes Poincaré-invariant and transformations (3.5) should coincide with those of Poincaré (between two frames in uniform relative motion with a velocity $\vec{v}$ plus transition of origin and arbitrary rotation), which are given by eqs.(1.7).

(iv). At the other limit, when N $\rightarrow$ 1, and the problem may be described by the one-body gravitational field solution (2.5), the transformations should coincide with that of Chandrasekhar–Contopulos, (1.12), for the uniform motion between the two RFs in the isolated one-body problem.

(v). For the gravitational theories, whose foundations are based upon the Equivalence Principle, the physical properties of constructed RFs should be generic for all the bodies in the system. Otherwise, the possible violation of this principle (which may be induced by the possible dependence of the gravitational coupling on the shape/size/composition of the bodies) should be taken into account while the proper RF is constructed.

3.2.2 The Inverse Transformations.

The transformations given by eq.(3.5) transform space and time coordinates from the barycentric space-time RF (x^p) to space and time coordinates in the proper RF$_A$ (y^p_A). However, in practice one needs to make the comparison between the proper time and position in different RFs and, hence, it is necessary to have the inverse transformations to those of eq.(3.5) and the mutual transformations between the two proper quasi-inertial frames as well. The existence of the small parameters and the assumptions in (3.9) and (3.10) make it possible to generate these transformations in a general form as well as to construct the group of motion for the problems in the WFSMA. Thus, the general condition of the inreversibility of transformations (3.5) is given as usual:

$$\det ||\frac{\partial x^m}{\partial y^n_A}|| \neq 0. \tag{3.11a}$$

Expressions ($B5$) from Appendix B enable us to present this condition in an arbitrary RF obtained with the WFSMA as follows:

$$\det ||\frac{\partial x^m}{\partial y^n_A}|| = 1 + \frac{\partial}{\partial y^0_A} K_A(y^0_A, y^\nu_A) +$$

$$+ v^\lambda_{A_0}(y^0_A) v_{A_0\lambda}(y^0_A) + \frac{\partial}{\partial y^\mu_A} Q^\alpha_A(y^0_A, y^\nu_A) + \mathcal{O}(c^{-4}) \neq \mathcal{O}(c^{-4}). \tag{3.11b}$$

Note that this condition is satisfied for most of the problems in modern celestial mechanics. A similar analysis has been made by Brumberg & Kopejkin (1989) for the dynamics of the planets in the solar system. It was shown that the determinant vanishes at the distance $r^* \sim c^2/|a_E| \sim 7.5 \cdot 10^{20}$ cm from the center of mass of the Earth. From this it follows that, in spite of an initial construction of a geocentric RF in the region lying inside the lunar orbit, it is possible to smoothly (without intersecting) prolongate the spatial coordinate axes of the geocentric RF for much larger distances beyond the orbit of Pluto.

We will search for the post-Newtonian transformations that will be inversed to those of eq.(3.5) in the following form:

$$y^0_A = x^0 + c^{-2}\hat{K}_A(x^0, x^\epsilon) + c^{-4}\hat{L}_A(x^0, x^\epsilon) + \mathcal{O}(c^{-6})x^0, \tag{3.12a}$$

$$y^\alpha_A = x^\alpha - y^\alpha_{A_0}(x^0) + c^{-2}\hat{Q}^\alpha_A(x^0, x^\epsilon) + \mathcal{O}(c^{-4})x^\alpha, \tag{3.12b}$$

where the functions $\hat{K}_A, \hat{L}_A$, and $\hat{Q}^\alpha_A$ are unknown at the moment. One can show that in the WFSMA, these functions may be expressed in terms of functions K_A, L_A, and Q^α_A written in

coordinates (x^p) of the barycentric RF_0. In order to find the expressions for $\hat{K}_A$, $\hat{L}_A$, and $\hat{Q}^\alpha_A$, let us substitute relations (3.5) into eqs.(3.12) and then expand the obtained relations with respect to the small parameters: $G \sim c^{-2}$. Thus, for the spatial components we will obtain

$$x^\alpha = x^\alpha - y^\alpha_{A_0}(x^0) + c^{-2}\hat{Q}^\alpha_A(x^0, x^\epsilon) + y^\alpha_{A_0}\Big(y^0_A(x^0, x^\alpha)\Big) +$$
$$+c^{-2}Q^\alpha_A\Big(y^0_A(x^0, x^\alpha), y^\epsilon_A(x^0, x^\alpha)\Big) + O(c^{-4})y^\alpha_A. \tag{3.13}$$

This equation enables us to find the expression for $\hat{Q}^\alpha_A(x^0, x^\epsilon)$ in terms of functions Q^α_A and $\hat{K}_A$. By expressing the arguments of the transformation functions $y^\alpha_{A_0}\Big(y^0_A(x^0, x^\alpha)\Big)$ and $Q^\alpha_A\Big(y^0_A(x^0, x^\alpha), y^\epsilon_A(x^0, x^\alpha)\Big)$ in terms of coordinates (x^p) and expanding the obtained relations in the power series of the small parameter c^{-1}, we will get

$$Q^\alpha_A\Big(y^0_A(x^0, x^\alpha), y^\epsilon_A(x^0, x^\alpha)\Big) = Q^\alpha_A\Big(x^0, x^\alpha - y^\alpha_{A_0}(x^0)\Big) + \mathcal{O}(c^{-4})x^\alpha, \tag{3.14a}$$
$$y^\alpha_{A_0}\Big(y^0_A(x^0, x^\alpha)\Big) = y^\alpha_{A_0}\Big(x^0 + c^{-2}\hat{K}_A(x^0, x^\epsilon) + \mathcal{O}(c^{-4})x^0\Big) =$$
$$= y^\alpha_{A_0}(x^0) + v^\alpha_{A_0}(x^0)\cdot c^{-2}\hat{K}_A(x^0, x^\epsilon) + \mathcal{O}(c^{-4})x^\alpha, \tag{3.14b}$$

where

$$\frac{d}{dy^0_A}y^\alpha_{A_0}\Big(y^0_A(x^0, x^\alpha)\Big) = v^\alpha_{A_0}(x^0) + \mathcal{O}(c^{-4})x^\alpha.$$

Then, by substituting eqs.(3.14) into eqs.(3.13), we will obtain the expression for the function $\hat{Q}^\alpha_A(x^0, x^\epsilon)$:

$$\hat{Q}^\alpha_A(x^0, x^\epsilon) = -Q^\alpha_A\Big(x^0, x^\alpha - y^\alpha_{A_0}(x^0)\Big) - v^\alpha_{A_0}(x^0)\cdot\hat{K}_A(x^0, x^\epsilon) + \mathcal{O}(c^{-4})x^\alpha. \tag{3.15}$$

By repeating this procedure for the temporal components of transformations (3.12), we may obtain the expressions for functions $\hat{K}_A(x^0, x^\epsilon)$ and $\hat{L}_A(x^0, x^\epsilon)$ as well:

$$\hat{K}_A(x^0, x^\epsilon) = -K_A\Big(x^0, x^\epsilon - y^\epsilon_{A_0}(x^0)\Big) + \mathcal{O}(c^{-4})x^0, \tag{3.16}$$

$$\hat{L}_A(x^0, x^\epsilon) = -L_A\Big(x^0, x^\epsilon - y^\epsilon_{A_0}(x^0)\Big) -$$
$$-\Big[\Big(\frac{\partial}{\partial x^0} + v^\nu_{A_0}(x^0)\frac{\partial}{\partial x^\nu}\Big)\cdot K_A\Big(x^0, x^\epsilon - y^\epsilon_{A_0}(x^0)\Big)\Big]\cdot\hat{K}_A(x^0, x^\epsilon) -$$
$$-\frac{\partial}{\partial x^\nu}K_A\Big(x^0, x^\epsilon - y^\epsilon_{A_0}(x^0)\Big)\cdot\hat{Q}^\alpha_A(x^0, x^\epsilon) + \mathcal{O}(c^{-6})x^0. \tag{3.17}$$

Making use of the resulting expressions for functions $\hat{K}_A$, $\hat{L}_A$, and $\hat{Q}^\alpha_A$, which are given by relations (3.15)-(3.17), from equation (3.12) we finally obtain the inverse transformations between proper and barycentric kinematically non-rotating RFs in the most general form:

$$y_A^0 = x^0 - c^{-2}K_A\Big(x^0, x^\epsilon - y_{A_0}^\epsilon(x^0)\Big) + c^{-4}\Big[- L_A\Big(x^0, x^\epsilon - y_{A_0}^\epsilon(x^0)\Big) +$$

$$+\frac{\partial}{\partial x^0}K_A\Big(x^0, x^\epsilon - y_{A_0}^\epsilon(x^0)\Big)\cdot K_A\Big(x^0, x^\epsilon - y_{A_0}^\epsilon(x^0)\Big) +$$

$$+\frac{\partial}{\partial x^\nu}K_A\Big(x^0, x^\epsilon - y_{A_0}^\epsilon(x^0)\Big)\cdot Q_A^\nu\Big(x^0, x^\epsilon - y_{A_0}^\epsilon(x^0)\Big)\Big] + \mathcal{O}(c^{-6})x^0 \tag{3.18a}$$

$$y_A^\alpha = x^\alpha - y_{A_0}^\alpha(x^0) +$$

$$+c^{-2}\Big[v_{A_0}^\alpha(x^0)\cdot K_A\Big(x^0, x^\epsilon - y_{A_0}^\epsilon(x^0)\Big) - Q_A^\alpha\Big(x^0, x^\epsilon - y_{A_0}^\epsilon(x^0)\Big)\Big] + \mathcal{O}(c^{-4})x^\alpha. \tag{3.18b}$$

Note that the method used to derive expressions (3.18) corresponds to finding such coordinate transformations $y_A^p = y_A^p(x^q)$ which transform the space-time γ_{mn}^A of the proper RF_A to that of the barycentric inertial RF_0 with the Minkowski metric $\gamma_{mn}^{(0)}$ in Cartesian coordinates. The latter may be presented as follows: $ds^2 = \gamma_{mn}^A(y_A^p)dy_A^m dy_A^n = c^2dt^2 - d\vec{x}^2$.

3.2.3 The Coordinate Transformations Between the Two Proper RFs.

The ability to make the power expansion with respect to the small parameters allows us to organize the iterative procedure for constructing the mutual coordinate transformation between the two different proper RFs, namely RF_A and RF_B. The definition of the proper RF, (3.5), was given based on the clearly defined physical properties of the barycentric inertial RF_0 for the entire N-body system. The transformation functions connecting the two proper RFs are easy to find by applying the same procedure that was used for the construction of the inverse transformation, (3.18). Thus, by making use of expressions (3.5) and (3.18), we may find the following relations for the mutual coordinate transformation:

$$y_B^0 = y_A^0 + c^{-2}K_{BA}(y_A^0, y_A^\epsilon) + c^{-4}L_{BA}(y_A^0, y_A^\epsilon) + \mathcal{O}(c^{-6})y_A^0, \tag{3.19a}$$

$$y_B^\alpha = y_A^\alpha + y_{BA_0}^\alpha(y_A^0) + c^{-2}Q_{BA}^\alpha(y_A^0, y_A^\epsilon) + \mathcal{O}(c^{-4})y_A^\alpha, \tag{3.19b}$$

where functions K_{BA}, L_{BA}, and Q_{BA}^α are given as follows:

$$K_{BA}(y_A^0, y_A^\epsilon) = K_A(y_A^0, y_A^\epsilon) - K_B\Big(y_A^0, y_A^\epsilon + y_{BA_0}^\epsilon(y_A^0)\Big), \tag{3.20a}$$

$$Q_{BA}^\alpha(y_A^0, y_A^\epsilon) = Q_A^\alpha(y_A^0, y_A^\epsilon) - Q_B^\alpha\Big(y_A^0, y_A^\epsilon + y_{BA_0}^\epsilon(y_A^0)\Big) - v_{B_0}^\alpha(y_A^0)\cdot K_{BA}(y_A^0, y_A^\epsilon), \tag{3.20b}$$

$$L_{BA}(y_A^0, y_A^\epsilon) = L_A(y_A^0, y_A^\epsilon) - L_B\Big(y_A^0, y_A^\epsilon + y_{BA_0}^\epsilon(y_A^0)\Big) -$$

$$-\Big[\Big(\frac{\partial}{\partial y_A^0} - v_{BA_0}^\nu(y_A^0)\frac{\partial}{\partial y_A^\nu}\Big)\cdot K_B\Big(y_A^0, y_A^\epsilon + y_{BA_0}^\epsilon(y_A^0)\Big)\Big]\cdot K_{BA}(y_A^0, y_A^\epsilon) -$$

$$-\frac{\partial}{\partial y_A^\nu}K_B\Big(y_A^0, y_A^\epsilon + y_{BA_0}^\epsilon(y_A^0)\Big)\cdot Q_{BA}^\nu(y_A^0, y_A^\epsilon). \tag{3.20c}$$

Relations (3.5) and (3.18)–(3.20) represent the necessary expressions for developing the perturbation theory in the WFSMA for the problems of the dynamics of an astronomical gravitationally bounded system of N self-gravitating arbitrarily shaped extended bodies. The transformations are presented in a functionally parameterized form by the two scalar functions, K_A and L_A, and one three-vector function, Q^α_A. Assuming all the bodies in the system are described by the same model of matter, one may conclude that the form of all these functions should be the same for any RF. This property of the transformations reflects the fact that a proper RF may be defined in a general way for each body in the system. Moreover, one can see that expressions (3.19)–(3.20) represent the group of motion that preserves the form-invariancy of the metric tensor γ^A_{mn} of the background pseudo-Euclidean space-time for any proper RF. This means that the RFs, constructed this way, should be equivalent and, hence, the physical phenomena will behave exactly the same way in all of them.

3.2.4 Notes on an Arbitrary Rotation of the Spatial Axes.

In this subsection, we will show how one may generalize the results obtained on the case of the transformations between dynamically rotational coordinate RFs. The need for such a coordinate system may appear, for example, in the case when one will relate the VLBI, LLR, and the planetary ephemeris RFs as well as in the case of relating the celestial and terrestrial frames (Folkner *et al.*, 1994; Sovers & Jacobs, 1994). The most general form of post-Newtonian transformations between the coordinates (x^p) of the barycentric inertial RF_0 to those (y^p_A) of the proper RF_A, which are undergoing the rotational motion of the spatial axes with an arbitrary time-dependent rotational matrix $\mathcal{R}^{\mu\nu}_A(y^0_A)$, may be presented in the following form:

$$x^0 = y^0_A + c^{-2}K_A\Big(y^0_A, \mathcal{R}^\epsilon_{A\nu}(y^0_A)\cdot y^\nu_A\Big) + c^{-4}L_A\Big(y^0_A, \mathcal{R}^\epsilon_{A\nu}(y^0_A)\cdot y^\nu_A\Big) + \mathcal{O}(c^{-6})y^0_A, \tag{3.21a}$$

$$x^\alpha = y^\alpha_{A_0}(y^0_A) + \mathcal{R}^\alpha_{A\nu}(y^0_A)\cdot y^\nu_A + c^{-2}Q^\alpha_A\Big(y^0_A, \mathcal{R}^\epsilon_{A\nu}(y^0_A)\cdot y^\nu_A\Big) + \mathcal{O}(c^{-4})y^\alpha_A. \tag{3.21b}$$

The matrix $\mathcal{R}^{\mu\nu}_A(y^0_A)$ represents both the rotation and the time-dependent deformation of the spatial axes:

$$\mathcal{R}^{\mu\nu}_A(y^0_A) = \sigma^{\mu\nu}_A(y^0_A) + \omega^{\mu\nu}_A(y^0_A), \tag{3.22}$$

where the first term is symmetric, $\sigma^{\mu\nu}_A = \sigma^{\nu\mu}_A$, and it represent the rescaling of the coordinates with respect to time. The second term is anti-symmetric, $\omega^{\mu\nu}_A = -\omega^{\nu\mu}_A$, and it describes the rotation of the spatial axes of the coordinate grid in the proper RF_A. Besides this, the tensor $\omega^{\mu\nu}_A$ contains the information about the precession and nutation of the spatial coordinates (Kopejkin, 1988; Fukushima, 1991; Folkner *et al.*, 1994; Sovers & Jacobs, 1994).

In the case when $\det ||\mathcal{R}^{\mu\nu}_A|| \neq 0$, one may find the inverse transformations to those given by expressions (3.21). To do this, we may repeat the same iterative procedure discussed above. Making use of this method, one may easily obtain these inverse transformations in the following form:

$$y^0_A = x^0 - K_A\Big(x^0, x^\epsilon - y^\epsilon_{A_0}(x^0)\Big) + \hat{L}''_A(x^0, x^\epsilon) + \mathcal{O}(c^{-6})x^0, \tag{3.23a}$$

$$y_A^\alpha = (\mathcal{R}_A^{-1})_\nu^\alpha(x^0)\cdot\Big[x^\nu - y_{A_0}^\nu(x^0)+$$

$$+v_{A_0}^\nu(x^0)\cdot K_A\Big(x^0, x^\epsilon - y_{A_0}^\epsilon(x^0)\Big) - Q_A^\nu\Big(x^0, x^\epsilon - y_{A_0}^\epsilon(x^0)\Big)\Big]-$$

$$-\frac{d}{dx^0}(\mathcal{R}_A^{-1})_\nu^\alpha(x^0)\cdot\Big(x^\nu - y_{A_0}^\nu(x^0)\Big)\,K_A\Big(x^0, x^\epsilon - y_{A_0}^\epsilon(x^0)\Big) + \mathcal{O}(c^{-4})x^\alpha, \tag{3.23b}$$

where the function $\hat{L}_A''$ is given as

$$\hat{L}_A''(x^0, x^\epsilon) = -L_A\Big(x^0, x^\epsilon - y_{A_0}^\epsilon(x^0)\Big)+$$

$$+\mathcal{R}_\lambda^\nu(x^0)\cdot\frac{\partial}{\partial x^\nu}K_A\Big(x^0, x^\epsilon - y_{A_0}^\epsilon(x^0)\Big)\cdot Q_A^\lambda\Big(x^0, x^\epsilon - y_{A_0}^\epsilon(x^0)\Big)+$$

$$+\frac{1}{2}\mu(x^0, x^\epsilon)\cdot K_A^2\Big(x^0, x^\epsilon - y_{A_0}^\epsilon(x^0)\Big) + \mathcal{O}(c^{-6})x^0 \tag{3.23c}$$

with the differential operator $\mu(x^0, x^\epsilon)$ taking the form

$$\mu(x^0, x^\epsilon) = \frac{\partial}{\partial x^0} + \Big[v_{A_0}^\nu(x^0)\cdot(\delta_\nu^\mu - \mathcal{R}_A^{\mu\nu})-$$

$$-\frac{d}{dx^0}(\mathcal{R}_A^{-1})_{\nu\lambda}\cdot\mathcal{R}_A^{\mu\nu}\cdot\Big(x^\lambda - y_{A_0}^\lambda(x^0)\Big)\Big]\cdot\frac{\partial}{\partial x^\mu} + \mathcal{O}(c^{-2})\frac{\partial}{\partial x^0}. \tag{3.24a}$$

Note that if we neglect the rotation (*i.e.*, will take the rotation matrix in the form of the Kronekker symbol, $\mathcal{R}_{A\nu}^\mu(x^0) = \delta_\nu^\mu$), the differential operator eq.(3.24a) becomes

$$\mu(x^0, x^\epsilon) = (1 + \mathcal{O}(c^{-2}))\frac{\partial}{\partial x^0} \tag{3.24b}$$

and the coordinate transformations (3.23) coincide with those of eq.(3.18) for the dynamically non-rotating case.

For most practical applications in modern astronomy, one may neglect the effects due to the time-dependent deformation of the axes and assume that the body is undergoing rigid three-dimensional rotation with the rotational matrix taken in the form $\mathcal{R}_A^{\mu\nu}(y_A^0) = \omega_A^{\mu\nu}(y_A^0)$. In the proper RF$_A$ of an isolated rotating body, the following equation describes the dynamic properties of the tensor $\mathcal{R}_A^{\mu\nu}(y_A^0)$:

$$\frac{d}{dy_A^0}\mathcal{R}_A^{\mu\nu}(y_A^0) = \epsilon^{\mu\sigma\beta}\mathcal{V}_{A\sigma}(y_A^0)\cdot\mathcal{R}_{A\beta}{}^\nu(y_A^0), \tag{3.25}$$

where $\mathcal{V}_A^\sigma = \epsilon_{\rho\nu}^\sigma\omega_A^{\rho\nu}$ is the vector of the angular velocity of rotation of the body (A). Usually, for most of the problems in relativistic celestial mechanics, one assumes that the angular velocity of the rotation of the celestial bodies is of the following order of magnitude: $\mathcal{V}_A^\sigma \sim \mathcal{O}(c^{-2})v_{A_0}^\sigma$, where $v_{A_0}^\sigma$ is the barycentric velocity of the translational motion of the body (A) moving along its world line (DSX, 1991). Then, taking this condition into account, one may neglect the time derivative terms from the transformation matrix in relations (3.23) and make use of the standard theory of coordinate transformations with rigid spatial rotation of the proper RF$_A$. Otherwise, for a general case with an arbitrary rotation, one should keep these terms in the post-Newtonian parts of transformations (3.23).

Following the procedure depicted above, we may obtain the mutual transformations between the coordinates of two rotating RFs. Furthermore, one may extend the results obtained above to the case of the non-uniform rotation of an elastic body (B) with the rotational matrix taken in a general form, $\mathcal{R}_B^{\mu\nu}(y_B^0, y_B^\epsilon)$. However, for the problems of celestial mechanics in the WFSMA, this generality is not necessary. Moreover, in 1991 the IAU made the recommendation that, in order to avoid the appearance of the fictitious forces (Coriolis-like) acting on a observer in the proper RF, all coordinate transformations for astronomical applications should not introduce any rotation of the spatial axes at all (Fukushima, 1991; Brumberg, 1991; Klioner, 1993). For this reason, we will limit ourselves in our further discussion solely to the case of the non-rotational coordinate transformations, leaving the problem of rotation for other publications.

3.3 The Definition of the Proper RF.

In this subsection, we will finally present a way to find the transformation functions necessary for constructing a proper RF with well-defined physical properties. As one can see from expressions (3.4), in the WFSMA the main contribution to the geometrical properties of the proper RF$_A$ in the body's immediate vicinity comes from its own gravitational field, $h_{mn}^{(0)A}$. Then, based on the Principle of Equivalence, the external gravitational influence should vanish at least to the first order in the spatial coordinates (Synge, 1960; Manasse & Misner, 1963). The proper RF$_A$, constructed this way, should resemble the properties of a quasi-inertial (or Lorentzian) reference frame and, as such, will be well suited for discussing the physical experiments. Note that the tensors $h_{mn}^{(0)B}$ and h_{mn}^{int} represent the real gravitational field that no coordinate transformation can eliminate everywhere in the system. In the case of a massive monopole body, one can eliminate the influence of external field on the body's world line only. However, for an arbitrarily shaped extended body, the coupling of the body's intrinsic multipole moments to the surrounding gravitational field changes the physical picture significantly. This means that the definition of the proper RF for the extended body must take into account this non-linear gravitational coupling.

In order to suggest the procedure for the choice of the coordinate transformations to the physically adequate proper RF$_A$, let us discuss the general structure of the solution $g_{mn}^A(y_A^p)$ given by expression (3.4). Thus, in the expressions for g_{mn}^A, one may easily separate the four physically different terms. These terms are:

(i). The Riemann-flat contribution of the field of inertia γ_{mn}^A given by expression (3.7).

(ii). The contribution of the body's own gravitational field $h_{mn}^{(0)A}$.

(iii). The term due to the non-linear interaction of the proper gravitational field with an external field.[10]

(iv). The term describing the field of the external sources of gravity. This term comes from the transformed solutions $h_{mn}^{(0)B}$ and the interaction term h_{mn}^{int}.

The first contribution depends on the external field in the gravitational domain occupied by the body (A) and appears to be 'parametrized' by transformation functions (3.5). Note that for any choice of these functions, by the way they were constructed, the obtained metric g_{mn}^A

[10]This contribution is due to the Newtonian potential and the potential Φ_2 in expressions (2.5). These interaction terms show up as the coupling of the body's intrinsic multipole moments with the external field.

satisfies the gravitational field equations of the specific metric theory of gravity under study. Furthermore, based on the properties of the proper RF_A discussed above, one may expect that the functions K_A, L_A, and Q_A^α should form a background Riemann-flat inertial space-time γ_{mn}^A in this RF that will be tangent to the total gravitational field in the vicinity of the body (A)'s world line, γ_A. Moreover, the difference of these fields should vanish to first order with respect to the spatial coordinates (*i.e.*, the 'external' dipole moment equals zero (Thorne & Hartle, 1985)). These conditions, applied to moving test particles, are known as Fermi conditions (Fermi, 1922; Manasse & Misner, 1963; Misner *et al.*, 1973). We have extended the applicability of these conditions to the case of a system composed of N arbitrarily shaped extended celestial bodies.

In order to obtain the functions K_A, L_A, and Q_A^α for coordinate transformation eq.(3.5), we will introduce an iterative procedure that will be based on a multipole power expansion with respect to the unperturbed spherical harmonics. To demonstrate the use of these conditions, let us denote $H_{mn}^A(y_A^p)$ as the *local* gravitational field, *i.e.*, the field that is formed from contributions (ii) and (iii) above. The metric tensor in the *local* region in this case can be represented by the expression $g_{mn}^{(loc)}(y_A^p) = \gamma_{mn} + H_{mn}^A(y_A^p)$. Then the generalized Fermi conditions in the *local* region of body (A) (or in the immediate vicinity of its world line, γ_A) may be imposed on this *local* metric tensor by the following equations:

$$\lim_{\gamma\to\gamma_A} g_{mn}(y_A^p) = \left. g_{mn}^{(loc)}(y_A^p)\right|_{\gamma_A}, \tag{3.26a}$$

$$\lim_{\gamma\to\gamma_A} \Gamma_{mn}^k(y_A^p) = \left. \Gamma_{mn}^{k(loc)}(y_A^p)\right|_{\gamma_A}, \tag{3.26b}$$

where γ is the world line of the point of interest and the quantities $\Gamma_{mn}^{k(loc)}(y_A^p)$ are the Christoffel symbols calculated with respect to the *local* gravitational field, $g_{mn}^{(loc)}(y_A^p)$. Application of these conditions will determine the functions $K_A, L_A, and Q_A^\alpha$, which are as yet unknown. Moreover, this procedure will enable us to derive the second-order ordinary differential equations for the functions $y_{A_0}^\alpha(y_A^0)$ and $Q_A^\alpha(y_A^0, 0)$, or, in other words, to determine the equations of the perturbed motion of the center of the *local* field in the vicinity of body (A).

Relations (3.26) summarize our expectations based on the Equivalence Principle about the *local* gravitational environment of the self-gravitating bodies. By making use of these equations, we will be able to separate the *local* gravitational field from the external field in the immediate vicinity of the bodies. However, these conditions only allow us to determine the transformation functions for the free-falling massive monopoles (*i.e.* only up to the second order with respect to the spatial coordinates). Transformation functions (3.5) in this case will depend only on the leading contributions of the external gravitational potentials U_B and V_B^α and their first derivatives taken on the world line of body (A). The results obtained will not account for the contribution of the multipolar interaction of the proper gravity with the external field in the volume of the extended body. This accuracy is sufficient for taking into account the terms describing the interaction of the intrinsic quadrupole moments of the bodies with the surrounding gravitational field, but some more general condition, in addition to eq.(3.26), must be applied in order to account for the higher multipole structure of the bodies.

Thus, as we shall see later, conditions (3.26) enable one to obtain the complete solution for the Newtonian function K_A. Functions L_A and Q_A^α may be defined up to the second order with respect to the spatial point separation, namely $L_A, Q_A^\alpha \sim O(|y_A^\alpha|^3)$, so the arbitrariness

of higher orders ($k \geq 3$) in the spatial point separation will remain in the transformation. In order to get the corrections to these functions up to the kth order ($k \geq 3$) with respect to the powers of the spatial coordinate y_A^λ, one should use conditions that contain the spatial derivatives of the metric tensor to the order $(k-1)$. The mathematical methods of modern theoretical physics generally consider *local* geometrical quantities only and involve second-order differential equations. These equations alone may not be very helpful for constructing the remaining terms in functions L_A and Q_A^α up to the order $k \geq 2$. However, following Synge (1960), one may apply additional geometrical constructions, such as properties of the Riemann tensor and the Fermi–Walker transport law (Misner & Manasse, 1963; Ni, 1977; Ni & Zimmermann, 1978; Li & Ni, 1978, 1979a,b). Another possibility is to postulate the existence of so-called 'external multipole moments' (Thorne, 1980; Blanchet & Damour 1986; Brumberg & Kopejkin, 1988a; DSX, 1991-1994). However, those moments are defined through vacuum solutions of the Hilbert–Einstein field equations of general relativity in an inertial RF, while the influence of external sources of gravity are ignored. Defining the moments in this way is essentially equivalent to defining the structure of the proper RF for the body under question.

The most natural approach for defining the desirable properties of the proper quasi-inertial RFs for the system of extended and deformable bodies is to study the motion of this system in an arbitrary KLQ-parametrized frame. There are two different ways to do that, namely: (i) to study the infinitesimal motion of each element of the body, or (ii) to study the motion of a whole body with respect to an accelerated frame attached, for example, to the center of inertia of the *local* fields of matter, inertia, and gravity. In our method, we will use the second way and will study the dynamics of the body in its own RF. Our analysis will be directed toward finding the functions K_A, L_A, and Q_A^α with the condition that the Riemann-flat inertial space-time $\gamma_{mn}^A(y_A^p)$ corresponding to these functions will be tangent to the total Riemann metric $g_{mn}(y_A^p)$ of the entire system in the body's vicinity. Physically, one expects that this inertial space-time will produce a 'fictitious' (or inertial) force with density $\vec{f}_{KLQ}$ acting on the body in its proper RF. At the same time, the body is under the influence of the overall real force due to the *local* fields of matter and gravity with density $\vec{f}_0$. Thus, the condition for finding the transformation functions K_A, L_A and Q_A^α is conceptually simple; the difference between these two densities, $\vec{\mathcal{F}} = \vec{f}_0 - \vec{f}_{KLQ}$, should vanish after integration (or averaging) over the body's compact volume:

$$\delta\vec{F} = \int_A d^3y'_A \vec{\mathcal{F}} = \int_A d^3y'_A \left(\vec{f}_0 - \vec{f}_{KLQ}\right) = 0. \tag{3.27}$$

Note that the notion of 'the center of mass' in this case loses its practical value, and one should substitute instead 'the *local* center of inertia.' Thus, the force $\vec{f}_{KLQ}$ should provide the overall static equilibrium for the body under consideration in the *local* center of inertia, which is defined for all three fields present in the immediate vicinity of the body, namely: matter, inertia and gravity. Let us mention here that in practice it is not possible to separate these two forces, $\vec{f}_0$ and $\vec{f}_{KLQ}$, from each other. Fortunately, we will be able to obtain the difference between them, $\vec{\mathcal{F}}$. This will considerably simplify the further analysis.

In order to construct the necessary solution for functions K_A, L_A, and Q_A^α in a way that will be valid for a wide class of metric theories of gravity, one must first analyze the conservation laws in an arbitrary KLQ-parameterized RF. This could be done based on the conservation law for the density of the total energy-momentum tensor $\hat{T}^{mn}$ of the whole isolated N-body system:

$$\nabla_n^A \hat{T}^{mn}(y_A^p) = 0, \tag{3.28}$$

where ∇_n^A is the covariant derivative with respect to the total Riemannian metric $g_{mn}^A(y_A^p)$ in these coordinates. Then, by using a standard technique for integration with the Killing vectors, one will have to integrate this equation over the compact volume of the body (A) and one can obtain the equations of motion of the extended body (Fock, 1957; Chandrasekhar, 1965; Will, 1993). Then the necessary conditions, equivalent to those of (3.27), may be formulated as the requirement that the translational motion of the extended bodies vanishes in their own RFs. This corresponds to the following conditions applied to the dipole mass moment $m_A^\alpha \equiv I_A^{\{1\}}$:

$$\frac{d^2 m_A^\alpha}{dy_A^{0\,2}} = \frac{dm_A^\alpha}{dy_A^0} = m_A^\alpha = 0, \tag{3.29a}$$

where the quantity m_A^α is calculated based on the total energy-momentum tensor matter, inertia, and gravitational field taken jointly (similar to the condition of eq.(2.12)). These conditions may also be presented in a different form. Indeed, if we require that the total momentum P_A^α of the *local* fields of matter, inertia, and gravity in the vicinity of the extended body vanish, we will have the following physically equivalent condition:

$$\frac{dP_A^\alpha}{dy_A^0} = P_A^\alpha = 0. \tag{3.29b}$$

These conditions finalize the formulation of the basic principles of construction of the relativistic theory of celestial RFs in the WFSMA.

This method is demonstrated to be a useful tool in practical analytical and numerical calculations for a number of metric theories of gravity (Turyshev *et al.*, 1996). Thus, the properties in the derivation of the unperturbed solutions for a number of metric theories of gravity[11] may be used in order to produce the general solution for the problem of motion of an N-body system. In each particular case, for a specific theory of gravity there exists the common strategy for constructing the iterative procedure, which may be expressed as follows:

1. One should first choose the particular model of matter distribution, T^{mn}, and define the small parameters relevant to the particular problem under consideration. The next step is to perform the power expansion with respect to these parameters for all the functions and fields entering the gravitational field equations of a particular metric theory of gravity and, by using the standard methods of the WFSMA (Fock, 1955; Will, 1993), to find the unperturbed solution for an isolated distribution of matter, $h_{mn}^{(0)}$.

2. Then, by using the obtained unperturbed solutions and the WFSMA theory of the coordinate transformations (developed in Appendix B), construct the general form of the solution for the total metric tensor from the *anzatz* eqs.(3.1)–(3.4). Then, by using the generalized de-Donder harmonical gauge and the Fock–Sommerfeld boundary conditions, (3.8), construct the interaction term, h_{mn}^{int}, and present the solution in coordinates of inertial barycentric RF_0 and in an arbitrary, KLQ-parameterized quasi-inertial RF.

3. In order to find the functions K_A, L_A, and Q_A^α of the coordinate transformation to the coordinates of the proper RF_A and fix the remaining coordinate freedom, one should apply the procedure for constructing the proper RF. First of all, find the solution for these functions by implementing the conditions of eqs.(3.26) in a *local region* of the body. Then by

[11]The solutions for an isolated distribution of matter (the *global problem*) are well known, and one may find their general properties in Will (1993).

generalizing the obtained result on the case of an arbitrary extended body, integrate the *local* conservation law (3.28) over the body's volume, in order to obtain the general form of the coordinate transformations from conditions (3.29).

4. In order to obtain the final multipolar solution for the astronomical N-body problem, one should substitute the obtained transformations into the generalized gravitational potentials. Then, by making the expansion of these quantities in the triple power series with respect to small parameters (gravitational constant G, the inverse powers of the speed of light c^{-1}, and the parameter of the geodesic separation $\lambda_A \sim y_A^\nu/|y_{BA_0}|$), one will have obtained the desired representation for the metric tensor and the corresponding equations of motion.

In the following sections, we will discuss the application of the proposed perturbation formalism for the solution of the problem of motion of an arbitrary astronomical N-body system in the general theory of relativity.

4 General Relativity: Solutions for the Field Equations.

In this section, we will apply the iterative formalism discussed in the previous section to construction of solutions for the problem of motion of the system of N extended bodies in the theory of general relativity with a perfect fluid as a model for matter distribution. In this section, we will obtain the solution for the Hilbert–Einstein field equations and a perfect fluid model of matter distribution in its application for solving the problem of motion of N extended self-gravitating bodies in the WFSMA. We will present these solutions in both barycentric inertial and proper quasi-inertial RFs. To do this, we must obtain all the necessary transformation rules under the general coordinate transformations discussed in the previous section. In order to simplify the discussion in this section, all these rules were obtained in a general form and are presented in the appendices, which will be referred to as necessary.

The gravitational field equations of the general theory of relativity were discovered in 1915 and presented by Einstein (1915a,b) (for more details see Misner *et al.* (1973)) as follows:

$$\sqrt{-g}\, R_{mn} = -\frac{8\pi G}{c^4}\left(\hat{T}_{mn} - \frac{1}{2} g_{mn}\hat{T}\right). \tag{4.1}$$

Let us mention that these equations were independently obtained and studied also by Hilbert (1915). At the present time, there exists confidence that a relativistic theory of astronomical RFs must be founded on the equations of the general theory of relativity, (4.1). The mathematical elegance of the field equations as well as the simplicity of the physical foundations of this theory made it particularly easy to perform and analyze the relativistic gravitational experiments. Thus, general relativity has passed many serious tests both in the weak gravitational field of the solar system (Will, 1993) and the strong-gravitational-field test based on the data obtained from the continuous observations of the double pulsar PSR 1913+16 (Damour, 1987; Damour & Taylor, 1992). It should be noted that presently the analysis of high-precision measurements of the light deflection and the delay of propagation time of radio signals in the solar gravitational field confirms the WFSMA of the general theory of relativity with an accuracy on the order of 1.5% and 0.5%, respectively. Concerning the practical applications, we must mention that most of the modern methods for relativistic data reduction as well as the solar system ephemerides are based upon the predictions of equations (4.1) with the perfect fluid model of matter (2.2). This is why we begin the application of the new method for construction of the relativistic theory of the RFs in the WFSMA from the general theory of relativity.

4.1 The Solution for the Interaction Term.

Let us assume that the non-gravitational forces are absent, the bodies are well separated, and the bodies' matter may be described by the model of a perfect fluid with the density of energy-momentum tensor $\hat{T}^{mn}$, given by expressions (2.1)–(2.2). As we have previously discussed, all the field equations and the boundary and initial conditions for this problem are much better defined mathematically in the coordinates of the inertial RF_0, so it is quite natural to begin the discussion within this reference frame. In Section 2, we assumed that the general solution for the gravitational field equations g_{mn} in coordinates (x^p) of the barycentric inertial RF_0 may be written as follows:

$$g_{mn}(x^p) = \gamma_{mn}(x^p) + \sum_{B=1}^{N} \frac{\partial y_B^k}{\partial x^m}\frac{\partial y_B^l}{\partial x^n}\, h_{kl}^{(0)B}(y_B^q(x^p)) + h_{mn}^{int}(x^p). \tag{4.2}$$

At this point, we already have all the necessary 'tools' to construct the metric tensor $g_{mn}(x^p)$. Let us recollect all the gained knowledge, which is necessary to obtain this tensor, namely:

(i). The unperturbed solution for the Hilbert–Einstein gravitational field equations, $h_{kl}^{(0)B}$, for an isolated distribution of matter with the perfect fluid model of matter distribution presented by the energy-momentum tensor, T^{mn}, eq.(2.1), in coordinates of inertial RF_0 has a simple form and in terms of the tensor, $h_{mn}^{(0)B}$, it is given by expressions (2.5) with the conditions $\gamma = \beta = 1, \nu = \tau = 0$.

(ii). The general transformation rules of these solutions under the coordinate transformations (3.5) with the transformation matrix as in eqs.$(C9)$ are established in the form of relations $(D7)$.

(iii). The transformation properties of the gravitational potentials, which are defined in Appendix A, are given by expressions $(E9a), (E14a), (E15a)$, and $(E16a)$.

By substituting all these expressions into formula (4.2), we will obtain the following expressions for the metric tensor g_{mn} in the coordinates (x^p) of the barycentric inertial RF_0:

$$g_{00}(x^0, x^\nu) = 1 - 2\sum_B U_B(x^p) + \sum_B \Bigg(2U_B^2(x^p) + 2\Psi_B(x^p) + \frac{\partial^2}{\partial x^{0^2}} \chi_B(x^p) +$$

$$-2\int_B d^3x' \rho_B\Big(x^0, x'^\nu - y_{B_0}^\nu(x^0)\Big) \frac{\partial}{\partial x'^\lambda} \Big[\frac{Q_B^\lambda\Big(x^0, x^\nu - y_{B_0}^\nu(x^0)\Big) - Q_B^\lambda\Big(x^0, x'^\nu - y_{B_0}^\nu(x^0)\Big)}{|x^\nu - x'^\nu|} \Big] -$$

$$-2v_{B_0\lambda}(x^0) v_{B_0}^\lambda(x^0) \cdot U_B(x^p) + v_{B_0}^\lambda(x^0) v_{B_0}^\beta(x^0) \cdot \frac{\partial^2}{\partial x^\lambda \partial x^\beta} \chi_B(x^p) +$$

$$+a_{B_0}^\lambda(x^0) \cdot \frac{\partial}{\partial x^\lambda} \chi_B(x^p) + 4\frac{\partial}{\partial x^0} K_A\Big(x^0, x_A^\nu - y_{B_0}^\nu(x^0)\Big) \cdot U_B(x^p) \Bigg) + h_{00}^{int<4>}(x^p) + \mathcal{O}(c^{-6}), \quad (4.3a)$$

$$g_{0\alpha}(x^0, x^\nu) = 4\sum_B \gamma_{\alpha\lambda} V_B^\lambda(x^p) + \mathcal{O}(c^{-5}), \quad (4.3b)$$

$$g_{\alpha\beta}(x^0, x^\nu) = \gamma_{\alpha\beta}\Big(1 + 2\sum_B U_B(x^p)\Big) + \mathcal{O}(c^{-4}), \quad (4.3c)$$

where interaction term $h_{00}^{int<4>}$ is the only term that hasn't yet been specified. In order to find this term, one should use the Hilbert–Einstein field equations, eq.(4.1), written in the coordinates of inertial RF_0 and expanded with respect to the small parameter, c^{-1}.

The necessary expansions for the Ricci tensor, R_{mn}, eq.$(B9)$, and for the modified energy-momentum tensor, S_{mn}, which is defined by eqs.$(B12–B13)$, are given correspondingly by the expressions $(D3)$ and $(D11)$ in this CS. By making use of these expressions, one may obtain the linearized Hilbert–Einstein field equations for an N-body system. Finally, by equating the expressions with the same orders of magnitude, with respect to powers of the small parameter, c^{-1}, we will obtain the following equations:

$$\gamma^{\nu\lambda}\frac{\partial^2}{\partial x^\nu \partial x^\lambda} g_{00}^{<2>}(x^p) = -8\pi \sum_B \rho_B(y_B^q(x^p)) + \mathcal{O}(c^{-4}), \tag{4.4a}$$

$$\gamma^{\nu\lambda}\frac{\partial^2}{\partial x^\nu \partial x^\lambda} g_{\alpha\beta}^{<2>}(x^p) = 8\pi\gamma_{\alpha\beta} \sum_B \rho_B(y_B^q(x^p)) + \mathcal{O}(c^{-4}), \tag{4.4b}$$

$$\gamma^{\nu\lambda}\frac{\partial^2}{\partial x^\nu \partial x^\lambda} g_{0\alpha}^{<3>}(x^p) = -16\pi\, \gamma_{\alpha\mu} \sum_B \rho_B(y_B^q(x^p)) \cdot v^\mu(x^p) + \mathcal{O}(c^{-5}), \tag{4.4c}$$

$$\gamma^{\nu\lambda}\frac{\partial^2}{\partial x^\nu \partial x^\lambda} g_{00}^{<4>}(x^p) - \gamma^{\lambda\mu}\gamma^{\nu\delta} g_{\lambda\nu}^{<2>}(x^p)\frac{\partial^2}{\partial x^\mu \partial x^\delta} g_{00}^{<2>}(x^p) +$$

$$+\frac{\partial^2}{\partial x^{0^2}} g_{00}^{<2>}(x^p) - \gamma^{\lambda\nu}\frac{\partial}{\partial x^\lambda} g_{00}^{<2>}(x^p)\frac{\partial}{\partial x^\nu} g_{00}^{<2>}(x^p) =$$

$$= -8\pi \sum_B \rho_B(y_B^q(x^p))\Big(\Pi \; - \; 2\sum_{B'} U_{B'} - 2v^\mu(x^p)v_\mu(x^p) + \frac{3p}{\rho}\Big) + \mathcal{O}(c^{-6}). \tag{4.4d}$$

By substituting into these equations the expressions for the metric tensor $g_{mn}(x^p)$ given by relations (4.3), one may see that the first three equations from (4.4) are automatically satisfied for the components $g_{00}^{<2>}(x^p), g_{\alpha\beta}^{<2>}(x^p)$, and $g_{0\alpha}^{<3>}(x^p)$ of the metric tensor. However, the last equation from this system, eq.(4.4d), written for the component $g_{00}^{<4>}$, produces the necessary equation for the determination of the interaction term $h_{00}^{int<4>}$ as follows:

$$\gamma^{\mu\nu}\frac{\partial^2}{\partial x^\mu \partial x^\nu}\Bigg[h_{00}^{int<4>}(x^p) - \sum_B \Bigg(2v_{B_0\lambda}(x^0)v_{B_0}^\lambda(x^0)\cdot U_B(x^p) -$$

$$-v_{B_0}^\lambda(x^0)v_{B_0}^\beta(x^0)\cdot\frac{\partial^2}{\partial x^\lambda \partial x^\beta}\chi_B(x^p) - a_{B_0}^\lambda(x^0)\cdot\frac{\partial}{\partial x^\lambda}\chi_B(x^p) +$$

$$+2\int_B d^3x'\rho_B\Big(x^0, x'^\nu - y_{B_0}^\nu(x^0)\Big)\times$$

$$\times\frac{\partial}{\partial x'^\lambda}\Big[\frac{Q_B^\lambda\Big(x^0, x^\nu - y_{B_0}^\nu(x^0)\Big) - Q_B^\lambda\Big(x^0, x'^\nu - y_{B_0}^\nu(x^0)\Big)}{|x^\nu - x'^\nu|}\Big] -$$

$$-4\frac{\partial}{\partial x^0}K_A(x^0, x^\nu - y_{B_0}^\nu(x^0))\cdot U_B(x^p)\Bigg)\Bigg] =$$

$$= 4\gamma^{\mu\nu}\sum_B \frac{\partial}{\partial x^\mu}U_B(x^p)\sum_{B'}\frac{\partial}{\partial x^\nu}U_{B'}(x^p). \tag{4.5}$$

The general solution to this equation is easy to obtain and it may be written as follows:

$$h_{00}^{int<4>}(x^0,x^\nu)=\sum_B\Bigg[2v_{B_0\lambda}(x^0)v^\lambda_{B_0}(x^0)\cdot U_B(x^p)-$$

$$-v^\lambda_{B_0}(x^0)v^\beta_{B_0}(x^0)\cdot\frac{\partial^2}{\partial x^\lambda\partial x^\beta}\chi_B(x^p)-$$

$$-a^\lambda_{B_0}(x^0)\cdot\frac{\partial}{\partial x^\lambda}\chi_B(x^p)-4\frac{\partial}{\partial x^0}K_B\Big(x^0,x^\nu-y^\nu_{B_0}(x^0)\Big)\cdot U_B(x^p)+$$

$$+2\int_B d^3x'\rho_B\Big(x^0,x'^\nu-y^\nu_{B_0}(x^0)\Big)\frac{\partial}{\partial x'^\lambda}\Big[\frac{Q^\lambda_B\Big(x^0,x^\nu-y^\nu_{B_0}(x^0)\Big)-Q^\lambda_B\Big(x^0,x'^\nu-y^\nu_{B_0}(x^0)\Big)}{|x^\nu-x'^\nu|}\Big]+$$

$$+2\sum_{B'}\Bigg(U_B(x^p)U_{B'}(x^p)-\int_B\frac{d^3x'}{|x^\nu-x'^\nu|}\Big[\rho_B\Big(x^0,x'^\nu-y^\nu_{B_0}(x^0)\Big)U_{B'}(x^0,x'^\nu)+$$

$$+\rho_{B'}\Big(x^0,x'^\nu-y^\nu_{B_0}(x^0)\Big)U_B(x^0,x'^\nu)\Big]\Bigg)\Bigg]+W_{00}^{<4>}(x^0,x^\nu)+\mathcal{O}(c^{-6}),\qquad(4.6)$$

where summations over both (B) and (B′) are from 1 to N. The only requirement on the arbitrary function $W_{00}^{<4>}$ is that it should satisfy the ordinary Laplace equation:

$$\gamma^{\mu\nu}\frac{\partial^2}{\partial x^\mu\partial x^\nu}W_{00}^{<4>}(x^0,x^\nu)=\mathcal{O}(c^{-6}).\qquad(4.7)$$

The solution to equation (4.7) has terms with both possible asymptotics: one falling off at infinity $\sim 1/r^k$ and the other divergent, *i.e.*, $\sim r^k$. The choice of the solution should be made in order to account for cosmological, galactic, or gravitational wave contributions to the behavior of the metric tensor g_{mn} at large distances from the system. If there is no incoming radiation falling on the system from outer space and the background metric is accepted as having been satisfied for the cosmological conditions of the PPN gauge,[12] then the Fock–Sommerfeld boundary conditions of eq.(3.8) enable us to choose the past-stationary and asymptotically Minkowskian solution to the field equations of general relativity (Damour, 1983). However, for further calculations we will retain the function $W_{00}^{<4>}$ as unspecified.

By substituting the obtained result for the interaction term $h_{00}^{int<4>}$ in the expression for the temporal component of the metric tensor, eq.(4.3*a*), we could write the final solution for the Hilbert–Einstein field equations in coordinates (x^p) of the inertial barycentric RF_0 as follows:

$$g_{00}(x^0,x^\nu)=1-2\sum_B U_B(x^p)+2\Big(\sum_B U_B(x^p)\Big)^2+$$

$$+\sum_B\Bigg(-4\Phi_{1B}(x^p)-2\Phi_{3B}(x^p)-6\Phi_{4B}(x^p)+\frac{\partial^2}{\partial x^{0^2}}\chi_B(x^p)-$$

$$-4\int_B\frac{d^3x'}{|x^\nu-x'^\nu|}\rho_B\Big(x^0,x'^\nu-y^\nu_{B_0}(x^0)\Big)\sum_{B'}U_{B'}(x^0,x'^\nu)\Bigg)+W_{00}^{<4>}(x^0,x^\nu)+\mathcal{O}(c^{-6}),\qquad(4.8a)$$

[12]The main requirement is that the cosmological evolution of the background metric be described by the Robertson–Walker cosmological solution at large distances from the system of the bodies under consideration (Will, 1993).

$$g_{0\alpha}(x^0, x^\nu) = 4\sum_B \gamma_{\alpha\lambda} V_B^\lambda(x^p) + \mathcal{O}(c^{-5}), \tag{4.8b}$$

$$g_{\alpha\beta}(x^0, x^\nu) = \gamma_{\alpha\beta}\Big(1 + 2\sum_B U_B(x^p)\Big) + \mathcal{O}(c^{-4}). \tag{4.8c}$$

The obtained expressions, (4.8), are the usual form of the general solution for the *global* problem in general relativity of the isolated distribution of matter, which was first obtained by Fock (1957) (see also Fock, 1955; Damour, 1986; Kopejkin, 1989; Will, 1993). It is easy to see that the general solution for the N-body problem in the barycentric inertial RF_0, eqs.(4.8), demonstrates the property of linear superposition of unperturbed fields $h_{mn}^{(0)B}$ boosted by transformations (3.18) in components $g_{00}^{<2>}(x^p), g_{\alpha\beta}^{<2>}(x^p)$, and $g_{0\alpha}^{<3>}(x^p)$ of the metric tensor. The non-linear contribution due to the motion of the bodies and their gravitational interaction with each other appears beginning in component $g_{00}^{<4>}(x^p)$ through the interaction term $h_{00}^{int<4>}$, which is given by relation (4.6). One may note that the interaction term contains three groups of terms with physically different origins, namely:

(i). The first seven terms are due to the boost of the isolated unperturbed solutions $h_{mn}^{(0)B}$ by transformations (3.18).

(ii). The eighth term is due to the mutual gravitational interaction between the bodies in the system.

(iii). The last term, $W_{00}^{<4>}$, is caused by the possible inhomogeneity of the background space-time.

It is clear that the terms of the first group are frame dependent (or coordinate dependent). Hence, these terms are responsible for the coordinate dependence of the quantity $h_{00}^{int<4>}$ in general. This implies that this term depends on the properties of the proper coordinate system chosen for description of the *internal* problem in the vicinity of a body (B) in the system. We can continue the analysis of these terms in the barycentric inertial RF_0. However, for further calculations, it will be more convenient to shift the discussion to the proper RF_A.

The transformation properties of the interaction term are given by the relations ($D9$). These relations suggest that, in the first post-Newtonian approximation, the form of the interaction term in the coordinates (y_A^p) of the proper RF_A could be obtained by taking into account the transformation properties of the gravitational potentials only. Thus, by making use of the direct transformations (3.5) with the transformation matrix (C1), one may write the interaction term $h_{00}^{int<4>}$ in the coordinates of the proper RF_A as follows:

$$h_{00}^{int<4>}(y_A^0, y_A^\nu) = h_A^{int<4>}(y_A^p) + h_{AB}^{int<4>}(y_A^p) +$$

$$+ h_B^{int<4>}(y_A^p) + h_{BB'}^{int<4>}(y_A^p) + W_{00}^{<4>}(y_A^p) + \mathcal{O}(c^{-6}), \tag{4.9a}$$

where the following notations have been accepted:

$$h_A^{int<4>}(y_A^p) = 2\int_A d^3y'_A \rho_A(y_A^0, y'^\nu_A)\frac{\partial}{\partial y'^\lambda_A}\Big[\frac{Q_A^\lambda(y_A^0, y_A^\nu) - Q_A^\lambda(y_A^0, y'^\nu_A)}{|y_A^\nu - y'^\nu_A|}\Big] -$$

$$-2v_{A_0\lambda}(y_A^0)v^\lambda_{A_0}(y_A^0)\cdot U_A(y_A^p) - v^\lambda_{A_0}(y_A^0)v^\beta_{A_0}(y_A^0)\cdot\frac{\partial^2}{\partial y_A^\lambda \partial y_A^\beta}\chi_A(y_A^p) -$$

$$-a^\lambda_{A_0}(y_A^0)\cdot\frac{\partial}{\partial y_A^\lambda}\chi_A(y_A^p) - 4\frac{\partial}{\partial y_A^0}K_A(y_A^0, y_A^\nu)\cdot U_A(y_A^p) + \mathcal{O}(c^{-6}), \tag{4.9b}$$

$$h_{AB}^{int<4>}(y_A^p) = 4\sum_{B\neq A}\Bigg(U_A(y_A^p)U_B(y_A^p) -$$

$$-\int_{A,B}\frac{d^3y'_A}{|y_A^\nu - y'^\nu_A|}\Big[\rho_A(y_A^0, y'^\nu_A)U_B(y_A^0, y'^\nu_A) + \rho_B(y_A^0, y'^\nu_A)U_A(y_A^0, y'^\nu_A)\Big]\Bigg) + \mathcal{O}(c^{-6}), \tag{4.9c}$$

$$h_B^{int<4>}(y_A^p) = \sum_{B\neq A}\Bigg(2\int_B d^3y'_A\rho_B\Big(y_A^0, y'^\nu_A + y^\nu_{BA_0}(y_A^0)\Big)\times$$

$$\times\frac{\partial}{\partial y'^\lambda_A}\Big[\frac{Q_B^\lambda\Big(y_A^0, y_A^\nu + y^\nu_{BA_0}(y_A^0)\Big) - Q_B^\lambda\Big(y_A^0, y'^\nu_A + y^\nu_{BA_0}(y_A^0)\Big)}{|y_A^\nu - y'^\nu_A|}\Big] +$$

$$+2v_{B_0\lambda}(y_A^0)\Big(v^\lambda_{B_0}(y_A^0) - 2v^\beta_{A_0}(y_A^0)\Big)\cdot U_B(y_A^p) -$$

$$-v^\lambda_{B_0}(y_A^0)v^\beta_{B_0}(y_A^0)\cdot\frac{\partial^2}{\partial y_A^\lambda\partial y_A^\beta}\chi_B(y_A^p) - a^\lambda_{B_0}(y_A^0)\cdot\frac{\partial}{\partial y_A^\lambda}\chi_B(y_A^p) -$$

$$-4\frac{\partial}{\partial y_A^0}K_B\Big(y_A^0, y_A^\nu - y^\nu_{BA_0}(y_A^0)\Big)\cdot U_B(y_A^p)\Bigg) + \mathcal{O}(c^{-6}), \tag{4.9d}$$

$$h_{BB'}^{int<4>}(y_A^p) = 4\sum_{B\neq A}\sum_{B'\neq B}\Bigg(U_B(y_A^p)U_{B'}(y_A^p) -$$

$$-\int_B\frac{d^3y'_A}{|y_A^\nu - y'^\nu_A|}\Big[\rho_B\Big(y_A^0, y'^\nu_A - y^\nu_{BA_0}(y_A^0)\Big)U_{B'}(y_A^0, y'^\nu_A) +$$

$$+\rho_{B'}\Big(y_A^0, y'^\nu_A - y^\nu_{B'A_0}(y_A^0)\Big)U_B(y_A^0, y'^\nu_A)\Big]\Bigg) + \mathcal{O}(c^{-6}). \tag{4.9e}$$

The physical meaning of these new functions is quite clear. The functions $h_A^{int<4>}$ and $h_B^{int<4>}$ are the post-Newtonian contributions of the unperturbed solutions $h_{mn}^{(0)}$ for body (A) and all the rest of the bodies (B≠A) in the system, boosted by transformations (3.5) and (3.20). The function $h_{AB}^{int<4>}$ is the contribution describing the gravitational interaction of the body (A) with the rest of the bodies in the system. And the last term, $h_{BB'}^{int<4>}$, is the function, physically analogous to the previous one, but describing the gravitational field generated by the gravitational interaction of the rest of the bodies in the system (B, B′ ≠A) with each other in the vicinity of the body (A).

The advantage of using conditions (3.8) is that they provide an opportunity to determine the interaction term $h_{mn}^{int}(x^p)$ in a unique way. It should be stressed that the corresponding solution

$g_{mn}(x^p)$ in the barycentric inertial RF_0 resembles the form of the solution for an isolated one-body problem, (2.5). The only change that should be made is to take into account the number of bodies in the system: $\rho \to \sum_B \rho_B$, where ρ_B is the compact-support mass density of a body (B) from the system. However, both the interaction term $h^{int}_{mn}(y^p_a)$ and the total solution for the metric tensor g_{mn} in the coordinates (y^p_A) appear to be 'parameterized' by the arbitrary functions K_A, L_A, and Q^α_A. This result reflects the covariancy of the gravitational field equations as well as the well defined transformation properties of the gauge conditions, (3.6), used to derive the total solution. This arbitrariness suggests that one could choose any form of these functions in order to describe the dynamics of the extended bodies in the system. However, as we noticed earlier, the unsuccessful choice of the proper RF_A (or, equivalently, the functions K_A, L_A, and Q^α_A) may cause an unreasonable complication in the future physical interpretations of the results obtained.

4.2 The Solution of the Field Equations in the Proper RF.

Once the interaction term $h^{int<4>}_{00}$ has been defined, one may easily obtain the form of the general solution to the Hilbert–Einstein field equations $g_{mn}(y^p_A)$ in the coordinates of the proper RF_A. This solution may be obtained directly from the tensor $g_{mn}(x^p)$ by the regular tensorial transformation law as follows:

$$g_{mn}(y^p_A) = \frac{\partial x^k}{\partial y^m_A}\frac{\partial x^l}{\partial y^n_A}\, g_{kl}(x^s(y^p_A)) = \frac{\partial x^k}{\partial y^m_A}\frac{\partial x^l}{\partial y^n_A}\gamma_{kl}(x^s(y^p_A)) +$$

$$+ h^{(0)A}_{mn}(y^p_A) + \sum_{B\neq A}\frac{\partial y^k_B}{\partial y^m_A}\frac{\partial y^l_B}{\partial y^n_A}\, h^{(0)B}_{kl}(y^s_B(y^p_A)) + \frac{\partial x^k}{\partial y^m_A}\frac{\partial x^l}{\partial y^n_A} h^{int}_{kl}(x^s(y^p_A)). \tag{4.10}$$

In order to obtain the final result for the metric tensor g_{mn} in the coordinates of the proper RF_A, we should establish and then make use of the transformation properties of all the quantities presented in expression (4.10). These quantities were obtained in the appendices as follows:

(i). The transformation properties of the background Riemann-flat metric γ^A_{mn} in the coordinates (y^p_A) are given by relations $(C5)$.

(ii). The transformations of the unperturbed solutions $h^{(0)B}_{mn}$ from the coordinates (y^p_B) of the proper RF_B to those of the RF_A are presented by relations $(D8)$.

(iii). The transformation properties of the interaction term h^{int}_{kl} were established and discussed in the previous subsection, where they were given by relations (4.6) and (4.9).

(iv). The transformation properties of all the potentials, which enter the above-named formulae, are given by eqs.$(E9b), (E14b), (E15b)$, and $(E16b)$.

By substituting all these quantities into relations (4.10), we will obtain the components of the metric tensor $g_{mn}(y^p_A)$ in the coordinates of the proper RF_A as follows:

$$g_{00}(y^p_A) = 1 + 2\frac{\partial}{\partial y^0_A}K_A(y^0_A, y^\nu_A) + v_{A_0\beta}(y^0_A)v^\beta_{A_0}(y^0_A) - 2\sum_B U_B(y^p_A) +$$

$$+2\frac{\partial}{\partial y^0_A}L_A(y^0_A, y^\nu_A) + \Big(\frac{\partial}{\partial y^0_A}K_A(y^0_A, y^\nu_A)\Big)^2 + 2v_{A_0\beta}(y^0_A)\frac{\partial}{\partial y^0_A}Q^\beta_A(y^0_A, y^\nu_A) +$$

$$+H^{<4>}_{00}(y^0_A, y^\nu_A) + \mathcal{O}(c^{-6}), \tag{4.11a}$$

$$g_{0\alpha}(y_A^p) = \frac{\partial}{\partial y_A^\alpha} L_A(y_A^0, y_A^\nu) - v_{A_0\alpha}(y_A^0) \frac{\partial}{\partial y_A^0} K_A(y_A^0, y_A^\nu) +$$

$$+ v_{A_0\nu}(y_A^0) \frac{\partial}{\partial y_A^\alpha} Q_A^\nu(y_A^0, y_A^\nu) + \gamma_{\alpha\nu} \frac{\partial}{\partial y_A^0} Q_A^\nu(y_A^0, y_A^\nu) + 4 \sum_B \gamma_{\alpha\lambda} V_B^\lambda(y_A^p) + \mathcal{O}(c^{-5}), \qquad (4.11b)$$

$$g_{\alpha\beta}(y_A^p) = \gamma_{\alpha\beta} + v_{A_0\alpha}(y_A^0) v_{A_0\beta}(y_A^0) +$$

$$+ \gamma_{\alpha\nu} \frac{\partial}{\partial y_A^\beta} Q_A^\nu(y_A^0, y_A^\nu) + \gamma_{\beta\nu} \frac{\partial}{\partial y_A^\alpha} Q_A^\nu(y_A^0, y_A^\nu) + 2 \sum_B \gamma_{\alpha\beta} U_B(y_A^p) + \mathcal{O}(c^{-4}), \qquad (4.11c)$$

where the post-Newtonian term $H_{00}^{<4>}$ in the component $g_{00}(y_A^p)$ of equation (4.11a) denotes the following expression:

$$H_{00}^{<4>}(y_A^0, y_A^\nu) = h_{00A}^{(0)<4>}(y_A^p) + h_A^{int<4>}(y_A^p) + h_{AB}^{int<4>}(y_A^p) +$$

$$+ \sum_{B \neq A} \Big[\frac{\partial y_B^k}{\partial y_A^0} \frac{\partial y_B^l}{\partial y_A^0} h_{kl}^{(0)B}(y_B^s(y_A^p)) \Big]^{<4>} + h_B^{int<4>}(y_A^p) + h_{BB'}^{int<4>}(y_A^p) + W_{00}^{<4>}(y_A^p). \qquad (4.12a)$$

The latter expression may be presented in terms of the generalized gravitational potentials as follows:

$$H_{00}^{<4>}(y_A^0, y_A^\nu) = 2\Big(\sum_B U_B(y_A^p)\Big)^2 + \sum_B \Bigg(-4\Phi_{1B}(y_A^p) - 2\Phi_{3B}(y_A^p) - 6\Phi_{4B}(y_A^p) -$$

$$-4 \int_B \frac{d^3 y'_A}{|y_A^\nu - y_A'^\nu|} \rho_B\Big(y_A^0, y_A'^\nu + y_{BA_0}^\nu(y_A^0)\Big) \cdot \sum_{B'} U_{B'}(y_A^0, y_A'^\nu) + \frac{\partial^2}{\partial {y_A^0}^2} \chi_B(y_A^p) +$$

$$+2 \int_B d^3 y'_A \rho_B\Big(y_A^0, y_A'^\nu + y_{BA_0}^\nu(y_A^0)\Big) \frac{\partial}{\partial y_A'^\lambda} \Big[\frac{Q_A^\lambda(y_A^0, y_A^\nu) - Q_A^\lambda(y_A^0, y_A'^\nu)}{|y_A^\nu - y_A'^\nu|} \Big] -$$

$$-2 v_{A_0\lambda}(y_A^0) v_{A_0}^\lambda(y_A^0) \cdot U_B(y_A^p) - v_{A_0}^\lambda(y_A^0) v_{A_0}^\beta(y_A^0) \cdot \frac{\partial^2}{\partial y_A^\lambda \partial y_A^\beta} \chi_B(y_A^p) -$$

$$-a_{A_0}^\lambda(y_A^0) \cdot \frac{\partial}{\partial y_A^\lambda} \chi_B(y_A^p) - 4 \frac{\partial}{\partial y_A^0} K_A(y_A^0, y_A^\nu) \cdot U_B(y_A^p) \Bigg) + W_{00}^{<4>}(y_A^p) + \mathcal{O}(c^{-6}). \qquad (4.12b)$$

The first three terms in expression (4.12a) describe both the unperturbed gravitational field of the body (A), boosted by the coordinate transformations (the terms $h_{00A}^{(0)<4>}$ and $h_A^{int<4>}$), and the gravitational field produced by the interaction of this field with one produced by the rest of the bodies in the system (the term $h_{AB}^{int<4>}$). These are the terms that govern the *local* gravitational environment in the immediate vicinity of the body (A), producing the major contribution to the equations of motion of the test particles orbiting this body. The next three terms in expression (4.12*a*) are the terms that are due to the boosted unperturbed gravitational fields produced by the rest of the bodies in the system and the gravitational field caused by their interaction with each other, presented in the coordinates of the proper RF_A. This external gravitational field should appear in the equations of motion of the test particles around the body (A), written in the coordinates of the proper RF_A, in the form of a tidal interaction only (Synge, 1960). Note that the approach discussed here is the generalization of the concept of the neutral test particle

freely falling in the external gravitational field. It is known that, up to these tidal corrections, a freely falling test particle will behave as if external gravity is absent (Bertotti & Grishchuk, 1990). In our case, the extended body (A) is not moving freely; instead, as we will see later, its internal multipole moments are couples to the external gravitational field through the terms $h_A^{int<4>}$ and $h_{AB}^{int<4>}$. This coupling produces a force that results in the deviation of the center of mass of this body from the support geodetic line along which it would move if it were a neutral test particle (Denisov & Turyshev, 1989). The presence of this term and its significance for solving the *local* problem has been pointed out by a number of authors (see, for instance, Thorne & Hartle (1985); Kopejkin (1987)); however, to our knowledge, the interaction term has never been previously presented in a closed relativistic form.

By straightforward calculation, one may check that the obtained metric tensor $g_{mn}(y_A^p)$ satisfies the Hilbert–Einstein field equations written in the coordinates of the proper RF$_A$. To do this, let us note that the covariant de Donder gauge is singling out these coordinates according to conditions $(C2)$. This gives the expressions for the Ricci tensor R_{mn} in the form of eqs.$(C4)$. The modified energy-momentum tensor S_{mn} in this coordinate system is given by expressions $(C12)$. By collecting all these expressions together, one may obtain the linearized Hilbert–Einstein field equations, eq.(4.1), presented in the coordinates of the proper RF$_A$. Finally, the substitution of the relations of eqs.(4.11) in the obtained linearized equations will complete the proof of the correspondence between the metric tensor $g_{mn}(y_A^p)$ and the field equations.

Thus, metric (4.11) is the KLQ parameterized solution of the Hilbert–Einstein gravitational field equations in the coordinates of the proper RF$_A$. The nature of this result is basically the post-Newtonian boost of solution (4.8) (obtained in the inertial RF$_0$) to the new non-inertial coordinate system defined in the vicinity of an arbitrary body (A). It is well known that the Riemann metric tensor $g_{mn}(y_A^p)$ contains ten degrees of freedom and could not be transformed to the Minkowski tensor for the entire space-time by any choice of a coordinate transformation that has only four degrees of freedom. This transformation could be done at one point of the space-time only (Eisenhart, 1926) or along the geodesic line (Manasse & Misner, 1963; Misner *et al.*, 1973; Landau & Lifshitz, 1988). Such an RF is called a quasi-inertial or 'locally Lorentzian frame.' Our future discussion will be based on the form of the metric tensor in the proper RF$_A$ given by relations (4.11). In the next section, we will implement the conditions for construction of a 'good' quasi-Lorentzian proper RF as discussed in Section 2, which will enable us to find the unknown transformation functions K_A, L_A, and Q_A^α.

4.3 Decomposition of the Fields in the Proper RF.

Concluding this section, we would like to emphasize that the solution to the Hilbert–Einstein field equations g_{mn} in the vicinity of the body's (A) world line in the coordinates (y_A^p) of its proper RF$_A$ in the first WFSMA may be decomposed into the following three major groups:

$$g_{mn}(y_A^p) = \gamma_{mn}^A(y_A^p) + H_{mn}^A(y_A^p) + H_{mn}^B(y_A^p), \tag{4.13}$$

where the notations for these groups and their meaning are as presented below.

(i). The first term, γ_{mn}^A, is the *local* inertial (or Riemann-flat) field that is presented by eqs.$(B4)$. This term is also convenient to split into two parts as shown by the relation

$$\gamma_{mn}^A(y_A^p) = \frac{\partial x^k}{\partial y_A^m}\frac{\partial x^l}{\partial y_A^n}\gamma_{kl}(x^s(y_A^p)) = \gamma_{mn}^{(0)}(y_A^p) + \gamma_{mn}^{<PN>}(y_A^p), \tag{4.14}$$

where $\gamma^{(0)}_{mn}$ is the usual Minkowski metric in the coordinates of the proper RF$_A$. The second term here, $\gamma^{<PN>}_{mn}$, is the KLQ-parameterized post-Newtonian contribution to this *local* inertial field at the vicinity of the body's (A) world line.

(ii). The second term in eq.(4.13), H^A_{mn}, is the *local* gravitational field, which is given as follows:

$$H^A_{00}(y^p_A) = h^{(0)A}_{00} + h^{int<4>}_{00A} + h^{int<4>}_{00AB} + \mathcal{O}(c^{-6}),$$

$$H^A_{0\alpha}(y^p_A) = h^{(0)A}_{0\alpha} + \mathcal{O}(c^{-5}), \qquad H^A_{\alpha\beta}(y^p_A) = h^{(0)A}_{\alpha\beta} + \mathcal{O}(c^{-4}), \tag{4.15a}$$

where the terms $h^{(0)A}_{mn}$ are the components of the unperturbed proper gravitational field of the body (A), the term $h^{int<4>}_{00A}$ (given by eq.(4.9b)) is the contribution due to the boost of this unperturbed field to the accelerated coordinates of the proper quasi-inertial RF$_A$, and the last term, $h^{int<4>}_{AB}$ (which is presented by eq.(4.9c)), is caused by the interaction of the proper unperturbed gravitational field with the external gravitation. Thus, the component $H^{A<4>}_{00}$ has the form

$$H^{A<4>}_{00}(y^p_A) = 2U^2_A(y^p_A) + 2\Psi_A(y^p_A) + \frac{\partial^2}{\partial {y^0_A}^2}\chi_A(y^p_A) +$$

$$+2\int_A d^3y'_A \rho_A(y^0_A, y'^\nu_A)\frac{\partial}{\partial y'^\lambda_A}\Big[\frac{Q^\lambda_A(y^0_A, y^\nu_A) - Q^\lambda_A(y^0_A, y'^\nu_A)}{|y^\nu_A - y'^\nu_A|}\Big] -$$

$$-2v_{A_0\lambda}(y^0_A)v^\lambda_{A_0}(y^0_A)\cdot U_A(y^p_A) - v^\lambda_{A_0}(y^0_A)v^\beta_{A_0}(y^0_A)\cdot\frac{\partial^2}{\partial y^\lambda_A \partial y^\beta_A}\chi_A(y^p_A) -$$

$$-a^\lambda_{A_0}(y^0_A)\cdot\frac{\partial}{\partial y^\lambda_A}\chi_A(y^p_A) - 4\frac{\partial}{\partial y^0_A}K_A(y^0_A, y^\nu_A)\cdot U_A(y^p_A) +$$

$$+4\sum_{B\neq A}\left(U_A(y^p_A)U_B(y^p_A) - \int_A \frac{d^3y'_A}{|y^\nu_A - y'^\nu_A|}\rho_A(y^0_A, y'^\nu_A)U_B(y^0_A, y'^\nu_A)\right) + \mathcal{O}(c^{-6}), \tag{4.15b}$$

where the subscript (A) for the integral sign means that the integration is performed over the volume of that body for which mass density is integrated, namely: $\int_A d^3y'_A\rho_B = \delta_{AB}$.

(iii). The last term in eq. (4.13), H^B_{mn}, is the *external* gravitational field, presented as follows:

$$H^B_{00}(y^p_A) = \sum_{B\neq A}\Big[\frac{\partial y^k_B}{\partial y^0_A}\frac{\partial y^l_B}{\partial y^0_A}h^{(0)B}_{kl}(y^s_B(y^p_A))\Big]^{<4>} +$$

$$+h^{int<4>}_{00B}(y^p_A) + h^{int<4>}_{00BB'}(y^p_A) + W^{<4>}_{00}(y^p_A) + \mathcal{O}(c^{-6}),$$

$$H^B_{\alpha 0}(y^p_A) = \sum_{B\neq A}\frac{\partial y^k_B}{\partial y^\alpha_A}\frac{\partial y^l_B}{\partial y^0_A}h^{(0)B}_{kl}(y^s_B(y^p_A)) + \mathcal{O}(c^{-5}),$$

$$H^B_{\alpha\beta}(y^p_A) = \sum_{B\neq A}h^{(0)B}_{\alpha\beta}(y^s_B(y^p_A)) + \mathcal{O}(c^{-4}), \tag{4.16a}$$

where the first two terms in the component H^B_{00} are the result of the boost (see eq. (4.9*d*)) to the coordinates (y^p_A) of the RF$_A$ of the unperturbed solutions $h^{(0)B}_{kl}$ for the bodies (B)

in the system (besides (A)); the third term, $h^{int<4>}_{00BB'}$ (given by eq.(4.9e)), is due to the mutual gravitational interactions of these external bodies with each other; and, finally, the last term is due to existing inhomogeneity of the background space-time in which the considered system is embedded. The component $H^{B<4>}_{00}(y^p_A)$ may be given as follows:

$$H^{B<4>}_{00}(y^p_A) = \sum_{B\neq A}\Bigg(2U_B(y^p_A)\sum_{B'\neq A}U_{B'}(y^p_A) + 2\Psi_B(y^p_A) + \frac{\partial^2}{\partial y^{0\,2}_A}\chi_B(y^p_A) +$$

$$+2\int_B d^3y'_A\rho_B\Big(y^0_A, y'^\nu_A + y^\nu_{BA_0}(y^0_A)\Big)\frac{\partial}{\partial y'^\lambda_A}\Big[\frac{Q^\lambda_A(y^0_A, y^\nu_A) - Q^\lambda_A(y^0_A, y'^\nu_A)}{|y^\nu_A - y'^\nu_A|}\Big] -$$

$$-2v_{A_0\lambda}(y^0_A)v^\lambda_{A_0}(y^0_A)\cdot U_B(y^p_A) - v^\lambda_{A_0}(y^0_A)v^\beta_{A_0}(y^0_A)\cdot\frac{\partial^2}{\partial y^\lambda_A\partial y^\beta_A}\chi_B(y^p_A) -$$

$$-a^\lambda_{A_0}(y^0_A)\cdot\frac{\partial}{\partial y^\lambda_A}\chi_B(y^p_A) - 4\frac{\partial}{\partial y^0_A}K_A(y^0_A, y^\nu_A)\cdot U_B(y^p_A)\Bigg) + \mathcal{O}(c^{-6}), \qquad (4.16b)$$

where the potential $\Phi_{2B}(y^p_A)$ entering the term $\Psi_B(y^p_A)$ is defined as

$$\Phi_{2B}(y^p_A) = \int_B \frac{d^3y'_A}{|y^\nu_A - y'^\nu_A|}\rho_B\Big(y^0_A, y'^\nu_A + y^\nu_{BA_0}(y^0_A)\Big)\sum_C U_C(y^0_A, y'^\nu_A) + \mathcal{O}(c^{-6}). \qquad (4.16c)$$

The decomposition presented by eqs.(4.13)–(4.16) may be successfully continued to the next 'post-post-Newtonian' order; however, the obtained accuracy is quite sufficient for most modern astronomical applications. The results obtained in this section will become a useful tool in the next section for constructing a proper RF with well-defined physical properties.

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5 General Relativity: Transformations to the Proper RF.

In this section, we will present the construction of a 'good' proper RF for an arbitrary body (A). This procedure should enable one to obtain the yet unknown transformation functions K_A, L_A, and Q^α_A. It is clear that one may choose any form of these functions for the description of the gravitational environment around the body under question. The analysis presented in the previous section shows by the results in eqs.(4.12) that, in order to solve the *local* problem, it is permissible to separate the contributions in the metric tensor $g_{mn}(y^p_A)$ into several terms. The first contribution is due to the inertial sector of the *local* space-time, the second is produced by the body itself, the third term is caused by the external sources of the gravitational field, and the last one is due to the interaction of the body's multipole moments with this external gravitational field. It is well known that if the body (A) is a neutral monopole test particle, this external gravitational field will define the parameters of the geodesic line that this test body will follow (Einstein *et al.*, 1938; Fock, 1957; Will, 1993). The equations of motion for spinning bodies are different from the latter by additional terms due to coupling of the body's spin to the external gravitational field (Papapetrou, 1948, 1951). It was noted that the presence of non-vanishing internal multipole moments of extended bodies significantly changes their equations of motion, and several attempts have made to account for these effects (see, for example, Ashby & Bertotti, 1986; Shahid-Salees *et al.*, 1991; Brumberg & Kopejkin, 1988a; DSX, 1991-94). In this report, we will introduce a new approach based on the KLQ parameterization discussed in the previous section.

The general idea for constructing the 'good' RF$_A$ in terms of the functions K_A, L_A, and Q^α_A is to choose these functions in such a way that the corresponding Riemann-flat inertial space-time γ^A_{mn} (which is the background space-time for the proper RF$_A$) will be tangent to the total metric tensor g_{mn} in the vicinity of the world line of the body (A). These conditions, when applied to inertially moving test particles, are known as the Fermi conditions (Misner *et al.*, 1973). We would like to extend the applicability of these conditions to the case of a system of extended self-gravitating and arbitrarily shaped celestial bodies. To do this, let us recall that the relation for the *local* gravitational field $g^{(loc)}_{mn}(y^p_A)$, which is based on the decomposition eqs.(4.13), may be given as follows:

$$g^{(loc)}_{mn}(y^p_A) = \gamma^{(0)}_{mn}(y^p_A) + H^A_{mn}(y^p_A). \tag{5.1}$$

Then the generalized Fermi conditions in the *local* region of body (A) (or in the immediate vicinity of its world-line γ_A) may be introduced by equations (3.26) as follows:

$$g_{mn}(y^p_A)\Big|_{\gamma_A} = g^{(loc)}_{mn}(y^p_A)\Big|_{\gamma_A} + \mathcal{O}(|y^\alpha_A|^2), \tag{5.2a}$$

$$\Gamma^k_{mn}(y^p_A)\Big|_{\gamma_A} = \Gamma^{k(loc)}_{mn}(y^p_A)\Big|_{\gamma_A} + \mathcal{O}(|y^\alpha_A|), \tag{5.2b}$$

where the quantities $\Gamma^{k(loc)}_{mn}(y^p_A)$ are the Christoffel symbols calculated with respect to the *local* gravitational field $g^{(loc)}_{mn}(y^p_A)$ given by eq.(5.1). These relations summarize our expectations based on the Equivalence Principle about the *local* gravitational environment of self-gravitating and arbitrarily shaped extended bodies. These conditions enable us to separate the *local* gravitational field from the *external* gravitation in the immediate vicinity of the body (A). This separation is possibly due to the remaining arbitrariness of the transformation functions K_A, L_A, and Q^α_A. The conditions of eqs.(5.2) will give the differential equations for these functions, the solutions of which will correspond to the specific choice of the background inertial space-time in the proper

RF_A. To obtain these equations, one should substitute the relations for the metric tensor in the form (4.11) in the expressions for the Christoffel symbols $(F2)$ and then make use of conditions (5.1).

5.1 Finding the Functions K_A and Q^α_A.

5.1.1 Equations for the Functions K_A and Q^α_A.

To obtain the equation for the function K_A, one should substitute into conditions (5.1) the relation for the component $\Gamma^0_{00}(y^p_A)$ of the connection coefficients given by eq.$(F2a)$. This will give the following result:

$$\Big[\frac{\partial}{\partial y^0_A}\Big(\frac{\partial}{\partial y^0_A}K_A(y^0_A, y^\nu_A) + \frac{1}{2}v_{A_0\epsilon}v^\epsilon_{A_0} - \sum_{B\neq A} U_B(y^0_A, y^\nu_A)\Big)\Big]\Big|_{\gamma_A} = \mathcal{O}(c^{-5}). \tag{5.3}$$

The components $\Gamma^0_{0\alpha}(y^p_A)$ and $\Gamma^\alpha_{00}(y^p_A)$, which are given by eqs.$(F2b) and (F2d)$, correspondingly, will provide us with the following equation:

$$\Big[a_{A_0\alpha} + \sum_{B\neq A}\frac{\partial}{\partial y^\alpha_A}U_B(y^0_A, y^\nu_A)\Big]\Big|_{\gamma_A} = \mathcal{O}(c^{-4}). \tag{5.4}$$

From the components $\Gamma^\alpha_{\beta\omega}(y^p_A)$ of the connection coefficients that are given by eq.$(F2f)$, one may obtain the first equation for the function Q^α_A:

$$\Big[\frac{\partial^2}{\partial y^\beta_A \partial y^\omega_A}Q^\alpha_A(y^0_A, y^\nu_A) + \sum_{B\neq A}\Big(\delta^\alpha_\beta\frac{\partial}{\partial y^\omega_A}U_B(y^0_A, y^\nu_A) +$$

$$+\delta^\alpha_\omega\frac{\partial}{\partial y^\beta_A}U_B(y^0_A, y^\nu_A) - \gamma_{\beta\omega}\gamma^{\alpha\lambda}\frac{\partial}{\partial y^\lambda_A}U_B(y^0_A, y^\nu_A)\Big)\Big]\Big|_{\gamma_A} = \mathcal{O}(c^{-4}). \tag{5.5}$$

The components $\Gamma^\alpha_{0\beta}(y^p_A)$ of eq.$(F2e)$ will give the second and last equation for the second unknown transformation function:

$$\Big[\frac{\partial^2}{\partial y^0_A \partial y^\beta_A}Q^\alpha_A(y^0_A, y^\nu_A) + \ v^\alpha_{A_0}a_{A_0\beta} +$$

$$+\sum_{B\neq A}\Big(2\frac{\partial}{\partial y^\beta_A}V^\alpha_B(y^0_A, y^\nu_A) - 2\frac{\partial}{\partial y^\mu_{A\alpha}}V_{B\beta}(y^0_A, y^\nu_A) + \delta^\alpha_\beta\frac{\partial}{\partial y^0_A}U_B(y^0_A, y^\nu_A)\Big]\Big)\Big|_{\gamma_A} = \mathcal{O}(c^{-5}). \tag{5.6}$$

5.1.2 The Solution for the Function K_A.

In order to find the solutions to the differential equations above, let us first denote the limiting operation from expressions (5.2) for any non-singular function $f(y^p_A)$ as follows:

$$\big\langle f\big\rangle_0 \equiv \lim_{|\vec{y}_A|\to 0} f(y^0_A, y^\nu_A) = f(y^0_A, y^\nu_A)\Big|_{\gamma_A}. \tag{5.7}$$

It is important to note that operation (5.7) commutes with the time derivative but not with the spatial derivative.

Then, using this new notation, we may formally integrate eq.(5.3) over time y_A^0 as follows:

$$\Big[\frac{\partial}{\partial y_A^0}K_A(y_A^0, y_A^\nu) + \frac{1}{2}v_{A_0\mu}v_{A_0}^\mu - \sum_{B\neq A} U_B(y_A^0, y_A^\nu)\Big]\Big|_{\gamma_A} = \zeta_{1A}(y_A^\nu), \tag{5.8}$$

where ζ_{1A} is an arbitrary function of the spatial coordinates y_A^ν. To continue the solution, let us recall the relation for the function K_A given by eq.($C5b$) in the following form:

$$K_{A_{[0]}}(y_A^0, y_A^\nu) = P_A(y_A^0) - v_{A_{[0]}\mu}(y_A^0)\cdot y_A^\mu + \mathcal{O}(c^{-4})y_A^0, \tag{5.9}$$

where the subscript ($_{[0]}$) denotes that the operation (5.7) was used to derive the result (5.9) for the functions K_A and $v_{A_{[0]}}$. One may notice that the dependence on the spatial coordinate in this relation for K_A disappears completely on a world line of the body, so the function $\zeta_{1A}(y_A^\nu)$ is a true constant, *i.e.*, $\zeta_{1A}(y_A^\nu) = \zeta_{1A} = const$. Then, from these two relations, (5.8) and (5.9), one may obtain the differential equation for the function $P_A(y_A^0)$ as follows:

$$\frac{\partial}{\partial y_A^0}P_A(y_A^0) = \frac{d}{dy_A^0}P_A(y_A^0) = \sum_{B\neq A}\Big\langle U_B\Big\rangle_0 - \frac{1}{2}v_{A_{[0]}\beta}v_{A_{[0]}}^\beta + \zeta_{1A}. \tag{5.10}$$

If we formally integrate this last equation over time y_A^0 and for the function K_A, we will obtain the following final solution:

$$K_{A_{[0]}}(y_A^0, y_A^\nu) = \int^{y_A^0} dt'\Big[\sum_{B\neq A}\Big\langle U_B\Big\rangle_{0'} - \frac{1}{2}v_{A_{[0]}\nu}v_{A_{[0]}}^\nu + \zeta_1^A\Big] -$$

$$-v_{A_{[0]}\nu}\cdot y_A^\nu + \mathcal{O}(c^{-4})y_A^0. \tag{5.11}$$

Equation (5.4) provides us with the usual relation for the Newtonian acceleration $a_{A_{[0]}}^\alpha$ of the center of inertia of a body (A) as follows:

$$a_{A_{[0]}}^\alpha(y_A^0) = -\ \gamma^{\alpha\nu}\sum_{B\neq A}\Big\langle\frac{\partial U_B}{\partial y_A^\nu}\Big\rangle_0 + \mathcal{O}(c^{-4}). \tag{5.12}$$

Thus, we have obtained the form of the first transformation function K_A, eq.(5.11), which describes the Newtonian corrections to the proper time y_A^0. These corrections should be made in order to take into account the external gravitational field and the Lorentzian time contraction caused by the motion of the origin of the proper RF$_A$ with the velocity $v_{A_{[0]}}^\nu$ relative to the inertial barycentric RF$_0$. This correction was first obtained by D'Eath (1975a,b) by the method of matched asymptotic expansions while studying the motion of black holes. In astronomical applications for the relativistic VLBI measurements, this effect was independently obtained and studied by Hellings (1986). The only new term in the expression in eq.(5.11) is the constant ζ_1^A, which is the free parameter entering the post-Poincaré group of motion. This parameter represents the possibility of the time shift in proper RF$_A$ and is responsible for the energy conservation in the immediate vicinity of the massive test particle moving along the geodesic. The acceleration, eq.(5.12), is the contribution of the monopole into the equation of motion of the extended body. The contributions of the other multipoles to the results in (5.11) and (5.12) will be obtained and discussed further.

5.1.3 The Solution for the Function Q_A^α.

The solution for the function Q_A^α requires slightly more sophisticated calculations. One may expect that the function Q_A^α behaves at least quadratically while approaching to the origin of the body's world line (*i.e.*, $Q_A^\alpha \sim y^\mu y^\nu \cdot f(y_A^0)$, where $f(y_A^0)$ is some time-dependent function). Let us look at the solution to the equation (5.5) for the function Q_A^α in the following form:

$$Q_A^\alpha(y_A^0, y_A^\nu) = -\sum_{B\neq A}\Big[c_1 \cdot y_A^\alpha y_A^\mu \cdot \Big\langle\frac{\partial U_B}{\partial y_A^\mu}\Big\rangle_0 + c_2\gamma^{\alpha\sigma}\cdot y_{A\mu}y_A^\mu \cdot \Big\langle\frac{\partial U_B}{\partial y_A^\sigma}\Big\rangle_0\Big] + \Omega_A^\alpha(y_A^0, y_A^\nu), \qquad (5.13)$$

where c_1 and c_2 are the constants, unknown for the moment. The function $\Omega_A^\alpha(y_A^0, y_A^\nu)$ behaves linearly in the vicinity of the body's world line: $\Omega_A^\alpha(y_A^0, y_A^\nu) \sim y^\mu \cdot f(y_A^0)$. By substituting the expression of eq.(5.13) into equation (5.5), we will find that these constants are $c_1 = 1$ and $c_2 = -1/2$ and that the function Ω_A^α should satisfy the equation

$$\frac{\partial^2}{\partial y_A^\beta \partial y_A^\lambda}\Omega_A^\alpha(y_A^0, y_A^\nu) = 0. \qquad (5.14)$$

By making use of these results, we may write the solution to eq.(5.5) as follows:

$$Q_A^\alpha(y_A^0, y_A^\nu) = -\sum_{B\neq A}\Big[y_A^\alpha y_A^\beta \cdot \Big\langle\frac{\partial U_B}{\partial y_A^\beta}\Big\rangle_0 - \frac{1}{2}\gamma^{\alpha\sigma}y_{A\beta}y_A^\beta \cdot \Big\langle\frac{\partial U_B}{\partial y_A^\sigma}\Big\rangle_0\Big] + \Omega_A^\alpha(y_A^0, y_A^\nu) \qquad (5.15)$$

Further calculations require somewhat more sophisticated approach. After some algebra, eq.(5.6) might be rewritten as follows:

$$\Big[\frac{\partial}{\partial y_A^0}\Big(\gamma_{\alpha\nu}\frac{\partial}{\partial y_A^\beta}Q_A^\nu(y^0, y^\nu) + \gamma_{\beta\nu}\frac{\partial}{\partial y_A^\alpha}Q_A^\nu(y^0, y^\nu) +$$

$$+ v_{A_0\alpha}v_{A_0\beta} + 2\gamma_{\alpha\beta}\sum_{B\neq A}U_B(y_A^0, y_A^\nu)\Big)\Big]\Big|_{\gamma_A} = \mathcal{O}(c^{-5}). \qquad (5.16)$$

By integrating equation (5.16) over time y_A^0, we obtain

$$g_{\alpha\beta}^{A<2>}(y_A^0, y_A^\nu)\Big|_{\gamma_A} = \Big[v_{A_0\alpha}v_{A_0\beta} + 2\gamma_{\alpha\beta}\sum_{B\neq A}U_B(y_A^0, y_A^\nu) +$$

$$+\gamma_{\alpha\nu}\frac{\partial}{\partial y_A^\beta}Q_A^\nu(y^0, y^\nu) + \gamma_{\beta\nu}\frac{\partial}{\partial y_A^\alpha}Q_A^\nu(y^0, y^\nu)\Big]\Big|_{\gamma_A} = \sigma_{\alpha\beta}^A = \text{const}. \qquad (5.17)$$

Then the function Ω_A^α from eq.(5.15) may be represented in the following form:

$$\Omega_A^\alpha(y_A^0, y_A^\nu) = -\sum_{B\neq A}y_A^\alpha \cdot \Big\langle U_B\Big\rangle_0 - \frac{1}{2}v_{A_{[0]}}^\alpha v_{A_{[0]}\mu}\cdot y_A^\mu + F_A^\alpha(y_A^0, y_A^\nu), \qquad (5.18)$$

where F_A^α is some unknown function. By substituting the expression of eq.(5.18) into eq.(5.17), we will define the function Q_A^α as follows:

$$Q^{\alpha}_{A}(y^0_A, y^{\nu}_A) = -\sum_{B\neq A}\Big[y^{\alpha}_A y^{\beta}_A \cdot \Big\langle \frac{\partial U_B}{\partial y^{\beta}_A}\Big\rangle_0 - \frac{1}{2}\gamma^{\alpha\sigma} y_{A\beta} y^{\beta}_A \cdot \Big\langle \frac{\partial U_B}{\partial y^{\sigma}_A}\Big\rangle_0 + y^{\alpha}_A \cdot \Big\langle U_B\Big\rangle_0\Big] -$$

$$-\frac{1}{2} v^{\alpha}_{A_{[0]}} v_{A_{[0]}\beta} \cdot y^{\beta}_A + F^{\alpha}_A(y^0_A, y^{\nu}_A), \tag{5.19}$$

with the condition on the function $F^{\alpha}_A(y^0_A, y^{\nu}_A)$:

$$\gamma_{\nu\beta}\frac{\partial}{\partial y^{\alpha}_A} F^{\nu}_A(y^0_A, y^{\nu}_A) + \gamma_{\nu\alpha}\frac{\partial}{\partial y^{\beta}_A} F^{\nu}_A(y^0_A, y^{\nu}_A) = \sigma^A_{\alpha\beta}. \tag{5.20}$$

From the expressions of eqs.(5.6),(5.18), and (5.20), one may write the equation for the function F^{ν}_A in the vicinity of the body's (A) world line as

$$\frac{\partial^2}{\partial y^0_A \partial y^{\beta}_A} F^{\alpha}_A(y^0_A, y^{\nu}_A) = \frac{1}{2}\Big[a^{\alpha}_{A_{[0]}} v_{A_{[0]}\beta} - v^{\alpha}_{A_{[0]}} a_{A_{[0]}\beta}\Big] + 2\sum_{B\neq A}\Big[\Big\langle \partial^{\alpha} V_{B\beta}\Big\rangle_0 - \Big\langle \partial_{\beta} V^{\alpha}_B\Big\rangle_0\Big]. \tag{5.21}$$

This last equation, eq.(5.21), may be solved together with eq.(5.20) as follows:

$$F^{\alpha}_A(y^0_A, y^{\nu}_A) = y_{A\beta}\int^{y^0_A} dt'\Big[\frac{1}{2} a^{[\alpha}_{A_{[0]}} v^{\beta]}_{A_{[0]}} + 2\sum_{B\neq A}\Big\langle \partial^{[\alpha} V^{\beta]}_B\Big\rangle_0\Big] +$$

$$+ f^{\alpha}_{A\beta} \cdot y^{\beta}_A + w^{\alpha}_{A_{[0]}}(y^0_A), \tag{5.22}$$

where the constants $\sigma^A_{\alpha\beta}$ and $f^A_{\alpha\beta}$ are connected as $f^A_{\alpha\beta} + f^A_{\beta\alpha} = \sigma^A_{\alpha\beta}$. The time-dependent function $w^{\alpha}_{A_{[0]}}(y^0_A)$ is unknown at the moment.

Finally, by collecting the obtained relations from eqs.(5.19) and (5.22), we will obtain the final solution for the second transformation function, Q^{α}_A, as follows:

$$Q^{\alpha}_{A_{[0]}}(y^0_A, y^{\nu}_A) = -\sum_{B\neq A}\Big[y^{\alpha}_A y^{\beta}_A \cdot \Big\langle \frac{\partial U_B}{\partial y^{\beta}_A}\Big\rangle_0 - \frac{1}{2}\gamma^{\alpha\sigma} y_{A\beta} y^{\beta}_A \cdot \Big\langle \frac{\partial U_B}{\partial y^{\sigma}_A}\Big\rangle_0 + y^{\alpha}_A \cdot \Big\langle U_B\Big\rangle_0\Big] +$$

$$+ y_{A\beta}\int^{y^0_A} dt'\Big[\frac{1}{2} a^{[\alpha}_{A_{[0]}} v^{\beta]}_{A_{[0]}} + 2\sum_{B\neq A}\Big\langle \partial^{[\alpha} V^{\beta]}_B\Big\rangle_0\Big] -$$

$$-\frac{1}{2} v^{\alpha}_{A_{[0]}} v_{A_{[0]}\beta} \cdot y^{\beta}_A + f^{\alpha}_{A\beta} \cdot y^{\beta}_A + w^{\alpha}_{A_{[0]}}(y^0_A) + \mathcal{O}(c^{-4}) y^{\alpha}_A + \mathcal{O}(|y^{\alpha}_A|^3). \tag{5.23}$$

Thus, we have obtained the second transformation function, Q^{α}_A, which is the first function to describe the post-Newtonian coordinate transformation to the proper RF of a moving massive monopole body. The only function that is still unknown in expression (5.23) is the function $w^{\alpha}_{A_{[0]}}$, which defines the post-Newtonian correction to the radius vector $y^{\alpha}_{A_{[0]}}$. This time-dependent function will be obtained later. Besides the usual Lorentzian terms of the length contraction (caused by the velocity of motion of the coordinate origin), the expression above contains terms caused purely by gravity. The first two terms are due to the acceleration

of the proper RF_A caused by the external gravitational field. The third term is the length contraction caused by the external gravitational field. The fourth term with the integral sign is the generalization of the expression for geodesic and Thomas precession of the coordinate axis (see Thomas, 1927). A similar expression was obtained by D'Eath (1975a,b). In astronomical practice, this result was introduced by Brumberg & Kopejkin (1988) (see also Ries *et al.*, 1991; DSX, 1991). The obtained relation is different from the previous results in that it contains a generalized representation of the term containing the precessions. In particular, the obtained relation is defined explicitly and does not contain an arbitrary multiplier q as in the Brumberg–Kopejkin method. This suggests that the precession term should always be present in the expressions for the coordinate transformations and neglecting this term will correspond to the RF, which is deviating from the geodesic world line even for the massless test particles, and will lead to the SEP violation. In addition to this, expression (5.23) has an arbitrary group parameter $f^{\alpha}_{A\beta}$. This parameter represents the angular momentum conservation law at the immediate vicinity of the world line of the body (A) in its proper RF_A. Besides this, we have studied separately the post-Newtonian part of the radius vector of the body (A), $w^{\alpha}_{A_{[0]}}$, which was never done before. The contributions of the other multipoles to the result in (5.23) will be obtained and discussed in the next section in a manner similar to the case of the function K_A, (5.11), and the Newtonian eq.m., (5.12),.

5.2 Finding the Function L_A.

In this subsection, we will consider the problem of finding the function L_A, which is the last unknown function for transformations (3.5). This function corresponds to the post-Newtonian correction to the transformation of barycentric time to time in the proper RF. As we shall see, this function will depend on the model of matter distribution taken to describe the internal structure of the bodies in the system. In contrast to the functions K_A and Q^{α}_A, the analog of the function L_A has never previously been obtained, which makes the results here particularly interesting.

5.2.1 Equations for the Function L_A.

The relations in $(F2)$ and conditions in eqs.(5.1) enable us to obtain the equations for the function L_A. Thus, from the components $\Gamma^0_{\alpha\beta}(y^p_A)$, which are given by eq.$(F2c)$, we will have the first equation for this function as follows:

$$\Big[\frac{\partial^2}{\partial y^{\alpha}_A \partial y^{\beta}_A}\Big(L_A(y^0_A, y^{\nu}_A) \; + \; v_{A_0\lambda} Q^{\lambda}_A(y^0_A, y^{\nu}_A)\Big) +$$

$$+ \sum_{B\neq A}\Big(2\gamma_{\beta\lambda}\frac{\partial}{\partial y^{\alpha}_A} V^{\lambda}_B(y^0_A, y^{\nu}_A) + 2\gamma_{\alpha\lambda}\frac{\partial}{\partial y^{\beta}_A} V^{\lambda}_B(y^0_A, y^{\nu}_A) -$$

$$-\gamma_{\alpha\beta}\frac{\partial}{\partial y^0_A} U_B(y^0_A, y^{\nu}_A)\Big)\Big]\Big|_{\gamma_A} = \mathcal{O}(c^{-6}). \tag{5.24}$$

The second necessary equation may be obtained from the expression for the components $\Gamma^0_{0\alpha}(y^p_A)$, eq.$(F2b)$, by simply making use of the solution for the function K_A given by eq.(5.11) and the result of the acceleration of the center of mass eq.(5.12). This equation has the following form:

$$\Big[\frac{\partial}{\partial y_A^\alpha}\Big(\frac{\partial}{\partial y_A^0}L_A(y_A^0,y_A^\nu)+\frac{1}{2}\Big(\frac{\partial}{\partial y_A^0}K_A(y_A^0,y_A^\nu)\Big)^2+$$

$$+v_{A_0\epsilon}\frac{\partial}{\partial y_A^0}Q_A^\epsilon(y_A^0,y_A^\nu)+\frac{1}{2}\sum_{B\neq A}H_{00}^{B<4>}(y_A^0,y_A^\nu)+$$

$$+2\zeta_1^A\cdot U_A(y_A^0,y_A^\nu)\Big)\Big]\Big|_{\gamma_A}=\mathcal{O}(c^{-6}), \tag{5.25}$$

where the function $H_{00}^{B<4>}$ is given by eq.(4.16b). From the relations for the components $\Gamma_{00}^\alpha(y_A^p)$, eq.($F2d$), and with the help of the expressions in eqs.(5.11), (5.12), (5.17), and eq.(5.24), one may obtain

$$\Big[\frac{\partial}{\partial y_A^0}\Big(\gamma^{\alpha\lambda}\frac{\partial}{\partial y_A^\lambda}\Big(L_A(y_A^0,y_A^\nu)+v_{A_0\nu}Q_A^\nu(y_A^0,y_A^\nu)\Big)-$$

$$-v_{A_0}^\alpha\frac{\partial}{\partial y_A^0}K_A(y_A^0,y_A^\nu)+\frac{\partial}{\partial y_A^0}Q_A^\alpha(y_A^0,y_A^\nu)+$$

$$+4\sum_{B\neq A}V_B^\alpha(y_A^0,y_A^\nu)\Big)\Big]\Big|_{\gamma_A}=\Big(\sigma_A^{\alpha\mu}-2\gamma^{\alpha\mu}\zeta_1^A\Big)\cdot\Big\langle\frac{\partial U_A}{\partial y_A^\lambda}\Big\rangle_0+\mathcal{O}(c^{-6}). \tag{5.26}$$

This last equation may be formally integrated over time as follows:

$$\Big[\gamma^{\alpha\lambda}\frac{\partial}{\partial y_A^\lambda}\Big(L_A(y_A^0,y_A^\nu)+v_{A_0\nu}Q_A^\nu(y_A^0,y_A^\nu)\Big)-v_{A_0}^\alpha\frac{\partial}{\partial y_A^0}K_A(y_A^0,y_A^\nu)+$$

$$+\frac{\partial}{\partial y_A^0}Q_A^\alpha(y_A^0,y_A^\nu)+4\sum_{B\neq A}V_B^\alpha(y_A^0,y_A^\nu)\Big]\Big|_{\gamma_A}=$$

$$=\Big(\sigma_A^{\alpha\mu}-2\gamma^{\alpha\mu}\zeta_1^A\Big)\cdot\int^{y_A^0}dt'\cdot\Big\langle\frac{\partial U_A}{\partial y_A^\lambda}\Big\rangle_{0'}+\sigma_A^\alpha+\mathcal{O}(c^{-6}), \tag{5.27}$$

where we have separated the integrating constant σ_A^α. Using the relations for the components $\Gamma_{00}^0(y_A^p)$, eq.($F2a$), and the solutions (5.11),(5.12), and (5.27), one may obtain the last equation for the function L_A as given below:

$$\Big[\frac{\partial}{\partial y_A^0}\Big(\frac{\partial}{\partial y_A^0}L_A(y_A^0,y_A^\nu)+\frac{1}{2}\Big(\frac{\partial}{\partial y_A^0}K_A(y_A^0,y_A^\nu)\Big)^2+$$

$$+v_{A_0\epsilon}\frac{\partial}{\partial y_A^0}Q_A^\epsilon(y_A^0,y_A^\nu)+\frac{1}{2}\sum_{B\neq A}H_{00}^{B<4>}(y_A^0,y_A^\nu)\Big)\Big]\Big|_{\gamma_A}=$$

$$=\Big(\sigma_A^{\mu\lambda}-2\gamma^{\mu\lambda}\zeta_1^A\Big)\Big\langle\frac{\partial U_A}{\partial y_A^\mu}\Big\rangle_0\int^{y_A^0}dt'\cdot\Big\langle\frac{\partial U_A}{\partial y_A^\lambda}\Big\rangle_{0'}+$$

$$+\sigma_A^\mu\cdot\Big\langle\frac{\partial U_A}{\partial y_A^\mu}\Big\rangle_0-2\zeta_1^A\cdot\frac{\partial}{\partial y_A^0}\Big\langle U_A\Big\rangle_0+\mathcal{O}(c^{-7}). \tag{5.28}$$

Thus, we have obtained four equations necessary to determine the last unknown transformation function L_A, namely, eqs.(5.24)–(5.26) and (5.28).

5.2.2 The Solution for the Function L_A.

The determination of functions K_A and Q_A^α helps us find the solution for the function L_A as well. In order to do this, let us look for the function L_A in the following form:

$$L_A(y_A^0, y_A^\nu) = \sum_{B\neq A}\Big[k_1 \cdot y_{A\beta} y_A^\beta \cdot \Big\langle \frac{\partial U_B}{\partial y_A^0}\Big\rangle_0 + k_2 \cdot y_A^\lambda y_A^\beta \cdot \Big\langle \partial_\lambda V_{B\beta}\Big\rangle_0 +$$

$$+k_3 \cdot v_{A_{[0]}\beta}\Big(y_A^\beta y_A^\lambda \cdot \Big\langle \frac{\partial U_B}{\partial y_A^\lambda}\Big\rangle_0 - \frac{1}{2}\gamma^{\beta\sigma} y_{A\lambda} y_A^\lambda \cdot \Big\langle \frac{\partial U_B}{\partial y_A^\sigma}\Big\rangle_0\Big)\Big] + B_1^A(y_A^0, y_A^\nu). \tag{5.29}$$

Then from eq.(5.24), one may easily obtain the unknown constants $k_1, k_2, and k_3$ and the condition on the function B_1^A as follows:

$$k_1 = \frac{1}{2}, \qquad k_2 = -2, \qquad k_3 = 1;$$

$$\frac{\partial^2}{\partial y_A^\beta \partial y_A^\lambda} B_1^A(y_A^0, y_A^\nu) = 0. \tag{5.30}$$

The unknown function B_1^A may be determined from eq.(5.27) by making use of the expressions of eqs.(5.11),(5.23), and (5.24). Thus, the intermediate solution for the function L_A may be presented as follows:

$$L_A(y_A^0, y_A^\nu) = \sum_{B\neq A}\Big[\frac{1}{2} y_{A\beta} y_A^\beta \cdot \Big\langle \frac{\partial U_B}{\partial y_A^0}\Big\rangle_0 - 2 y_A^\lambda y_A^\beta \cdot \Big\langle \partial_\lambda V_{B\beta}\Big\rangle_0 +$$

$$+v_{A_{[0]}\beta}\Big(y_A^\beta y_A^\lambda \cdot \Big\langle \frac{\partial U_B}{\partial y_A^\lambda}\Big\rangle_0 - \frac{1}{2}\gamma^{\beta\sigma} y_{A\lambda} y_A^\lambda \cdot \Big\langle \frac{\partial U_B}{\partial y_A^\sigma}\Big\rangle_0\Big)\Big] +$$

$$+v_{A_{[0]}\beta} y_{A_\lambda} \int^{y_A^0} dt' \Big[\frac{1}{2} a_{A_{[0]}}^{[\lambda} v_{A_{[0]}}^{\beta]} + 2\sum_{B\neq A}\Big\langle \partial^{[\lambda} V_B^{\beta]}\Big\rangle_{0'}\Big] +$$

$$+y_{A_\beta}\Big[2 v_{A_{[0]}}^\beta \sum_{B\neq A}\Big\langle U_B\Big\rangle_0 - 4\sum_{B\neq A}\Big\langle V_B^\beta\Big\rangle_0 - \dot{w}_{A_{[0]}}^\beta(y_A^0) + v_{A_{[0]}}^\beta \cdot \zeta_{1A} +$$

$$+\Big(\sigma_A^{\beta\lambda} - 2\gamma^{\beta\lambda}\zeta_1^A\Big)\int^{y_A^0} dt' \cdot \Big\langle \frac{\partial U_A}{\partial y_A^\lambda}\Big\rangle_{0'} + \sigma_A^\beta - v_{A_{[0]}\lambda} \cdot f_A^{\beta\lambda}\Big] + B_2^A(y_A^0), \tag{5.31}$$

where σ_A is a constant, and the unknown time-dependent function B_2^A may be obtained from eq.(5.28). In order to do this, let us first integrate eq.(5.28) over time y_A^0:

$$\Big[\frac{\partial}{\partial y_A^0}L_A(y_A^0,y_A^\nu)+\frac{1}{2}\Big(\frac{\partial}{\partial y_A^0}K_A(y_A^0,y_A^\nu)\Big)^2+$$

$$+v_{A_0\epsilon}\frac{\partial}{\partial y_A^0}Q_A^\epsilon(y_A^0,y_A^\nu)+\frac{1}{2}H_{00}^{B<4>}(y_A^0,y_A^\nu)\Big]\Big|_{\gamma_A}=$$

$$=\Big(\sigma_A^{\mu\lambda}-2\gamma^{\mu\lambda}\zeta_1^A\Big)\int^{y_A^0}dt'\cdot\Big\langle\frac{\partial U_A}{\partial y_A^\mu}\Big\rangle_{0'}\int^{t'}dt''\cdot\Big\langle\frac{\partial U_A}{\partial y_A^\lambda}\Big\rangle_{0''}+$$

$$+\sigma_A^\mu\cdot\int^{y_A^0}dt'\Big\langle\frac{\partial U_A}{\partial y_A^\mu}\Big\rangle_{0'}-2\zeta_1^A\cdot\Big\langle U_A\Big\rangle_0+\zeta_2^A+\mathcal{O}(c^{-7}). \tag{5.32}$$

Then, the function B_2^A may be determined from equation (5.32) with the help of eqs.(5.11), (5.23), and (5.31) in the following form:

$$B_{2A}(y_A^0)=\int^{y_A^0}dt'\Big[-\frac{1}{2}\sum_{B\neq A}\Big\langle H_{00}^{B<4>}\Big\rangle_{0'}-\frac{1}{2}\Big(\sum_{B\neq A}\Big\langle U_B\Big\rangle_{0'}-\frac{1}{2}v_{A_{[0]}\beta}v^\beta_{A_{[0]}}+\zeta_1^A\Big)^2+$$

$$+\Big(\sigma_A^{\mu\lambda}-2\gamma^{\mu\lambda}\zeta_1^A\Big)\int^{t'}dt''\Big\langle\frac{\partial U_A}{\partial y_A^\mu}\Big\rangle_{0''}\int^{t''}dt'''\Big\langle\frac{\partial U_A}{\partial y_A^\lambda}\Big\rangle_{0'''}+$$

$$+\sigma_A^\mu\cdot\int^{t'}dt''\Big\langle\frac{\partial U_A}{\partial y_A^\mu}\Big\rangle_{0''}-2\zeta_1^A\cdot\Big\langle U_A\Big\rangle_{0'}+\zeta_2^A-v_{A_{[0]}\mu}\dot{w}^\mu_{A_{[0]}}(t')\Big]. \tag{5.33}$$

Finally, by collecting the results in obtained eqs.(5.31) and (5.33) together, we get the following expression for the transformation function L_A in the coordinates (y_A^p) of quasi-inertial RF$_A$:

$$L_A(y_A^0,y_A^\nu)=\sum_{B\neq A}\Big[\frac{1}{2}y_{A\beta}y_A^\beta\cdot\Big\langle\frac{\partial U_B}{\partial y_A^0}\Big\rangle_0-2y_A^\lambda y_A^\beta\cdot\Big\langle\partial_\lambda V_{B\beta}\Big\rangle_0+$$

$$+v_{A_{[0]}\beta}\Big(y_A^\beta y_A^\lambda\cdot\Big\langle\frac{\partial U_B}{\partial y_A^\lambda}\Big\rangle_0-\frac{1}{2}\gamma^{\beta\sigma}y_{A\lambda}y_A^\lambda\cdot\Big\langle\frac{\partial U_B}{\partial y_A^\sigma}\Big\rangle_0\Big)\Big]+$$

$$+v_{A_{[0]}\beta}y_{A\lambda}\int^{y_A^0}dt'\Big[\frac{1}{2}a^{[\lambda}_{A_{[0]}}v^{\beta]}_{A_{[0]}}+2\sum_{B\neq A}\Big\langle\partial^{[\lambda}V_B^{\beta]}\Big\rangle_{0'}\Big]+$$

$$+y_{A\beta}\Big[2v^\beta_{A_{[0]}}\sum_{B\neq A}\Big\langle U_B\Big\rangle_0-4\sum_{B\neq A}\Big\langle V_B^\beta\Big\rangle_0-\dot{w}^\beta_{A_{[0]}}+v^\beta_{A_{[0]}}\cdot\zeta_{1A}+$$

$$+\sigma_A^\beta-v_{A_{[0]}\lambda}\cdot f_A^{\beta\lambda}+\Big(\sigma_A^{\beta\lambda}-2\gamma^{\beta\lambda}\zeta_1^A\Big)\int^{y_A^0}dt'\cdot\Big\langle\frac{\partial U_A}{\partial y_A^\lambda}\Big\rangle_{0'}\Big]+$$

$$+\int^{y_A^0}dt'\Big[-\frac{1}{2}\sum_{B\neq A}\Big\langle H_{00}^{B<4>}\Big\rangle_{0'}-\frac{1}{2}\Big(\sum_{B\neq A}\Big\langle U_B\Big\rangle_{0'}-\frac{1}{2}v_{A_{[0]}\beta}v^\beta_{A_{[0]}}+\zeta_1^A\Big)^2-$$

$$-v_{A_{[0]}\mu}\dot{w}^\mu_{A_{[0]}}(t')+\Big(\sigma_A^{\mu\lambda}-2\gamma^{\mu\lambda}\zeta_1^A\Big)\int^{t'}dt''\Big\langle\frac{\partial U_A}{\partial y_A^\mu}\Big\rangle_{0''}\int^{t''}dt'''\Big\langle\frac{\partial U_A}{\partial y_A^\lambda}\Big\rangle_{0'''}+$$

$$+\zeta_2^A-2\zeta_1^A\cdot\Big\langle U_A\Big\rangle_{0'}+\sigma_A^\mu\cdot\int^{t'}dt''\Big\langle\frac{\partial U_A}{\partial y_A^\mu}\Big\rangle_{0''}\Big]+\mathcal{O}(c^{-6})y_A^0+\mathcal{O}(|y_A^\alpha|^3). \tag{5.34}$$

Thus, we have obtained the last function, L_A, for the post-Newtonian transformation in the WFSMA. Notice that this function is the only one that depends on the model of matter chosen for description of the bodies in the system through the term $H_{00}^{B<4>}$. This function contains two new parameters of the group of motion, namely: the parameter ζ_2^A, which is the extension of the Newtonian parameter ζ_1^A on the post-Newtonian order, and the parameter σ_A^α, which represents the time-dependent Poincaré rotation. The function L_A demonstrates the non-linear character of the obtained group of motion. This non-linearity is due to the interaction of the proper gravitational field of the body (A) with the external gravitation. Thus, the Newtonian potential U_A and its gradients influence the dynamic of the proper RF_A in the case when some of the parameters from the ten parametric group $(\zeta_1^A, \zeta_2^A; \sigma_A^\alpha; f_A^{\alpha\beta})$ are not zero.

It is worth noting that some parts of the expression (5.34) were obtained by D'Eath (1975a,b), whose method has been used in the Brumberg–Kopejkin formalism (Brumberg & Kopejkin, 1988a,b). However, this is the first time the function L_A has been obtained in the form of the expression above. This function describes the post-Newtonian corrections to the proper time and, besides the usual Lorentzian contributions, it contains the purely gravitational terms caused by the external gravitational field. The only unknown function in this expression is the function $w_{A_{[0]}}^\alpha$, which will be discussed in the following subsection. Let us mention that knowledge of the function L_A will be required for analyzing the results of the proposed post-Newtonian redshift experiment planned for the *Solar Probe* mission (Anderson, 1989). This effect on the necessary accuracy was studied by Krisher (1993), who had formulated the frequency shift of the spacecraft clock to the order of c^{-4}. However, his formulation appears to be very simplified and does not include the dynamical effects due to proper accelerated motion of the spacecraft in close proximity to the Sun, which is the crucial phase of the experiment. We believe that the correct derivation of the corresponding effect should be based upon the relativistic theory of the astronomical RFs, so that the function L_A, (5.34), will provide one with all the required corrections, including both kinematical and dynamical effects. Moreover, in Section 7, we will obtain the parameterized form of this function which will enable one to include in the analysis alternative tensor-scalar theories of gravity.

5.3 Equations of Motion for the Massive Bodies.

By finding the form of the function L_A, we determined almost all of the functions for the coordinate transformation between RFs. However, one quantity still remains unspecified: the function $w_{A_{[0]}}^\alpha$ in expressions (5.23) and (5.34). This function might be obtained from the last unused equation, namely eq.(5.25). By substituting the relations obtained for functions $K_A and Q_A^\alpha$ given by eqs.(5.11) and (5.23) into eq.(5.25), and making use of the expression for the function L_A given by eqs.(5.34), one obtains the following ordinary differential equation for the last unknown function $w_{A_{[0]}}^\alpha$:

$$\ddot{w}_{A_{[0]}}^\alpha(y_A^0) = \sum_{B\neq A}\Big(\frac{1}{2}\gamma^{\alpha\sigma}\Big\langle\frac{\partial H_{00}^{B<4>}}{\partial y_A^\sigma}\Big\rangle_0 + v_{A_{[0]}}^\alpha\Big\langle\frac{\partial U_B}{\partial y_A^0}\Big\rangle_0 - 4\Big\langle\frac{\partial V_B^\alpha}{\partial y_A^0}\Big\rangle_0\Big) -$$

$$-\frac{1}{2}v_{A_{[0]}}^\alpha v_{A_{[0]}\beta}a_{A_{[0]}}^\beta + a_{A_{[0]}}^\alpha\sum_{B\neq A}\Big\langle U_B\Big\rangle_0 +$$

$$+a_{A_{[0]}\lambda}\int^{y_A^0}dt'\Big[\frac{1}{2}a_{A_{[0]}}^{[\alpha}v_{A_{[0]}}^{\lambda]} + 2\sum_{B\neq A}\Big\langle\partial^{[\alpha}V_B^{\lambda]}\Big\rangle_{0'}\Big] - a_{A_{[0]}\lambda}\cdot f_A^{\alpha\lambda} + \sigma_A^{\alpha\mu}\cdot\Big\langle\frac{\partial U_A}{\partial y_A^\mu}\Big\rangle_0 + \mathcal{O}(c^{-6}). \quad (5.35)$$

We may check that this equation is the post-Newtonian part of the acceleration $A^{\alpha}_{A_{[0]}}$ of the center of the field of the body (A) with respect to the barycenter written in its proper coordinate system. If we perform the coordinate transformation from the coordinates (y^p_A) of the proper RF_A to those (x^p) of the inertial barycentric RF_0 of all the functions and potentials entering in eq.(5.35), we obtain the well-known geodesic equation for the test body written in coordinates (x^p) of the barycentric inertial RF_0. To do this, let us first combine the two parts of the acceleration $A^{\alpha}_{A_{[0]}}$ as follows:

$$A^{\alpha}_{A_{[0]}}(y^0_A) = a^{\alpha}_{A_{[0]}}(y^0_A) + \ddot{w}^{\alpha}_{A_{[0]}}(y^0_A) + O(c^{-6}), \tag{5.36}$$

where the terms in this equation are given by relations (5.12) and (5.35). Then, by using the transformation rules from Appendix E, we may obtain the following result for acceleration $A^{\alpha}_{A_{[0]}}$ transformed into the coordinates of the inertial barycentric RF_0:

$$\begin{aligned} A^{\alpha}_{A_{[0]}}(x^p) = \sum_{B\neq A}\Big[& -\partial^{\alpha}U_B(x^p)\cdot\Big(1 - v_{A_{[0]}\beta}v^{\beta}_{A_{[0]}} - 4\sum_{B'\neq A}U_{B'}(x^p)\Big) - \\ & -3v^{\alpha}_{A_{[0]}}\partial_0 U_B(x^p) - 4v^{\alpha}_{A_{[0]}}v^{\lambda}_{A_{[0]}}\partial_{\lambda}U_{B'}(x^p) - 4\partial_0 V^{\alpha}_B(x^p) + \\ & +4v_{A_{[0]}\lambda}\Big(\partial^{\alpha}V^{\lambda}_B(x^p) - \partial^{\lambda}V^{\alpha}_B(x^p)\Big) - 2\partial^{\alpha}\Phi_{1B}(x^p) - 2\partial^{\alpha}\Phi_{2B}(x^p) - \\ & -\partial^{\alpha}\Phi_{3B}(x^p) - 3\partial^{\alpha}\Phi_{4B}(x^p) + \frac{1}{2}\gamma^{\alpha\nu}\frac{\partial^3}{\partial x^{\nu}\partial x^{0^2}}\chi_B(x^p)\Big]\Big|_{\gamma_A} + \mathcal{O}(c^{-6}), \end{aligned} \tag{5.37}$$

where the quantities in the right-hand side of this expression are taken at the world line of the test body (A). Equation (5.37) is the usual form of the geodesic equation (Will, 1993; Brumberg, 1991) in the coordinates of an inertial RF_0. This result proves the previous conclusion that relation (5.35) is also the geodesic equation, simply written in the coordinates of the proper quasi-inertial RF_A.

5.4 The Proper RF of the Small Self-Gravitating Body.

In this subsection, we will discuss the transformation functions for the massive rotating test body with the small proper dimensions obtained in the previous parts of this section. In order to do this, let us note that the generalized Fermi conditions, eqs.(5.2), involve the first derivatives from the metric tensor, which gave us the differential equations of the second order on the transformation functions K_A, L_A, and Q^{α}_A. The expected form of the post-Newtonian expansions of the metric tensor in the proper RF_A, which resulted in condition $(B3a)$, enabled us (with the help of conditions (5.2)) to obtain the complete solution for the function K_A. However, the functions L and Q^{α} were only defined up to the second order with respect to the spatial point separation, namely: $L_A, Q^{\alpha}_A \sim \mathcal{O}(|y^{\alpha}_A|^3)$. This means that the arbitrariness due to the highest orders of the spatial point separation caused by the multipoles of higher orders than quadrupole $(k \geq 3)$ should be included in the expressions for these functions. Taking these notes into account, we should include in the final expressions for these functions the higher-order terms with respect to the spatial point separation. Then, the solutions for these functions, presented by relations (5.23) and (5.34), respectively, should be extended as follows:

$$\widehat{Q}^{\alpha}_{A_{[0]}}(y^0_A, y^{\nu}_A) = Q^{\alpha}_{A_{[0]}}(y^0_A, y^{\nu}_A) + \sum_{l\geq 3}^{k} Q^{\alpha}_{A\{L\}}(y^0_A)\cdot y^{\{L\}}_A + \mathcal{O}(|y^{\nu}_A|^{k+1}), \tag{5.38}$$

$$\widehat{L}_{A_{[0]}}(y_A^0, y_A^\nu) = L_{A_{[0]}}(y_A^0, y_A^\nu) + \sum_{l\geq 3}^{k} L_{A\{L\}}(y_A^0)\cdot y_A^{\{L\}} + \mathcal{O}(|y_A^\nu|^{k+1}). \tag{5.39}$$

As a result, the post-Newtonian dynamically non-rotating coordinate transformations from the coordinates of barycentrical inertial RF_0 to those of the proper quasi-inertial RF_A will take the following form:

$$x^0 = y_A^0 + c^{-2}K_{A_{[0]}}(y_A^0, y_A^\epsilon) + c^{-4}\widehat{L}_{A_{[0]}}(y_A^0, y_A^\epsilon) + \mathcal{O}(c^{-6})y_A^0, \tag{5.40a}$$

$$x^\alpha = y_A^\alpha + y_{A_{[0]}}^\alpha(y_A^0) + c^{-2}\widehat{Q}_{A_{[0]}}^\alpha(y_A^0, y_A^\epsilon) + \mathcal{O}(c^{-4})y_A^\alpha. \tag{5.40b}$$

The transformation functions K_A, Q_A^α, and L_A are given as follows:

$$K_{A_{[0]}}(y_A^0, y_A^\nu) = \int^{y_A^0} dt' \Big[\sum_{B\neq A}\Big\langle U_B\Big\rangle_{0'} - \frac{1}{2}v_{A_{[0]}\nu}v_{A_{[0]}}^\nu + \zeta_1^A\Big] - v_{A_{[0]}\nu}\cdot y_A^\nu + \mathcal{O}(c^{-4})y_A^0, \tag{5.41a}$$

$$\begin{aligned}
\widehat{Q}_{A_{[0]}}^\alpha(y_A^0, y_A^\nu) = &-\sum_{B\neq A}\Big[y_A^\alpha y_A^\beta\cdot\Big\langle\frac{\partial U_B}{\partial y_A^\beta}\Big\rangle_0 - \frac{1}{2}\gamma^{\alpha\sigma}y_{A\beta}y_A^\beta\cdot\Big\langle\frac{\partial U_B}{\partial y_A^\sigma}\Big\rangle_0 + y_A^\alpha\cdot\Big\langle U_B\Big\rangle_0\Big] + \\
&+y_{A\beta}\int^{y_A^0}dt'\Big[\frac{1}{2}a_{A_{[0]}}^{[\alpha}v_{A_{[0]}}^{\beta]} + 2\sum_{B\neq A}\Big\langle\partial^{[\alpha}V_B^{\beta]}\Big\rangle_{0'}\Big] - \frac{1}{2}v_{A_{[0]}}^\alpha v_{A_{[0]}\beta}\cdot y_A^\beta + \\
&+f_{A\beta}^\alpha\cdot y_A^\beta + w_{A_{[0]}}^\alpha(y_A^0) + \sum_{l\geq 3}^{k}Q_{A\{L\}}^\alpha(y_A^0)\cdot y_A^{\{L\}} + \mathcal{O}(|y_A^\nu|^{k+1}) + \mathcal{O}(c^{-4})y_A^\alpha,
\end{aligned} \tag{5.41b}$$

$$\begin{aligned}
\widehat{L}_{A_{[0]}}(y_A^0, y_A^\nu) = &\sum_{B\neq A}\Big(\frac{1}{2}y_{A\beta}y_A^\beta\cdot\Big\langle\frac{\partial U_B}{\partial y_A^0}\Big\rangle_0 - 2y_A^\lambda y_A^\beta\cdot\Big\langle\partial_\lambda V_{B\beta}\Big\rangle_0 + \\
&+v_{A_{[0]}\beta}\Big[y_A^\beta y_A^\lambda\cdot\Big\langle\frac{\partial U_B}{\partial y_A^\lambda}\Big\rangle_0 - \frac{1}{2}\gamma^{\beta\sigma}y_{A_\lambda}y_A^\lambda\cdot\Big\langle\frac{\partial U_B}{\partial y_A^\sigma}\Big\rangle_0\Big]\Big) + \\
&+v_{A_{[0]}\beta}y_{A_\lambda}\int^{y_A^0}dt'\Big[\frac{1}{2}a_{A_{[0]}}^{[\lambda}v_{A_{[0]}}^{\beta]} + 2\sum_{B\neq A}\Big\langle\partial^{[\lambda}V_B^{\beta]}\Big\rangle_{0'}\Big] + \\
&+y_{A_\beta}\Big[2v_{A_{[0]}}^\beta\sum_{B\neq A}\Big\langle U_B\Big\rangle_0 - 4\sum_{B\neq A}\Big\langle V_B^\beta\Big\rangle_0 - \dot{w}_{A_{[0]}}^\beta(y_A^0) + v_{A_{[0]}}^\beta\cdot\zeta_1^A + \\
&+\sigma_A^\beta - v_{A_{[0]}\lambda}\cdot f_A^{\beta\lambda} + \Big(\sigma_A^{\beta\lambda} - 2\gamma^{\beta\lambda}\zeta_1^A\Big)\int^{y_A^0}dt'\cdot\Big\langle\frac{\partial U_A}{\partial y_A^\lambda}\Big\rangle_{0'}\Big] + \\
&+\int^{y_A^0}dt'\Big[-\sum_{B\neq A}\Big\langle W_B\Big\rangle_{0'} - \frac{1}{2}\Big(\sum_{B\neq A}\Big\langle U_B\Big\rangle_{0'} - \frac{1}{2}v_{A_{[0]}\beta}v_{A_{[0]}}^\beta + \zeta_1^A\Big)^2 - \\
&-v_{A_{[0]}\mu}\dot{w}_{A_{[0]}}^\mu(t') + \Big(\sigma_A^{\mu\lambda} - 2\gamma^{\mu\lambda}\zeta_1^A\Big)\int^{t'}dt''\Big\langle\frac{\partial U_A}{\partial y_A^\mu}\Big\rangle_{0''}\int^{t''}dt'''\Big\langle\frac{\partial U_A}{\partial y_A^\lambda}\Big\rangle_{0'''} + \\
&+\zeta_2^A - 2\zeta_1^A\cdot\Big\langle U_A\Big\rangle_{0'} + \sigma_A^\mu\cdot\int^{t'}dt''\Big\langle\frac{\partial U_A}{\partial y_A^\mu}\Big\rangle_{0''}\Big] + \\
&+\sum_{l\geq 3}^{k}L_{A\{L\}}(y_A^0)\cdot y_A^{\{L\}} + \mathcal{O}(|y_A^\nu|^{k+1}) + \mathcal{O}(c^{-6}).
\end{aligned} \tag{5.41c}$$

with the equations for both time-dependent functions $y^{\alpha}_{A_{[0]}}(y^0_A)$ and $w^{\alpha}_{A_{[0]}}(y^0_A)$ defined by equations (5.12) and (5.35), respectively.

At this point, we are ready to present the general form of the metric tensor in the proper RF_A defined with the generalized Fermi conditions. Thus, by substituting the solutions obtained for the functions K_A, L_A, and Q^{α}_A into the general form of the metric tensor $g_{mn}(y^p_A)$ in a proper RF_A given by the relations in eqs.(4.11), we will obtain this tensor in the following form:

$$
\begin{aligned}
g^A_{00}(y^p_A) = \; & 1 - 2\Big(\sum_B U_B(y^0_A, y^\nu_A) - \sum_{B\neq A} \Big[y^\mu_A \Big\langle \frac{\partial U_B}{\partial y^\mu_A} \Big\rangle_0 + \Big\langle U_B \Big\rangle_0 \Big] + \zeta_{1A} \Big) + \\
& +2\Big(\sum_B W_B(y^0_A, y^\nu_A) - \sum_{B\neq A} \Big[y^\mu_A \Big\langle \frac{\partial W_B}{\partial y^\mu_A} \Big\rangle_0 + \Big\langle W_B \Big\rangle_0 \Big] + \zeta_{2A} \Big) + \\
& + y^\delta_A y^\beta_A \cdot \Big[\gamma_{\delta\beta}\, a_{A_{[0]}\lambda} a^\lambda_{A_{[0]}} - a_{A_{[0]}\delta} a_{A_{[0]}\beta} + \sum_{B\neq A} \frac{\partial}{\partial y^0_A} \Big(\gamma_{\delta\beta} \Big\langle \frac{\partial U_B}{\partial y^0_A} \Big\rangle_0 - 4 \Big\langle \frac{\partial V_{B\beta}}{\partial y^\delta_A} \Big\rangle_0 \Big) \Big] + \\
& +2 \sum_{l\geq 3}^{k} \Big(\frac{\partial}{\partial y^0_A} L_{A\{L\}}(y^0_A) + v_{A_{[0]}\beta} \cdot \frac{\partial}{\partial y^0_A} Q^\beta_{A\{L\}}(y^0_A) \Big) \cdot y^{\{L\}}_A + \\
& + \Big(\sigma^{\mu\lambda}_A - 2\gamma^{\mu\lambda}\zeta^A_1 \Big) \Big(y_{A\mu} \cdot \Big\langle \frac{\partial U_A}{\partial y^\lambda_A} \Big\rangle_0 + \int^{y^0_A} dt' \Big\langle \frac{\partial U_A}{\partial y^\mu_A} \Big\rangle_{0'} \int^{t'} dt'' \Big\langle \frac{\partial U_A}{\partial y^\lambda_A} \Big\rangle_{0''} \Big) + \\
& + \sigma^\mu_A \cdot \int^{y^0_A} dt' \Big\langle \frac{\partial U_A}{\partial y^\mu_A} \Big\rangle_{0'} - 2\zeta^A_1 \cdot \Big\langle U_A \Big\rangle_0 + \mathcal{O}(|y^\nu_A|^{k+1}) + \mathcal{O}(c^{-6}),
\end{aligned}
\tag{5.42a}
$$

$$
\begin{aligned}
g^A_{0\alpha}(y^p_A) = \; & 4\gamma_{\alpha\epsilon} \Big(\sum_B V^\epsilon_B(y^0_A, y^\nu_A) - \sum_{B\neq A} \Big[y^\mu_A \Big\langle \frac{\partial V^\epsilon_B}{\partial y^\mu_A} \Big\rangle_0 + \Big\langle V^\epsilon_B \Big\rangle_0 \Big] + \sigma^\epsilon_A \Big) - \\
& - \frac{1}{2} \Big(\gamma_{\alpha\epsilon}\delta^\beta_\lambda + \gamma_{\alpha\lambda}\delta^\beta_\epsilon - \gamma_{\epsilon\lambda}\delta^\beta_\alpha \Big)\, y^\epsilon_A y^\lambda_A \cdot \sum_{B\neq A} \frac{\partial}{\partial y^0_A} \Big\langle \frac{\partial U_B}{\partial y^\beta_A} \Big\rangle_0 + \\
& + \sum_{l\geq 3}^{k} \Big(\gamma_{\alpha\lambda} \frac{\partial}{\partial y^0_A} Q^\lambda_{A\{L\}}(y^0_A) + \Big[L_{A\{L\}}(y^0_A) + v_{A_{[0]}\beta} \cdot Q^\beta_{A\{L\}}(y^0_A) \Big] \frac{\partial}{\partial y^\alpha_A} \Big) \cdot y^{\{L\}}_A + \\
& + \gamma_{\alpha\mu} \Big(\sigma^{\mu\lambda}_A - 2\gamma^{\mu\lambda}\zeta^A_1 \Big) \int^{y^0_A} dt' \Big\langle \frac{\partial U_A}{\partial y^\lambda_A} \Big\rangle_{0'} + \mathcal{O}(|y^\nu_A|^{k+1}) + \mathcal{O}(c^{-5}),
\end{aligned}
\tag{5.42b}
$$

$$
\begin{aligned}
g^A_{\alpha\beta}(y^p_A) = \; & \gamma_{\alpha\beta} + 2\gamma_{\alpha\beta} \Big(\sum_B U_B(y^0_A, y^\nu_A) - \sum_{B\neq A} \Big[y^\mu_A \Big\langle \frac{\partial U_B}{\partial y^\mu_A} \Big\rangle_0 + \Big\langle U_B \Big\rangle_0 \Big] \Big) + \sigma_{A\alpha\beta} + \\
& + \sum_{l\geq 3}^{k} \Big(\gamma_{\alpha\lambda} Q^\lambda_{A\{L\}}(y^0_A) \frac{\partial}{\partial y^\beta_A} + \gamma_{\beta\lambda} Q^\lambda_{A\{L\}}(y^0_A) \frac{\partial}{\partial y^\alpha_A} \Big) \cdot y^{\{L\}}_A + \mathcal{O}(|y^\nu_A|^{k+1}) + \mathcal{O}(c^{-4}),
\end{aligned}
\tag{5.42c}
$$

where the subscript (A) for the components of the metric tensor specifies that this tensor was obtained by making use of the specifically defined transformation functions (5.41). The expressions for the functions W_A and W_B were obtained by substituting the solutions for the transformation

functions into the relations for $H_{00}^{A<4>}$ and $H_{00}^{B<4>}$ given by eqs.(4.14) and (4.16), correspondingly. These functions have the following form:

$$W_A(y_A^p) = U_A^2(y_A^p) + \Psi_A(y_A^p) + \frac{1}{2}\frac{\partial^2}{\partial y_A^{0\,2}}\chi_A(y_A^p) + 2a_{A_{[0]}}^{\lambda} \cdot \frac{\partial}{\partial y_A^{\lambda}}\chi_A(y_A^p) +$$

$$+2\sum_{B\neq A}\Big(U_A(y_A^p)U_B(y_A^p) - \int_A \frac{d^3y'_A}{|y_A^{\nu} - y_A'^{\nu}|}\rho_A(y_A^0, y_A'^{\nu})U_B(y_A^0, y_A'^{\nu})\Big) +$$

$$+\sum_{l\geq 3}^{k} Q_{A\{L\}}^{\lambda}(y_A^0)\int_A d^3y'_A\rho_A(y_A^0, y_A'^{\nu})\frac{\partial}{\partial y_A'^{\lambda}}\Big[\frac{y_A^{\{L\}} - y_A'^{\{L\}}}{|y_A^{\nu} - y_A'^{\nu}|}\Big] -$$

$$-f_A^{\lambda\beta} \cdot \frac{\partial^2}{\partial y_A^{\lambda}\partial y_A^{\beta}}\chi_A(y_A^0, y_A^{\nu}) - 2\zeta_{1A} \cdot U_A(y_A^0, y_A^{\nu}) + O(|y_A^{\nu}|^{k+1}) + \mathcal{O}(c^{-6}). \qquad (5.43a)$$

$$W_B(y_A^p) = U_B(y_A^p)\sum_{B'\neq A} U_{B'}(y_A^p) + \Psi_B(y_A^p) + \frac{1}{2}\frac{\partial^2}{\partial y_A^{0\,2}}\chi_B(y_A^p) + 2a_{A_{[0]}}^{\lambda}(y_A^0) \cdot \frac{\partial}{\partial y_A^{\lambda}}\chi_B(y_A^0, y_A^{\nu}) -$$

$$-2\int_B \frac{d^3y'_A}{|y_A^{\nu} - y_A'^{\nu}|}\rho_B(y_A^0, y_A'^{\nu})\sum_{B'} U_{B'}(y_A^0, y_A'^{\nu}) +$$

$$+\sum_{l\geq 3}^{k} Q_{A\{L\}}^{\lambda}(y_A^0) \cdot \int_B d^3y'_A\rho_B\Big(y_A^0, y_A'^{\nu} + y_{BA_0}^{\nu}(y_A^0)\Big)\frac{\partial}{\partial y_A'^{\lambda}}\Big[\frac{y_A^{\{L\}} - y_A'^{\{L\}}}{|y_A^{\nu} - y_A'^{\nu}|}\Big] -$$

$$-f_A^{\lambda\beta} \cdot \frac{\partial^2}{\partial y_A^{\lambda}\partial y_A^{\beta}}\chi_B(y_A^0, y_A^{\nu}) - 2\zeta_{1A} \cdot U_B(y_A^0, y_A^{\nu}) + O(|y_A^{\nu}|^{k+1}) + \mathcal{O}(c^{-6}). \qquad (5.43b)$$

Expressions (5.42) are the general solution for the field equations of the general theory of relativity, which satisfies the generalized Fermi conditions, eqs.(5.2), in the immediate vicinity of body (A). This solution reflects the geometrical features of the proper RF_A with respect to the special properties of the motion of the kth multipoles of the unknown functions $L_{A\{L\}}(y_A^0)$ and $Q_{A\{L\}}^{\nu}(y_A^0)$ for $l \geq 3$, which will be discussed further.

The transformation functions in eq.(5.41) correspond to non-rotating coordinate transformations between different RFs in the WFSMA. They were obtained by applying the generalized Fermi conditions in eqs.(5.2). The set of the resulting formulae, eqs.(5.41) together with eqs.(5.12) and (5.35), represents the generalization of the Poincaré group of motion to the problem of practical celestial mechanics. The arbitrary constants $\zeta_A = c^{-2}\zeta_1^A + c^{-4}\zeta_2^A$, σ_A^{α}, and $f_A^{\alpha\beta}$ correspond to the maximal number of Killing vectors ($M = 10$) in the background pseudo-Euclidean space-time, and the expressions (5.40)–(5.41) represent the ten-parameter group of motion constructed for the dynamic of the celestial bodies in the WFSMA. The non-zero parameters describe the shift of the origin of the coordinate system, the constant spatial rotation of the axes, and the relativistic Poincaré rotation. These parameters represent the offset of the origin of the coordinate system from the center of the field of the body under consideration, which may vary from body to body. Moreover, these parameters lead to the appearance of the proper gravitational potential

U_A and its gradients $\partial^\alpha U_A$ in the function L_A (5.41c). A contribution of this sort could be a useful tool for some practical applications of the atomic time comparison (Brumberg, 1991a). This dependence indicates the fact that the constant part of the proper gravity of the body (A) is also affecting the definition of its world line. This contribution may be neglected if one will choose these constants in such a way that this influence of the proper field will vanish. In addition, let us mention that the component of the metric tensor g^A_{00} also becomes dependent on these quantities describing the proper gravitational field, which violates the conditions on the metric tensor and the coordinate transformations to the proper RF$_A$ given in Section 1. Therefore, without losing a generality, in our future calculations, we will eliminate this offset and will set all of these parameters to be zero:

$$\zeta_A = c^{-2}\zeta^A_1 + c^{-4}\zeta^A_2 = \sigma^\alpha_A = f^{\alpha\beta}_A = 0. \tag{5.44}$$

In order to find the unknown functions $Q^\alpha_{A\{L\}}(y^0_A)$ and $L_{A\{L\}}(y^0_A)$ up to the kth ($k \geq 3$) order, one should use the conditions that will contain the spatial derivatives from the metric tensor of the $(k-1)$ order. Moreover, one expects to obtain the recurrent formulae that would connect the features of transformation of an arbitrary k^{th} term with those for the previous $(k-1)$ terms. Thus, following Synge (1960), one may want to apply some non-local geometrical constructions, such as Jacobi equations (Manasse & Misner, 1963) or both Jacobi equations and the Fermi–Walker transport (Li & Ni, 1979a,b). However, these constraints generally are not related to the particular theory under consideration, so their application should be justified for the particular theory of gravity under question. Another method is to use the 'external' multipole moments as they were defined for the gravitational wave theory by Thorne (1980) or Blanchet & Damour (1986, 1989). Indeed, one could show that the functions $Q^\alpha_{A\{L\}}(y^0_A)$ and $L_{A\{L\}}(y^0_A)$ in the WFSMA may be chosen in such a way that the metric tensor in a proper RF$_A$, eqs.(4.11), corresponding to this choice will accept the desired form. The presentation of the transformation functions in terms of the 'external' multipole moments simply corresponds to the specific RF for which KLQ dynamical parameterization is strictly defined by this choice.

5.5 The Fermi-Normal-Like Coordinates.

As we noticed above, in order to determine the metric up to the kth multipole contribution, one should apply some additional conditions that enable us to define the specific properties of the reference frame with which we will be dealing. For example, we might obtain these functions for the case of the motion of the monopole test particle up to the second order of a spatial point separation. Assuming the motion of that particle is described by the geodesic equation and the deviation of geodesics is governed by the Jacobi equation, we might easily obtain the metric tensor in the generalized Fermi normal coordinates (Misner and Manasse, 1963; Li & Ni, 1979; Dolgov, Khriplovich 1983; Ashby & Bertotti, 1986; Marzlin, 1994) up to the second order of the spatial separation and present it as follows:

$$g^{\mathcal{F}}_{00}(y^p_A) = 1 + H^A_{00}(y^p_A) + \left\langle R^B_{0\mu0\nu}\right\rangle_0 \cdot y^\mu_A y^\nu_A + \mathcal{O}(c^{-6}) + \mathcal{O}(|y^\nu_A|^3), \tag{5.45a}$$

$$g^{\mathcal{F}}_{0\alpha}(y^p_A) = H^A_{0\alpha}(y^p_A) + \frac{2}{3}\left\langle R^B_{0\mu\alpha\nu}\right\rangle_0 \cdot y^\mu_A y^\nu_A + \mathcal{O}(c^{-5}) + \mathcal{O}(|y^\nu_A|^3), \tag{5.45b}$$

$$g^{\mathcal{F}}_{\alpha\beta}(y^p_A) = \gamma_{\alpha\beta} + H^A_{\alpha\beta}(y^p_A) + \frac{1}{3}\left\langle R^B_{\alpha\mu\beta\nu}\right\rangle_0 \cdot y^\mu_A y^\nu_A + \mathcal{O}(c^{-4}) + \mathcal{O}(|y^\nu_A|^3), \tag{5.45c}$$

where $\left\langle R^B_{mnkl}\right\rangle_0$ are the components of the Riemann tensor, eqs.($G9$), which is calculated with respect to the external gravitational field H^B_{mn} and taken on the world line γ_A of the body (A) under consideration.

Let us mention that if the proper gravitational field may be neglected and the effects due to acceleration of the proper RF$_A$ are also negligible, the obtained metric tensor (5.45) will correspond to that of, so-called, Fermi normal coordinates constructed in the immediate vicinity of the world line of an inertial observer (Misner *et al.*, 1973). However, for the general case of non-vanishing contributions of the proper gravitational field and accelerated barycentric motion, the form of the metric tensor, $g^{\mathcal{F}}_{mn}$ (5.45), and the corresponding proper RF is what will be referred to as the Fermi-normal-like coordinates. From these expressions for the metric tensor $g^{\mathcal{F}}_{mn}$, one may see that, in order to obtain this form of the metric tensor, it is necessary to perform the coordinate transformation that should contain the terms with the third order of the spatial point separation (Li & Ni, 1979a,b; Zhang, 1985, 1986). We will obtain the necessary equations on these functions by making use of the components of the Riemann tensor $R_{mnkl}(y^p_A)$ expanded with respect to the spatial separation from the world line of the body (A) and then equating the coefficients proportional to $\sim y^\mu_A y^\nu_A$.

Thus, the components of the Riemann tensor calculated with respect to the external gravitational field H^B_{mn} from the relations in eqs.($G9$) might be presented on a world line of the body (A) as follows:

$$\left\langle R^B_{0\mu0\nu}\right\rangle_0 = \sum_{B\neq A}\Bigg(-\Big\langle\frac{\partial^2 U_B}{\partial y^\mu_A\partial y^\nu_A}\Big\rangle_0 + \gamma_{\mu\nu}\frac{\partial}{\partial y^0_A}\Big[\Big\langle\frac{\partial U_B}{\partial y^0_A}\Big\rangle_0 - 2\Big(\Big\langle\frac{\partial V_{B\mu}}{\partial y^\nu_A}\Big\rangle_0 + \Big\langle\frac{\partial V_{B\nu}}{\partial y^\mu_A}\Big\rangle_0\Big)\Big] +$$

$$+\Big\langle\frac{\partial^2 W_B}{\partial y^\mu_A\partial y^\nu_A}\Big\rangle_0\Bigg) + \gamma_{\mu\nu}\, a_{A_{[0]}\lambda}a^\lambda_{A_{[0]}} - a_{A_{[0]}\mu}a_{A_{[0]}\nu} + \mathcal{O}(c^{-6}), \tag{5.46a}$$

$$\left\langle R^B_{0\mu\alpha\nu}\right\rangle_0 = \gamma_{\mu\alpha}\dot{a}_{A_{[0]}\nu} - \gamma_{\mu\nu}\dot{a}_{A_{[0]}\alpha} +$$

$$+2\sum_{B\neq A}\Big(\gamma_{\alpha\lambda}\Big\langle\frac{\partial^2 V^\lambda_B}{\partial y^\mu_A\partial y^\nu_A}\Big\rangle_0 - \gamma_{\nu\lambda}\Big\langle\frac{\partial^2 V^\lambda_B}{\partial y^\mu_A\partial y^\alpha_A}\Big\rangle_0\Big) + \mathcal{O}(c^{-5}), \tag{5.46b}$$

$$\left\langle R^B_{\alpha\mu\beta\nu}\right\rangle_0 = \sum_{B\neq A}\Big(\gamma_{\alpha\beta}\Big\langle\frac{\partial^2 U_B}{\partial y^\mu_A\partial y^\nu_A}\Big\rangle_0 + \gamma_{\mu\nu}\Big\langle\frac{\partial^2 U_B}{\partial y^\alpha_A\partial y^\beta_A}\Big\rangle_0 -$$

$$-\gamma_{\beta\mu}\Big\langle\frac{\partial^2 U_B}{\partial y^\alpha_A\partial y^\nu_A}\Big\rangle_0 - \gamma_{\alpha\nu}\Big\langle\frac{\partial^2 U_B}{\partial y^\mu_A\partial y^\beta_A}\Big\rangle_0\Big) + \mathcal{O}(c^{-4}). \tag{5.46c}$$

To find the necessary corrections of the third order of the transformation functions Q^α_A and L_A, let us look in the following form:

$$\delta_{\nu_3}Q^\alpha_{A_{[0]}}(y^0_A, y^\nu_A) =$$

$$= \sum_{B\neq A}\Big[c_1\cdot\gamma^{\alpha\sigma}y^\mu_A y_{A\mu}y^\nu_A\cdot\Big\langle\frac{\partial^2 U_B}{\partial y^\nu_A\partial y^\sigma_A}\Big\rangle_0 + c_2\cdot y^\alpha_A y^\mu_A y^\lambda_A\cdot\Big\langle\frac{\partial^2 U_B}{\partial y^\mu_A\partial y^\lambda_A}\Big\rangle_0\Big] + \mathcal{O}(|y^\nu_A|^4), \tag{5.47a}$$

$$\delta_{\nu_3} L_{A_{[0]}}(y_A^0, y_A^\nu) = \sum_{B\neq A} \Big[q_1 \cdot y_A^\mu y_{A\mu} y_A^\lambda \Big\langle \frac{\partial^2 U_B^*}{\partial y_A^0 \partial y_A^\lambda} \Big\rangle_0 + q_2 \cdot y_A^\mu y_A^\nu y_A^\lambda \Big\langle \frac{\partial^2 V_{B\lambda}}{\partial y_A^\mu \partial y_A^\nu} \Big\rangle_0 \Big] -$$

$$- v_{A_{[0]}\beta} \cdot \delta_{\nu_3} Q^\beta_{A_{[0]}}(y_A^0, y_A^\nu) + \mathcal{O}(|y_A^\nu|^4), \tag{5.47b}$$

where the constants c_1, c_2 and q_1, q_2 are unknown at the moment.

The expressions for the components of the metric tensor $g^{\mathcal{F}}_{mn}$, eqs.(5.45), and those for the Riemann tensor, eqs.(5.46), will enable us to obtain the equations for the determination of the constants c_1, c_2 and q_1, q_2. Thus, from the component $g^{\mathcal{F}}_{\alpha\beta}$, eq.(5.45$c$), and the relation for the $\Big\langle R^B_{\alpha\mu\beta\nu} \Big\rangle_0$, eq.(5.46$c$), we will have

$$2c_1 + 1 = \frac{1}{3}, \quad 2c_2 = \frac{1}{3}, \qquad 2(c_1 + c_2) = -\frac{1}{3},$$

which will give the following values for these constants:

$$c_1 = -\frac{1}{3}, \qquad c_2 = \frac{1}{6}. \tag{5.48}$$

Analogously, from the component $g^{\mathcal{F}}_{0\alpha}$, eq.(5.45$b$); the relation for $\Big\langle R^B_{0\mu\alpha\sigma} \Big\rangle_0$, eq.(5.46$b$); and the solution for function $\delta_{\nu_3} Q^\alpha_{A_{[0]}}$ given by eq.(5.46a), with the obtained c_1 and c_2, eq.(5.47), we obtain

$$2q_1 - 1 = -\frac{2}{3}; \quad q_1 + \frac{1}{2} = \frac{2}{3} \quad \Rightarrow \quad q_1 = \frac{1}{6},$$

$$2q_2 = -\frac{4}{3}; \quad q_2 + 2 = \frac{4}{3} \quad \Rightarrow \quad q_2 = -\frac{2}{3}. \tag{5.49}$$

Taking these results into account, the corrections up to the third order with respect to the spatial point separation to the solutions for Q^α_A and L_A, presented by eqs.(5.47), will take the following form:

$$\delta_{\nu_3} Q^\alpha_{A_{[0]}}(y_A^0, y_A^\nu) = \frac{1}{6} \sum_{B\neq A} \Big[\gamma^{\alpha\sigma} y_A^\mu y_{A\mu} y_A^\nu \cdot \Big\langle \frac{\partial^2 U_B}{\partial y_A^\nu \partial y_A^\sigma} \Big\rangle_0 -$$

$$-2 \cdot y_A^\alpha y_A^\mu y_A^\lambda \cdot \Big\langle \frac{\partial^2 U_B}{\partial y_A^\mu \partial y_A^\lambda} \Big\rangle_0 \Big] + \mathcal{O}(|y_A^\nu|^4), \tag{5.50a}$$

$$\delta_{\nu_3} L_{A_{[0]}}(y_A^0, y_A^\nu) = \frac{1}{6} \sum_{B\neq A} \Big(y_A^\mu y_{A\mu} y_A^\lambda \Big\langle \frac{\partial^2 U_B}{\partial y_A^0 \partial y_A^\lambda} \Big\rangle_0 - 4 \cdot y_A^\mu y_A^\nu y_A^\lambda \Big\langle \frac{\partial^2 V_{B\lambda}}{\partial y_A^\mu \partial y_A^\nu} \Big\rangle_0 \Big) -$$

$$- v_{A_{[0]}\beta} \Big(\gamma^{\beta\sigma} y_A^\mu y_{A\mu} y_A^\nu \cdot \Big\langle \frac{\partial^2 U_B^*}{\partial y_A^\nu \partial y_A^\sigma} \Big\rangle_0 - 2 \cdot y_A^\beta y_A^\mu y_A^\lambda \cdot \Big\langle \frac{\partial^2 U_B^*}{\partial y_A^\mu \partial y_A^\lambda} \Big\rangle_0 \Big) + \mathcal{O}(|y_A^\nu|^4). \tag{5.50b}$$

By substituting these solutions into the expressions of eqs.(5.42), one might get the metric tensor in a proper RF_A of the moving extended body (A) with accuracy up to the second order of the spatial point separation. Thus, assuming that all the integration constants satisfy eq.(5.44), one may get the following form of the metric tensor in the generalized Fermi normal coordinates:

$$g^{\mathcal{F}}_{00}(y^p_A) = 1 - 2U_A(y^p_A) + 2W_A(y^p_A) +$$

$$+\Big(\sum_{B\neq A}\Big[-\Big\langle\frac{\partial^2 U_B}{\partial y^\mu_A \partial y^\nu_A}\Big\rangle_0 + \Big\langle\frac{\partial^2 W_B}{\partial y^\mu_A \partial y^\nu_A}\Big\rangle_0 + \frac{\partial}{\partial y^0_A}\Big(\gamma_{\mu\nu}\Big\langle\frac{\partial U_B}{\partial y^0_A}\Big\rangle_0 - 4\Big\langle\frac{\partial V_{B\mu}}{\partial y^\nu_A}\Big\rangle_0\Big)\Big] +$$

$$+\gamma_{\mu\nu}\, a_{A_{[0]}\lambda} a^\lambda_{A_{[0]}} - a_{A_{[0]}\mu} a_{A_{[0]}\nu}\Big)\cdot y^\mu_A y^\nu_A + \mathcal{O}(c^{-6}) + \mathcal{O}(|y^\nu_A|^3), \tag{5.51a}$$

$$g^{\mathcal{F}}_{0\alpha}(y^p_A) = 4\gamma_{\alpha\epsilon} V^\epsilon_A(y^p_A) + \frac{2}{3}\Big(2\sum_{B\neq A}\Big[\gamma_{\alpha\lambda}\Big\langle\frac{\partial^2 V^\lambda_B}{\partial y^\mu_A \partial y^\nu_A}\Big\rangle_0 - \gamma_{\nu\lambda}\Big\langle\frac{\partial^2 V^\lambda_B}{\partial y^\mu_A \partial y^\alpha_A}\Big\rangle_0\Big] +$$

$$+\gamma_{\mu\alpha}\dot{a}_{A_{[0]}\nu} - \gamma_{\mu\nu}\dot{a}_{A_{[0]}\alpha}\Big)\cdot y^\mu_A y^\nu_A + \mathcal{O}(|y^\nu_A|^3) + \mathcal{O}(c^{-5}), \tag{5.51b}$$

$$g^{\mathcal{F}}_{\alpha\beta}(y^p_A) = \gamma_{\alpha\beta} + 2\gamma_{\alpha\beta} U_A(y^p_A) + \frac{1}{3}\Big(\sum_{B\neq A}\Big[\gamma_{\alpha\beta}\Big\langle\frac{\partial^2 U_B}{\partial y^\mu_A \partial y^\sigma_A}\Big\rangle_0 + \gamma_{\mu\sigma}\Big\langle\frac{\partial^2 U_B}{\partial y^\alpha_A \partial y^\beta_A}\Big\rangle_0 -$$

$$-\gamma_{\beta\mu}\Big\langle\frac{\partial^2 U_B}{\partial y^\alpha_A \partial y^\sigma_A}\Big\rangle_0 - \gamma_{\alpha\sigma}\Big\langle\frac{\partial^2 U_B}{\partial y^\mu_A \partial y^\beta_A}\Big\rangle_0\Big]\Big)\cdot y^\mu_A y^\sigma_A + \mathcal{O}(|y^\nu_A|^3) + \mathcal{O}(c^{-4}). \tag{5.51c}$$

Thus we have obtained the form of the metric tensor in Fermi-normal-like coordinates and the coordinate transformation, which leads to this form as well. These transformations are defined up to the third order with respect to the spatial point separation.

A more detailed analysis of the coordinate transformation for the extended self-gravitating bodies will be performed in the next section.

6 General Relativity: The Proper RF for the Extended Body.

In this section, we will generalize the results obtained for the relativistic coordinate transformations (5.40) and will extend their applicability to the problem of motion of a system of N extended bodies in the WFSMA. The relations (5.40) were obtained by using the generalized Fermi conditions (3.26) and, hence, they are well suited to describe the motion of the system of N self-gravitating bodies, omitting only the lowest intrinsic multipole moments. To generalize these results in the case of arbitrarily shaped extended bodies, we must use the more general definition of the proper RF given by expressions (3.29). This definition is based on the study of the existence of integral conservation laws for metric theories of gravity, (3.28). The studies of the existence of the conservation laws in general relativity were performed by a number of scientists, notably by Fock (1955) and Chandrasekhar (1965), whose methods were developed in application to the motion of the more general N-body systems in the framework of the PPN formalism (Lee *et al.*, 1974; Denisov & Turyshev, 1989; Will, 1993). It should be noted that the search for the conservation laws in these methods was performed in the barycentric inertial RF_0 and, in particular, it was shown that general relativity in the WFSMA has all ten conservation laws for the closed system of fields corresponding to energy of the system, its momentum, and angular momentum. The difference of the present research from that cited above is in the fact that we will study the problem of existence of the integral conservation laws in an accelerated arbitrary KLQ-parameterized proper RF_A. As a result of our study, we should find the conditions necessary to impose on the transformation functions K_A, L_A, and Q^α_A, so that the general relativity in the coordinates of this RF will preserve the existent conservation laws for the entire system under consideration.

6.1 The Extended-Body Generalization.

It is well known that in all metric theories of gravity the Lagrangian density of matter is the same functional of metric of Riemann space-time g_{mn} and the other fields of matter ψ_A. Then the application of the method of infinitesimal displacements (Bogolyubov & Shirkov, 1984; Logunov, 1987) to the action function of matter in these theories, together with the condition that the eq.m. for the fields ψ_A are satisfied, leads to the same covariant equation for the conservation of density of the energy-momentum tensor of matter in Riemann space-time:

$$\nabla_k \hat{T}^{mk} = \partial_k \hat{T}^{mk} + \Gamma^m_{lp}\hat{T}^{lp} = 0. \tag{6.1}$$

Note that this result is independent of the choice of RF. In the case of a system of bodies formed from an ideal fluid with the individual density of energy-momentum tensor $\hat{T}^{mn}_B$ of an arbitrary body (B), which is given in the coordinates of its proper RF_B by the expression (2.1) as

$$\hat{T}^{mn}_B(y^p_B) = \sqrt{-g}_B\Big(\Big[\rho_{B_0}(1+\Pi)+p\Big]u^m u^n - p g^{mn}_B\Big), \tag{6.2a}$$

the total density of the energy-momentum tensor of the system of N bodies in the coordinates (y^p_A) of the proper RF_A of a particular body (A) may then be composed as follows:

$$\hat{T}^{mn}(y^p_A) = \sum_B J_B(y^p_A)\frac{\partial y^m_A}{\partial y^k_B}\frac{\partial y^n_A}{\partial y^l_B}\hat{T}^{kl}_B(y^q_B(y^p_A)), \tag{6.2b}$$

where J_B is the Jacobian of the corresponding coordinate transformation:

$$J_B(y^p_A) = \det||\frac{\partial y^m_B}{\partial y^k_A}||. \tag{6.2c}$$

In addition, from equation (6.1) for an ideal fluid model, (6.2), we may also obtain a covariant equation of continuity in the coordinates (y_A^p) as follows:

$$\nabla_k^A\Big[\sum_B \rho_{B_0} u^k\Big] = \frac{1}{\sqrt{-g}_A}\Big[\frac{\partial\overline{\rho}}{\partial y_A^0} + \frac{\partial(\overline{\rho}v^\mu)}{\partial y_A^\mu}\Big] = 0, \tag{6.3}$$

where ∇_k^A is the covariant derivative with respect to metric tensor g_{mn}^A of the proper RF_A. The total conserved mass density of the entire system in coordinates (y_A^p) is denoted as

$$\overline{\rho}(y_A^p) = \sum_B \rho_{B_0}\sqrt{-g}_B u_B^0 = \sum_B \hat{\rho}_B(y_A^p) J_B^{-1}\frac{dy_B^0}{dy_A^0}, \tag{6.4}$$

where $\hat{\rho}_B$ is the conserved mass density of the body (B) and all the quantities on the right-hand side of this expression are transformed to the coordinates (y_A^p) using the standard rules of relativistic transformations of the mechanics of Poincaré (Fock, 1955). Equations (6.1) and (6.3) together with the metric tensor give all the expressions necessary for the construction of the eq.m. of the extended bodies composed from ideal fluid and for analysis of various general questions.

In order to generalize the results obtained in the previous section, in the case of arbitrarily composed extended bodies, we shall first construct the components of the density of the energy-momentum tensor of matter $\hat{T}^{mn}$ to the required accuracy. Thus, from the definition in (6.2), one may get these components in the Newtonian approximation as follows:

$$\hat{T}^{00}(y_A^p) = \overline{\rho}\Big(1 + \mathcal{O}(c^{-2})\Big), \tag{6.5a}$$

$$\hat{T}^{0\alpha}(y_A^p) = \overline{\rho}v^\alpha\Big(1 + \mathcal{O}(c^{-2})\Big), \tag{6.5b}$$

$$\hat{T}^{\alpha\beta}(y_A^p) = \overline{\rho}v^\alpha v^\beta - \gamma^{\alpha\beta}p + \hat{\rho}\mathcal{O}(c^{-4}). \tag{6.5c}$$

As a result, the covariant conservation equation (6.1) for $m = \alpha$ transforms into the Euler equation for an ideal fluid, while for $m = 0$ it transforms into the equation for the internal energy Π of the *local* fields in the vicinity of the body (A):

$$\overline{\rho}\frac{dv^\alpha}{dy_A^0} = -\overline{\rho}\partial^\alpha\overline{U} + \partial^\alpha p + \overline{\rho}\partial^\alpha\mathcal{O}(c^{-4}), \tag{6.6a}$$

$$\overline{\rho}\frac{d\Pi}{dy_A^0} = -p\partial_\mu v^\mu + \overline{\rho}\mathcal{O}(c^{-5}), \tag{6.6b}$$

where the total time derivative with respect to the proper time y_A^0 is given by the usual relation: $d/dy_A^0 = \partial/\partial y_A^0 + v^\mu\partial/\partial y_A^\mu$. The total Newtonian potential of the system in these coordinates was denoted as $\overline{U}(y_A^p)$.

In order to apply the conditions (3.29), one must substitute the expression for the total Newtonian potential $\overline{U}$ into (6.6a) and integrate this equation over the body (A)'s compact volume. However, if we do so for the potential from the solution in (5.42), the conditions in

(3.29) will not be satisfied. Indeed, the total Newtonian potential $\overline{U}_{[0]}$ may be identified as the terms of order c^{-2} in expression (5.42a) for the g_{00} component of the metric tensor as follows:

$$\overline{U}_{[0]} = \sum_B U_B(y_A^0, y_A^\nu) - \sum_{B\neq A} \Big[y_A^\mu \Big\langle \frac{\partial U_B}{\partial y_A^\mu} \Big\rangle_0 + \Big\langle U_B \Big\rangle_0 \Big]. \tag{6.7}$$

If one substitutes this potential into equation (6.6a) and integrates the resultant expression over the body (A)'s compact volume, one obtains

$$\ddot{m}^\alpha_{A_{[0]}} = -\gamma^{\alpha\sigma} \sum_{B\neq A} \int_A d^3y'_A \hat{\rho}_A \Big[\frac{\partial U_B}{\partial y'^\sigma_A} - \Big\langle \frac{\partial U_B}{\partial y^\sigma_A} \Big\rangle_0 \Big] + \mathcal{O}(c^{-4}) \neq 0. \tag{6.8a}$$

By expanding the integrand in the expression above in the Taylor series with respect to the spatial deviation from the supporting world line γ_A (which is given as $\lambda_A \sim \vec{y}_A/|y_{BA_0}|$), one may bring this result to the following form:

$$\ddot{m}^\alpha_{A_{[0]}} = -\gamma^{\alpha\sigma} \sum_{B\neq A} \sum_{l\geq 1}^{k} \frac{1}{l!} \Big\langle \frac{\partial^{\{L+1\}} U_B}{\partial y^\sigma_A y_A^{\{L\}}} \Big\rangle_0 \int_A d^3y' \hat{\rho}_A y'^{\{L\}}_A + \mathcal{O}(c^{-4}). \tag{6.8b}$$

It is easy to see that this result does not satisfy the requirement for the 'good' proper RF even in the Newtonian order. The origin of the RF, defined this way, coincides with the center of inertia of the *local* fields in the vicinity of the body under question in one particular moment of time only and will drift away from it as time progresses. Exactly the same situation was encountered with the solutions in both the Brumberg–Kopejkin (Brumberg & Kopejkin, 1988a,b) and the Damour–Soffel–Xu (DSX, 1991-1994; Damour & Vokrouhlický, 1995) formalisms. In both of these methods, the translational motion of extended bodies in their proper RFs does not vanish in the Newtonian limit, but rather non-linearly depends on the coupling of the intrinsic multipole moments with the external gravitational field. To solve this problem, the authors of both formalisms have introduced 'external' multipole moments in order to compensate for the terms on the right-hand side of expression (6.8b). However, this substitution may not be considered as a satisfactory solution to this problem. The reason for this is that the authors in both approaches were trying to describe the motion of extended bodies using methods that were developed to treat the motion of point-like test bodies. As we already know, to overcome this problem, we should develop a microscopic treatment of the matter, the gravitational field, and the field of inertia in the immediate vicinity of the bodies (*i.e.*, in their *local* region) in the system.

In our method, the only step we have to make in order to the take into account the extent of the bodies is to change the limiting procedure $\langle .. \rangle_0$ defined by expression (5.7) to an averaging over the bodies' volumes.[13] We define this new procedure $\langle .. \rangle_A$, which, being applied to any function $f(y_A^p)$, will denote an averaging of this function over the body (A)'s three-dimensional compact volume in accord with the following formula:

$$\Big\langle f(y_A^p) \Big\rangle_A \equiv \hat{f}(y_A^0) = \frac{1}{m_A} \int_A d^3y'_A \, \hat{t}_A^{00}(y'^p_A) f(y_A^0, y'^\nu_A), \tag{6.9a}$$

$$m_A = \int_A d^3y'_A \hat{t}_A^{00}(y'^p_A) + \mathcal{O}(c^{-4}), \tag{6.9b}$$

[13]Note that this situation is similar to that from the electrodynamics of continuous media, where one has to average the field over the body's volume (Landau & Lifshitz, 1987).

where $\hat{t}^{00}(y_A'^p)$ is the component of the conserved density of the energy-momentum tensor of matter, inertia, and gravitational field in the *local* region of the body (A) taken jointly. It is easy to see that in the case of a system of N massive particles with the total mass density taken to be $\overline{\rho}(y_A^p) = \sum_B m_B \delta(y_A^0, \vec{y}_A + \vec{y}_{BA_0})$, this new procedure coincides with the procedure $\langle .. \rangle_0$ defined by the expression in eq.(5.7). Note that the new operation $\langle .. \rangle_A$ given by eq.(6.9), contrary to that of eq.(5.7), does not commutate with the operation of time differentiation.

Because of this change, the total gravitational potential $\overline{U}(y_A^p)$, which in the vicinity of the body (A) is composed from the *local* Newtonian potential generated by the body (A) itself and the tidal gravitational potential produced by external sources of gravity, will now have the form

$$\overline{U}(y_A^p) = \sum_B U_B(y_B^q(y_A^p)) - \sum_{B \neq A} \Big[y_A^\beta \Big\langle \frac{\partial U_B}{\partial y_A^\beta} \Big\rangle_A + \langle U_B \rangle_A \Big] + \mathcal{O}(c^{-4}). \tag{6.10}$$

One may make sure that expression (6.10) is what we need in order to have the origin of the proper RF_A coincide with the *local* center of inertia. Indeed, by substituting this result for the total Newtonian potential $\overline{U}$ into equation (6.6a) and integrating the resultant expression over the body (A)'s compact volume, one finds that $\ddot{m}_{A_0}^\alpha = \mathcal{O}(c^{-4})$. Thus, the center of inertia of the *local* fields, defined as the dipole moment of the fields in the immediate vicinity of the body (A), moves along a straight line as given by the formula $m_{A_0}^\alpha(y_A^0) = \mathcal{A}^\alpha + \mathcal{B}^\alpha y_A^0 + \mathcal{O}(c^{-4})$, where $\mathcal{A}^\alpha and \mathcal{B}^\alpha$ are constants. One may perform an additional infinitesimal post-Galilean transformation (similar to that of (1.12)) in order to make them vanish: $\mathcal{A}^\alpha = \mathcal{B}^\alpha = 0$. This means that the origin of the proper RF_A will coincide with the center of inertia of the *local* fields and, hence, the constructed frame will satisfy the definition of a 'good' proper RF discussed in Section 3.

As a result, the general form of the coordinate transformations between the coordinates (x^p) of RF_0 and those (y_A^p) of a proper quasi-inertial RF_A of an arbitrary body (A) for the problem of motion of the N-extended-body system in the WFSMA may be presented as follows:

$$x^0 = y_A^0 + c^{-2} K_A(y_A^0, y_A^\epsilon) + c^{-4} L_A(y_A^0, y_A^\epsilon) + \mathcal{O}(c^{-6}), \tag{6.11a}$$

$$x^\alpha = y_A^\alpha + y_{A_0}^\alpha(y_A^0) + c^{-2} Q_A^\alpha(y_A^0, y_A^\epsilon) + \mathcal{O}(c^{-4}), \tag{6.11b}$$

where the barycentric radius vector $r_{A_0}^\alpha$ of the body (A) in the coordinates of the proper RF_A is decomposed into Newtonian and post-Newtonian parts, which are given as follows:

$$\Big\langle r_{A_0}^\alpha(y_A^p) \Big\rangle_A = y_{A_0}^\alpha(y_A^0) + \frac{1}{m_A c^2} \int_A d^3 y_A' \hat{t}^{00}(y_A'^p) Q_A^\alpha(y_A^0, y_A'^\epsilon) + \mathcal{O}(c^{-4}). \tag{6.12}$$

The transformation functions K_A, Q_A^α, and L_A, in this case will take the following form:

$$K_A(y_A^0, y_A^\nu) = \int^{y_A^0} dt' \Big(\sum_{B \neq A} \Big\langle U_B \Big\rangle_A - \frac{1}{2} v_{A_0\nu} v_{A_0}^\nu \Big) - v_{A_0\nu} \cdot y_A^\nu + \mathcal{O}(c^{-4}) y_A^0, \tag{6.13a}$$

$$\begin{aligned} Q_A^\alpha(y_A^0, y_A^\nu) = & - \sum_{B \neq A} \Big(y_A^\alpha y_A^\beta \cdot \Big\langle \partial_\beta U_B \Big\rangle_A - \frac{1}{2} y_{A\beta} y_A^\beta \Big\langle \partial^\alpha U_B \Big\rangle_A + y_A^\alpha \Big\langle U_B \Big\rangle_A \Big) + \\ & + y_{A\beta} \int^{y_A^0} dt' \Big(\frac{1}{2} a_{A_0}^{[\alpha} v_{A_0}^{\beta]} + 2 \sum_{B \neq A} \Big[\Big\langle \partial^{[\alpha} V_B^{\beta]} \Big\rangle_A + \Big\langle \partial^{[\alpha} U_B v^{\beta]} \Big\rangle_A \Big] \Big) - \\ & - \frac{1}{2} v_{A_0}^\alpha v_{A_0}^\beta y_{A\beta} + \underline{w}_{A_0}^\alpha(y_A^0) + \sum_{l=3}^{k} Q_{A\{L\}}^\alpha(y_A^0) \cdot y_A^{\{L\}} + \mathcal{O}(|y_A^\nu|^{k+1}) + \mathcal{O}(c^{-4}) y_A^\alpha, \end{aligned} \tag{6.13b}$$

$$L_A(y_A^0, y_A^\nu) = \sum_{B\neq A}\Big(\frac{1}{2}\, y_{A\beta}y_A^\beta \cdot \frac{\partial}{\partial y_A^0}\Big\langle U_B\Big\rangle_A - 2\, y_A^\lambda y_A^\beta \cdot \Big[\Big\langle \partial_\lambda V_{B\beta}\Big\rangle_A + \Big\langle v_\beta \partial_\lambda U_B\Big\rangle_A\Big] +$$

$$+ v_{A_0\beta}\Big[y_A^\beta y_A^\lambda \cdot \Big\langle \partial_\lambda U_B\Big\rangle_A - \frac{1}{2} y_{A_\lambda} y_A^\lambda \cdot \Big\langle \partial^\beta U_B\Big\rangle_A\Big]\Big) +$$

$$+ y_{A_\lambda} v_{A_0\beta} \int^{y_A^0} dt' \Big(\frac{1}{2} a_{A_0}^{[\lambda} v_{A_0}^{\beta]} + 2 \sum_{B\neq A}\Big[\Big\langle \partial^{[\lambda} V_B^{\beta]}\Big\rangle_A + \Big\langle v^{[\beta}\partial^{\lambda]} U_B\Big\rangle_A\Big]\Big) +$$

$$+ y_{A_\beta}\Big[2\, v_{A_0}^\beta \sum_{B\neq A}\Big\langle U_B\Big\rangle_A - 4 \sum_{B\neq A}\Big\langle V_B^\beta\Big\rangle_A - \underline{\dot{w}}_{A_0}^\beta(y_A^0)\Big] -$$

$$- \int^{y_A^0} dt' \Big[\sum_{B\neq A}\Big\langle W_B\Big\rangle_A + \frac{1}{2}\Big(\sum_{B\neq A}\Big\langle U_B\Big\rangle_A - \frac{1}{2} v_{A_0\beta} v_{A_0}^\beta\Big)^2 + v_{A_0\mu}\underline{\dot{w}}_{A_0}^\mu(t')\Big] +$$

$$+ \sum_{l=3}^{k} L_{A\{L\}}(y_A^0)\cdot y_A^{\{L\}} + \mathcal{O}(|y_A^\nu|^{k+1}) + \mathcal{O}(c^{-6}). \tag{6.13c}$$

One may verify that in the case of the free-falling massive test particle with conserved mass density given as $\hat{\rho}_A(y_A^p) = m_A\delta(\vec{y}_A)$, functions (6.13) will correspond to the coordinate transformations to the proper RF defined on the geodesic world line, (5.41).

Note that we have changed the notation $f_{A_{[0]}}$ to f_{A_0} in the new expressions, eqs.(6.11)–(6.13). This is because all these quantities are now defined with the procedure eq.(6.9), which takes into account the internal structure of the bodies. As a result, the Newtonian acceleration of the extended body (A) with respect to the barycentric RF_0 now is given as

$$a_{A_0}^\alpha(y_A^0) = -\gamma^{\alpha\mu} \sum_{B\neq A}\Big\langle \frac{\partial U_B}{\partial y_A^\mu}\Big\rangle_A + \mathcal{O}(c^{-4}). \tag{6.14}$$

Furthermore, in order to take into account the extent of the bodies and the influence of this extent on the post-Newtonian dynamics of the N-body system, the time-dependent function $w_{A_0}^\alpha$ has been replaced by the new function $\underline{w}_{A_0}^\alpha$:

$$\underline{w}_{A_0}^\alpha(y_A^0) = w_{A_0}^\alpha + \delta w_{A_0}^\alpha, \tag{6.15a}$$

where the function $w_{A_0}^\alpha$ is determined as the solution of the following differential equation:

$$\ddot{w}_{A_0}^\alpha(y_A^0) = \sum_{B\neq A}\Big(\gamma^{\alpha\mu}\Big\langle \frac{\partial W_B}{\partial y_A^\mu}\Big\rangle_A + v_{A_0}^\alpha \frac{\partial}{\partial y_A^0}\Big\langle U_B\Big\rangle_A - 4\,\frac{\partial}{\partial y_A^0}\Big\langle V_B^\alpha\Big\rangle_A\Big) -$$

$$- \frac{1}{2} v_{A_0}^\alpha v_{A_0\beta} a_{A_0}^\beta + a_{A_0}^\alpha \sum_{B\neq A}\Big\langle U_B\Big\rangle_A +$$

$$+ a_{A_0\beta}\int^{y_A^0} dt' \Big(\frac{1}{2} a_{A_0}^{[\alpha} v_{A_0}^{\beta]} + 2\sum_{B\neq A}\Big[\Big\langle \partial^{[\alpha} V_B^{\beta]}\Big\rangle_A + \Big\langle v^{[\beta}\partial^{\alpha]} U_B\Big\rangle_A\Big]\Big) + \mathcal{O}(c^{-6}), \tag{6.15b}$$

and the function $\delta w_{A_0}^\alpha$ is unknown at the moment. This function will be determined later, when we will apply conditions (3.29) in order to make the total momentum of the matter, the inertia,

and the gravitational field calculated in the coordinates of the proper RF$_A$ vanish in the volume of the body (A).

As a result, the 'averaged' components of the metric tensor g^A_{mn} in the coordinates (y^p_A) of the proper RF$_A$ take the following form:

$$g^A_{00}(y^p_A) = 1 - 2\overline{U} + 2\overline{W} + y^\mu_A y^\beta_A \cdot \Big\{\gamma_{\mu\beta}\, a_{A_0\lambda} a^\lambda_{A_0} - a_{A_0\mu} a_{A_0\beta} +$$

$$+ \sum_{B\neq A} \frac{\partial}{\partial y^0_A}\Big(\gamma_{\mu\beta}\frac{\partial}{\partial y^0_A}\Big\langle U_B\Big\rangle_A - 2\Big[\Big\langle \partial_{(\mu} V_{B\beta)}\Big\rangle_A + \Big\langle v_{(\beta}\partial_{\mu)} U_B\Big\rangle_A\Big]\Big)\Big\} +$$

$$+2\sum_{l\geq 3}^{k}\Big[\partial_0 L_{A\{L\}}(y^0_A) + v_{A_0\beta}\partial_0 Q^\beta_{A\{L\}}(y^0_A)\Big]\cdot y^{\{L\}}_A + \mathcal{O}(|y^\nu_A|^{k+1}) + \mathcal{O}(c^{-6}), \tag{6.16a}$$

$$g^A_{0\alpha}(y^p_A) = 4\,\gamma_{\alpha\epsilon}\overline{V}^\epsilon - \frac{1}{5}\Big(y_{A\alpha} y_{A\beta} + \frac{1}{2}\gamma_{\alpha\beta} y_{A\mu} y^\mu_A\Big)\cdot \dot{a}^\beta_{A_0} +$$

$$+\sum_{l\geq 3}^{k}\Big[\gamma_{\alpha\lambda}\partial_0 Q^\lambda_{A\{L\}}(y^0_A) + \Big(L_{A\{L\}}(y^0_A) + v_{A_0\beta}\cdot Q^\beta_{A\{L\}}(y^0_A)\Big)\frac{\partial}{\partial y^\alpha_A}\Big]\cdot y^{\{L\}}_A +$$

$$+\mathcal{O}(|y^\nu_A|^{k+1}) + \mathcal{O}(c^{-5}), \tag{6.16b}$$

$$g^A_{\alpha\beta}(y^p_A) = \gamma_{\alpha\beta}\Big(1 + 2\overline{U}\Big) +$$

$$+\sum_{l\geq 3}^{k}\Big[\gamma_{\alpha\lambda} Q^\lambda_{A\{L\}}(y^0_A)\frac{\partial}{\partial y^\beta_A} + \gamma_{\beta\lambda} Q^\lambda_{A\{L\}}(y^0_A)\frac{\partial}{\partial y^\alpha_A}\Big]\cdot y^{\{L\}}_A + \mathcal{O}(|y^\nu_A|^{k+1}) + \mathcal{O}(c^{-4}), \tag{6.16c}$$

where the total gravitational potential $\overline{U}$ at the vicinity of the body (A) is composed of the *local* Newtonian potential generated by the body (A) itself, and the tidal gravitational potential produced by the external sources of gravity is given by expression (6.10). This potential may now be obtained from (6.13a) as follows:

$$\overline{U}(y^p_A) = \sum_B U_B(y^q_B(y^p_A)) - \frac{\partial K_A(y^p_A)}{\partial y^0_A} - \frac{1}{2} v_{A_0\mu} v^\mu_{A_0} =$$

$$= \sum_B U_B(y^q_B(y^p_A)) - \sum_{B\neq A}\Big[y^\beta_A\Big\langle\frac{\partial U_B}{\partial y^\beta_A}\Big\rangle_A + \langle U_B\rangle_A\Big] + \mathcal{O}(c^{-4}). \tag{6.17}$$

This potential is the solution of the corresponding Poisson equation in the coordinates (y^p_A):

$$\gamma^{\mu\nu}\frac{\partial^2\overline{U}}{\partial y^\mu_A \partial y^\nu_A} = 4\pi\overline{\rho}(y^p_A), \tag{6.18}$$

which is searched for together with the following integral boundary conditions:

$$\langle\overline{U}\rangle_A = \int_A d^3y'_A\,\hat{\rho}_A(y'^p_A)\overline{U}(y'^p_A) = \int_A d^3y'_A\,\hat{\rho}_A(y'^p_A) U_A(y'^p_A) \tag{6.19a}$$

$$\Big\langle\frac{\partial\overline{U}}{\partial y^\mu_A}\Big\rangle_A = \int_A d^3y'_A\,\hat{\rho}_A(y'^p_A)\frac{\partial\overline{U}(y'^p_A)}{\partial y'^\mu_A} = 0. \tag{6.19b}$$

The quantity $\overline{V}^{\alpha}(y_A^p)$ in the expressions in (6.16b) is the total vector potential produced by all the bodies in the system as seen in the coordinates (y_A^p) of the RF$_A$. The averaging procedure (6.9) enables one to define this potential as follows:

$$\overline{V}^{\alpha}(y_A^p) = \sum_B V_B^{\alpha}(y_B^q(y_A^p)) - \sum_{B\neq A}\left(y_A^{\mu}\Big[\Big\langle\frac{\partial V_B^{\alpha}}{\partial y_A^{\mu}}\Big\rangle_A + \Big\langle v^{\alpha}\frac{\partial U_B}{\partial y_A^{\mu}}\Big\rangle_A\Big] + \langle V_B^{\alpha}\rangle_A\right) +$$

$$+\frac{1}{10}\big(3y_A^{\alpha}y_A^{\lambda} - \gamma^{\alpha\lambda}y_{A\mu}y_A^{\mu}\big)\dot{a}_{A_0\lambda} + \mathcal{O}(c^{-4}). \tag{6.20}$$

It is interesting to note that the vector potential now depends on the coupling of the intrinsic motion of matter in the body (A) to the gradient of the external gravitational field. Thus, it can be seen from expression (6.13) for the function Q_A^{α} that this coupling contributes to the corresponding precession term of the coordinates in this RF relative to the barycentric inertial frame. This potential also satisfies the usual Poisson equation of the form

$$\gamma^{\mu\nu}\frac{\partial^2\overline{V}^{\alpha}}{\partial y_A^{\mu}\partial y_A^{\nu}} = -4\pi\overline{\rho}(y_A^p)v^{\alpha}(y_A^p). \tag{6.21}$$

Moreover, due to the covariant equation of continuity, (6.3), both quantities (6.17) and (6.21) are connected by the following relation:

$$\frac{\partial\overline{U}}{\partial y_A^0} = \frac{\partial\overline{V}^{\mu}}{\partial y_A^{\mu}}. \tag{6.22}$$

Another quantity we have introduced in expressions (6.16) is $\overline{W}(y_A^p)$. This is the post-Newtonian contribution to the component g_{00} of the effective metric tensor in coordinates (y_A^p) given by (6.16a). This contribution is given as follows:

$$\overline{W}(y_A^p) = \sum_B W_B(y_B^q(y_A^p)) - \sum_{B\neq A}\Big[y_A^{\mu}\Big\langle\frac{\partial W_B}{\partial y_A^{\mu}}\Big\rangle_A + \langle W_B\rangle_A\Big] + \mathcal{O}(c^{-6}). \tag{6.23a}$$

The solution (6.23a) repeats the structure of the tidal representation of the Newtonian potential (6.17), so it could be considered as the generalized post-Newtonian potential in this RF. The functions W_A and W_B in expression (6.23a) are given by relations (5.43), and they fully represent the non-linearity of the total post-Newtonian gravitational field in the proper RF$_A$. These functions contain contributions of two sorts: (i) the gravitational field produced by the external bodies in the system (B$\neq$A), and (ii) the field of inertia caused by the accelerated and non-geodesic motion of the proper RF$_A$. This happens due to the coupling of the proper multipole moments of the body (A) to the external gravitational field as well as to the self-action contributions that are given by the terms with $Q_{A\{L\}}^{\lambda}$ in expressions (5.43). One may obtain the corresponding Poisson-like equation for this potential as well. Thus, directly from the gravitational field equation (4.4d), this last equation will take the form

$$\gamma^{\mu\nu}\frac{\partial^2\overline{W}}{\partial y_A^{\mu}\partial y_A^{\nu}} = -8\pi\overline{\rho}\Big(\Pi - 2v_{\mu}v^{\mu} + \frac{3p}{\overline{\rho}}\Big) + 2\sum_B\Big[\partial_{00}^2U_B + 2\partial_{\mu}U_B\Big(2a_{A_0}^{\mu} + \sum_C\partial^{\mu}U_C\Big)\Big] -$$

$$-2\sum_B\sum_{l\geq 3}^{k}Q_{A\{L\}}^{\mu}(y_A^0)\Big[2\partial_{\mu\lambda}^2U_B\cdot\partial^{\lambda} + \partial_{\mu}U_B\cdot\partial_{\lambda}\partial^{\lambda}\Big]y_A^{\{L\}} + \mathcal{O}(|y_A^{\nu}|^{k+1}) + \mathcal{O}(c^{-6}). \tag{6.23b}$$

We have not yet presented the last function that is necessary to complete the coordinate transformation for the extended bodies, namely: the function $\delta w^{\alpha}_{A_0}$ from (6.15). To find this function, one needs to apply the procedure for constructing a 'good' proper RF with full post-Newtonian accuracy. In order to do this, one must perform the study of the existence of conservation laws in the proper RF_A and define the conserved quantities that will correspond to the energy, momentum, and angular momentum of the *local* fields. Then, after integrating these quantities over the body's compact volume, one must find the form of the eq.m. for the extended bodies in their proper RFs. These equations will contain the time derivatives of the only unknown function, $\delta w^{\alpha}_{A_0}$, which should be chosen in such a way that conditions (3.29) will be satisfied.

6.2 Conservation Laws in the Proper RF.

As we have stated before, our goal is to construct a formalism that will be useful for calculations in a number of the metric theories of gravity. This is the reason why in our further discussion we will use the method developed for analysis of the conservation laws in parametrized post-Newtonian gravity developed by Fock and Chandrasekhar (Fock, 1955; Ehlers, 1967; Denisov & Turyshev, 1989; Will, 1993). It is known that the most important question for any metric theory of gravity is the presence or absence of laws of conservation of energy, momentum, and angular momentum for the closed system of interacting fields. Strictly speaking, the solution to this question requires detailed information regarding the structure of each metric theory of gravitation. It is necessary to know what geometric object has been chosen to describe the gravitational field, what geometry is natural for it, and what is the form of the equation connecting the gravitational field and the metric of the Riemann space-time. Using the standard methods of theoretical physics, it is then possible to give an exhaustive answer to this question. However, such an analysis cannot be carried out in a general form for all metric theories of gravity at once. This leaves us with only one option: attempt to obtain some information regarding the possibility of the existence of conservation laws in these theories by proceeding only from the eq.m. of matter in the WFSMA. It should be noted that conditions obtained in this way are necessary but not sufficient to prove the existence of integral conservation laws for matter and the gravitational field taken jointly in a particular metric theory of gravitation. It is altogether possible that, although the necessary conditions are satisfied for some theory of gravitation, there nevertheless may not be conservation laws for a closed system of interacting fields. The reason for this situation is that quantities that do not depend on time, obtained on the basis of post-Newtonian equations of motion, may not have the character of integrals of motion for a closed system and hence also have no physical meaning. Therefore, in resolving the question of whether or not conservation laws are present in a particular theory of gravitation, the last word can be said only after a complete analysis of the theory has been performed.

It is known that general relativity in the WFSMA possesses the integral conservation laws for the energy-momentum tensor of matter and the gravitational field taken jointly. It means that the covariant equation of conservation of the energy-momentum tensor of matter in Riemann space-time, (6.1), can be identically represented as the covariant conservation law of the sum of symmetric energy-momentum tensors of the gravitational field t_g^{mn} and matter t_M^{mn} in space-time of a constant curvature:

$$\nabla_k \hat{\mathcal{T}}^{mk} = 0 \Rightarrow \mathcal{D}_k\left(t_g^{mk} + t_M^{mk}\right) = 0. \tag{6.24}$$

It should be especially emphasized that, since in an arbitrary Riemann space-time the operation of integrating tensors (with the exception of scalar density) is meaningless from a math-

ematical point of view, it follows that the presence of some differential conservation equations in this case does not guarantee the possibility of obtaining corresponding integral conservation laws. The possibility of obtaining integral conservation laws in a Riemann space-time is entirely predetermined by its geometry and closely connected with the existence of Killing vectors of the given space-time. Namely, only an equation of the form of (6.24) guarantees the existence of all ten integral conservation laws for a closed system of interacting fields. Indeed, since, in a space-time of constant curvature the Killing equations $\mathcal{D}_m\eta_n + \mathcal{D}_n\eta_m = 0$ are completely integrable and their solutions contain the maximal possible number, M= 10, of arbitrary parameters (Eisenhart, 1926), we have ten independent Killing vectors in this case. Multiplying (6.24) successively by each of these vectors η_k, we obtain

$$\mathcal{D}_k\Big[\eta_m\Big(t_g^{mk} + t_M^{mk}\Big)\Big] = \frac{1}{\sqrt{-\gamma}}\partial_k\Big[\eta_m\Big(t_g^{mk} + t_M^{mk}\Big)\sqrt{-\gamma}\Big] = 0. \tag{6.25}$$

Since the left side of this expression is a scalar, we can integrate it over a three-dimensional volume (Logunov, 1987) and obtain all ten (the number of independent Killing vectors) integral conservation laws for a system consisting of matter, inertia, and a gravitational field taken jointly.

Thus, in general relativity, which possesses the integral conservation laws, expressions for the integrals of motion of an isolated system can be determined also from the equation of motion of matter, eq.(6.1). We shall find a necessary condition that the post-Newtonian expansions of this theory in the proper quasi-inertial RF must satisfy and obtain post-Newtonian expansions of integrals of the motion required for subsequent computation. For this we should transform the covariant conservation equation (6.1) to the form of eq.(6.24), after which, multiplying this relation by the corresponding Killing vectors of a space of constant curvature and integrating over the volume, we may easily obtain the desired expressions. Since the metric tensor of Riemann space-time in the absence of matter ($\rho_0 = p = 0$) should have as its limit the pseudo-Euclidean Minkowski metric, the covariant conservation equation (6.1) should be transformed to the conservation law (6.24) just in the pseudo-Euclidean space-time. Then in the quasi-Cartesian coordinates of the barycentric inertial RF_0, expression (6.24) will take the form

$$\nabla_k\hat{T}^{mk} = \partial_k\Big(t_g^{mk} + t_M^{mk}\Big) = 0. \tag{6.26}$$

We expect that the 'good' proper RF will resemble the properties of the inertial RF_0; then in the coordinates of this proper RF, the expression, analogous to that of (6.24), should take the form of the conservation law of the total energy-momentum tensor of the fields of inertia, matter, and gravity taken jointly:

$$\nabla_k\hat{T}^{mk}(y_A^p) = \frac{\partial}{\partial y_A^k}\Big(t_i^{mk} + t_g^{mk} + t_M^{mk}\Big) = 0. \tag{6.27}$$

Knowledge of the metric (6.16) to a post-Newtonian degree of accuracy makes it possible to determine the components of the energy-momentum tensor in the next approximation. Indeed, using the definition for $\hat{T}^{mn}$, (6.2); the metric (6.16); the expressions for the four-velocity, eqs.($E4$) and ($E13b$); and also the covariant components of the metric tensor, ($B5a$), we obtain the following expressions for the components of the density of the energy-momentum tensor in the post-Newtonian approximation in the coordinates of the proper RF_A:

$$\hat{T}^{00}(y_A^p) = \overline{\rho}\Big[1 + \Pi - \frac{1}{2}v_\mu v^\mu + \overline{U} + \mathcal{O}(c^{-4})\Big], \tag{6.28a}$$

$$\hat{T}^{0\alpha}(y_A^p) = \overline{\rho} v^\alpha \Big[1 + \Pi - \frac{1}{2} v_\mu v^\mu + \frac{p}{\overline{\rho}} + \overline{U} + \mathcal{O}(c^{-4})\Big], \tag{6.28b}$$

$$\hat{T}^{\alpha\beta}(y_A^p) = \overline{\rho} v^\alpha v^\beta \Big[1 + \Pi - \frac{1}{2} v_\mu v^\mu + \frac{p}{\overline{\rho}} + \overline{U}\Big] - p\gamma^{\alpha\beta} +$$

$$+ p \sum_{l\geq 3}^{k} Q^\mu_{A\{L\}}(y_A^0)\Big(\delta^\alpha_\mu \partial^\beta + \delta^\beta_\mu \partial^\alpha - \gamma^{\alpha\beta}\partial_\mu\Big) y_A^{\{L\}} + \overline{\rho}\mathcal{O}(c^{-4}) + \mathcal{O}(|y_A^\nu|^{k+1}), \tag{6.28c}$$

where the total conserved mass density of the entire system $\overline{\rho}$ is given by (6.4).

Furthermore, by using the solutions for the transformation functions (6.11) and (6.13), from the expressions $(F2)$ one may obtain the Christoffel symbols of the Riemann metric in the proper RF_A in the form

$$\Gamma^0_{00}(y_A^p) = -\frac{\partial \overline{U}}{\partial y_A^0} + \mathcal{O}(c^{-5}), \qquad \Gamma^0_{0\alpha}(y_A^p) = -\frac{\partial \overline{U}}{\partial y_A^\alpha} + \mathcal{O}(c^{-6}), \qquad \Gamma^0_{\alpha\beta}(y_A^p) = \mathcal{O}(c^{-3}),$$

$$\Gamma^\alpha_{00}(y_A^p) = \gamma^{\alpha\mu}\frac{\partial}{\partial y_A^\mu}\Big[\overline{U} - \overline{W} - \overline{U}^2\Big] + 4\frac{\partial \overline{V}^\alpha}{\partial y_A^0} - \frac{1}{5}\Big(y_A^\alpha y_A^\lambda + \frac{1}{2}\gamma^{\alpha\lambda} y_A^\mu y_{A\mu}\Big)\ddot{a}_{A_0\lambda} + \tag{6.29}$$

$$+\Big\{a^\alpha_{A_0} a_{A_0\lambda} - \delta^\alpha_\lambda \cdot a^\mu_{A_0} a_{A_0\mu} + \sum_{B\neq A}\frac{\partial}{\partial y_A^0}\Big[2\Big\langle \frac{\partial V_B^{\alpha)}}{\partial y_A^{(\lambda}}\Big\rangle_A + 2\Big\langle v^{(\alpha}\frac{\partial U_B}{\partial y_A^{\lambda)}}\Big\rangle_A - \delta^\alpha_\lambda \frac{\partial}{\partial y_A^0}\Big\langle U_B\Big\rangle_A\Big]\Big\} y_A^\lambda +$$

$$+\delta \ddot{w}^\alpha_{A_0} + \sum_{l\geq 3}^{k}\Big[\partial^2_{00} Q^\alpha_{A\{L\}}(y_A^0) + a_{A_0\lambda} Q^\lambda_{A\{L\}}(y_A^0)\partial^\alpha\Big] y_A^{\{L\}} -$$

$$-\frac{\partial \overline{U}}{\partial y_A^\mu}\sum_{l\geq 3}^{k}\Big[Q^\mu_{A\{L\}}(y_A^0)\partial^\alpha + Q^\alpha_{A\{L\}}(y_A^0)\partial^\mu\Big] y_A^{\{L\}} + \mathcal{O}(|y_A^\nu|^{k+1}) + \mathcal{O}(c^{-6}),$$

$$\Gamma^\alpha_{0\beta}(y_A^p) = \delta^\alpha_\beta \frac{\partial \overline{U}}{\partial y_A^0} + 2\Big(\frac{\partial \overline{V}^\alpha}{\partial y_A^\beta} - \frac{\partial \overline{V}_\beta}{\partial y_{A\alpha}}\Big) +$$

$$+\sum_{l\geq 3}^{k}\partial_0 Q^\alpha_{A\{L\}}(y_A^0)\partial_\beta y_A^{\{L\}} + \mathcal{O}(|y_A^\nu|^{k+1}) + \mathcal{O}(c^{-5}),$$

$$\Gamma^\alpha_{\beta\omega}(y_A^p) = \delta^\alpha_\beta \frac{\partial \overline{U}}{\partial y_A^\omega} + \delta^\alpha_\omega \frac{\partial \overline{U}}{\partial y_A^\beta} - \gamma_{\beta\omega}\gamma^{\alpha\sigma}\frac{\partial \overline{U}}{\partial y_A^\sigma} + \sum_{l\geq 3}^{k} Q^\alpha_{A\{L\}}(y_A^0)\partial^2_{\beta\omega} y_A^{\{L\}} + \mathcal{O}(|y_A^\nu|^{k+1}) + \mathcal{O}(c^{-4}).$$

Writing (6.1) for $m = 0$ and substituting (6.28) and (6.29) into it, we obtain

$$\frac{\partial}{\partial y_A^0}\Big[\overline{\rho}\Big(1 + \Pi - \frac{1}{2}v_\mu v^\mu + \overline{U}\Big)\Big] + \frac{\partial}{\partial y_A^\mu}\Big[\overline{\rho} v^\mu\Big(1 + \Pi - \frac{1}{2}v_\mu v^\mu + \overline{U} + \frac{p}{\overline{\rho}}\Big)\Big] -$$

$$-\overline{\rho}\frac{\partial \overline{U}}{\partial y_A^0} - 2\overline{\rho} v^\mu \frac{\partial \overline{U}}{\partial y_A^\mu} = \overline{\rho}\mathcal{O}(c^{-5}). \tag{6.30}$$

Thus, to bring this relation to the form of (6.27), it is necessary to transform the last two terms by extracting from them the partial derivatives with respect to time and the three-dimensional divergence. Such a transformation cannot be carried out in a unique manner. Therefore, using

equations (6.18) and (6.21), we rewrite the given terms in the most general form reflecting this ambiguity:

$$\overline{\rho}\frac{\partial\overline{U}}{\partial y_A^0}+2\overline{\rho}v^\mu\frac{\partial\overline{U}}{\partial y_A^\mu}=\frac{\partial}{\partial y_A^0}\Big(a_1\overline{\rho}\overline{U}+\frac{2a_1-3}{8\pi}\partial_\mu\overline{U}\partial^\mu\overline{U}\Big)+$$

$$+\frac{\partial}{\partial y_A^\mu}\Big(\frac{1-a_1}{4\pi}\partial^\mu\overline{U}\frac{\partial\overline{U}}{\partial y_A^0}+\frac{a_2}{4\pi}\overline{U}\partial^\mu\frac{\partial\overline{U}}{\partial y_A^0}+(a_1+a_2)\overline{\rho}\overline{U}v^\mu+$$

$$+\frac{2-a_1-a_2}{4\pi}\partial_\nu\overline{U}[\partial^\nu\overline{V}^\mu-\partial^\mu\overline{V}^\nu]\Big),\tag{6.31}$$

where a_1 and a_2 are arbitrary numbers. With consideration of this relation, and collecting like terms in (6.30), we get

$$\frac{\partial}{\partial y_A^0}\Big(t_i^{00}+t_g^{00}+t_M^{00}\Big)+\frac{\partial}{\partial y_A^\beta}\Big(t_i^{0\beta}+t_g^{0\beta}+t_M^{0\beta}\Big)=\overline{\rho}\mathcal{O}(c^{-5}),\tag{6.32a}$$

with the following expressions for the (00) and (0α) components of the density of the total energy-momentum tensor:

$$t_i^{00}+t_g^{00}+t_M^{00}=\overline{\rho}\Big(1+\Pi-\frac{1}{2}v_\mu v^\mu+(1-a_1)\overline{U}\Big)+\frac{3-2a_1}{8\pi}\partial_\mu\overline{U}\partial^\mu\overline{U}+\overline{\rho}\mathcal{O}(c^{-4}),\tag{6.32b}$$

$$t_i^{0\alpha}+t_g^{0\alpha}+t_M^{0\alpha}=\overline{\rho}v^\alpha\Big(1+\Pi-\frac{1}{2}v_\mu v^\mu+(1-a_1-a_2)\overline{U}+\frac{p}{\overline{\rho}}\Big)+$$

$$+\frac{a_1-1}{4\pi}\partial^\alpha\overline{U}\frac{\partial\overline{U}}{\partial y_A^0}-\frac{a_2}{4\pi}\overline{U}\partial^\alpha\frac{\partial\overline{U}}{\partial y_A^0}+$$

$$+\frac{a_1+a_2-2}{4\pi}\partial_\nu\overline{U}[\partial^\nu\overline{V}^\alpha-\partial^\alpha\overline{V}^\nu]\Big)+\overline{\rho}\mathcal{O}(c^{-5}).\tag{6.32c}$$

Writing expression (6.1) for $n=\alpha$ and substituting (6.28) and (6.29) into it, we have

$$\frac{\partial}{\partial y_A^0}\hat{T}^{\alpha 0}+\frac{\partial}{\partial y_A^\mu}\hat{T}^{\alpha\mu}+\overline{\rho}\partial^\alpha\overline{U}-\overline{\rho}\partial^\alpha\overline{W}+\overline{\rho}\partial^\alpha\overline{U}\Big(\Pi-\frac{3}{2}v_\mu v^\mu+\frac{p}{\overline{\rho}}-\overline{U}\Big)+$$

$$+4\overline{\rho}\Big[\frac{\partial\overline{V}^\alpha}{\partial y_A^0}-\frac{1}{20}\ddot{a}_{A_0\lambda}\Big(y_A^\alpha y_A^\lambda+\frac{1}{2}\gamma^{\alpha\lambda}y_A^\mu y_{A\mu}\Big)\Big]+$$

$$+\overline{\rho}\Big\{a_{A_0}^\alpha a_{A_0\lambda}-\delta_\lambda^\alpha\cdot a_{A_0\mu}a_{A_0}^\mu+\sum_{B\neq A}\frac{\partial}{\partial y_A^0}\Big[2\Big\langle\frac{\partial V_B^{\alpha)}}{\partial y_A^{(\lambda}}\Big\rangle_A+2\Big\langle v^{(\alpha}\frac{\partial U_B}{\partial y_A^{\lambda)}}\Big\rangle_A-\delta_\lambda^\alpha\frac{\partial}{\partial y_A^0}\Big\langle U_B\Big\rangle_A\Big]\Big\}y_A^\lambda+$$

$$+4\overline{\rho}v^\mu\Big(\partial_\mu\overline{V}^\alpha-\partial^\alpha\overline{V}_\mu\Big)+2\overline{\rho}v^\alpha\frac{\partial\overline{U}}{\partial y_A^0}+2\overline{\rho}v^\alpha v^\mu\partial_\mu\overline{U}+$$

$$+\overline{\rho}\Big(\delta\ddot{w}_{A_0}^\alpha(y_A^0)+\sum_{l\geq 3}^{k}\Big[\partial_{00}^2Q_{A\{L\}}^\alpha(y_A^0)+a_{A_0\lambda}Q_{A\{L\}}^\lambda(y_A^0)\partial^\alpha\Big]y_A^{\{L\}}\Big)-$$

$$-\overline{\rho}\frac{\partial\overline{U}}{\partial y_A^\mu}\sum_{l\geq 3}^{k}\Big[Q_{A\{L\}}^\mu(y_A^0)\partial^\alpha+Q_{A\{L\}}^\alpha(y_A^0)\partial^\mu\Big]y_A^{\{L\}}+2\overline{\rho}v^\mu\sum_{l\geq 3}^{k}\partial_0Q_{A\{L\}}^\alpha(y_A^0)\partial_\mu y_A^{\{L\}}+$$

$$+(\overline{\rho}v^\mu v^\lambda-\gamma^{\mu\lambda}p)\sum_{l\geq 3}^{k}Q_{A\{L\}}^\alpha(y_A^0)\partial_{\mu\lambda}^2y_A^{\{L\}}=\overline{\rho}\mathcal{O}(|y_A^\nu|^{k+1})+\overline{\rho}\mathcal{O}(c^{-6}).\tag{6.33}$$

One may note that these expressions are not dependent on the function $L_{A\{L\}}$ with $l \geq 3$. This means that, in the post-Newtonian order, the function $Q^{\alpha}_{A\{L\}}$ with $l \geq 3$ only is responsible for the existence of the integrals of motion in the RF_A under consideration.

To reduce this equation to the form of (6.27), we use the identities presented in Appendix H. Substituting these into (6.33) and collecting the like terms, we obtain

$$\frac{\partial}{\partial y^0_A}\Big(t^{\alpha 0}_i + t^{\alpha 0}_g + t^{\alpha 0}_M\Big) + \frac{\partial}{\partial y^{\beta}_A}\Big(t^{\alpha\beta}_i + t^{\alpha\beta}_g + t^{\alpha\beta}_M\Big) =$$

$$= -\overline{\rho}\frac{d^2}{dy^{0\,2}_A}\sum_{l\geq 3}^{k} Q^{\alpha}_{A\{L\}}(y^0_A)y^{\{L\}}_A + \overline{\rho}\mathcal{O}(|y^{\nu}_A|^{k+1}) + \overline{\rho}\mathcal{O}(c^{-5}), \tag{6.34}$$

with the following expressions for the $(\alpha 0)$ and $(\alpha\beta)$ components of the density of the total energy-momentum tensor:

$$t^{\alpha 0}_i + t^{\alpha 0}_g + t^{\alpha 0}_M = \overline{\rho}v^{\alpha}\Big(1 + \Pi - \frac{1}{2}v_{\mu}v^{\mu} + 3\overline{U} + \frac{p}{\rho}\Big) -$$

$$-\frac{3}{4\pi}\partial^{\alpha}\overline{U}\frac{\partial\overline{U}}{\partial y^0_A} + \frac{1}{\pi}\partial_{\nu}\overline{U}[\partial^{\alpha}\overline{V}^{\nu} - \partial^{\nu}\overline{V}^{\alpha}]\Big) + \overline{\rho}\mathcal{O}(c^{-5}), \tag{6.35}$$

$$4\pi\Big(t^{\alpha\beta}_i + t^{\alpha\beta}_g + t^{\alpha\beta}_M\Big) = \Gamma^{\alpha\beta}\Bigg[\overline{U} - \overline{W} + \overline{U}^2 + \Big(y^{\mu}_A y^{\nu}_A - \gamma^{\mu\nu}y_{A\lambda}y^{\lambda}_A\Big)a_{A_0\mu}a_{A_0\nu} + \Big(y_{A\mu}\delta\ddot{w}^{\mu}_{A_0}\Big) +$$

$$+\sum_{B\neq A}\frac{\partial}{\partial y^0_A}\Big(4y^{\lambda}_A y^{\epsilon}_A\Big[\Big\langle\frac{\partial V_{B\epsilon}}{\partial y^{\lambda}_A}\Big\rangle_A + \Big\langle v_{\epsilon}\frac{\partial U_B}{\partial y^{\lambda}_A}\Big\rangle_A\Big] - y_{A\mu}y^{\mu}_A\frac{\partial}{\partial y^0_A}\Big\langle U_B\Big\rangle_A\Big) + 2a_{A_0\mu}\sum_{l\geq 3}^{k}Q^{\mu}_{A\{L\}}(y^0_A)y^{\{L\}}_A\Bigg] +$$

$$+4\Bigg(\partial^{\alpha}\overline{U}\frac{\partial\overline{V}^{\beta}}{\partial y^0_A} + \partial^{\beta}\overline{U}\frac{\partial\overline{V}^{\alpha}}{\partial y^0_A} - \gamma^{\alpha\beta}\partial_{\lambda}\overline{U}\frac{\partial\overline{V}^{\lambda}}{\partial y^0_A} + [\partial^{\nu}\overline{V}^{\alpha} - \partial^{\alpha}\overline{V}^{\nu}][\partial_{\nu}\overline{V}^{\beta} - \partial^{\beta}\overline{V}_{\nu}] -$$

$$-\gamma^{\alpha\beta}\Big[\partial^{\mu}\overline{V}^{\nu}\partial_{\mu}\overline{V}_{\nu} - \partial^{\mu}\overline{V}^{\nu}\partial_{\nu}\overline{V}_{\mu}\Big]\Bigg) + \frac{3}{2}\gamma^{\alpha\beta}\Big(\frac{\partial\overline{U}}{\partial y^0_A}\Big)^2 + 4\pi(1 + 2\overline{U})\hat{T}^{\alpha\beta} +$$

$$+\frac{1}{5}\Big(\gamma^{\alpha\beta}\gamma_{\lambda\mu} - \delta^{\alpha}_{\lambda}\delta^{\beta}_{\mu} - \delta^{\alpha}_{\mu}\delta^{\beta}_{\lambda}\Big)\Big(y^{\mu}_A y^{\nu}_A + \frac{1}{2}\gamma^{\mu\nu}y_{A\epsilon}y^{\epsilon}_A\Big)\ddot{a}_{A_0\nu}\partial^{\lambda}\overline{U} +$$

$$+\Big[\delta^{\alpha}_{\lambda}\delta^{\beta}_{\mu}\gamma_{\sigma\epsilon} - (\delta^{\alpha}_{\lambda}\delta^{\beta}_{\epsilon} + \delta^{\alpha}_{\epsilon}\delta^{\beta}_{\lambda})\gamma_{\mu\sigma} + \gamma^{\alpha\beta}(\gamma_{\lambda\epsilon}\gamma_{\mu\sigma} - \frac{1}{2}\gamma_{\lambda\mu}\gamma_{\sigma\epsilon})\Big]\partial^{\lambda}\overline{U}\partial^{\mu}\overline{U}\sum_{l\geq 3}^{k}Q^{\sigma}_{A\{L\}}(y^0_A)\partial^{\epsilon}y^{\{L\}}_A +$$

$$+\frac{1}{2}\partial_{\mu}\overline{U}\partial^{\mu}\overline{U}\sum_{l\geq 3}^{k}Q^{\beta}_{A\{L\}}(y^0_A)\partial^{\alpha}y^{\{L\}}_A - \partial^{\alpha}\overline{U}\partial^{\mu}\overline{U}\sum_{l\geq 3}^{k}Q^{\beta}_{A\{L\}}(y^0_A)\partial_{\mu}y^{\{L\}}_A -$$

$$-p\sum_{l\geq 3}^{k}Q^{\alpha}_{A\{L\}}(y^0_A)\partial^{\beta}y^{\{L\}}_A + \overline{\rho}\mathcal{O}(|y^{\nu}_A|^{k+1}) + \overline{\rho}\mathcal{O}(c^{-6}). \tag{6.36}$$

It can be shown that the expression on the right-hand side of relation (6.34) cannot be represented as four-dimensional divergence of any combination of generalized gravitational potentials and characteristics of the ideal fluid. Then for arbitrary functions $Q^{\alpha}_{A\{L\}}$, the expression (6.33)

cannot be reduced to the form of (6.27). However, since general relativity possesses all conservation laws, such a reduction is always possible, and it follows that we must require that all the functions $Q^{\alpha}_{A\{L\}}$ with $l \geq 3$ vanish:

$$Q^{\alpha}_{A\{L\}}(y^0_A) = 0, \qquad \forall l \geq 3. \tag{6.37a}$$

In addition to this, as we have noticed earlier, the functions $L_{A\{L\}}$ with ($l \geq 3$) do not enter the eq.m. in (6.33) at all and any choice of these functions will not affect the dynamics of the system of the extended bodies in the WFSMA. This suggests that these functions may be considered as the infinitesimal gauge functions and, without losing generality of the description, we may set these functions to be zero:

$$L_{A\{L\}}(y^0_A) = 0, \qquad \forall l \geq 3. \tag{6.37b}$$

Moreover, in correspondence with the definition in (6.27), in metric theories of gravitation that possess all conservation laws, expression (6.36) must then contain the components of the complete energy-momentum tensor of matter and gravitational field in pseudo-Euclidean space-time. Since below we shall mainly be interested in the components $t^{\alpha 0}$ of this tensor, comparing the expressions for it given by (6.32c) and (6.35), we can see that $t^{0\alpha}_i + t^{0\alpha}_g + t^{0\alpha}_M \neq t^{\alpha 0}_i + t^{\alpha 0}_g + t^{\alpha 0}_M$. Therefore, although it is possible to obtain the conservation laws of energy and momentum, it is not yet sufficient for obtaining the remaining conservation laws for which it is required that the components of the complete energy-momentum tensor of the system be symmetric. For our purposes, in order to ensure the symmetry of the complete energy-momentum tensor of the system, we should set

$$a_1 = -2, \qquad a_2 = 0. \tag{6.38}$$

Thus, a necessary (but not sufficient) condition for the existence of all conservation laws in any metric theory of gravitation is that relations (6.37) and (6.38) should hold.

With consideration of these equalities, the component t^{00} of (6.32a) of the complete energy-momentum tensor will have the form

$$t^{00}_i + t^{00}_g + t^{00}_M = \overline{\rho}\Big(1 + \Pi - \frac{1}{2}v_\mu v^\mu + 3\overline{U}\Big) + \frac{7}{8\pi}\partial_\mu \overline{U}\partial^\mu \overline{U} + \overline{\rho}\mathcal{O}(c^{-4}). \tag{6.39}$$

This expression can be used to describe the energy distribution of the system in space, while the component $t^{\alpha 0}$ of (6.35) can be used to describe the density of momentum. Integrating expression (6.39) for the energy-momentum tensor over the body (A)'s volume space and using the trivial relation

$$\int_A d^3y'_A \partial_\mu \overline{U}\partial^\mu \overline{U} = -4\pi \int_A d^3y'_A \overline{\rho}\overline{U} + \oint_A dS^\mu_A\, \overline{U}\partial_\mu \overline{U}, \tag{6.40}$$

we obtain the following expression for the energy P^0 of the system of matter, inertia, and gravitational field defined in the vicinity of the body (A) as usual:

$$P^0 \equiv m_A = \int_A d^3y'_A \Big(t^{00}_i + t^{00}_g + t^{00}_M\Big). \tag{6.41a}$$

This corresponds to the following result for the total mass of the fields in this RF_A:

$$m_A = \int_A d^3y'_A \overline{\rho}\Big(1 + \Pi - \frac{1}{2}v_\mu v^\mu - \frac{1}{2}\overline{U}\Big) + \frac{7}{8\pi}\oint_A dS^\mu_A\, \overline{U}\partial_\mu \overline{U} =$$

$$= \int_A d^3y'_A \hat{\rho}_A\Big(1 + \Pi - \frac{1}{2}v_\mu v^\mu - \frac{1}{2}U_A\Big) + \frac{7}{16\pi}\oint_A dS^\mu_A\, \partial_\mu \overline{U}^2 + m_A\mathcal{O}(c^{-3}). \qquad (6.41b)$$

The obtained result may be presented in terms of the unperturbed mass $m_{A(0)}$ of the body (A) as follows:

$$m_A = m_{A(0)} + \frac{7}{16\pi}\oint_A dS^\mu_A\, \partial_\mu \overline{U}^2 + m_A\mathcal{O}(c^{-3}), \qquad (6.42)$$

where the second term represents the contribution of the coupling of the proper gravitational field of the body under study to the external gravity. This term is zero in the case of an isolated body, because one may move the boundary of integration to infinite distance. Taking into account that the integrand behaves as r^{-3}, one makes the conclusion that this integral is zero. One loses this useful option in the case of the N-body system, and, due to this reason, we must take into account such 'surface' effects in order to correctly describe the perturbed motion of the bodies in the system.

The momentum P^α_A of the system of fields in the coordinates of this RF$_A$ is determined in an entirely analogous way: by integrating component $t^{\alpha 0}$ of (6.35) of the complete energy-momentum tensor over the compact volume of the body (A),

$$P^\alpha_A = \int_A d^3y'_A\Big(t^{\alpha 0}_i + t^{\alpha 0}_g + t^{\alpha 0}_M\Big). \qquad (6.43a)$$

Then for momentum P^α_A we obtain the following expression:

$$P^\alpha_A = \int_A d^3y'_A\Big[\overline{\rho}v^\alpha\Big(1 + \Pi - \frac{1}{2}v_\mu v^\mu + 3\overline{U} + \frac{p}{\rho}\Big) -$$

$$-\frac{3}{4\pi}\partial^\alpha\overline{U}\frac{\partial\overline{U}}{\partial y^0_A} + \frac{1}{\pi}\partial_\nu\overline{U}\Big(\partial^\alpha\overline{V}^\nu - \partial^\nu\overline{V}^\alpha\Big)\Big] + \overline{\rho}\mathcal{O}(c^{-5}). \qquad (6.43b)$$

Finally, the requirement of (3.29) may be fulfilled by integrating equation (6.34) over the volume of the body (A) and choosing the function $\delta\ddot{w}_{A_0}$ such that the corresponding momentum P^α_A in the RF$_A$ will vanish for all times. However, as we will see later, this requirement is not easy to satisfy. The problem one is faced with is that the system of the fields and matter overlapping the body (A) is not a closed system. This system is a part of a bigger ensemble of celestial bodies that was initially taken to be a closed N-body system. The definitions for the energy and momentum of the system may not be given in the *local* form; instead these quantities are non-zero in all regions of the system. As a result of such a non-locality, one loses the possibility of eliminating the integrals from the three-divergences. Thus, in the analysis of the conservational laws in the gravitational one-body problem, one can integrate such divergences by using the Stokes theorem and moving the surface of integration at the infinite distance (Fock, 1959; Denisov & Turyshev, 1989; Will, 1993). In the case of coordinates originated with the quasi-inertial proper RF, such an integration is meaningless. Instead, one may integrate the corresponding quantities on the surface of the body under consideration. As a result, one may see from expressions (6.32) that the mass in the proper RF is not a constant anymore. Thus, by integrating expression (6.32a) over the body (A)'s compact volume, we obtain

$$\frac{dm_A}{dy^0_A} = \frac{1}{4\pi}\oint_A dS^\beta_A\Big[3\partial_\beta\overline{U}\frac{\partial\overline{U}}{\partial y^0_A} + 4\partial_\nu\overline{U}\Big(\partial^\nu\overline{V}_\beta - \partial_\beta\overline{V}^\nu\Big)\Big] + m_A\mathcal{O}(c^{-5}). \qquad (6.44)$$

The integral on the right-hand side of the expression above vanishes in the case of an isolated distribution of matter, but for the N-body problem in the quasi-inertial RF it depends on the magnitude of the fields on the surface of the body under study. Analysis of the conservation laws is the only way to correctly define the important physical quantities, such as the mass, momentum, and angular momentum of the field in the *local* region of the body. One expects that, in the immediate vicinity of the origin of the coordinate system in the 'good' proper RF, the form of these laws should resemble that which was developed by Fock (1955) and Chandrasekhar (1965) for the inertial frames. Therefore, we will use the technique that was developed for the barycentric approach by modifying it for the case under consideration.

Here we must mention the following circumstance. It follows from eq.(6.35) and eq.(6.43) that, in the post-Newtonian approximation, the density of the total momentum of the system, in contrast to the barycentric RF_0, can be written in the coordinates of the proper RF only in the non-local form of (6.35) when the components $t^{\alpha 0}$ are non-zero, generally speaking, in the entire space. Unfortunately, this expression cannot be written in the *local* form that would be nonzero only in the region occupied by the body (A) because of the presence of external sources of gravity. Comparing (6.32*a*) and (6.40), we can draw an analogous conclusion regarding the energy density of the system. Since the total momentum and total energy of the system in the post-Newtonian approximation do not depend on the form in which one chooses to write them, the momentum and energy of the gravitational field, which are non-local by their nature, can be effectively considered in this approximation by adding *local* terms to the energy density of matter. The latter circumstance is especially convenient in computing the motion of complex systems, since it lets us distinguish in explicit form the total momentum and energy of each of the bodies of the system.

Therefore, we shall henceforth use the following expression for the density of the total momentum of matter, inertia, and the gravitational field in the volume occupied by the body:

$$\hat{t}^{\alpha 0} = \overline{\rho} v^{\alpha}\Big(1 + \Pi - \frac{1}{2} v_{\mu} v^{\mu} + 3\overline{U} + \frac{p}{\rho}\Big) -$$

$$-\frac{3}{4\pi}\partial^{\alpha}\overline{U}\frac{\partial \overline{U}}{\partial y_A^0} + \frac{1}{\pi}\partial_{\nu}\overline{U}\Big(\partial^{\alpha}\overline{V}^{\nu} - \partial^{\nu}\overline{V}^{\alpha}\Big)\Big] + \overline{\rho}\mathcal{O}(c^{-5}), \tag{6.45}$$

and for the total energy density, we shall use the expression

$$\hat{t}^{00} = \overline{\rho}\Big(1 + \Pi - \frac{1}{2} v_{\mu} v^{\mu} - \frac{1}{2}\overline{U}\Big) + \frac{7}{8\pi}\partial_{\mu}\Big[\overline{U}\partial^{\mu}\overline{U}\Big] + \overline{\rho}\mathcal{O}(c^{-4}). \tag{6.46}$$

The relationships obtained will be used in order to define the eq.m. of the extended bodies' forms with respect to the coordinates of the proper RF_A. Note that by integrating expressions (6.45) and (6.46) over the compact volumes of the bodies in the system, one may obtain the mass and the momentum of these bodies measured with respect to the proper RF_A. Such relative quantities may be very important in the analysis of the relativistic gravitational experiments in the solar system that we will discuss in the next section. In order to complete the formulation of the coordinate transformations to the 'good' proper RF_A, we should present the function that was not yet determined, namely the function $\delta w^{\alpha}_{A_0}$.

6.3 The Solution for the Function $\delta w^{\alpha}_{A_0}$.

To obtain the equation of motion of extended bodies in the gravitational field, we must first bring the covariant conservation equation (6.33) to the form

$$\frac{\partial}{\partial y^0_A}\hat{t}^{\alpha 0}(y^p_A) = \mathcal{F}^{\alpha}(y^p_A), \tag{6.47}$$

where $\hat{t}^{\alpha 0}$ is defined by (6.45), and $\mathcal{F}^{\alpha}$ represents the entire remaining part of (6.33) and can be considered the force density acting on matter. This is exactly the force we have mentioned in Section 3 while discussing expressions (3.27). After performing identity transformations using eqs.(6.3) and (6.6), we obtain from eq.(6.33):

$$\mathcal{F}^{\alpha}(y^p_A) = -\overline{\rho}\partial^{\alpha}\overline{U} + \overline{\rho}\partial^{\alpha}\overline{W} - \overline{\rho}\partial^{\alpha}\overline{U}\Big(\Pi - \frac{3}{2}v_{\mu}v^{\mu} + \frac{3p}{\overline{\rho}} + \overline{U}\Big) +$$

$$+\overline{\rho}\Big(y^{\alpha}_A y^{\lambda}_A + \frac{1}{2}\gamma^{\alpha\lambda}y_{A\mu}y^{\mu}_A\Big)\frac{1}{5}\ddot{a}_{A_0\lambda} + \frac{1}{4\pi}\frac{\partial}{\partial y^0_A}\Big[\partial^{\alpha}\overline{U}\frac{\partial\overline{U}}{\partial y^0_A}\Big] -$$

$$-\overline{\rho}y^{\lambda}_A\left(a^{\alpha}_{A_0}a_{A_0\lambda} - \delta^{\alpha}_{\lambda}\cdot a^{\mu}_{A_0}a_{A_0\mu} + \sum_{B\neq A}\frac{\partial}{\partial y^0_A}\Big[2\Big\langle\frac{\partial V^{\alpha)}_B}{\partial y^{(\lambda}_A}\Big\rangle_A + 2\Big\langle v^{(\alpha}\frac{\partial U_B}{\partial y^{\lambda)}_A}\Big\rangle_A - \delta^{\alpha}_{\lambda}\frac{\partial}{\partial y^0_A}\Big\langle U_B\Big\rangle_A\Big]\right) -$$

$$-\frac{1}{\pi}\frac{\partial}{\partial y^{\beta}_A}\left(\partial^{\alpha}\overline{U}\frac{\partial\overline{V}^{\beta}}{\partial y^0_A} + \partial^{\beta}\overline{U}\frac{\partial\overline{V}^{\alpha}}{\partial y^0_A} - \gamma^{\alpha\beta}\partial_{\nu}\overline{U}\frac{\partial\overline{V}^{\nu}}{\partial y^0_A} + \partial^{[\nu}\overline{V}^{\alpha]}\partial_{[\nu}\overline{V}^{\beta]} -\right.$$

$$\left.-\gamma^{\alpha\beta}\partial^{[\mu}\overline{V}^{\nu]}\partial_{[\mu}\overline{V}_{\nu]} + \frac{1}{2}\gamma^{\alpha\beta}\Big(\frac{\partial\overline{U}}{\partial y^0_A}\Big)^2 + \pi(1+2\overline{U})\hat{T}^{\alpha\mu}\right) - \overline{\rho}\delta\ddot{w}^{\alpha}_{A_0}(y^0_A) + \overline{\rho}\mathcal{O}(c^{-6}). \tag{6.48}$$

The eq.m. of each body can be obtained if (6.44) is integrated over the volume occupied by this body. In order to find the function $\delta w^{\alpha}_{A_0}$, we should start with finding the eq.m. of the body (A) relative to its own RF. Integrating (6.48) over the V_A, we obtain

$$\frac{dP^{\alpha}_A}{dy^0_A} = F^{\alpha}_A(y^0_A), \tag{6.49}$$

where P^{α}_A is given by expression (6.43b) and

$$F^{\alpha}_A(y^0_A) = \int_A d^3y'_A\mathcal{F}^{\alpha}(y^0_A, y'^{\nu}). \tag{6.50}$$

In order to define the function $\delta w_{A_0}(y^0_A)$, we will require that the momentum of the body (A) in its proper RF will vanish. This requirement may be fulfilled if the equation for $\delta\ddot{w}_{A_0}(y^0_A)$ is chosen in the following form:

$$\delta\ddot{w}_{A_0}(y^0_A) = -\Big\langle\partial^{\alpha}\overline{U}\Big(\Pi - \frac{3}{2}v_{\mu}v^{\mu} + \frac{3p}{\overline{\rho}} + \overline{U}\Big)\Big\rangle_A +$$

$$+\frac{1}{5}\ddot{a}_{A_0\mu}\int_A d^3y'^{\nu}_A\hat{\rho}_A\Big(y'^{\alpha}_A y'^{\mu}_A + \frac{1}{2}\gamma^{\alpha\mu}y'_{A_\lambda}y'^{\lambda}_A\Big) -$$

$$-\frac{1}{\pi m_A}\oint_A dS^{\beta}_A\left(\partial^{\alpha}\overline{U}\frac{\partial\overline{V}_{\beta}}{\partial y^0_A} + \partial_{\beta}\overline{U}\frac{\partial\overline{V}^{\alpha}}{\partial y^0_A} - \delta^{\alpha}_{\beta}\partial_{\lambda}\overline{U}\frac{\partial\overline{V}^{\alpha}}{\partial y^0_A} + \partial^{[\nu}\overline{V}^{\alpha]}\partial_{[\nu}\overline{V}_{\beta]} -\right.$$

$$\left.-\frac{1}{2}\delta^{\alpha}_{\beta}\partial^{[\mu}\overline{V}^{\nu]}\partial_{[\mu}\overline{V}_{\nu]} + \frac{1}{2}\delta^{\alpha}_{\beta}\Big(\frac{\partial\overline{U}}{\partial y^0_A}\Big)^2\right) + \mathcal{O}(c^{-6}). \tag{6.51}$$

As a result, one may obtain the differential equation for the total post-Newtonian acceleration $\underline{\ddot{w}}^{\alpha}_{A_0}$ from (6.13), which is necessary to apply in order to hold the extended body in the state of equilibrium in its proper RF. Thus, with the help of expressions (6.15) and (6.51), we obtain the following equation:

$$\underline{\ddot{w}}^{\alpha}_{A_0}(y^0_A) = \sum_{B\neq A}\Big(\gamma^{\alpha\mu}\Big\langle\frac{\partial W_B}{\partial y^{\mu}_A}\Big\rangle_A + v^{\alpha}_{A_0}\frac{\partial}{\partial y^0_A}\Big\langle U_B\Big\rangle_A - 4\frac{\partial}{\partial y^0_A}\Big\langle V^{\alpha}_B\Big\rangle_A\Big) -$$

$$+a_{A_0\beta}\int^{y^0_A} dt'\Big(\frac{1}{2}a^{[\alpha}_{A_0}v^{\beta]}_{A_0} + 2\sum_{B\neq A}\Big[\Big\langle\partial^{[\alpha}V^{\beta]}_B\Big\rangle_A + \Big\langle v^{[\beta}\partial^{\alpha]}U_B\Big\rangle_A\Big]\Big) -$$

$$-\frac{1}{2}v^{\alpha}_{A_0}v_{A_0\beta}a^{\beta}_{A_0} + a^{\alpha}_{A_0}\sum_{B\neq A}\Big\langle U_B\Big\rangle_A - \Big\langle\partial^{\alpha}\overline{U}\Big(\Pi - \frac{3}{2}v_{\mu}v^{\mu} + \frac{3p}{\overline{\rho}} + \overline{U}\Big)\Big\rangle_A +$$

$$+\frac{1}{5}\ddot{a}_{A_0\mu}\int_A d^3y'^{\nu}_A\hat{\rho}_A\Big(y'^{\alpha}_A y'^{\mu}_A + \frac{1}{2}\gamma^{\alpha\mu}y'_{A_\lambda}y'^{\lambda}_A\Big) -$$

$$-\frac{1}{\pi m_A}\oint_A dS^{\beta}_A\Big(\partial^{\alpha}\overline{U}\frac{\partial\overline{V}_{\beta}}{\partial y^0_A} + \partial_{\beta}\overline{U}\frac{\partial\overline{V}^{\alpha}}{\partial y^0_A} - \delta^{\alpha}_{\beta}\partial_{\lambda}\overline{U}\frac{\partial\overline{V}^{\alpha}}{\partial y^0_A} + \partial^{[\nu}\overline{V}^{\alpha]}\partial_{[\nu}\overline{V}_{\beta]} -$$

$$-\frac{1}{2}\delta^{\alpha}_{\beta}\partial^{[\mu}\overline{V}^{\nu]}\partial_{[\mu}\overline{V}_{\nu]} + \frac{1}{2}\delta^{\alpha}_{\beta}\Big(\frac{\partial\overline{U}}{\partial y^0_A}\Big)^2\Big) + \mathcal{O}(c^{-6}). \tag{6.52}$$

For practical purposes, one may find the value of the surface integrals in expression (6.52) by performing the iteration procedure. Thus, it is easy to show that the lowest multipole moments of the bodies will not contribute to this surface integration. However, the general results will fully depend on the non-linear interaction of the intrinsic multipole moments with the external gravity in the *local* region at the vicinity of the body under consideration. This additional iterative option will make all the results obtained with the proposed formalism easy to use in the practical applications.

As one may see, we have reconstructed the post-Newtonian non-linear group of motion for the WFSMA. Thus, the straight transformation is given by eqs.(6.11)–(6.13). Substitution of the results obtained for the transformation functions in relations (3.18) will give the inverse transformation. Finally, the common element of this group may be obtained by making use of relations (3.19)–(3.20). These results generalize and specify those obtained by Chandrasekhar & Contopulos (1967) and given by (1.12). In this previous work, the post-Galilean transformations that preserve the invariancy of the metric tensor were obtained. In contrast to these, our transformations, eqs.(6.11)–(6.13), in general, transform the coordinates in different non-inertial RFs and were specifically defined for a system of self-gravitating extended and arbitrarily shaped bodies. Comparison with the Poincaré group of motion, (1.7), expanded similarly in inverse powers of c shows the following:

(i). The spatial part of the transformations up to the terms $\mathcal{O}(c^{-2})$ includes the Lorentzian terms and allows, in addition, infinitesimal rotations, uniform motion, the shift of the origin, and the terms due to the gravitational coupling of the internal multipoles of the extended bodies with the external gravitation.

(ii). The temporal part of the transformation includes the Lorentzian terms up to $\mathcal{O}(c^{-4})$ plus additional terms of a purely gravitational nature, as well as the terms due to precession of the spatial axes. It is the presence of these gravitational terms in both spatial and temporal components of the transformation that gives the transformation its non-Lorentzian character.

As one can see, the obtained coordinate transformations are in general the non-local ones. As such, they represent an important and powerful way to study the nature of the multipolar structure of a system of extended bodies and their gravitational interaction in the WFSMA of the general theory of relativity. In the next section, we will discuss the generalization of the obtained results to the case of the scalar-tensor theories of gravity.

7 Parameterized Proper RF.

In this section, we will further generalize the results obtained for the coordinate transformations and the metric tensor in the proper RF, which were obtained in the previous section. In order to generalize the results obtained, we have applied the presented formalism to the scalar-tensor theories of gravity. It should be noted that considerable interest has recently been shown in the physical processes occurring in the strong gravitational field regime. However, many modern theoretical models that include general relativity as a standard gravity theory are faced with the problem of the unavoidable appearance of space-time singularities. It is well known that the classical description, provided by general relativity, breaks down in a domain where the curvature is large, and, hence, a proper understanding of such regions requires new physics (Horowitz & Myers, 1995). The tensor-scalar theories of gravity, in which the usual general relativity tensor field coexists together with one or several long-range scalar fields, are believed to be the most interesting extension of the theoretical foundation of modern gravitational theory. The superstring, many-dimensional Kaluza–Klein and inflationary cosmology theories have revived interest in so-called 'dilaton fields,' *i.e.*, neutral scalar fields whose background values determine the strength of the coupling constants in the effective four-dimensional theory. However, although the scalar field naturally arises in theory, its existence leads to a violation of the strong equivalence principle and modification of large-scale gravitational phenomena (Damour *et al.*, 1990; Damour & Taylor, 1992; Damour & Esposito–Farese, 1992; Damour & Nordtvedt, 1993; Berkin & Hellings, 1994; Turyshev, 1996). Moreover, the presence of the scalar field affects the equations of motion of the other matter fields as well (Turyshev, 1996), which makes it interesting to study the opportunities for advanced dynamical tests of these theories in the WFSMA before they will be applied to the strong-field-regime research. Therefore, the motivation for the present work was to perform a similar full-scale analysis of the WFSMA for some tensor-scalar theories of gravity in order to generalize the results obtained previously.

7.1 Parameterized Coordinate Transformations.

In this subsection, we will present the results of the relativistic study of the Brans–Dicke theory of gravity (Will, 1993). However, due to the length of the expressions and also in order to avoid unreasonable complication of the discussion in this section, we will not present here the details of these calculations. Instead, we have introduced the two Eddington parameters (γ, β) in order to present the obtained relations in a more compact form valid for a number of modern metric theories of gravity. This gives us a chance to present the final results only. One may repeat the necessary calculations using the technique of the general post-Newtonian power expansions in the WFSMA developed in the appendices.

By taking into account the properties of scalar-tensor theories of gravity, and by applying the rules for constructing the proper RF presented in Section 2, one may obtain the set of differential equations on the transformation functions $\mathcal{K}_A$, $\mathcal{Q}^\alpha_A$, and $\mathcal{L}_A$. As a result, the relativistic coordinate transformation between coordinates (x^p) of RF_0 to those (y^p_A) of the proper quasi-inertial RF_A of an arbitrary body (A) may be given as follows:

$$x^0 = y^0_A + c^{-2}\mathcal{K}_A(y^0_A, y^\epsilon_A) + c^{-4}\mathcal{L}_A(y^0_A, y^\epsilon_A) + \mathcal{O}(c^{-6}), \tag{7.1a}$$

$$x^\alpha = y^\alpha_A + y^\alpha_{A_0}(y^0_A) + c^{-2}\mathcal{Q}^\alpha_A(y^0_A, y^\epsilon_A) + \mathcal{O}(c^{-4}). \tag{7.1b}$$

One may obtain the necessary corrections of the third order with respect to spatial separation for functions (7.2) in a manner analogous to that used for the derivations in Section 5. Then the

parameterized coordinate transformation functions $\mathcal{K}_A$, $\mathcal{Q}^\alpha_A$, and $\mathcal{L}_A$ may be given as follows:

$$\mathcal{K}_A(y^0_A, y^\nu_A) = \int^{y^0_A} dt' \Big(\sum_{B\neq A} \big\langle U_B \big\rangle_A - \frac{1}{2} v_{A_0\nu} v^\nu_{A_0} \Big) - v_{A_0\nu} \cdot y^\nu_A + \mathcal{O}(c^{-4}) y^0_A, \tag{7.2a}$$

$$\begin{aligned}
\mathcal{Q}^\alpha_A(y^0_A, y^\nu_A) = -\gamma \sum_{B\neq A} \Big(& y^\alpha_A y^\beta_A \cdot \big\langle \partial_\beta U_B \big\rangle_A - \frac{1}{2} y_{A\beta} y^\beta_A \big\langle \partial^\alpha U_B \big\rangle_A + y^\alpha_A \big\langle U_B \big\rangle_A \Big) + \\
& - \frac{1}{2} v^\alpha_{A_0} v^\beta_{A_0} y_{A\beta} + w^\alpha_A(y^0_A) + \\
& + y_{A\beta} \int^{y^0_A} dt' \Big(\frac{1}{2} a^{[\alpha}_{A_0} v^{\beta]}_{A_0} + (\gamma+1) \sum_{B\neq A} \Big[\big\langle \partial^{[\alpha} V^{\beta]}_B \big\rangle_A + \big\langle \partial^{[\alpha} U_B v^{\beta]} \big\rangle_A \Big] \Big) - \\
& + \frac{1}{6} \gamma \sum_{B\neq A} y^\mu_A y^\nu_A y^\beta_A \cdot \Big(\eta^{\alpha\lambda} \eta_{\mu\nu} \big\langle \partial^2_{\beta\lambda} U_B \big\rangle_0 - 2\delta^\alpha_\beta \big\langle \partial^2_{\mu\nu} U_B \big\rangle_0 \Big) + \mathcal{O}(|y^\nu_A|^4) + \mathcal{O}(c^{-4}) y^\alpha_A,
\end{aligned} \tag{7.2b}$$

$$\begin{aligned}
\mathcal{L}_A(y^0_A, y^\nu_A) = \sum_{B\neq A} \Big(& \frac{1}{2} \gamma\, y_{A\beta} y^\beta_A \cdot \frac{\partial}{\partial y^0_A} \big\langle U_B \big\rangle_A - (\gamma+1)\, y^\lambda_A y^\beta_A \cdot \Big[\big\langle \partial_\lambda V_{B\beta} \big\rangle_A + \big\langle v_\beta \partial_\lambda U_B \big\rangle_A \Big] + \\
& + \gamma\, v_{A_0\beta} \Big[y^\beta_A y^\lambda_A \cdot \big\langle \partial_\lambda U_B \big\rangle_A - \frac{1}{2} y_{A\lambda} y^\lambda_A \cdot \big\langle \partial^\beta U_B \big\rangle_A \Big] \Big) + \\
& + y_{A\lambda} v_{A_0\beta} \int^{y^0_A} dt' \Big(\frac{1}{2} a^{[\lambda}_{A_0} v^{\beta]}_{A_0} + (\gamma+1) \sum_{B\neq A} \Big[\big\langle \partial^{[\lambda} V^{\beta]}_B \big\rangle_A + \big\langle v^{[\beta} \partial^{\lambda]} U_B \big\rangle_A \Big] \Big) + \\
& + y_{A\beta} \Big[(\gamma+1)\, v^\beta_{A_0} \sum_{B\neq A} \big\langle U_B \big\rangle_A - 2(\gamma+1) \sum_{B\neq A} \big\langle V^\beta_B \big\rangle_A - \dot{w}^\beta_{A_0}(y^0_A) \Big] - \\
& - \int^{y^0_A} dt' \Big[\sum_{B\neq A} \big\langle W_B \big\rangle_A + \frac{1}{2} \Big(\sum_{B\neq A} \big\langle U_B \big\rangle_A - \frac{1}{2} v_{A_0\beta} v^\beta_{A_0} \Big)^2 + v_{A_0\mu} \dot{w}^\mu_{A_0}(t') \Big] + \\
& + \frac{1}{6} \sum_{B\neq A} y^\mu_A y^\nu_A y^\beta_A \cdot \Big(\gamma\, \eta_{\mu\nu} \big\langle \partial^2_{0\beta} U_B \big\rangle_0 - 2(\gamma+1) \big\langle \partial^2_{\mu\nu} V_{B\beta} \big\rangle_0 - \\
& - \gamma\, v_{A_0\sigma} \Big[\eta^{\sigma\lambda} \eta_{\mu\nu} \big\langle \partial^2_{\beta\lambda} U_B \big\rangle_0 - 2\delta^\sigma_\beta \big\langle \partial^2_{\mu\nu} U_B \big\rangle_0 \Big] \Big) + \mathcal{O}(|y^\nu_A|^4) + \mathcal{O}(c^{-6}).
\end{aligned} \tag{7.2c}$$

Note that, in order to distinguish between the PPN parameter γ and the Minkowski tensor γ_{mn}, we are using new notation for this tensor, namely: $\eta_{mn} \equiv \gamma_{mn} = \mathrm{diag}(1, -1, -1, -1)$. The time-dependent functions $Q^\alpha_{A\{L\}}$ and $L_{A\{L\}}$ in expressions (7.2) are the contributions coming from the higher multipoles ($l \geq 3$) (both mass and current induced) of the external gravitational field generated by the bodies (B≠A) in the system. These functions enable one to take into account the geometric features of the proper RF_A with respect to three-dimensional spatial rotation. The form of these functions may be chosen arbitrarily. This freedom enables one to choose any coordinate dependence for the terms with $l \geq 3$ in order to describe the motion of the highest monopoles. Moreover, one may show that, even though the total solution to the metric tensor $g_{mn}(x^p)$ in the barycentric inertial RF_0 resembles the form of the one-body solution, eqs.(2.5), if one expresses this solution through the proper multipole moments of the bodies, it will contain the contributions coming from the functions $Q^\alpha_{A\{L\}}(y^0_A)$ and $L_{A\{L\}}(y^0_A)$. As a result, the metric

tensor in the proper RF_A fully represents the tidal nature of the external gravity in the coordinates of this frame.

The quasi-Newtonian acceleration of the body (A) with respect to the barycentric RF_0 may be described as follows:

$$a^{\alpha}_{A_0}(y^0_A) = -\eta^{\alpha\mu} \sum_{B\neq A} \Big\langle \frac{\partial U_B}{\partial y^{\mu}_A} \Big\rangle_A + \mathcal{O}(c^{-6}). \tag{7.3a}$$

With the accuracy necessary for future analysis, we present the equation for the time-dependent function $\underline{w}^{\alpha}_{A_0}(y^0_A)$ with respect to the time y^0_A as follows:

$$\begin{aligned}
\underline{\ddot{w}}^{\alpha}_{A_0}(y^0_A) = \sum_{B\neq A} \Big(\eta^{\alpha\mu}\Big\langle \frac{\partial W_B}{\partial y^{\mu}_A}\Big\rangle_0 + v^{\alpha}_{A_0}\frac{\partial}{\partial y^0_A}\Big\langle U_B\Big\rangle_0 - 2(\gamma+1)\,\frac{\partial}{\partial y^0_A}\Big\langle V^{\alpha}_B\Big\rangle_0\Big) - \\
-\frac{1}{2}v^{\alpha}_{A_0}v_{A_0\beta}a^{\beta}_{A_0} + \gamma\, a^{\alpha}_{A_0}\sum_{B\neq A}\Big\langle U_B\Big\rangle_0 + \\
+a_{A_0\beta}\int^{y^0_A} dt'\Big(\frac{1}{2}a^{[\alpha}_{A_0}v^{\beta]}_{A_0} + (\gamma+1)\sum_{B\neq A}\Big[\Big\langle \partial^{[\alpha}V^{\beta]}_B\Big\rangle_A + \Big\langle v^{[\beta}\partial^{\alpha]}U_B\Big\rangle_A\Big]\Big) + \\
+\frac{1}{5}\ddot{a}_{A_0\mu}\int_A d^3y'_A\hat{\rho}_A\Big(y'^{\alpha}_A y'^{\mu}_A + \frac{1}{2}\gamma^{\alpha\mu}y'_{A_\lambda}y'^{\lambda}_A\Big) + \mathcal{O}(c^{-6}).
\end{aligned} \tag{7.3b}$$

Expressions (7.3) are the two parts of the force necessary to keep the body (A) in its orbit (world tube) in the N-body system. These expressions are written in the proper time and, if one performs the coordinate transformation from coordinates (y^p_A) to those of (x^p) for all the functions and potentials entering both equations, (7.3), and takes into account the lowest intrinsic multipole moments of the bodies only, one will obtain the simplified equations of motion for the extended bodies, (2.14)–(2.20), written in coordinates (x^p) of the barycentric inertial RF_0.

Finally, the metric tensor, corresponding to the coordinate transformations (7.1)–(7.3), will take the form of the Fermi-normal-like proper RF_A chosen to study the physical processes in the vicinity of the body (A):

$$\begin{aligned}
g^{\mathcal{F}}_{00}(y^p_A) = 1 - 2\overline{U}(y^p_A) + 2W_A(y^p_A) + \\
+\Bigg(\sum_{B\neq A}\Bigg[\Big\langle \partial^2_{\mu\nu}W_B\Big\rangle_0 + \frac{\partial}{\partial y^0_A}\Big(\gamma\,\eta_{\mu\nu}\Big\langle\frac{\partial U_B}{\partial y^0_A}\Big\rangle_0 - (\gamma+1)\Big[\Big\langle\partial_{(\nu}V_{B\mu)}\Big\rangle_0 + \Big\langle\partial_{(\nu}U_B v_{\mu)}\Big\rangle_A\Big]\Big)\Bigg] + \\
+\gamma\,\eta_{\mu\nu}\,a_{A_0\lambda}a^{\lambda}_{A_0} - (2\gamma-1)\,a_{A_0\mu}a_{A_0\nu}\Bigg)\cdot y^{\mu}_A y^{\nu}_A + \mathcal{O}(|y^{\nu}_A|^3) + \mathcal{O}(c^{-6}),
\end{aligned} \tag{7.4a}$$

$$\begin{aligned}
g^{\mathcal{F}}_{0\alpha}(y^p_A) = 2(\gamma+1)\eta_{\alpha\epsilon}\Big[V^{\epsilon}_A(y^p_A) + \sum_{B\neq A} y^{\mu}_A\Big\langle v^{\epsilon}\partial_{\mu}U_B\Big\rangle_A\Big] + \frac{2}{3}\Big(\gamma\,\Big(\eta_{\alpha\mu}\dot{a}_{A_0\nu} - \eta_{\mu\nu}\dot{a}_{A_0\alpha}\Big) + \\
+(\gamma+1)\sum_{B\neq A}\Big[\eta_{\alpha\lambda}\Big\langle\partial^2_{\mu\nu}V^{\lambda}_B\Big\rangle_0 - \eta_{\nu\lambda}\Big\langle\partial^2_{\mu\alpha}V^{\lambda}_B\Big\rangle_0\Big]\Big)\cdot y^{\mu}_A y^{\nu}_A + \mathcal{O}(|y^{\nu}_A|^3) + \mathcal{O}(c^{-5}),
\end{aligned} \tag{7.4b}$$

$$g^{\mathcal{F}}_{\alpha\beta}(y^p_A) = \eta_{\alpha\beta}\Big(1 + 2\gamma U_A(y^p_A)\Big) + \frac{1}{3}\gamma \sum_{B\neq A}\Big[\eta_{\alpha\beta}\Big\langle \partial^2_{\mu\nu}U_B\Big\rangle_0 + \eta_{\mu\nu}\Big\langle \partial^2_{\alpha\beta}U_B\Big\rangle_0 -$$

$$-\eta_{\beta\mu}\Big\langle \partial^2_{\alpha\nu}U_B\Big\rangle_0 - \eta_{\alpha\nu}\Big\langle \partial^2_{\beta\mu}U_B\Big\rangle_0\Big]\cdot y^\mu_A y^\nu_A + \mathcal{O}(|y^\nu_A|^3) + \mathcal{O}(c^{-4}), \tag{7.4c}$$

where the total Newtonian potential $\overline{U}$ in the vicinity of the body (A) is given by expression (6.17). Both functions W_A and W_B are parameterized by the two Eddington parameters γ and β as follows:

$$W_A(y^p_A) = \beta\, U^2_A(y^p_A) + \Psi_A(y^p_A) + 2a^\lambda_{A_0}\cdot\frac{\partial}{\partial y^\lambda_A}\chi_A(y^p_A) + \frac{1}{2}\frac{\partial^2}{\partial {y^0_A}^2}\chi_A(y^p_A) +$$

$$+\sum_{B\neq A}\left(2\beta\, U_A(y^p_A)U_B(y^p_A) - (3\gamma + 1 - 2\beta)\int_A \frac{d^3y'_A}{|y^\nu_A - y'^\nu_A|}\rho_A(y^0_A, y'^\nu_A)U_B(y^0_A, y'^\nu_A)\right) +$$

$$+\frac{1}{6}\gamma\sum_{B\neq A}\left(\eta^{\alpha\lambda}\eta_{\mu\nu}\Big\langle \partial^2_{\beta\lambda}U_B\Big\rangle_0 - 2\delta^\alpha_\beta\Big\langle \partial^2_{\mu\nu}U_B\Big\rangle_0\right)\times$$

$$\times\int_A d^3y'_A\rho_A(y^0_A, y'^\nu_A)\frac{\partial}{\partial y'^\lambda_A}\Big[\frac{y^\mu_A y^\nu_A y^\beta_A - y'^\mu_A y'^\nu_A y'^\beta_A}{|y^\nu_A - y'^\nu_A|}\Big] + \mathcal{O}(|y^\nu_A|^4) + \mathcal{O}(c^{-4})y^\alpha_A, \tag{7.5a}$$

$$W_B(y^p_A) = \beta U_B(y^p_A)\sum_{C\neq A}U_C(y^p_A) + \Psi_B(y^p_A) + 2a^\lambda_{A_0}\cdot\frac{\partial}{\partial y^\lambda_A}\chi_B(y^p_A) + \frac{1}{2}\frac{\partial^2}{\partial {y^0_A}^2}\chi_B(y^p_A) -$$

$$-(3\gamma + 1 - 2\beta)\int_B \frac{d^3y'_A}{|y^\nu_A - y'^\nu_A|}\rho_B(y^0_A, y'^\nu_A)\sum_{B'}U_{B'}(y^0_A, y'^\nu_A) +$$

$$+\frac{1}{6}\gamma\sum_{B\neq A}\left(\eta^{\alpha\lambda}\eta_{\mu\nu}\Big\langle \partial^2_{\beta\lambda}U_B\Big\rangle_0 - 2\delta^\alpha_\beta\Big\langle \partial^2_{\mu\nu}U_B\Big\rangle_0\right)\times$$

$$\times\int_A d^3y'_A\rho_B\Big(y^0_A, y'^\nu_A + y^\nu_{BA_0}(y^0_A)\Big)\frac{\partial}{\partial y'^\lambda_A}\Big[\frac{y^\mu_A y^\nu_A y^\beta_A - y'^\mu_A y'^\nu_A y'^\beta_A}{|y^\nu_A - y'^\nu_A|}\Big] + \mathcal{O}(|y^\nu_A|^4) + \mathcal{O}(c^{-4})y^\alpha_A. \tag{7.5b}$$

The functions W_A and W_B fully represent the non-linearity of the total post-Newtonian gravitational field in the Fermi-normal-like coordinates of the proper RF. As a result, we have obtained the metric tensor in the Fermi-normal-like coordinates and the coordinate transformations leading to this form. These transformations are defined up to the third order with respect to the spatial coordinates. Let us note that, as a partial result of the analysis presented in the previous section, we have shown that the Fermi-normal-like coordinates do not provide one with the conservation laws of the joint density of the energy-momentum of matter, inertia, and the gravitational field in the immediate vicinity of the body under consideration. However, taking into account the expected accuracy of the radio-tracking data from the future *Mercury Orbiter* mission, we can neglect the influence of the corresponding effects and, therefore, use the Fermi-normal-like coordinates for our theoretical studies. As a result, we will analyze the motion of the spacecraft in orbit around Mercury from the position of parameterized relativistic gravity.

7.2 Equations of the Spacecraft Motion.

We will now obtain the equations of the spacecraft motion in a Hermean-centric RF. To do this, we consider a Riemann space-time whose metric coincides with the metric of N moving extended bodies. We shall study the motion of a point body in the neighborhood of the body (A). The expression for the acceleration of the point body $a^{\alpha}_{(0)}$ can be obtained in two ways—either by using the equations of geodesics of Riemann space-time $du^n/ds + \Gamma^n_{mk}u^m u^k = 0$ or by computing the acceleration of the center of mass of the extended body and then letting all quantities characterizing its internal structure and proper gravitational field tend to zero. In either case, one obtains the same result (Denisov & Turyshev, 1990).

In order to obtain the Hermean-centric equations of the satellite motion, we will write out the equations of geodesics to the required degree of accuracy. For $n = \alpha$, we have

$$\frac{du^\alpha}{ds} + \Gamma^\alpha_{00}u^0u^0 + 2\Gamma^\alpha_{0\beta}u^0u^\beta + \Gamma^\alpha_{\mu\beta}u^\mu u^\beta = \mathcal{O}(c^{-6}). \tag{7.6}$$

We consider the metric tensor of Riemann space-time to be given by expressions (7.4) in this case. It is then possible to find the connection components of Riemann space-time needed for subsequent computations:

$$\Gamma^\alpha_{00}(y^p_A) = \eta^{\alpha\lambda}\Big[\frac{\partial\overline{U}}{\partial y^\lambda_A} - \frac{\partial W_A}{\partial y^\lambda_A} - \gamma\frac{\partial U^2_A}{\partial y^\lambda_A}\Big] + 2(\gamma+1)\frac{\partial V^\alpha_A}{\partial y^0_A} +$$

$$+\Bigg((2\gamma-1)a^\alpha_{A_0}a_{A_0\mu} - \gamma\delta^\alpha_\mu\cdot a^\lambda_{A_0}a_{A_0\lambda} - \eta^{\alpha\lambda}\sum_{B\neq A}\Big[\Big\langle\partial^2_{\lambda\mu}W_B\Big\rangle_0 + 2\gamma U_A\Big\langle\partial^2_{\lambda\mu}U_B\Big\rangle_0\Big] +$$

$$+\sum_{B\neq A}\frac{\partial}{\partial y^0_A}\Bigg[(\gamma+1)\Big[\Big\langle\partial_{(\mu}V^{\alpha)}_B\Big\rangle_0 + \Big\langle v^{(\alpha}\partial_{\mu)}U_B\Big\rangle_A\Big] - \gamma\delta^\alpha_\lambda\Big\langle\frac{\partial U_B}{\partial y^0_A}\Big\rangle_0\Bigg]\Bigg)\cdot y^\mu_A + \mathcal{O}(|y^\nu_A|^2) + \mathcal{O}(c^{-6}), \tag{7.7a}$$

$$\Gamma^\alpha_{0\beta}(y^p_A) = \gamma\delta^\alpha_\beta\frac{\partial U_A}{\partial y^0_A} + (\gamma+1)\delta^\mu_{[\beta}\frac{\partial V^{\alpha]}_A}{\partial y^\mu_A} + \gamma\eta_{\beta\mu}\dot{a}^{[\mu}_{A_0}y^{\alpha]}_A +$$

$$+(\gamma+1)\sum_{B\neq A}\Big[\Big\langle\partial^2_{[\beta\mu}V^{\alpha]}_B\Big\rangle_0 - \Big\langle v^{[\alpha}\partial_{\beta]}U_B\Big\rangle_A\Big]\cdot y^\mu_A + \mathcal{O}(|y^\nu_A|^2) + \mathcal{O}(c^{-5}), \tag{7.7b}$$

$$\Gamma^\alpha_{\beta\omega}(y^p_A) = \gamma\Big(\delta^\alpha_\beta\frac{\partial U_A}{\partial y^\omega_A} + \delta^\alpha_\omega\frac{\partial U_A}{\partial y^\beta_A} - \eta_{\beta\omega}\eta^{\alpha\lambda}\frac{\partial U_A}{\partial y^\lambda_A}\Big) +$$

$$+\frac{1}{3}\gamma\sum_{B\neq A}\Big[\eta_{\beta\mu}\Big\langle\partial^\alpha\partial_\omega U_B\Big\rangle_0 + \eta_{\omega\mu}\Big\langle\partial^\alpha\partial_\beta U_B\Big\rangle_0 + \delta^\alpha_\omega\Big\langle\partial^2_{\beta\mu}U_B\Big\rangle_0 + \delta^\alpha_\beta\Big\langle\partial^2_{\omega\mu}U_B\Big\rangle_0 -$$

$$-2\eta_{\beta\omega}\Big\langle\partial^\alpha\partial_\mu U_B\Big\rangle_0 - 2\delta^\alpha_\mu\Big\langle\partial^2_{\beta\omega}U_B\Big\rangle_0\Big]\cdot y^\mu_A + \mathcal{O}(|y^\nu_A|^2) + \mathcal{O}(c^{-4}). \tag{7.7c}$$

To reduce the equation of geodesic motion, (7.6), we shall use both the expressions above and the definition for the four-vector of velocity in the form

$$u^n = \frac{dy_A^n}{dy_A^0}\Big(g_{00} + 2g_{0\mu}v^\mu + g_{\mu\lambda}v^\mu v^\lambda\Big)^{-1/2}.$$

Then by taking into account that $d/ds = u^0 d/dy_A^0$ (with the components of the three-dimensional velocity vector of the point body denoted as $v_{(0)}^\alpha = dy_A^\alpha/dy_A^0$) and by using the Newtonian equation of motion of a point body as

$$a_{(0)}^\alpha = \frac{dv_{(0)}^\alpha}{dy_A^0} = -\eta^{\alpha\mu}\frac{\partial\overline{U}}{\partial y_A^\mu} + \mathcal{O}(c^{-4}),$$

we may make the following simplification:

$$v_{(0)}^\alpha \frac{d\ln u^0}{dy_A^0} = v_{(0)}^\alpha\Big(\frac{\partial\overline{U}}{\partial y_A^0} + 2v_{(0)}^\mu\frac{\partial\overline{U}}{\partial y_A^\mu} + \mathcal{O}(c^{-5})\Big).$$

Substituting this relation into the equations of motion, (7.6), we find the acceleration $a_{(0)}^\alpha$ of the point body:

$$\begin{aligned}
a_{(0)}^\alpha = {} & -\eta^{\alpha\mu}\frac{\partial\overline{U}}{\partial y_A^\mu}(1-2\gamma U_A) + \partial^\alpha W_A + \sum_{B\neq A}\Big\langle\partial^\alpha\partial_\mu W_B\Big\rangle_0\cdot y_A^\mu - \\
& -(2\gamma+1)v_{(0)}^\alpha\frac{\partial U_A}{\partial y_A^0} - 2(\gamma+1)\frac{\partial V_A^\alpha}{\partial y_A^0} - 2(\gamma+1)v_{(0)\mu}\Big[\partial^{[\mu}V_A^{\alpha]} + \sum_{B\neq A}\Big\langle v^{[\mu}\partial^{\alpha]}U_B\Big\rangle_0\Big] + \\
& +\gamma v_{(0)\lambda}v_{(0)}^\lambda\Big[\partial^\alpha U_A + \frac{2}{3}\sum_{B\neq A}\Big\langle\partial^\alpha\partial_\mu U_B\Big\rangle_0\cdot y_A^\mu\Big] - \\
& -v_{(0)}^\alpha v_{(0)}^\lambda\Big[2(\gamma+1)\partial_\lambda U_A + \frac{2}{3}(\gamma+3)\sum_{B\neq A}\Big\langle\partial_\lambda\partial_\mu U_B\Big\rangle_0\cdot y_A^\mu\Big] + \\
& +y_A^\mu\Bigg(\gamma\delta_\mu^\alpha a_{A_0\lambda}a_{A_0}^\lambda - (2\gamma-1)a_{A_0}^\alpha a_{A_0\mu} + 2\gamma v_{(0)}^\lambda[\eta_{\lambda\mu}\dot{a}_{A_0}^\alpha - \delta_\mu^\alpha\dot{a}_{A_0\lambda}] + \\
& + \sum_{B\neq A}\Bigg(\frac{\partial}{\partial y_A^0}\Big[\gamma\delta_\mu^\alpha\Big\langle\frac{\partial U_B}{\partial y_A^0}\Big\rangle_0 - (\gamma+1)\Big(\Big\langle\partial_{(\mu}V_B^{\alpha)}\Big\rangle_0 + \Big\langle v_\mu\partial^\alpha U_B\Big\rangle_0 + 3\Big\langle v^\alpha\partial_\mu U_B\Big\rangle_0\Big)\Big] + \\
& +2(\gamma+1)v_{(0)}^\lambda\Big\langle\partial^{[\alpha}\partial_\mu V_{B\lambda]}\Big\rangle_0 + \frac{2}{3}\gamma v_{(0)}^\lambda v_{(0)}^\beta\delta_\mu^{[\alpha}\Big\langle\partial^2_{\lambda]\beta}U_B\Big\rangle_0\Bigg)\Bigg) + \mathcal{O}(|y_A^\nu|^2) + \mathcal{O}(c^{-6}).
\end{aligned} \tag{7.8}$$

By expanding all the potentials in (7.8) in power series of $1/y_{BA_0}$ and retaining terms with $\sim y^\alpha/|y_{BA_0}|$ only to the required accuracy, we then obtain

$$a_{(0)}^\alpha = -\eta^{\alpha\mu}\frac{\partial\overline{U}}{\partial y_A^\mu} + \delta_A a_{(0)}^\alpha + \delta_{AB}a_{(0)}^\alpha + \delta_B a_{(0)}^\alpha + \delta_{BC}a_{(0)}^\alpha + \mathcal{O}(|y_A^\nu|^2) + \mathcal{O}(c^{-6}), \tag{7.9}$$

where the post-Newtonian acceleration $\delta_A a_{(0)}^\alpha$ due to the gravitational field of the body (A) only may be given as

$$\delta_A a^\alpha_{(0)} = 2(\gamma+\beta)U_A\partial^\alpha U_A - (\gamma+\frac{1}{2})\partial^\alpha\Phi_{1A} + (2\beta-\frac{3}{2})\partial^\alpha\Phi_{2A}+$$

$$+(1-\gamma)\partial^\alpha\Phi_{4A} + \frac{1}{2}\partial^\alpha A_A - (2\gamma+1)v^\alpha_{(0)}\partial_\mu V^\mu_A + \gamma v_{(0)_\mu}v^\mu_{(0)}\partial^\alpha U_A-$$

$$-2(\gamma+1)v^\alpha_{(0)}v^\mu_{(0)}\partial_\mu U_A - 2(\gamma+1)v_{(0)_\mu}\Big[\partial^\mu V^\alpha_A - \partial^\alpha V^\mu_A\Big]-$$

$$-2(\gamma+1)\int_A d^3y'_A\hat\rho_A v'^\alpha_A v'_{A\mu}\frac{(y^\mu_A - y'^\mu_A)}{|y^\nu_A - y'^\nu_A|} - \frac{1}{2}(4\gamma+3)\int_A d^3y'_A\frac{\hat\rho_A\partial^\alpha U_A}{|y^\nu_A - y'^\nu_A|}+$$

$$+\frac{1}{2}\int_A d^3y'_A\hat\rho_A\partial_\mu U_A\frac{(y^\alpha_A - y'^\alpha_A)(y^\mu_A - y'^\mu_A)}{|y^\nu_A - y'^\nu_A|} + \mathcal{O}(c^{-6}). \tag{7.10}$$

This term is known and reasonably well understood (Denisov & Turyshev, 1990). The term $\delta_{AB}a^\alpha_{(0)}$ is the acceleration due to the interaction of the gravitational field of the extended body (A) with the external gravitation in the N-body system:

$$\delta_{AB}a^\alpha_{(0)} = \sum_{B\neq A}\Bigg((4\beta-3\gamma-1)\frac{m_A m_B}{y_{BA_0}}\frac{n^\alpha}{y^2} + 2(\beta-1)\frac{m_A m_B}{y}\frac{N^\mu_{BA_0}}{y^2_{BA_0}}(\delta^\alpha_\mu + n^\alpha n_\mu)+$$

$$+\frac{m_A m_B}{y^3_{BA_0}}\mathcal{P}_{\epsilon\lambda}\Big((2\beta+\frac{5}{3}\gamma)\eta^{\alpha\epsilon}n^\lambda + (\beta-\frac{1}{6})n^\alpha n^\epsilon n^\lambda\Big) - (\gamma+1)\frac{m_B}{y^3_{BA_0}}\mathcal{P}_{\epsilon\lambda}S^{\alpha\epsilon}_A v^\lambda_{(0)}+$$

$$+\frac{m_A m_B}{y^4_{BA_0}}\Big((2\beta+\gamma-1)\delta^\alpha_\mu + 2(3\beta+\gamma-1)N^\alpha_{BA_0}N_{BA_0\mu}\Big)y^\mu_A+$$

$$+3\beta\frac{m_A m_B}{y^4_{BA_0}}|y^\nu_A|n^\epsilon n^\lambda\Big(2\delta^\alpha_\epsilon N_{BA_0\lambda} + N^\alpha_{BA_0}(\eta_{\epsilon\lambda} + 5N_{BA_0\epsilon}N_{BA_0\lambda})\Big)\Bigg) + \mathcal{O}(|y^\nu_A|^2) + \mathcal{O}(c^{-6}), \tag{7.11}$$

where $S^{\alpha\epsilon}_A$ is the reduced spin moment of the body (A) and $\mathcal{P}_{\epsilon\lambda} = \eta_{\epsilon\lambda} + 3N_{BA_0\epsilon}N_{BA_0\lambda}$ is the polarizing operator. Note that the combination of the post-Newtonian parameters in the first term of expression (7.11) differs from that for the well-known Nordtvedt effect (Nordtvedt, 1968b,c; Will, 1993). This may provide an independent test for the parameters involved. The reason that our third term in this expression differs from the analogous term derived in Ashby & Bertotti (1986) is that, in order to obtain this result, (7.9)–(7.11), we used the consistent definitions for the conserved mass density in the proper RF$_A$. Moreover, in constructing the Fermi normal coordinates previous authors used incomplete expressions for the spatial coordinate transformations, which differ from eq.(7.2) (specifically in the third order of the spatial coordinates). Note that if we decide to use our definitions, the result cited above will take the form of (7.11). The next term, $\delta_B a^\alpha_{(0)}$, is the post-Newtonian acceleration caused by the other bodies in the system on the orbit of the body (A) (the effect of the post-Newtonian tidal forces):

$$\delta_B a^\alpha_{(0)} = \sum_{B\neq A} y^\mu_A\Bigg(-\frac{3}{2}\frac{m^2_B}{y^4_{BA_0}}\Big(\frac{1}{3}(14\gamma-4\beta-7)\delta^\alpha_\mu + 3N^\alpha_{BA_0}N_{BA_0\mu}\frac{1}{9}(42\gamma-16\beta-17)\Big)+$$

$$+\frac{m_B}{y^3_{BA_0}}\mathcal{P}^\alpha_\mu\Big[(\gamma+\frac{1}{2})v_{BA_0\lambda}v^\lambda_{BA_0} + \frac{3}{2}(v_{BA_0\lambda}N^\lambda_{BA_0})^2 + (4\beta-\gamma-3)E_B\Big]+$$

$$+\gamma\frac{m_B}{y^3_{BA_0}}\mathcal{P}_{\epsilon\lambda}\Big[\delta^\alpha_\mu v^\epsilon_{BA_0}v^\lambda_{BA_0} - v^\alpha_{BA_0}v^\epsilon_{BA_0}\delta^\lambda_\mu - v^\epsilon_{BA_0}v_{BA_0\mu}\eta^{\alpha\lambda}\Big]-$$

$$-\frac{m_B}{y^3_{BA_0}}\Big(\delta^\alpha_\epsilon\eta_{\lambda\mu}-3N^\alpha_{BA_0}N_{BA_0\mu}N_{BA_0\epsilon}N_{BA_0\lambda}\Big)v^\epsilon_{BA_0}v^\lambda_{BA_0}+$$

$$+\frac{m_B}{y^3_{BA_0}}\mathcal{P}_{\epsilon\lambda}\Big[\frac{2}{3}\gamma v_{(0)\beta}v^\beta_{(0)}\eta^{\alpha\epsilon}\delta^\lambda_\mu-\frac{2}{3}(\gamma+3)v^\alpha_{(0)}v^\epsilon_{(0)}\delta^\lambda_\mu+$$

$$+\frac{2}{3}\gamma v^\beta_{(0)}v^\lambda_{(0)}[\delta^\alpha_\mu\delta^\epsilon_\beta-\eta_{\beta\mu}\eta^{\alpha\epsilon}]+2\gamma v^\beta_{(0)}v^\lambda_{BA_0}[\delta^\alpha_\mu\delta^\epsilon_\beta-\eta_{\beta\mu}\eta^{\alpha\epsilon}]+$$

$$+2(\gamma+1)v^\beta_{(0)}\delta^\lambda_\mu[v_{BA_0\beta}\eta^{\alpha\epsilon}-v^\alpha_{BA_0}\delta^\epsilon_\beta]\Big]\Big)+\mathcal{O}(|y^\nu_A|^2)+\mathcal{O}(c^{-6}). \tag{7.12}$$

Finally, the last term in expression (7.9), $\delta_{BC}a^\alpha_{(0)}$, is the contribution to the equation of motion of the non-linear gravitational interaction of the external bodies with each other, given as follows:

$$\delta_{BC}a^\alpha_{(0)}=-\sum_{B\neq A}\sum_{C\neq A,B}y^\mu_A\Big(\frac{m_Bm_C}{y^2_{BA_0}y^2_{CA_0}}\Big[3(\gamma-1)\mathcal{P}^\alpha_\mu N_{BA_0\lambda}N^\lambda_{CA_0}+$$

$$+(2\beta-1)N^\alpha_{BA_0}N_{CA_0\mu}-N^\alpha_{CA_0}N_{BA_0\mu}\Big]+\frac{m_Bm_C}{y^3_{BA_0}y_{CA_0}}\Big[(2\beta-3\gamma+\frac{1}{2})\mathcal{P}^\alpha_\mu+$$

$$+\frac{1}{2}\gamma\mathcal{P}_{\epsilon\lambda}\Big(\eta^{\alpha\epsilon}N_{CA_0\mu}N^\lambda_{CA_0}+\delta^\epsilon_\mu N^\alpha_{CA_0}N^\lambda_{CA_0}-(\delta^\alpha_\mu+N^\alpha_{CA_0}N_{CA_0\mu})N^\epsilon_{CA_0}N^\lambda_{CA_0}\Big)\Big]+$$

$$+\frac{m_Bm_C}{y^2_{BA_0}y^2_{CB_0}}\Big(\frac{1}{2}\mathcal{P}^\alpha_\mu+\gamma\delta^\alpha_\mu\Big)N_{BA_0\lambda}N^\lambda_{CB_0}-$$

$$-(\gamma+\frac{1}{2})(N^\alpha_{BA_0}N_{CB_0\mu}+N_{BA_0\mu}N^\alpha_{CB_0})\Big]+(2\beta-\frac{3}{2})\mathcal{P}^\alpha_\mu\frac{m_Bm_C}{y^3_{BA_0}y_{CB_0}}\Big)+$$

$$+\mathcal{O}(|y^\nu_A|^2)+\mathcal{O}(c^{-6}). \tag{7.13}$$

Thus, the equations presented in this subsection are represent the motion of a test body in the Fermi-normal-like coordinates chosen in the proper RF of a body (A). Together with the coordinate transformations, (7.1)–(7.3), this is the general solution of the gravitational N body problem.

We present here the restricted version of the equations, which is consistent with the expected accuracy for ESA's *Mercury Orbiter* mission. This limited accuracy permits us to completely neglect contributions proportional to the spatial coordinates y^μ_A. The planeto-centric equations of satellite motion around Mercury can be represented by a series of $1/|y_{BA_0}|$ as follows:

$$a^\alpha_{(0)}=-\eta^{\alpha\mu}\Big(\frac{\partial U_A}{\partial y^\mu_A}+\sum_{B\neq A}\Big[\frac{\partial U_B}{\partial y^\mu_A}-\Big\langle\frac{\partial U_B}{\partial y^\mu_A}\Big\rangle_A\Big]\Big)+\delta_Aa^\alpha_{(0)}+$$

$$+\sum_{B\neq A}\Bigg((4\beta-3\gamma-1)\frac{m_Am_B}{y_{BA_0}}\frac{n^\alpha}{y^2}+2(\beta-1)\frac{m_Am_B}{y}\frac{N^\mu_{BA_0}}{y^2_{BA_0}}(\delta^\alpha_\mu+n^\alpha n_\mu)+$$

$$+\frac{m_Am_B}{y^3_{BA_0}}\mathcal{P}_{\epsilon\lambda}\Big[(2\beta+\frac{5}{3}\gamma)\eta^{\alpha\epsilon}n^\lambda+(\beta-\frac{1}{6})n^\alpha n^\epsilon n^\lambda\Big]\Bigg)+\mathcal{O}(|y^\nu_A|)+\mathcal{O}(c^{-6}), \tag{7.14}$$

where index (A) denotes the planet Mercury, and the post-Newtonian acceleration $\delta_A a^{\alpha}_{(0)}$ is due to the gravitational field of Mercury only.

Thus, the formalism presented in this section could significantly simplify the general analysis of the tracking data for the *Mercury Orbiter* mission. We have presented the Hermean-centric equations of satellite motion, the barycentric equations of the planet's motion in the solar system barycentric RF, and the coordinate transformations that link these equations together. In particular, our analysis has shown that in a proper Hermean-centric RF the corresponding equations of the satellite motion depend on Mercury's gravitational field only. This set of equations is well known and widely in use for studying the dynamics of test bodies in the isolated gravitational one-body problem. The existence of the external gravitational field manifests itself in the form of the tidal forces only and also determines the dynamic properties of the constructed Hermean-centric proper RF. Note that within the accuracy expected for the future *Mercury Orbiter* mission, one may completely neglect the post-Newtonian tidal terms. However, while constructing this RF, we went further than the expected accuracy of the future experiments. Indeed, the last term in equation (7.14) is due to the coordinate transformation to the Fermi-normal-like RF, which may be chosen in the planet's vicinity. One may neglect this term for the solar system motion; however, if one applies the presented formalism to the problems of motion with a more intensive gravitational environment, one will find that this term may play a significant role. The application of the results obtained here to the problems of motion of the double pulsars is currently under study and will be reported elsewhere.

It should be noted at once that the coefficients in front of the two terms in the second line of the expression in eq.(7.14) prove the correctness of the decomposition of the *local* fields in the proper RF_A, which we performed at the end of section 4. Indeed, if even one of these terms had had a non-zero value, this would have meant that the metric tensor of the *local* problem would not depend on the external gravity through the relativistic tidal-like potential, which is of the second order with respect to the spatial coordinate $\sim y_A^{\{2\}}$, but instead this dependence would be at least of the order $\sim y_A^{\{1\}}$. As a result, this new dependence may lead to a violation of the Strong Equivalence Principle (SEP). There are certainly no worries for the general theory of relativity for which the PPN parameters have the values $\gamma = \beta = 1$. However, the theories having a different means of γ and β may predict new effects in motion of the satellites due to the corresponding SEP violation. At this point, we have all the necessary equations in order to discuss this and other gravitational experiments with the future *Mercury Orbiter* mission.

7.3 Gravitational Experiment for Post-2000 Missions.

Mercury is the closest to the Sun of all the planets of the terrestrial group, and because of its unique location and orbital parameters, it is well suited to relativistic gravitational experiments. The short period of its solar orbit allows experiments over several orbital revolutions, and its high eccentricity and inclination allow various effects to be well separated. In this section, we will discuss possible gravitational experiments for the *Mercury Orbiter* mission. Analysis performed in this section is directed toward the future mission, so we will show which relativistic effects may be measured and how accurately.

It is generally considered that the processing of data from orbiters is more complicated than that from landers. This is because of the need to convert from the measured Earth–spacecraft distance to the desired Earth–planet distance. This involves determining the orbit of the space-

craft about the planetary center of mass, which requires solving from the tracking data for a number of spatial harmonics of the gravitational field and for radiation pressure and other such effects. The other non-gravitational perturbations, such as firing attitude control jets, which have unbalanced forces, are also frequently present, which further complicates the analysis. The orbit determination of the *Mariner 9*, for example, was substantially affected not only by these factors, but also by the fact that the spacecraft was placed on the 12 hr period orbit with low periapsis. Thus, in order to precisely describe the motion of the *Mercury Orbiter* relative to Earth, one should solve two problems, namely: (i) the problem of the satellite motion about Mercury's center of inertia in the Hermean-centric frame, and (ii) the relative motion of both planets—Earth and Mercury—in the solar system barycentric RF_0. Our analysis is intended to provide a complete solution of these two problems.

In order to study the relativistic effects in the motion of the *Mercury Orbiter* satellite, we separate these effects into the three following groups:

(i). The effects due to Mercury's motion with respect to the solar system barycentric RF_0.

(ii). Effects on the satellite's motion with respect to the Hermean-centric RF.

(iii). Effects due to the dragging of the inertial frames.

The effects of the first group are standard and, with the accuracy anticipated for the future *Mercury Orbiter* mission, most of them may be obtained directly from the Lagrangian function (2.9) or from the equations of motion, (2.14)–(2.20). The effects of the second group can be discussed based on equations (7.14). And finally, the effects of the last group can be discussed based on the coordinate transformation rules given by the eqs.(7.2). In the last case, however, we employ a simplified version of these transformations, due to the limited expected accuracy (~ 1 m) of the Mercury ranging data. Thus, in the future discussion, we will use the following expression for the temporal components:

$$x^0(y_A^0, y_A^\mu) = y_A^0 + c^{-2}\Bigg(\int^{y_A^0}\Big[\sum_{B\neq A}\frac{m_B}{y_{BA_0}}\Big(1 + (I_A^{\lambda\mu} + I_B^{\lambda\mu})\frac{N_{BA_0\lambda}N_{BA_0\mu}}{y_{BA_0}^2}\Big) - \\ -\frac{1}{2}v_{A_0\mu}v_{A_0}^\mu\Big]dt' - v_{A_0\mu}y_A^\mu\Bigg) + \mathcal{O}(c^{-4}), \tag{7.15}$$

where $I_C^{\mu\nu}$ represents the STF intrinsic quadrupole moments of the bodies. Note that the terms contained in the function $\mathcal{L}_A \sim \mathcal{O}(c^{-4})$ will contribute to the post-Newtonian redshift. However, it will not be possible to perform the redshift experiment with the accuracy anticipated for the mission, and therefore this term was omitted. The corresponding expression for the spatial components of the coordinate transformation is given by

$$x^\alpha(y_A^0, y_A^\mu) = y_{A_0}^\alpha(y_A^0) + y_A^\alpha + c^{-2}\Bigg(y_{A\mu}\Big[\int^{y_A^0}\Omega_A^{\alpha\mu}(t')dt' - \frac{1}{2}v_{A_0}^\alpha v_{A_0}^\mu - \gamma\eta^{\alpha\mu}\sum_{B\neq A}\frac{m_B}{y_{BA_0}}\Big] - \\ -\sum_{B\neq A}\frac{m_B}{y_{BA_0}^2}\Big[y_A^\alpha y_{A\mu}N_{BA_0}^\mu - \frac{1}{2}y_{A\mu}y_A^\mu N_{BA_0}^\alpha\Big] + w_A^\alpha(y_A^0)\Bigg) + \mathcal{O}(|y_A|^3) + \mathcal{O}(c^{-4}), \tag{7.16a}$$

with the precession angular velocity tensor $\Omega_A^{\alpha\beta}$ given as follows:

$$\Omega_A^{\alpha\beta}(y_A^0) = \sum_{B\neq A}\Big[(\gamma+\frac{1}{2})\frac{m_B}{y_{BA_0}^2}N_{BA_0}^{[\alpha}v_{A_0}^{\beta]} - (\gamma+1)\frac{m_B}{y_{BA_0}^2}N_{BA_0}^{[\alpha}v_{b_0}^{\beta]} +$$

$$+(\gamma+1)\frac{m_B}{2y_{BA_0}^3}\mathcal{P}_\lambda^{[\alpha}(S_A^{\beta]\lambda}+S_B^{\beta]\lambda})\Big], \qquad (7.16b)$$

where $S_C^{\mu\nu}$ is the STF intrinsic spin moment of the bodies.

We mention that, by means of a topographic Legendre expansion complete through the second degree and order, the systematic error in Mercury radar ranging has been reduced significantly (Anderson *et al.*, 1995). However, a *Mercury Orbiter* is required before significant improvements in relativity tests become possible. Currently, the precession rate of Mercury's perihelion, in excess of the 530 arcsec per century ($''$/cy) from planetary perturbations, is 43.13 $''$/cy with a realistic standard error of 0.14 $''$/cy (Anderson *et al.*, 1991). After taking into account a small excess precession from solar oblateness, Anderson *et al.* find that this result is consistent with general relativity. Pitjeva (1993) has obtained a similar result but with a smaller estimated error of 0.052 $''$/cy. Similarly, attempts to detect a time variation in the gravitational constant G using Mercury's orbital motion have been unsuccessful, again consistent with general relativity. The current result (Pitjeva, 1993) is $\dot{G}/G = (4.7 \pm 4.7) \times 10^{-12}$ yr^{-1}.

7.3.1 Mercury's Perihelion Advance.

Based on Mercury's barycentric equations of motion, one may study the phenomenon of Mercury's perihelion advance. The secular trend in Mercury's perihelion[14] depends on the linear combination of the PPN parameters γ and β and the solar quadrupole coefficient $J_{2\odot}$ (Nobili & Will, 1986; Heimberger *et al.*, 1990; Will, 1993):

$$\dot{\pi} = (2+2\gamma-\beta)\frac{\mu_\odot n_M}{a_M(1-e_M^2)} + \frac{3}{4}\Big(\frac{R_\odot}{a_M}\Big)^2\frac{J_{2\odot}n_M}{(1-e_M^2)^2}(3\cos^2 i_M - 1), \quad ''/\text{cy} \qquad (7.17a)$$

where a_M, n_M, i_M, and e_M are the mean distance, mean motion, inclination, and eccentricity of Mercury's orbit. The parameters $\mu_\odot$ and $R_\odot$ are the solar gravitational constant and radius, respectively. For Mercury's orbital parameters, one obtains

$$\dot{\pi} = 42''.98\left[\frac{1}{3}(2+2\gamma-\beta) + 0.296\cdot J_{2\odot}\times 10^4\right], \quad ''/\text{cy} \qquad (7.17b)$$

Thus, the accuracy of the relativity tests on the *Mercury Orbiter* mission will depend on our knowledge of the solar gravity field. The major source of uncertainty in these measurements is the solar quadrupole moment $J_{2\odot}$. As evidenced by the oblateness of the photosphere (Brown *et al.*, 1989) and perturbations in frequencies of solar oscillations, the internal structure of the Sun is slightly aspherical. The amount of this asphericity is uncertain. It has been suggested that it could be significantly larger than calculated for a simply rotating star and that the internal rotation rate varies with the solar cycle (Goode & Dziembowski, 1991). Solar oscillation data suggest that most of the Sun rotates slightly slower than the surface with the possible exception of a more rapidly rotating core (Duvall & Harvey, 1984). An independent measurement of $J_{2\odot}$

[14]It should be noted that the *Mercury Orbiter* itself, being placed in orbit around Mercury, will experience the phenomenon of periapse advance as well. However, we expect that uncertainties in Mercury's gravity field will mask the relativistic precession, at least at the level of interest for ruling out alternative gravitational theories.

performed with the *Mercury Orbiter* would provide a valuable direct confirmation of the indirect helioseismology value $(2 \pm 0.2) \times 10^{-7}$. Furthermore, there are suggestions of a rapidly rotating core, but helioseismology determinations are limited by uncertainties at depths below 0.4 solar radius (Libbrecht & Woodard, 1991).

The *Mercury Orbiter* will help us understand this asphericity and independently will enable us to gain some important data on the properties of the solar interior and the features of its rotational motion. Preliminary analysis of a *Mercury Orbiter* mission suggests that $J_{2\odot}$ would be measurable to at best $\sim 10^{-9}$ (Ashby *et al.*, 1995) or about 1% of the expected $J_{2\odot}$ value. This should be compared with the present 10% solar oscillation determination (Brown *et al.*, 1989).

7.3.2 The Redshift Experiment.

Another important experiment that could be performed on a *Mercury Orbiter* mission is a test of the solar gravitational redshift. This would require that a stable frequency standard be flown on the spacecraft. The experiment would provide a fundamental test of the theory of general relativity and the Equivalence Principle upon which it and other metric theories of gravity are based (Will, 1993). Because in general relativity the gravitational redshift of an oscillator or clock depends upon its proximity to a massive body (or more precisely the size of the Newtonian potential at its location), a frequency standard at the location of Mercury would experience a large, measurable redshift due to the Sun. With the result for the function $\mathcal{K}_A$ given by eqs. (7.2*a*) and (7.15) in hand, one can obtain the corresponding Newtonian proper frequency variation between the barycentric standard of time and that of the satellite (the terms with magnitude up to 10^{-12}), given as

$$\frac{dx^0}{dy^0_{(0)}} = 1 + \frac{\mu_\odot}{R_M} + \frac{\mu_M}{y_{(0)}} + \frac{1}{2c^2}(\vec{v}_M + \vec{v}_{(0)})^2 - \frac{\mu_\odot}{R_M^3}(\vec{R}_M \vec{y}_{(0)}) + \mathcal{O}(c^{-4}), \tag{7.18a}$$

where $(y^0_{(0)}, \vec{y}_{(0)})$ are the four coordinates of the spacecraft in the Hermean-centric RF and $\vec{v}_{(0)}$ is the spacecraft orbital velocity. One can see that the eccentricity of Mercury's orbit would be highly effective in varying the solar potential at the clock, thereby producing a distinguishing signature in the redshift. The anticipated frequency variation between perihelion and aphelion is to the first order in eccentricity:

$$\left(\frac{\delta f}{f_0}\right)_{e_M} = \frac{2\mu_\odot e_M}{a_M}. \tag{7.18b}$$

This contribution is quite considerable and is calculated to be $(\delta f/f_0)_{e_M} = 1.1 \times 10^{-8}$. Its magnitude, for instance, at a radio wavelength $\lambda_0 = 3$ cm ($f_0 = 10$ GHz) is $(\delta f)_{e_M} = 110$ Hz. We would also benefit from the short orbital period of Mercury, which would permit the redshift signature of the Sun to be measured several times over the duration of the mission. If the spacecraft tracking and modelling are of sufficient precision to determine the spacecraft position relative to the Sun to 100 m (a conservative estimate), then a frequency standard with 10^{-15} fractional frequency stability $\delta f/f = 10^{-15}$ would be able to measure the redshift to 1 part in 10^7 or better. This stability is within the capability of the proposed spaceborne trapped-ion (Prestage *et al.*, 1992) or H-maser clocks (Vessot *et al.*, 1980; Walsworth *et al.*, 1994).

7.3.3 The SEP Violation Effect.

Besides the Nordtvedt effect (for more details see Anderson, Turyshev *et al.* (1996)), there exists an interesting possibility for testing the SEP violation effect by studying spacecraft motion in

orbit around Mercury. The corresponding equation of motion is given by eq.(7.14). As one can see, the two terms in the second line of this equation vanish for general relativity, but for scalar-tensor theories, they become responsible for small deviations of the spacecraft motion from the support geodesic. Both of these effects, if they exist, are due to non-linear coupling of the gravitational field of Mercury to external gravity. They come from the expression for W_A given by eq.(7.5a), which is the *local* post-Newtonian contribution to the g_{00} component of the metric tensor in the proper RF.

The first of these terms may be interpreted as a dependence of the locally measured gravitational constant on the external gravitational environment and may be expressed in the vicinity of body (A) as follows:

$$G_A = G_0\Big[1 - (4\beta - 3\gamma - 1)\sum_{B\neq A}\frac{m_B}{y_{BA_0}}\Big]. \tag{7.19}$$

In the case of a satellite around Mercury, the main contribution to this effect comes from the Sun.[15] Because of the high eccentricity of Mercury's orbit, the periodic changing of the Sun's *local* gravitational potential may produce an observable effect, which can be modeled by a periodic time variation in the effective *local* gravitational constant:

$$\Big[\frac{\dot{G}}{G}\Big]_{\text{period}} = (4\beta - 3\gamma - 1)\Big[\frac{\mu_\odot}{a_M(1-e_M^2)}\Big]^{\frac{3}{2}}\frac{ce_M\sin\phi(t)}{a_M(1-e_M^2)}(1 + e_M\cos\phi(t))^2, \tag{7.20}$$

which gives the following estimate for this effect on Mercury's orbit:

$$\Big[\frac{\dot{G}}{G}\Big]_{\text{period}} \approx (4\beta - 3\gamma - 1)\times 1.52\times 10^{-7}\sin\phi(t) \qquad \text{yr}^{-1}. \tag{7.21}$$

Note that this effect in eq.(7.21) is fundamentally different from that introduced by Dirac's hypothesis of possible time dependence of the gravitational constant (Pitjeva, 1993). As one can see from expression (7.21), the characteristic time in this case is Mercury's siderial period. This short period may be considered as an advantage from the experimental point of view. In addition, the results of the redshift experiment could help in confident studies of this effect. Recently a different combination of the post-Newtonian parameters entering in the Nordtvedt effect, $\eta = 4\beta - \gamma - 3$, was measured at $\eta \leq 10^{-4}$ (Dickey *et al.*, 1994). This means that, in order to obtain comparable accuracy for the combination of parameters in eq.(7.21), one should perform the Mercury gravimetric measurements on a level no less precise than $[\dot{G}/G]_{\text{period}} \approx 10^{-11}$ yr^{-1}. Recently a group at the University of Colorado analyzed a number of gravitational experiments possible with future Mercury missions (Ashby *et al.*, 1995). Using a modified worst-case error analysis, this group suggests that after one year of ranging between Earth and Mercury (and assuming a 6 cm rms error), the fractional accuracy of determination of the Sun's gravitational constant, $m_\odot G$, is expected to be of the order $\sim 2.1\times 10^{-11}$. Moreover, even higher accuracy could be achieved with a Mercury lander as proposed by Ashby *et al.* (1995). This suggests that the experiment for determination of the effect in eq.(7.21) may be feasible with the *Mercury Orbiter* mission.

[15]Note that this combination of PPN parameters differs from the one presented for a similar effect in (Will, 1993). The reason for this is that, in this case, the transformations in the form of eqs.(7.2) let us define transformation rules of the metric tensor between the barycentric and proper planeto-centric RFs and, hence, obtain the correct and complete equations of geodesic motion in the quasi-inertial Hermean-centric RF$_M$.

Another interesting effect on the satellite's orbit may be derived from eq.(7.14) in the form of the following acceleration term:

$$\delta\vec{a}_{(0)SEP} = 2(\beta - 1)\frac{m_M m_\odot}{yR_M^2}(\vec{N}_M - \vec{n}(\vec{n}\cdot\vec{N}_M)), \tag{7.22}$$

where R_M is Mercury's heliocentric radius vector and $\vec{N}_M$ is the unit vector along this direction. This effect is very small for the orbit proposed for the ESA's *Mercury Orbiter* mission. However, one can show that there exist two resonant orbits for a satellite around Mercury, either with the orbital frequency $\omega_{(0)}$ equal to Mercury's siderial frequency ω_M, $\omega_{(0)} \approx \omega_M$, or at one-third of this frequency, $\omega_{(0)} \approx \omega_M/3$. For these resonant orbits, the corresponding experiment could provide an independent direct test of the parameter β.

7.3.4 The Precession Phenomena.

In addition to the perihelion advance, while constructing the Hermean proper RF$_M$, one should take into account several precession phenomena included in transformation function $\mathcal{Q}_A^\alpha$ and associated with the angular momentum of the bodies. As one may see directly from eqs.(7.2*b*) and (7.16*a*), besides the obvious special relativistic contributions, the post-Newtonian transformation of the spatial coordinates contains terms due to the non-perturbative influence of the gravitational field. This non-Lorentzian behavior of the post-Newtonian transformations was discussed first by Chandrasekhar & Contopulos (1967) for the case of post-Galilean transformations. Our derivations differ from the latter by taking into account the acceleration of the proper RF and by including the infinitesimal precession of the coordinate axes with the angular velocity tensor $\Omega_M^{\alpha\beta}$ given as follows:

$$\Omega_M^{\alpha\beta} = \sum_{B\neq M}\Big[(\gamma+\frac{1}{2})\frac{\mu_B}{y_{BM_0}^2}N_{BM_0}^{[\alpha}v_{M_0}^{\beta]} - (\gamma+1)\frac{\mu_B}{y_{BM_0}^2}N_{BM_0}^{[\alpha}v_{B_0}^{\beta]} +$$
$$+(\gamma+1)\frac{\mu_B}{2y_{BM_0}^3}\mathcal{P}_\lambda^{[\alpha}(S_M^{\beta]\lambda} + S_B^{\beta]\lambda})\Big], \tag{7.23}$$

where, as before, the subscript (M) stands for Mercury and summation is performed over the bodies of the solar system. This expression rederives and generalizes the result for the precession of the spin of a gyroscope $\vec{s_0}$ attached to a test body orbiting a gravitating primary. Previously this result was obtained from the theory of the Fermi–Walker transport (Will, 1993). Indeed, in accord with eq.(7.16*a*), this spin (or coordinate axes of a proper Hermean RF$_M$) will precess with respect to a distant standard of rest, such as quasars or distant galaxies. The motion of the spin vector of a gyroscope can be described by the relation

$$\frac{d\vec{s_0}}{dt} = [\vec{\Omega}_M \times \vec{s_0}]. \tag{7.24}$$

By keeping the leading contributions only and neglecting the influence of Mercury's intrinsic spin moment, we obtain from expression (7.23) the angular velocity $\vec{\Omega}_M$ in the following form:

$$\vec{\Omega}_M = (\gamma+\frac{1}{2})\frac{\mu_\odot}{R_M^3}[\vec{R}_M \times \vec{v}_{M_0}] - (\gamma+1)\frac{\mu_\odot}{R_M^3}[\vec{R}_M \times \vec{v}_\odot] +$$
$$+(\gamma+1)\frac{\mu_\odot}{2R_M^3}\Big(\vec{S_\odot} - 3(\vec{S_\odot}\vec{N}_M)\vec{N}_M\Big), \tag{7.25}$$

where $\vec{v}_{M_0}$ and $\vec{v}_\odot$ are Mercury's and the Sun's barycentric orbital velocities and $\vec{S}_\odot$ is the solar intrinsic spin moment.

The first term in eq.(7.25) is known as geodetic precession (De-Sitter, 1916). This term arises in any non-homogeneous gravitational field because of the parallel transport of a direction defined by $\vec{s}_0$ in (7.24). It can be viewed as spin precession caused by a coupling between the particle velocity $\vec{v}_{M_0}$ and the static part of the space-time geometry. For Mercury orbiting the Sun, this precession has the form

$$\vec{\Omega}_G = (\gamma + \frac{1}{2})\frac{\mu_\odot}{R_M^3}(\vec{R}_M \times \vec{v}_{M_0}). \tag{7.26}$$

This effect could be studied for the *Mercury Orbiter*, which, being placed in orbit around Mercury, is in effect a gyroscope orbiting the Sun. Thus, if we introduce the angular momentum per unit mass, $\vec{L} = \vec{R}_M \times \vec{v_{M_0}}$, of Mercury in solar orbit, equation (7.26) shows that $\vec{\Omega}_G$ is directed along the pole of the ecliptic, in the direction of $\vec{L}$. The vector $\vec{\Omega}_G$ has a constant part

$$\vec{\Omega}_0 = \frac{1}{2}(1+2\gamma)\frac{\mu_\odot \omega_M}{a_M} = \frac{1+2\gamma}{3} \cdot 0.205 \quad ''/\text{yr}, \tag{7.27a}$$

with a significant correction due to the eccentricity e_M of Mercury's orbit:

$$\vec{\Omega}_1 \cos\omega_M t = \frac{3}{2}(1+2\gamma)\frac{\mu_\odot \omega_M}{a_M} e_M \cos\omega_M t_0 = \frac{1+2\gamma}{3} \cdot 0.126 \cos\omega_M t_0 \quad ''/\text{yr}, \tag{7.27b}$$

where ω_M is Mercury's siderial frequency, t_0 is reckoned from a perihelion passage, and a_M is the semimajor axis of Mercury's orbit.

Geodetic precession has been studied for the motion of lunar perigee, and its existence was first confirmed with an accuracy of 10% (Bertotti *et al.*, 1987). Two other groups have analyzed the lunar laser-ranging data more completely to estimate the deviation of the lunar orbit from the predictions of general relativity (Shapiro *et al.*, 1988; Dickey *et al.*, 1989). Geodetic precession has been confirmed within a standard deviation of 2%. The precession of the orbital plane proposed for the ESA's *Mercury Orbiter* (periherm at 400 km altitude, apherm at 16,800 km, period of 13.45 hr, and latitude of periherm at +30 deg) would include a contribution on the order of 0.205 ″/yr from the geodetic precession. We recommend this precession be included in future studies of the *Mercury Orbiter* mission.

The third term in expression (43) is known as Lense–Thirring precession, $\vec{\Omega}_{LT}$. This term gives the relativistic precession of the gyroscope's spin, $\vec{s}_0$, caused by the intrinsic angular momentum $\vec{S}$ of the central body. This effect is responsible for a small perturbation in the orbits of artificial satellites around the Earth (Tapley *et al.*, 1972; Ries *et al.*, 1991). However, our preliminary studies indicate that this effect is so small for the satellite's orbit around Mercury that will be masked by uncertainties in the orbit's inclination.

8 Discussion: Relativistic Astronomical RFs.

In this section, we will discuss some questions of the practical application of the results presented in this report. Let us mention that presently radio sources seem to be able to provide one with the more stable and precise reference measurements needed for reliable navigation in outer space. This makes it reasonable to construct the future astronomical RFs based upon the radio source catalogues that are expected to be an essential part of future relativistic navigation in the solar system and beyond (Standish, 1995). Moreover, as we know, the accuracy of VLBI timing measurements has improved rapidly over the last few years and is presently a few tens of picoseconds (ps). It is important in precise measurements such as these that inadequate modelling not contribute to the inaccuracy of the results. We believe that the results obtained in this report are ready to be used directly in application to this and many other problems of relativistic observations in the solar system.

The KLQ parameterized theory of astronomical RFs discussed in this report enables one to perform the necessary calculations in the most arbitrary form valid for many theories of gravity. The different physical aspects of choosing a well-defined *local* RF in a curved space-time has been discussed in many publications. In summary, in modern astronomical practice there are two physically different types of relativistic RFs that are extensively in use, namely:

1. The set of inertial RFs, which includes
   - (i). The asymptotic inertial RF.
   - (ii). The barycentric inertial RF.
2. The set of the observer's quasi-inertial proper RFs, which consists of
   - (i). The bodycentric RF, constructed for a particular extended body in the system.
   - (ii). The satellite RF, defined on the geodetic world line of a test particle orbiting the body under consideration.
   - (iii). The topocentric RF, which is defined on the surface of the body under study.

The main difference between these two classes of RFs is that, unlike the frames of the first type, which are inertial, the observer's frame is, in general, non-inertial. Such a hierarchy of frames in the WFSMA, if needed, may be extended to a larger scale of motion. The barycentric RF_0 could also be used (with some cosmological assumptions) as an analog of the rest frame of the universe for the description of the galactic and extra-galactic motion. One may find a more detailed discussion of this hierarchy of the RFs in applications to problems of modern astronomical practice in (Brumberg, 1991; Voinov, 1994; Folkner *et al.*, 1994). Theoretically, the RF and the set of coordinates selected may be arbitrary. The relativistic terms in the equations of motion, the light time equations, and the transformation from coordinate time to physically measurable time will vary with the RF and coordinates selected. In general, the numerical values of various constants, obtained by fitting the theory to observations, will also vary. However, the numerical values of the computed observable are independent of the RF and the CS selected (Moyer, 1971).

While the properties of inertial RFs from the first set of the frames listed above are well understood and widely accepted in many areas of modern astronomical practice, below we shall

concentrate our attention on the properties of the relativistic RFs from the second set, namely, we will be interested in construction of the geocentric, the satellite, and the topocentric frames. The logic of construction of these frames is quite simple: Due to the fact that the geocentric frame was previously well justified physically and explicitly constructed from the mathematical standpoint, the construction of the two remaining frames will be made based on these established properties of the geocentric RF. Indeed, the proper RF_A of an extended body (A) contains all the information about the proper gravitational field of the body (A) as well as the explicit information about the external gravity. Then, considering that the properties of the geocentric RF are already known, we will give the definitions of the satellite and the topocentric RFs. Moreover, we will present the results generalized on the case of the scalar-tensor theories of gravity and will include in the analysis the two Eddington parameters (γ, β).

8.1 The Geocentric Proper RF.

The properties of construction of the geocentric RF were discussed in Section 6 of the present report, and below we will present the final results only. Thus, the form of the coordinate transformations between the coordinates (x^p) of the barycentric inertial RF_0 and those (y_A^p) of a proper quasi-inertial RF_A of an arbitrary body (A) for the problem of motion of the N-extended-body system in the WFSMA was obtained as follows:

$$x^0 = y_A^0 + c^{-2}K_A(y_A^0, y_A^\epsilon) + c^{-4}L_A(y_A^0, y_A^\epsilon) + \mathcal{O}(c^{-6}), \tag{8.1a}$$

$$x^\alpha = y_A^\alpha + y_{A_0}^\alpha(y_A^0) + c^{-2}Q_A^\alpha(y_A^0, y_A^\epsilon) + \mathcal{O}(c^{-4}). \tag{8.1b}$$

We will present the results corresponding to the coordinate transformations to the RF, which has all ten parameters ζ_A, σ_A^α and $f_A^{\alpha\beta}$ of the constructed group of motion vanish and which is given by eq.(5.44) as

$$\zeta_A = c^{-2}\zeta_1^A + c^{-4}\zeta_2^A = \sigma_A^\alpha = f_A^{\alpha\beta} = 0.$$

Moreover, we shall be interested in such RFs that preserve all ten existing conservation laws of the *local* gravity, inertia, and matter, so that we require that the conditions of eqs.(6.37) hold, namely:

$$Q_{A\{L\}}^\alpha(y_A^0) = L_{A\{L\}}(y_A^0) = 0, \qquad \forall l \geq 3.$$

With these conditions, the transformation functions K_A, Q_A^α, and L_A take the following form:

$$K_A(y_A^0, y_A^\nu) = \int^{y_A^0} dt' \Big[\sum_{B\neq A} \big\langle U_B \big\rangle_A - \frac{1}{2} v_{A_0\nu} v_{A_0}^\nu \Big] - v_{A_0\nu} \cdot y_A^\nu + \mathcal{O}(c^{-4}) y_A^0, \tag{8.2a}$$

$$\begin{aligned} Q_A^\alpha(y_A^0, y_A^\nu) = \gamma \sum_{B\neq A} \Big[\Big(\frac{1}{2}\eta^{\alpha\lambda} y_{A_\mu} y_A^\mu - y_A^\alpha y_A^\lambda\Big) \big\langle \partial_\lambda U_B \big\rangle_A - y_A^\alpha \big\langle U_B \big\rangle_A \Big] - \frac{1}{2} v_{A_0}^\alpha v_{A_0}^\mu y_{A_\mu} + \\ + y_{A\beta} \int^{y_A^0} dt' \Big(\frac{1}{2} a_{A_0}^{[\alpha} v_{A_0}^{\beta]} + (\gamma+1) \sum_{B\neq A} \Big[\big\langle \partial^{[\alpha} V_B^{\beta]} \big\rangle_A + \big\langle \partial^{[\alpha} U_B v^{\beta]} \big\rangle_A \Big] \Big) + \\ + w_{A_0}^\alpha(y_A^0) + \mathcal{O}(c^{-4}) y_A^\alpha, \end{aligned} \tag{8.2b}$$

$$L_A(y_A^0, y_A^\nu) = \sum_{B\neq A}\Bigg(\gamma\frac{1}{2}\,y_{A\beta}y_A^\beta\cdot\frac{\partial}{\partial y_A^0}\Big\langle U_B\Big\rangle_A - (\gamma+1)\,y_A^\lambda y_A^\beta\cdot\Big[\Big\langle\partial_\lambda V_{B\beta}\Big\rangle_A + \Big\langle v_\beta\partial_\lambda U_B\Big\rangle_A\Big]+$$

$$+\gamma v_{A_0\beta}\Big(y_A^\beta y_A^\lambda - \frac{1}{2}\eta^{\beta\lambda}y_{A_\mu}y_A^\mu\Big)\cdot\Big\langle\partial_\lambda U_B\Big\rangle_A\Bigg)+$$

$$+y_{A_\lambda}v_{A_0\beta}\int^{y_A^0}dt'\Big(\frac{1}{2}a_{A_0}^{[\lambda}v_{A_0}^{\beta]} + (\gamma+1)\sum_{B\neq A}\Big[\Big\langle\partial^{[\lambda}V_B^{\beta]}\Big\rangle_A + \Big\langle v^{[\beta}\partial^{\lambda]}U_B\Big\rangle_A\Big]\Big)+$$

$$+y_{A_\beta}\Big[(\gamma+1)\,v_{A_0}^\beta\sum_{B\neq A}\Big\langle U_B\Big\rangle_A - 2(\gamma+1)\sum_{B\neq A}\Big\langle V_B^\beta\Big\rangle_A - \dot{w}_{A_0}^\beta(y_A^0)\Big]-$$

$$-\int^{y_A^0}dt'\Big[\sum_{B\neq A}\Big\langle W_B\Big\rangle_A + \frac{1}{2}\Big(\sum_{B\neq A}\Big\langle U_B\Big\rangle_A - \frac{1}{2}v_{A_0\beta}v_{A_0}^\beta\Big)^2 + v_{A_0\mu}\dot{w}_{A_0}^\mu(t')\Big] + \mathcal{O}(c^{-6})y_A^0. \qquad (8.2c)$$

The equations for the functions $a_{A_0}^\alpha$ and $\ddot{w}_{A_0}$ were given previously by equations (7.3) and (6.52), respectively.

The transformations, eqs.(8.2), produce the metric tensor g_{mn}^A of the geocentric RF with the following components:

$$g_{00}^A(y_A^p) = 1 - 2\mathcal{U} + 2\mathcal{W} + \mathcal{O}(c^{-6}), \qquad (8.3a)$$

$$g_{0\alpha}^A(y_A^p) = 4\,\eta_{\alpha\epsilon}\mathcal{V}^\epsilon + \mathcal{O}(c^{-5}), \qquad (8.3b)$$

$$g_{\alpha\beta}^A(y_A^p) = \eta_{\alpha\beta}\Big(1 + 2\gamma\mathcal{U}\Big) + \mathcal{O}(c^{-4}), \qquad (8.3c)$$

where for the brevity of the future discussion we introduced the following notations for the generalized gravitational potentials in this *local* frame:

$$\mathcal{U}(y_A^p) \equiv \overline{U}(y^p) = \sum_B U_B(y_B^q(y_A^p)) - \sum_{B\neq A}\Big[y_A^\beta\Big\langle\frac{\partial U_B}{\partial y_A^\beta}\Big\rangle_A + \langle U_B\rangle_A\Big], \qquad (8.4a)$$

$$\mathcal{W}(y_A^p) = \sum_B W_B(y_B^q(y_A^p)) - \sum_{B\neq A}\Big[y_A^\mu\Big\langle\frac{\partial W_B}{\partial y_A^\mu}\Big\rangle_A + \langle W_B\rangle_A\Big]+$$

$$+\frac{1}{2}\,y_A^\mu y_A^\beta\Bigg(\gamma\eta_{\mu\beta}\,a_{A_0\lambda}a_{A_0}^\lambda - (2\gamma-1)\,a_{A_0\mu}a_{A_0\beta}+$$

$$+\sum_{B\neq A}\frac{\partial}{\partial y_A^0}\Big(\gamma\eta_{\mu\beta}\frac{\partial}{\partial y_A^0}\Big\langle U_B\Big\rangle_A - (\gamma+1)\Big[\Big\langle\partial_{(\mu}V_{B\beta)}\Big\rangle_A + \Big\langle v_{(\beta}\partial_{\mu)}U_B\Big\rangle_A\Big]\Big)\Bigg), \qquad (8.4b)$$

$$\mathcal{V}^\alpha(y_A^p) = \frac{1}{2}(\gamma+1)\Bigg(\sum_B V_B^\alpha(y_B^q(y_A^p))-$$

$$-\sum_{B\neq A}\Big[y_A^\mu\Big(\Big\langle\frac{\partial V_B^\alpha}{\partial y_A^\mu}\Big\rangle_A + \Big\langle v^\alpha\frac{\partial U_B}{\partial y_A^\mu}\Big\rangle_A\Big) + \langle V_B^\alpha\rangle_A\Big]\Bigg) + \gamma\frac{1}{4}\Big(y_A^\alpha y_A^\lambda - \frac{1}{2}\eta^{\alpha\lambda}y_{A_\mu}y_A^\mu\Big)\dot{a}_{A_0\lambda}, \qquad (8.4c)$$

where both functions W_A and W_B are given by the expressions of eqs.(7.5).

The expressions presented for the geocentric proper RF take into account the proper gravitational field of the body (A), the *external* gravity, and the dynamical properties of the inertial sector of the *local* space-time. This presentation of the *local* metric, eqs.(8.3), will enable us to simplify the discussion of the results obtained for the two other important quasi-inertial frames that are widely in use for many practical applications of modern astronomy—the satellite and the topocentric ones.

8.2 The Satellite Proper RF.

The motion of an artificial satellite may be presented as the motion of a test particle that is moving along the geodetic world line in the effective space-time with the metric tensor given by eqs.(8.3). This means that, in order to define the coordinate transformations and the metric tensor of a satellite $\mathrm{RF}_{(0)}$, we can apply the conditions of eqs.(3.26) or those of eqs.(5.2). By performing the same calculations as in Section 5 for the test particle, we can obtain the post-Newtonian dynamically non-rotating coordinate transformations linking together the coordinates (y_A^p) of the geocentric quasi-inertial RF_A and those (z^p) of the proper quasi-inertial $\mathrm{RF}_{(0)}$. These transformations may be obtained in the familiar form:

$$y_A^0 = z^0 + c^{-2}K_{(0)}(z^0, z^\nu) + c^{-4}L_{(0)}(z^0, z^\epsilon) + \mathcal{O}(c^{-6})z^0, \tag{8.5a}$$

$$y_A^\alpha = z^\alpha + z^\alpha_{(0)}(z^0) + c^{-2}Q^\alpha_{(0)}(z^0, z^\nu) + \mathcal{O}(c^{-4})z^\alpha. \tag{8.5b}$$

The solutions for the transformation functions $K_{(0)}, Q^\alpha_{(0)}$, and $L_{(0)}$ were chosen with the same conditions as those for the functions of eqs.(8.2), namely: the corresponding group parameters $\zeta_{(0)}, \sigma^\alpha_{(0)}$ and $f^{\alpha\beta}_{(0)}$ are taken to be zero and the requirement of preserving all the conservation laws in the satellite's *local* vicinity is fulfilled. The resultant functions were obtained as follows:

$$K_{(0)}(z^0, z^\nu) = \int^{z^0} dt' \Big[\big\langle \mathcal{U} \big\rangle_{(0)'} - \frac{1}{2} v_{(0)_\nu} v^\nu_{(0)}\Big] - v_{(0)_\nu} \cdot z^\nu + \mathcal{O}(c^{-4})z^0, \tag{8.6a}$$

$$Q^\alpha_{(0)}(z^0, z^\nu) = \gamma\Big(\frac{1}{2}\eta^{\alpha\beta} z_\mu z^\mu - z^\alpha z^\beta\Big)\Big\langle \frac{\partial \mathcal{U}}{\partial z^\beta}\Big\rangle_{(0)} - \gamma z^\alpha \big\langle \mathcal{U} \big\rangle_{(0)} - \frac{1}{2} v^\alpha_{(0)} v^\beta_{(0)} z_\beta +$$

$$+ z_\beta \int^{z^0} dt' \Big[\frac{1}{2} a^{[\alpha}_{(0)} v^{\beta]}_{(0)} + 2\big\langle \partial^{[\alpha} \mathcal{V}^{\beta]} \big\rangle_{(0)'}\Big] + w^\alpha_{(0)}(z^0) + \mathcal{O}(c^{-4})z^\alpha, \tag{8.6b}$$

$$L_{(0)}(z^0, z^\nu) = \gamma \frac{1}{2} z_\beta z^\beta \frac{\partial}{\partial z^0}\big\langle \mathcal{U} \big\rangle_{(0)} - 2 z^\lambda z^\beta \big\langle \partial_\lambda \mathcal{V}_\beta \big\rangle_{(0)} + \gamma v_{(0)_\beta}\Big(z^\beta z^\lambda - \frac{1}{2}\eta^{\beta\lambda} z_\mu z^\mu\Big)\Big\langle \frac{\partial \mathcal{U}}{\partial z^\lambda}\Big\rangle_{(0)} +$$

$$+ v_{(0)_\beta} z_\lambda \int^{z^0} dt' \Big[\frac{1}{2} a^{[\lambda}_{(0)} v^{\beta]}_{(0)} + 2\big\langle \partial^{[\lambda} \mathcal{V}^{\beta]} \big\rangle_{(0)'}\Big] + z_\beta\Big((\gamma+1) v^\beta_{(0)} \big\langle \mathcal{U} \big\rangle_{(0)} - 4\big\langle \mathcal{V}^\beta \big\rangle_{(0)} - \dot{w}^\beta_{(0)}\Big) -$$

$$- \int^{z^0} dt' \Big[\big\langle \mathcal{R} \big\rangle_{(0)'} + \frac{1}{2}\Big(\big\langle \mathcal{U} \big\rangle_{(0)'} - \frac{1}{2} v_{(0)_\beta} v^\beta_{(0)}\Big)^2 + v_{(0)_\mu} \dot{w}^\mu_{(0)}\Big] + \mathcal{O}(c^{-6})z^0, \tag{8.6c}$$

where the quantities $v^\alpha_{(0)}$ and $a^\alpha_{(0)}$ are the geocentric velocity and acceleration of the spacecraft, respectively. The notation $\big\langle f \big\rangle_{(0)}$, analogously to that of eq.(5.7), denotes the limiting procedure of taking the value of the function $f(z^p)$ on the geodetic world line of an artificial satellite, where $z^\alpha \to 0$. The equations for both time-dependent functions $z^\alpha_{(0)}$ and $w^\alpha_{(0)}$ may be determined

similarly to those presented in Section 5. Thus, eq.(5.4) provides us with the usual relation for the Newtonian acceleration $a^{\alpha}_{(0)}$ of the center of inertia of a test body:

$$a^{\alpha}_{(0)}(z^0) = -\eta^{\alpha\mu}\Big\langle\frac{\partial\mathcal{U}}{\partial z^{\mu}}\Big\rangle_{(0)} + \mathcal{O}(c^{-4}). \tag{8.7}$$

Analogously, the function $w^{\alpha}_{(0)}$ is determined as the solution of the equation below:

$$\ddot{w}^{\alpha}_{(0)}(z^0) = \eta^{\alpha\mu}\Big\langle\frac{\partial\mathcal{R}}{\partial z^{\mu}}\Big\rangle_{(0)} + v^{\alpha}_{(0)}\frac{\partial}{\partial z^0}\Big\langle\mathcal{U}\Big\rangle_{(0)} + a^{\alpha}_{(0)}\Big\langle\mathcal{U}\Big\rangle_{(0)} - 4\frac{\partial}{\partial z^0}\Big\langle\mathcal{V}^{\alpha}\Big\rangle_{(0)} -$$

$$-\frac{1}{2}v^{\alpha}_{(0)}v_{(0)\beta}a^{\beta}_{(0)} + a_{(0)\lambda}\int^{z^0}dt'\Big[\frac{1}{2}a^{[\alpha}_{(0)}v^{\lambda]}_{(0)} + 2\Big\langle\partial^{[\alpha}\mathcal{V}^{\lambda]}\Big\rangle_{(0)'}\Big] + \mathcal{O}(c^{-6}), \tag{8.8a}$$

where function $\mathcal{R} \sim \mathcal{O}(c^{-4})$ is defined in the same way as functions W_A and W_B in eqs.(5.43) from the result of the fields decomposition in the *local* quasi-inertial $\mathrm{RF}_{(0)}$ of a satellite. By repeating this decomposition as it was presented in Section 4, one may obtain this function as follows:

$$\mathcal{R}(z^p) = \mathcal{W}(z^p) + 2a^{\lambda}_{(0)}\sum_{B}\frac{\partial}{\partial z^{\lambda}}\chi_B(z^0, z^{\nu}) + \mathcal{O}(c^{-6}). \tag{8.8b}$$

At this point, we may present the form of the metric tensor in the proper $\mathrm{RF}_{(0)}$ of an artificial satellite defined on the geodetic world line with the generalized Fermi conditions (3.26). Thus, by substituting the solutions obtained for functions $K_{(0)}, Q^{\alpha}_{(0)}$, and $L_{(0)}$ into the general form of the metric tensor $g_{mn}(z^p)$ in the expressions for the metric in a proper $\mathrm{RF}_{(0)}$ given by the relations in eqs.(4.11), we will obtain this tensor in the following form:

$$g^{(0)}_{00}(z^p) = \ 1 - 2\overline{\mathcal{U}}_{(0)} + 2\overline{\mathcal{R}}_{(0)} + \mathcal{O}(c^{-6}), \tag{8.9a}$$

$$g^{(0)}_{0\alpha}(z^p) = 4\eta_{\alpha\epsilon}\overline{\mathcal{V}}^{\epsilon}_{(0)} + \mathcal{O}(c^{-5}), \tag{8.9b}$$

$$g^{(0)}_{\alpha\beta}(z^p) = \eta_{\alpha\beta}\Big(1 + 2\gamma\overline{\mathcal{U}}_{(0)}\Big) + \mathcal{O}(c^{-4}). \tag{8.9c}$$

Expressions (8.9) are the general solution for the field equations of the general theory of relativity, which satisfy the generalized Fermi conditions in the immediate vicinity of a dimensionless test body. By definition, the proper gravity of the test body is negligibly small, then the effective Newtonian potential in the vicinity of the satellite may be presented as follows:

$$\overline{\mathcal{U}}_{(0)}(z^p) = \mathcal{U}(z^0, z^{\nu}) - \Big[z^{\mu}\Big\langle\frac{\partial\mathcal{U}}{\partial z^{\mu}}\Big\rangle_{(0)} + \Big\langle\mathcal{U}\Big\rangle_{(0)}\Big]. \tag{8.10}$$

In addition, functions $\overline{\mathcal{R}}_{(0)}$ and $\overline{\mathcal{V}}^{\alpha}_{(0)}$ were obtained in the following form:

$$\overline{\mathcal{R}}_{(0)}(z^p) = \mathcal{R}(z^0, z^{\nu}) - \Big[z^{\mu}\Big\langle\frac{\partial\mathcal{R}}{\partial z^{\mu}}\Big\rangle_{(0)} + \Big\langle\mathcal{R}\Big\rangle_{(0)}\Big] + \frac{1}{2}z^{\mu}z^{\beta}\Big[\gamma\eta_{\mu\beta}\,a_{(0)\lambda}a^{\lambda}_{(0)} -$$

$$-(2\gamma - 1)a_{(0)\mu}a_{(0)\beta} + \frac{\partial}{\partial z^0}\Big(\gamma\eta_{\mu\beta}\frac{\partial}{\partial z^0}\Big\langle\mathcal{U}\Big\rangle_{(0)} - 2\Big\langle\partial_{(\mu}\mathcal{V}_{\beta)}\Big\rangle_{(0)}\Big)\Big], \tag{8.11}$$

and

$$\overline{\mathcal{V}}^{\alpha}_{(0)}(z^p) = \mathcal{V}^{\alpha}(z^0, z^{\nu}) - \Big[z^{\mu}\Big\langle\frac{\partial\mathcal{V}^{\alpha}}{\partial z^{\mu}}\Big\rangle_{(0)} + \Big\langle\mathcal{V}^{\alpha}\Big\rangle_{(0)}\Big] + \gamma\frac{1}{4}\Big(z^{\alpha}z_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}z_{\mu}z^{\mu}\Big)\dot{a}^{\beta}_{(0)}. \tag{8.12}$$

8.3 The Topocentric Proper RF.

The construction of the topocentric RF requires a little bit more sophisticated analysis. Thus, we have to specify where this frame is located on the surface of the extended body and what point will be considered as the origin of the coordinates. In order to find the dynamical conditions necessary for construction of the transformation functions (analogous to those given by eqs.(3.26)–(3.29)), one should make an explicit relativistic analysis of the constrained motion of the tracking station placed on the Earth's surface. This analysis should provide one with a detailed description of the problem of static equilibrium of a test particle on the surface of an extended body whose interior is characterized by the energy-momentum tensor T^{mn} and the corresponding equation of state $p(\rho)$. It is likely that the present accuracy of the topocentric radio-metric measurements does not require this level of generality. This permits us to neglect the geometry of the tracking station and its weight, and instead, to consider the law of relativistic motion of an atomic time standard only. Then the answer to the second part of the above question is simple: the origin of the coordinates of the topocentric RF coincides with the atomic time standard that is used as the physically measurable time τ. The world line of the clocks may be considered as the geodetic line of the massless test particle. This suggests that, in order to find the form of the corresponding coordinate transformation functions, one can apply the same generalized Fermi conditions (3.26).

As a result, the general form of the coordinate transformations between the coordinates (y^p_A) of the geocentric RF_A and those $(\zeta^p) \equiv (\tau, \zeta^\nu)$ of a topocentric one in the WFSMA may be presented as follows:

$$y^0_A = \tau + c^{-2}K_{S_0}(\tau,\zeta^\epsilon) + c^{-4}L_{S_0}(\tau,\zeta^\epsilon) + \mathcal{O}(c^{-6}), \tag{8.13a}$$

$$y^\alpha_A = \zeta^\alpha + \zeta^\alpha_{S_0}(\tau) + c^{-2}Q^\alpha_{S_0}(\tau,\zeta^\epsilon) + \mathcal{O}(c^{-4}), \tag{8.13b}$$

where we, as before, have neglected the associated group parameters $\zeta^{S_0}, \sigma^\alpha_{S_0}$, and $f^{\alpha\beta}_{S_0}$ and require that the constructed frame should preserve all the conservation laws in its immediate vicinity. The transformation functions $K_{S_0}, Q^\alpha_{S_0}$, and L_{S_0} in this case will take the following form:

$$K_{S_0}(\tau,\zeta^\nu) = \int^\tau dt' \Big[\big\langle \mathcal{U}\big\rangle_{S'_0} - \frac{1}{2}v_{S_0\nu}v^\nu_{S_0}\Big] - v_{S_0\nu}\cdot\zeta^\nu + \mathcal{O}(c^{-4})\tau, \tag{8.14a}$$

$$Q^\alpha_{S_0}(\tau,\zeta^\nu) = \gamma\Big(\frac{1}{2}\eta^{\alpha\beta}\zeta_\mu\zeta^\mu - \zeta^\alpha\zeta^\beta\Big)\big\langle\partial_\beta\mathcal{U}\big\rangle_{S_0} - \gamma\zeta^\alpha\big\langle\mathcal{U}\big\rangle_{S_0} - \frac{1}{2}v^\alpha_{S_0}v^\beta_{S_0}\zeta_\beta +$$

$$+\zeta_\beta\int^\tau dt'\Big[\frac{1}{2}a^{[\alpha}_{S_0}v^{\beta]}_{S_0} + 2\big\langle\partial^{[\alpha}\mathcal{V}^{\beta]}\big\rangle_{S'_0}\Big] + w^\alpha_{S_0}(\tau) + \mathcal{O}(c^{-4})\zeta^\alpha, \tag{8.14b}$$

$$L_{S_0}(\tau,\zeta^\nu) = \gamma\frac{1}{2}\zeta_\mu\zeta^\mu\frac{\partial}{\partial\tau}\big\langle\mathcal{U}\big\rangle_{S_0} - 2\zeta^\lambda\zeta^\beta\big\langle\partial_\lambda\mathcal{V}_\beta\big\rangle_{S_0} + \gamma v_{S_0\beta}\Big(\zeta^\beta\zeta^\lambda - \frac{1}{2}\eta^{\beta\lambda}\zeta_\mu\zeta^\mu\Big)\big\langle\partial_\lambda\mathcal{U}\big\rangle_{S_0} +$$

$$+\zeta_\lambda v_{S_0\beta}\int^\tau dt'\Big[\frac{1}{2}a^{[\lambda}_{S_0}v^{\beta]}_{S_0} + 2\big\langle\partial^{[\lambda}\mathcal{V}^{\beta]}\big\rangle_{S'_0}\Big] + \zeta_\beta\Big((\gamma+1)v^\beta_{S_0}\big\langle\mathcal{U}\big\rangle_{S_0} - 4\big\langle\mathcal{V}^\beta\big\rangle_{S_0} - \dot{w}^\beta_{S_0}\Big) -$$

$$-\int^\tau dt'\Big[\big\langle\mathcal{Z}\big\rangle_{S'_0} + \frac{1}{2}\Big(\big\langle\mathcal{U}\big\rangle_{S'_0} - \frac{1}{2}v_{S_0\beta}v^\beta_{S_0}\Big)^2 + v_{S_0\mu}\dot{w}^\mu_{S_0}\Big] + \mathcal{O}(c^{-6})\tau, \tag{8.14c}$$

where the new quantities $v^{\alpha}_{S_0}$ and $a^{\alpha}_{S_0}$ are the geocentric velocity and acceleration of a particular point S_0 on the surface of an extended body under question. The notation $\langle f \rangle_{S_0}$ reflects that this quantity was defined in a particular point, S_0, on the surface, S_A, of an extended body (A). The Newtonian acceleration of the clock with respect to the geocentric RF$_A$ is given as

$$a^{\alpha}_{S_0}(\tau) = -\eta^{\alpha\mu}\Big\langle \frac{\partial \mathcal{U}}{\partial \zeta^{\mu}} \Big\rangle_{S_0} + \mathcal{O}(c^{-4}). \tag{8.15}$$

Furthermore, the function $w^{\alpha}_{S_0}$ is determined as the solution of the following differential equation:

$$\ddot{w}^{\alpha}_{S_0}(\tau) = \eta^{\alpha\mu}\Big\langle \frac{\partial \mathcal{Z}}{\partial \zeta^{\mu}} \Big\rangle_{S_0} + v^{\alpha}_{S_0}\frac{\partial}{\partial \tau}\Big\langle \mathcal{U} \Big\rangle_{S_0} + a^{\alpha}_{S_0}\Big\langle \mathcal{U} \Big\rangle_{S_0} - 4\frac{\partial}{\partial \tau}\Big\langle \mathcal{V}^{\alpha} \Big\rangle_{S_0} -$$

$$-\frac{1}{2}v^{\alpha}_{S_0}v_{S_0\beta}a^{\beta}_{S_0} + a_{S_0\beta}\int^{\tau} dt' \Big[\frac{1}{2}a^{[\alpha}_{S_0}v^{\beta]}_{S_0} + 2\Big\langle \partial^{[\alpha}\mathcal{V}^{\beta]} \Big\rangle_{S'_0}\Big] + \mathcal{O}(c^{-6}), \tag{8.16}$$

where the function $\mathcal{Z} \sim \mathcal{O}(c^{-4})$ was defined similarly to the function $\mathcal{R}$ from eqs.(8.8b):

$$\mathcal{Z}(\zeta^p) = \mathcal{W}(\zeta^p) + 2a^{\lambda}_{S_0}\sum_B \frac{\partial}{\partial \zeta^{\lambda}}\chi_B(\tau, \zeta^{\nu}) + \mathcal{O}(c^{-6}). \tag{8.17}$$

As a result, the components of the metric tensor $g^{S_0}_{mn}$ in the coordinates $(\zeta^p) \equiv (\tau, \zeta^{\nu})$ of the topocentric RF take the following form:

$$g^{S_0}_{00}(\zeta^p) = 1 - 2\overline{\mathcal{U}}_{S_0} + 2\overline{\mathcal{Z}}_{S_0} + \mathcal{O}(c^{-6}), \tag{8.18a}$$

$$g^{S_0}_{0\alpha}(\zeta^p) = 4\,\eta_{\alpha\epsilon}\overline{\mathcal{V}}^{\epsilon}_{S_0} + \mathcal{O}(c^{-5}), \tag{8.18b}$$

$$g^{S_0}_{\alpha\beta}(\zeta^p) = \eta_{\alpha\beta}\Big(1 + 2\gamma\overline{\mathcal{U}}_{S_0}\Big) + \mathcal{O}(c^{-4}). \tag{8.18c}$$

The obtained expressions, (8.18), represent the general solution for the field equations of the general relativity on the surface of an extended body in the WFSMA. The effective Newtonian potential in the vicinity of the antenna may be presented as follows:

$$\overline{\mathcal{U}}_{S_0}(\zeta^p) = \mathcal{U}(\tau, \zeta^{\nu}) - \frac{\partial K_S(\tau, \zeta^{\nu})}{\partial \tau} - \frac{1}{2}v_{S_0\mu}v^{\mu}_{S_0} =$$

$$= \mathcal{U}(\tau, \zeta^{\nu}) - \Big[\zeta^{\mu}\Big\langle \frac{\partial \mathcal{U}}{\partial \zeta^{\mu}} \Big\rangle_{S_0} + \Big\langle \mathcal{U} \Big\rangle_{S_0}\Big]. \tag{8.19}$$

The functions $\overline{\mathcal{Z}}_{S_0}$ and $\overline{\mathcal{V}}^{\alpha}_{S_0}$ were obtained in the following form:

$$\overline{\mathcal{Z}}_{S_0}(\zeta^p) = \mathcal{R}(\tau, \zeta^{\nu}) - \Big[\zeta^{\mu}\Big\langle \frac{\partial \mathcal{R}}{\partial \zeta^{\mu}} \Big\rangle_{S_0} + \Big\langle \mathcal{R} \Big\rangle_{S_0}\Big] + \frac{1}{2}\zeta^{\mu}\zeta^{\beta}\Big[\gamma\eta_{\mu\beta}\, a_{S_0\lambda}a^{\lambda}_{S_0} -$$

$$-(2\gamma - 1)a_{S_0\mu}a_{S_0\beta} + \frac{\partial}{\partial \tau}\Big(\gamma\eta_{\mu\beta}\frac{\partial}{\partial \tau}\Big\langle \mathcal{U} \Big\rangle_{S_0} - 2\Big\langle \partial_{(\mu}\mathcal{V}_{\beta)} \Big\rangle_{S_0}\Big)\Big], \tag{8.20}$$

$$\overline{\mathcal{V}}^{\alpha}_{S_0}(\zeta^p) = \mathcal{V}^{\alpha}(\tau, \zeta^{\nu}) - \Big[\zeta^{\mu}\Big\langle \frac{\partial \mathcal{V}^{\alpha}}{\partial \zeta^{\mu}} \Big\rangle_{S_0} + \Big\langle \mathcal{V}^{\alpha} \Big\rangle_{S_0}\Big] + \gamma\frac{1}{4}\Big(\zeta^{\alpha}\zeta_{\beta} - \frac{1}{2}\delta^{\alpha}_{\beta}\zeta_{\mu}\zeta^{\mu}\Big)\dot{a}^{\beta}_{S_0}. \tag{8.21}$$

It should be stressed that more detailed analysis is necessary for the final solution of the problem of relativistic astronomical measurements performed from the topocentric RF. However, we believe that the presented general approach, incorporated in the new formalism, enables one to construct the topocentric proper reference frame with well-defined physical properties. Moreover, the accuracy of the theoretical expressions obtained here is far beyond that achieved in real astronomical practice. This suggests that the presented formulae could be used for quite a long time before practical needs will require theoreticians to reconsider the presented results in order to achieve higher accuracy of the physical modelling of the relativistic measurements.

8.4 Discussion.

It is generally understood that any RF is not a physical substance but rather a conventional artifact. The main reason we need RFs is that they are convenient in exchanging the observational data and one's discoveries and opinions, which are the starting points in doing scientific research. In this sense, the most important character of the RF is that it is widely accepted and is related clearly with the other existing references. In addition, it is desirable to represent the actual phenomenon precisely. If the first point is respected, what we should do in these days of an advanced electronic/computational environment is to move toward standardization, which never means the exclusion of other points of view. Rather it should be understood as only a scale that enables us to express observation/theoretical quantities in a concise manner.

An application of atomic frequency standards is the establishment of atomic time scales. International Atomic Time is the official basis by which events are dated. However, the need to distinguish between theoretical times and their realizations, the need for a relativistic treatment, and the survival of previous astronomical times generate a complex situation. Specific problems raised by time scales and the relationships they have with one another and with the successive definitions of seconds in different RFs should be examined in more detail. Thus, currently employed definitions of ephemeris astronomy and the system of astronomical constants are based on Newtonian mechanics with its absolute time and absolute space. To avoid any relativistic ambiguities in applying new IAU (1991) resolutions on RFs and time scales, one should specify the astronomical constructions and definitions of constants to make them consistent with general relativity. However, up to this time, the VSOP theories of the motion of the planets were constructed on the base of integration of Lagrange's differential equations (Brumberg *et al.*, 1993). The development of the perturbative function included the mutual perturbations of the bodies and was performed up to the third order of the perturbative masses using the Newtonian perturbative function. The relativistic contributions to the equations of motion were limited to the Schwarzschild problem. The accuracy reached by such solutions is only a few mas for the inner planets and less for the outer ones. Due to the fact that the present astrometric accuracies have reached the mas level, the mutual relativistic perturbations of the planets must be included in the ephemeris constructions.

In this report, we addressed these and other problems of modern astronomy and have presented the theoretical foundation necessary for conducting relativistic measurements in curved space-time in the WFSMA. Our approach naturally incorporates the general properties of the dynamical RF into the hierarchy of the relativistic RFs and the time scales. Moreover, we obtained the new relation between the time scales, which was obtained to the fourth order in $1/c$, c being the velocity of light in a vacuum. The accuracy of these expressions is at the ps level, which is the future requirement in many different applications. Thus, this formulation leads to improved relations between barycentric and geocentric quantities. These expressions will be

useful in converting the numerical values of some astronomical constants determined in the old IAU time scale to new scales. The obtained results naturally contain exhaustive information about the multipolar structure of the gravitational field in the N-body system and enable one to model the experimental situation with very high accuracy. Because of this, we anticipate that the results presented in this report may be immediately applied in the following important areas of modern astronomy and astrophysics:

(i). Precise VLBI timing measurements.

(ii). Precise radiometric navigation of future space missions and the corresponding data analysis.

(iii). More precise analysis of the binary system dynamics, including modelling of the coalescing experiments and studies of the gravitational wave physics.

Let us mention that there are some problems that remain to be solved. It is known that the rotational motion of extended bodies in general relativity is a complicated problem that has no satisfactory solution up to now. Moreover, modern observational accuracy of geodynamical observations makes it necessary to have a rigorous relativistic model of Earth's rotation. The currently employed solution for the Earth's rotation problem is valid for restricted intervals of time. Moreover, there is an urgent necessity to elaborate a theory of nutation-precession matching the accuracy of very modern techniques, such as VLBI and LLR. To do this, one would have to model the transfer function leading to theoretical determination of the nutation coefficients when including predominant geophysical characteristics (elastic mantle, coupling at core-mantle boundary, free core nutation, free inner core nutation, etc.). Furthermore, reductions of measurements included relativistic corrections, effects of propagation of electromagnetic signals in the Earth's troposphere and in the solar corona with simultaneous evaluation of the parameters of the corona model from general fitting. The presented formalism provides one with the necessary basis for studying this problem from very general positions and could serve as the foundation for future theoretical analysis.

As a result, an astronomical reference system may be defined as a set of the transformation functions and constants including the physically well-defined set of RFs and their mutual relationships, time arguments, ephemerides, and the standard constants and algorithms. The extragalactic, or radio, RF will be the basic frame for the development of the future ephemeris (Standish, 1995). Achieving milli- to micro-arcsecond accuracies at optical wavelengths will reduce the disparity between optical, radar, and radio RF determinations. Thus, the relationships and identifications of common sources should be much more accurate. Another significant change should be the ability to determine distances and, thus, space motions on a three-dimensional basis, rather than the current two-dimensional basis of proper motions. Improvements in ephemerides provide the opportunity to investigate the difference between atomic and dynamical time, the relationship between the dynamical and extragalactic RF, and the values of precession and nutation. Also, the relationships between the bright and faint optical catalogs, the infrared, and the extragalactic RFs should be better determined. The theory of relativistic astronomical RFs presented in this report was developed in order to serve exactly the above-mentioned needs, and it will be used the future analysis of these problems of fundamental importance.

In order to accomplish these goals, our future efforts will be directed to finalizing the transcription of the results obtained on the language of the practical applications. We will establish the necessary relativistic measurement models and will implement these results into existing computer software codes, as well as performing detailed analysis of the real data from the space gravitational experiments. The analysis of the above-mentioned problems from the new positions of the presented theory of relativistic astronomical RFs will be the subject for specific studies and future publications.

Appendix A: Generalized Gravitational Potentials.

The generalized gravitational potentials for the non-radiative problems in the WFSMA are given in Will (1993) as

$$U(z^p) = \int \frac{d^3z'\rho_0(z'^p)}{|z^\nu - z'^\nu|}, \qquad V^\alpha(z^p) = -\int d^3z' \frac{\rho_0(z'^p)v^\alpha(z'^p)}{|z^\nu - z'^\nu|},$$

$$W^\alpha(z^p) = \int d^3z'\rho_0(z'^p)v_\mu(z'^p)\frac{(z^\alpha - z'^\alpha)(z^\mu - z'^\mu)}{|z^\nu - z'^\nu|},$$

$$A(z^p) = \int d^3z'\rho_0(z'^p)\frac{[v^\mu(z'^p)(z^\mu - z'^\mu)]^2}{|z^\nu - z'^\nu|^3}, \qquad \chi(z^p) = -\int d^3z'\rho_0(z'^p)|z^\nu - z'^\nu|,$$

$$U^{\alpha\beta}(z^p) = \int d^3z'\rho_0(z'^p)\frac{(z^\alpha - z'^\alpha)(z^\beta - z'^\beta)}{|z^\nu - z'^\nu|^3},$$

$$\Psi(z^p) = -(\gamma+1)\Phi_1 - (3\gamma + 1 - 2\beta)\Phi_2 - \Phi_3 - 3\gamma\Phi_4,$$

where the other potentials are given as follows:

$$\Phi_1(z^p) = -\int d^3z' \frac{\rho_0(z'^p)v_\lambda(z'^p)v^\lambda(z'^p)}{|z^\nu - z'^\nu|}, \qquad \Phi_2(z^p) = \int d^3z' \frac{\rho_0(z'^p)U(z'^p)}{|z^\nu - z'^\nu|},$$

$$\Phi_3(z^p) = \int d^3z' \frac{\rho_0(z'^p)\Pi(z'^p)}{|z^\nu - z'^\nu|}, \qquad \Phi_4(z^p) = \int d^3z' \frac{\rho_0(z'^p)p(\rho(z'^p))}{|z^\nu - z'^\nu|},$$

$$\Phi_w(z^p) = \int\int d^3z'd^3z''\rho_0(z'^p)\rho_0(z''^p)\frac{(z^\beta - z'^\beta)}{|z^\nu - z'^\nu|^3}\Big[\frac{(z_\beta - z''^\beta)}{|z'^\nu - z''^\nu|} - \frac{(z'^\beta - z''^\beta)}{|z^\nu - z''^\nu|}\Big].$$

In order to indicate the functional dependence in the potentials introduced above, we have used the following notation: $(z^p) \equiv (z^0, z^\nu)$. Then for any function f, one will have $f(z^p) = f(z^0, z^\nu)$ and $f(z'^p) = f(z^0, z'^\nu)$.

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Appendix B: Power Expansions of the General Geometric Quantities.

In this appendix, we will present the expansion of some physical quantities with respect to powers of the small parameter c^{-1}. We will use these expansions for linearizing the gravitational field equations of the metric theories of gravity in the WFSMA.

B.1. Expansion for the Metric Tensor g_{mn}.

The post-Newtonian expansion for the metric tensor g_{mn} with respect to the powers of the small parameter c^{-1} in the coordinates (z^p) of an arbitrary RF (either a barycentric inertial RF_0 or a proper RF_A non-inertial one) may be presented as follows:

$$g_{00} = 1 + c^{-2} g_{00}^{<2>} + c^{-4} g_{00}^{<4>} + \mathcal{O}(c^{-6}), \qquad (B1a)$$

$$g_{0\alpha} = c^{-3} g_{0\alpha}^{<3>} + c^{-5} g_{0\alpha}^{<5>} + \mathcal{O}(c^{-7}), \qquad (B1b)$$

$$g_{\alpha\beta} = \gamma_{\alpha\beta} + c^{-2} g_{\alpha\beta}^{<2>} + c^{-4} g_{\alpha\beta}^{<4>} + \mathcal{O}(c^{-6}), \qquad (B1c)$$

where $\gamma_{\alpha\beta}$ is the spatial part of the background metric γ_{mn}. The notations $g_{mn}^{<k>}$, $(k = 1, 2, 3...)$ at the right-hand side of expressions $(B1)$ are the terms of the expansion of g_{mn} with the order of magnitude $\epsilon^k \sim c^{-k}$, respectively. In some calculations, we will omit the multipliers c^{-k} in order to achieve brevity in the expressions. It should be noted that reversing the sign of the time $z^0 \to -z^0$ corresponds to the change of the sign of the small parameter ϵ. Because of this, in the expressions for g_{00} $(B1a)$ and $g_{\alpha\beta}$ $(B1c)$, only the terms with even powers of the small parameter c^{-1} have been taken into account, and in the expressions for $g_{0\alpha}$ $(B1b)$, only the odd ones are used. The fact that in expression $(B1b)$ the term $g_{0\alpha}^{<1>}$ is absent is quite natural. Indeed, even the main expansion for $g_{0\alpha}$ (Newtonian) should not be less than the second order with respect to the small parameter c^{-1} (Will, 1993). In our further calculations, we will not be investigating the processes of generating the gravitational waves by the system of astronomical bodies, so our expressions for the component g_{00} in expressions $(B1)$, do not contain the term of order $\mathcal{O}(c^{-5})$. However, one may easily reconstruct all the calculations to account for this term as well.

B.2. Expansion for the $det\,[g_{mn}]$ and g^{mn}.

In some calculations, we will need the relations for the determinant of the metric tensor $g = det\,[g_{mn}]$ and the inverse metric g^{mn}. From the expressions in eqs.$(B1)$, one may obtain the following relations that are valid for any RF:

$$g = -1 - g_{00}^{<2>} + g_{11}^{<2>} + g_{22}^{<2>} + g_{33}^{<2>} - g_{00}^{<4>} + g_{11}^{<4>} + g_{22}^{<4>} + g_{33}^{<4>} +$$

$$+ g_{00}^{<2>}\left(g_{11}^{<2>} + g_{22}^{<2>} + g_{33}^{<2>}\right) - g_{11}^{<2>} g_{22}^{<2>} - g_{11}^{<2>} g_{33}^{<2>} -$$

$$- g_{22}^{<2>} g_{33}^{<2>} + g_{12}^{<2>2} + g_{13}^{<2>2} + g_{23}^{<2>2} + \mathcal{O}(c^{-6}), \qquad (B2)$$

and

$$g^{00} = 1 + g^{<2>00} + g^{<4>00} + \mathcal{O}(c^{-6}),$$

$$g^{0\alpha} = g^{<3>0\alpha} + g^{<5>0\alpha} + \mathcal{O}(c^{-7}),$$

$$g^{\alpha\beta} = \gamma^{\alpha\beta} + g^{<2>\alpha\beta} + g^{<4>\alpha\beta} + \mathcal{O}(c^{-6}), \qquad (B3)$$

where the components of the inverse metric $g^{<k>mn}$ are given as follows:

$$g^{<2>00} = -g_{00}^{<2>}, \qquad g^{<2>\alpha\beta} = -\gamma^{\alpha\mu}\gamma^{\beta\nu}g_{\mu\nu}^{<2>},$$

$$g^{<3>0\alpha} = -\gamma^{\alpha\nu}g_{0\nu}^{<3>}, \qquad g^{<4>00} = (g_{00}^{<2>})^2 - g_{00}^{<4>},$$

$$g^{<4>\alpha\beta} = -\gamma^{\alpha\mu}\gamma^{\beta\nu}g_{\mu\nu}^{<4>} + \gamma^{\alpha\sigma}\gamma^{\beta\lambda}\gamma^{\mu\nu}g_{\sigma\mu}^{<2>}g_{\lambda\nu}^{<2>},$$

$$g^{<5>0\alpha} = -\gamma^{\alpha\mu}g_{0\mu}^{<5>} + \gamma^{\alpha\mu}g_{00}^{<2>}g_{0\mu}^{<3>} + \gamma^{\alpha\lambda}\gamma^{\mu\nu}g_{\lambda\mu}^{<2>}g_{0\nu}^{<3>}. \qquad (B4)$$

B.3. Expansion for the $\hat{g}^{mn} = \sqrt{-g}g^{mn}$.

For some practical applications, we will need the expansions for the components of density of the metric tensor $\hat{g}^{mn} = \sqrt{-g}g^{mn}$ as well. One may easily obtain those from the expressions of eqs.$(B2)$–$(B4)$ in the following form:

$$\hat{g}^{00} = 1 + \hat{g}^{<2>00} + \hat{g}^{<4>00} + \mathcal{O}(c^{-6}),$$

$$\hat{g}^{0\alpha} = \hat{g}^{<3>0\alpha} + \hat{g}^{<5>0\alpha} + \mathcal{O}(c^{-7}),$$

$$\hat{g}^{\alpha\beta} = \hat{\gamma}^{\alpha\beta} + \hat{g}^{<2>\alpha\beta} + \hat{g}^{<4>\alpha\beta} + \mathcal{O}(c^{-6}), \qquad (B5a)$$

with the components of $\hat{g}^{mn}$ given as

$$\hat{g}^{<2>00} = g^{<2>00} + \frac{1}{2}A^{<2>},$$

$$\hat{g}^{<4>00} = g^{<4>00} + \frac{1}{2}g^{<2>00}A^{<2>} + \frac{1}{2}\Big(A^{<4>} - \frac{1}{4}(A^{<2>})^2\Big),$$

$$\hat{g}^{<3>0\alpha} = g^{<3>0\alpha}, \qquad \hat{g}^{<5>0\alpha} = g^{<5>0\alpha} + \frac{1}{2}g^{<3>0\alpha}A^{<2>},$$

$$\hat{g}^{<2>\alpha\beta} = g^{<2>\alpha\beta} + \frac{1}{2}\gamma^{\alpha\beta}A^{<2>},$$

$$\hat{g}^{<4>\alpha\beta} = g^{<4>\alpha\beta} + \frac{1}{2}g^{<2>\alpha\beta}A^{<2>} + \frac{1}{2}\gamma^{\alpha\beta}\Big(A^{<4>} - \frac{1}{4}(A^{<2>})^2\Big). \qquad (B5b)$$

In expressions $(B5b)$, we have introduced the following notations:

$$A^{<2>} = g_{00}^{<2>} - g_{11}^{<2>} - g_{22}^{<2>} - g_{33}^{<2>},$$

$$A^{<4>} = g_{00}^{<4>} - g_{11}^{<4>} - g_{22}^{<4>} - g_{33}^{<4>} -$$

$$-g_{00}^{<2>}\Big(g_{11}^{<2>} + g_{22}^{<2>} + g_{33}^{<2>}\Big) +$$

$$+g_{11}^{<2>}g_{22}^{<2>} + g_{11}^{<2>}g_{33}^{<2>} + g_{22}^{<2>}g_{33}^{<2>} -$$

$$-(g_{12}^{<2>})^2 - (g_{13}^{<2>})^2 - (g_{23}^{<2>})^2. \qquad (B6)$$

B.4. Expansion for the Gauge Conditions.

The covariant de Donder gauge conditions are given by eq.(3.6) as follows:

$$\mathcal{D}_m\Big(\sqrt{-g}g^{mn}(z^p)\Big) = 0, \qquad (B7a)$$

or equivalently

$$\frac{\partial}{\partial z^m}\Big(\sqrt{-g}g^{mn}(z^p)\Big) + \gamma_{kl}^n(z^p)\sqrt{-g}g^{kl}(z^p) = 0, \qquad (B7b)$$

where $\gamma_{kl}^n(z^p)$ is the Christoffel symbols with respect to the background metric $\gamma_{kl}(z^p)$ in coordinates (z^p) of an arbitrary RF. The relations of eqs.$(B5)$–$(B6)$ enable one to find the expressions for linearized gauge conditions $(B7b)$. Thus, for $n = 0$, we will have

$$\frac{1}{2}\frac{\partial}{\partial z^0}\Big(\gamma^{\epsilon\nu}g_{\epsilon\nu}^{<2>} - g_{00}^{<2>}\Big) - \gamma^{\epsilon\nu}\frac{\partial}{\partial z^\epsilon}g_{0\nu}^{<3>} + \gamma_{00}^{0\ <3>}(z^p) +$$

$$+\gamma^{\mu\nu}\gamma_{\mu\nu}^{0<3>}(z^p) = \mathcal{O}(c^{-5}). \qquad (B7c)$$

For $n = \alpha$, we will obtain

$$\frac{1}{2}\gamma^{\alpha\lambda}\frac{\partial}{\partial z^\lambda}\Big(g_{00}^{<2>} + \gamma^{\epsilon\nu}g_{\epsilon\nu}^{<2>}\Big) - \gamma^{\epsilon\nu}\gamma^{\mu\alpha}\frac{\partial}{\partial z^\epsilon}g_{\nu\mu}^{<3>} + \gamma_{00}^{\alpha<3>}(z^p) +$$

$$+\gamma^{\mu\nu}\gamma_{\mu\nu}^{\alpha<3>}(z^p) = \mathcal{O}(c^{-4}), \qquad (B7d)$$

where $\gamma_{kl}^m(z^p)$ is the components of the Christoffel symbols with respect to the Riemann-flat non-inertial background metric $\gamma_{mn}(z^p)$ in coordinates (z^p). One may easily see that for any non-inertial RF these components may produce a non-vanishing contribution to the gauge conditions $(B7)$. This property will be of use in order to write the field equations in an arbitrary RF.

B.5. Expansion of the Christoffel Symbols.

One could easily find the expansion of the connection components Γ_{mn}^k with respect to the small parameter c^{-1} and present those in terms of expansions $g_{mn}^{<k>}$. Thus, defining Γ_{mn}^k as usual:

$$\Gamma_{nm}^k(z^p) = \frac{1}{2}g^{kp}(z^p)\Big(\partial_n g_{mp}(z^p) + \partial_m g_{pn}(z^p) - \partial_p g_{mn}(z^p)\Big),$$

where $\partial_n = \partial/\partial z^n$ in coordinates (z^p) of an arbitrary RF from the relations $(B1) and (B3)$–$(B4)$, we will have the following expressions for the components of the Christoffel symbols with respect to the powers of the small parameter c^{-1}:

$$\Gamma^0_{00}(z^p) = \frac{1}{2}\partial_0 g^{<2>}_{00} + \frac{1}{2}\partial_0 g^{<4>}_{00} +$$

$$+\frac{1}{2}\Big(g^{<2>00}\partial_0 g^{<2>}_{00} - g^{<3>0\mu}\partial_\mu g^{<2>}_{00}\Big) + \mathcal{O}(c^{-7}), \qquad (B8a)$$

$$\Gamma^0_{0\alpha}(z^p) = \frac{1}{2}\partial_\alpha g^{<2>}_{00} + \frac{1}{2}\Big(\partial_\alpha g^{<4>}_{00} + g^{<2>00}\partial_\alpha g^{<2>}_{00}\Big) + \mathcal{O}(c^{-6}), \qquad (B8b)$$

$$\Gamma^0_{\alpha\beta}(z^p) = \frac{1}{2}\Big(\partial_\alpha g^{<3>}_{0\beta} + \partial_\beta g^{<3>}_{0\alpha} - \partial_0 g^{<2>}_{\alpha\beta}\Big) + \mathcal{O}(c^{-5}), \qquad (B8c)$$

$$\Gamma^\alpha_{00}(z^p) = -\frac{1}{2}\gamma^{\alpha\mu}\partial_\mu g^{<2>}_{00} - \frac{1}{2}\gamma^{\alpha\mu}\partial_\mu g^{<4>}_{00} + \gamma^{\alpha\mu}\partial_0 g^{<3>}_{0\mu} -$$

$$-\frac{1}{2}g^{<2>\alpha\mu}\partial_\mu g^{<2>}_{00} + \mathcal{O}(c^{-6}), \qquad (B8d)$$

$$\Gamma^\alpha_{0\beta}(z^p) = \frac{1}{2}\gamma^{\alpha\mu}\partial_\beta g^{<3>}_{0\mu} + \frac{1}{2}\gamma^{\alpha\mu}\partial_0 g^{<2>}_{\beta\mu} - \frac{1}{2}\gamma^{\alpha\mu}\partial_\mu g^{<3>}_{0\beta} + \mathcal{O}(c^{-5}), \qquad (B8e)$$

$$\Gamma^\alpha_{\beta\nu}(z^p) = \gamma^{\alpha(0)}_{\beta\nu} + \frac{1}{2}\gamma^{\alpha\mu}\Big(\partial_\beta g^{<2>}_{\mu\nu} + \partial_\nu g^{<2>}_{\mu\beta} - \partial_\mu g^{<2>}_{\beta\nu}\Big) + \mathcal{O}(c^{-4}), \qquad (B8f)$$

where $\gamma^{\alpha(0)}_{\beta\nu}$ is the Christoffel symbols in coordinates of the Galilean inertial RF. One may make them vanish by choosing quasi-Cartesian coordinates.

B.6. Expansion for the Ricci Tensor R_{mn}.

By making use of the expressions of eqs.$(B8)$, one may also find the relations for the expanded components of the Ricci tensor $R_{mn}(z^p)$ in coordinates (z^p) of an arbitrary RF. This tensor is defined as follows:

$$R_{mn}(z^p) = g^{kp}R_{kmnp}(z^p) = \partial_p\Gamma^p_{mn} - \partial_n\Gamma^p_{mp} + \Gamma^l_{mn}\Gamma^p_{lp} - \Gamma^l_{mp}\Gamma^p_{ln}.$$

Then, in quasi-Cartesian coordinates of an arbitrary RF, one may obtain the expanded components of the Ricci tensor as follows:

$$R_{00}(z^p) = -\frac{1}{2}\gamma^{\mu\lambda}\partial^2_{\mu\lambda}g^{<2>}_{00} - \frac{1}{2}\gamma^{\mu\lambda}\partial^2_{\mu\lambda}g^{<4>}_{00} + \gamma^{\mu\lambda}\partial^2_{0\mu}g^{<3>}_{0\lambda} -$$

$$-\frac{1}{2}\partial_\mu\Big(g^{<2>\mu\lambda}\partial_\lambda g^{<2>}_{00}\Big) - \frac{1}{2}\gamma^{\mu\lambda}\partial^2_{00}g^{<2>}_{\mu\lambda} -$$

$$-\frac{1}{4}\gamma^{\mu\lambda}\gamma^{\sigma\nu}\partial_\mu g^{<2>}_{00}\partial_\lambda g^{<2>}_{\sigma\nu} + \frac{1}{4}\gamma^{\mu\lambda}\partial_\mu g^{<2>}_{00}\partial_\lambda g^{<2>}_{00} + \mathcal{O}(c^{-6}), \qquad (B9a)$$

$$R_{0\alpha}(z^p) = \frac{1}{2}\gamma^{\mu\lambda}\partial^2_{0\mu}g^{<2>}_{\alpha\lambda} - \frac{1}{2}\gamma^{\mu\lambda}\partial^2_{0\alpha}g^{<2>}_{\mu\lambda} +$$

$$+\frac{1}{2}\gamma^{\mu\lambda}\partial^2_{\alpha\mu}g^{<3>}_{\lambda 0} - \frac{1}{2}\gamma^{\mu\lambda}\partial^2_{\mu\lambda}g^{<3>}_{0\alpha} + \mathcal{O}(c^{-5}), \qquad (B9b)$$

$$R_{\alpha\beta}(z^p) = -\frac{1}{2}\gamma^{\mu\lambda}\partial^2_{\mu\lambda}g^{<2>}_{\alpha\beta} + \frac{1}{2}\gamma^{\mu\lambda}\partial^2_{\mu\alpha}g^{<2>}_{\lambda\beta} + \frac{1}{2}\gamma^{\mu\lambda}\partial^2_{\mu\beta}g^{<2>}_{\lambda\alpha} -$$

$$-\frac{1}{2}\partial^2_{\alpha\beta}g^{<2>}_{00} - \frac{1}{2}\gamma^{\mu\lambda}\partial^2_{\alpha\beta}g^{<2>}_{\mu\lambda} + \mathcal{O}(c^{-4}). \qquad (B9c)$$

B.7. Expansion of an Arbitrary Energy-Momentum Tensor T^B_{mn}.

At this point, the precise definition for the energy-momentum tensor of the matter distribution T^B_{mn} is not important. For future analysis, we will accept the most general assumptions concerning this quantity. Namely, we will work with such energy-momentum tensors, T^B_{mn}, the temporal, the temporal-spatial, and the spatial components of which may be presented in terms of the order of magnitude as follows: $T^B_{mn}(y^p_B) = \Big(\mathcal{O}(1), \mathcal{O}(c^{-1}), \mathcal{O}(c^{-2})\Big)$.

The construction of the iterative scheme is required to perform the power expansion of the energy-momentum tensor of matter T^{mn} as well. Suppose that T^{mn} may be expanded with respect to the small parameter c^{-1} as follows:

$$T^{00} = T^{<0>00} + T^{<2>00} + \mathcal{O}(c^{-4}), \qquad (B10a)$$

$$T^{0\alpha} = T^{<1>0\alpha} + T^{<3>0\alpha} + \mathcal{O}(c^{-5}), \qquad (B10b)$$

$$T^{\alpha\beta} = T^{<2>\alpha\beta} + T^{<4>\alpha\beta} + \mathcal{O}(c^{-6}). \qquad (B10c)$$

Then, by taking into account expressions $(B1)$ and with the help of relations $(B10)$, we will get the inverse tensor T_{mn} as follows:

$$T_{00} = T^{<0>}_{00} + T^{<2>}_{00} + \mathcal{O}(c^{-4}), \qquad (B11a)$$

$$T_{0\alpha} = T^{<1>}_{0\alpha} + T^{<3>}_{0\alpha} + \mathcal{O}(c^{-5}), \qquad (B11b)$$

$$T_{\alpha\beta} = T^{<2>}_{\alpha\beta} + T^{<4>}_{\alpha\beta} + \mathcal{O}(c^{-6}), \qquad (B11c)$$

where

$$T^{<0>}_{00} = T^{<0>00}, \qquad T^{<2>}_{00} = T^{<2>00} + 2\, g^{<2>}_{00}\, T^{<0>00},$$

$$T^{<3>}_{0\alpha} = g^{<3>}_{0\alpha} T^{<0>00} + \Big(g^{<2>}_{\alpha\mu} + \gamma_{\alpha\mu} g^{<2>}_{00}\Big)\, T^{<1>0\beta},$$

$$T^{<1>}_{0\alpha} = \gamma_{\alpha\beta}\, T^{<1>0\beta}, \qquad T^{<2>}_{\alpha\mu} = \gamma_{\alpha\mu}\gamma_{\beta\lambda}\, T^{<2>\mu\lambda}. \qquad (B11d)$$

Concluding this part, we will present the expression for the right-hand side of the Hilbert–Einstein field equations, eqs.(4.1), which is given as follows:

$$S_{mn} = T_{mn} - \frac{1}{2} g_{mn}\, T. \tag{B12}$$

By substituting the expressions of eqs.$(B1) and (B11)$ into definition $(B12)$, we obtain the expansions for the quantity S_{mn} in the WFSMA:

$$S_{00} = \frac{1}{2} T^{<0>00} + \frac{1}{2}\Big(T^{<2>00} + 2\, g_{00}^{<2>}\, T^{<0>00} - \gamma_{\mu\lambda}\, T^{<2>\mu\lambda}\Big) + \mathcal{O}(c^{-4}), \tag{B13a}$$

$$S_{0\alpha} = \gamma_{\alpha\lambda}\, T^{<3>0\lambda} + \mathcal{O}(c^{-4}), \tag{B13b}$$

$$S_{\alpha\beta} = -\frac{1}{2}\gamma_{\alpha\beta}\, T^{<0>00} + \Big(\gamma_{\alpha\mu}\gamma_{\beta\lambda} - \frac{1}{2}\gamma_{\alpha\beta}\gamma_{\mu\lambda}\Big)\, T^{<2>\mu\lambda} -$$

$$-\frac{1}{2}\Big(\gamma_{\alpha\beta}\, T^{<2>00} + \gamma_{\alpha\beta}\, g_{00}^{<2>}\, T^{<0>00} + g_{\alpha\beta}^{<2>}\, T^{<0>00}\Big) + \mathcal{O}(c^{-4}). \tag{B13c}$$

Appendix C: Transformation Laws of the Coordinate Base Vectors.

In this appendix, we will present the transformation rules for the coordinate base vectors under the general post-Newtonian coordinate transformations, which were discussed in Section 3.

C.1. Direct Transformation of the Coordinate Base Vectors.

According to the transformation rules of the solutions of the field equations $h^{mn}_{(0)}$ and an arbitrary energy-momentum tensor T^{mn} given by eqs.(3.1)–(3.4), in order to develop consistent perturbation theory for the N-body problem in the WFSMA, one needs to have the post-Newtonian expansions for the following derivatives:

$$\frac{\partial x^k}{\partial y^m_A}, \quad \frac{\partial y^l_B}{\partial x^n}, \quad \frac{\partial y^k_B}{\partial y^m_A}.$$

These derivatives form the transformation matrix $\lambda^{\bar{k}}_l$ of the coordinate bases while the transition between the different coordinate systems is performed. Thus, for the transition from the barycentric RF_0 coordinate base $e_m = \partial/\partial x^n$ to the body-centric one $e^A_m = \partial/\partial y^m_A$, the transformation matrix is defined as usual: $e_m = e^A_{\bar{p}} \partial x^{\hat{p}}/\partial y^m_A = e^A_{\bar{p}} \lambda^{\bar{p}}_{Am}$. Then, making use of the transformations of eqs.(3.5), it is easy to get

$$\lambda^{\bar{0}}_{A0}(y^p_A) = \frac{\partial x^0}{\partial y^0_A} = 1 + \frac{\partial}{\partial y^0_A} K_A(y^0_A, y^\nu_A) + \frac{\partial}{\partial y^0_A} L_A(y^0_A, y^\nu_A) + \mathcal{O}(c^{-6}), \qquad (C1a)$$

$$\lambda^{\bar{0}}_{A\alpha}(y^p_A) = \frac{\partial x^0}{\partial y^\alpha_A} = \frac{\partial}{\partial y^\alpha_A} K_A(y^0_A, y^\nu_A) + \frac{\partial}{\partial y^\alpha_A} L_A(y^0_A, y^\nu_A) + \mathcal{O}(c^{-5}), \qquad (C1b)$$

$$\lambda^{\bar{\alpha}}_{A0}(y^p_A) = \frac{\partial x^\alpha}{\partial y^0_A} = v^\alpha_{A_0}(y^0_A) + \frac{\partial}{\partial y^0_A} Q^\alpha_A(y^0_A, y^\nu_A) + \mathcal{O}(c^{-5}), \qquad (C1c)$$

$$\lambda^{\bar{\alpha}}_{A\mu}(y^p_A) = \frac{\partial x^\alpha}{\partial y^\mu_A} = \delta^\alpha_\mu + \frac{\partial}{\partial y^\mu_A} Q^\alpha_A(y^0_A, y^\nu_A) + \mathcal{O}(c^{-4}). \qquad (C1d)$$

By using expressions $(C1)$, one could obtain the determinant of this transformation matrix as follows:

$$det\Big[\lambda^{\bar{p}}_{Am}(y^p_A)\Big] = 1 + \frac{\partial}{\partial y^0_A} K_A(y^0_A, y^\nu_A) -$$

$$-v^\lambda_{A_0} \frac{\partial}{\partial y^\lambda_A} K_A(y^0_A, y^\nu_A) + \frac{\partial}{\partial y^\mu_A} Q^\mu_A(y^0_A, y^\nu_A) + \mathcal{O}(c^{-4}). \qquad (C2)$$

The condition $det\Big[\lambda^{\bar{p}}_{Am}(y^p_A)\Big] = 0$ gives the boundary of validity of these transformations in their application for constructing a proper RF_A.

C.2. Transformation of the Background Metric γ_{mn}.

Relations $(C1)$ are a useful tool for calculating the metric tensor $\gamma^A_{kl}(y^p_A)$ of the background space-time in the non-inertial proper RF_A from the eqn.(3.4). The transformation rule for these components is given by the usual expression:

$$\gamma^A_{mn}(y^p_A) = \frac{\partial x^k}{\partial y^m_A} \frac{\partial x^l}{\partial y^n_A} \gamma_{kl}(x^s(y^p_A)). \qquad (C3)$$

Then, with the help of relations $(C1)$, the temporal-spatial components of the background metric could be presented as

$$\gamma^A_{0\alpha}(y^p_A) = \gamma^{A<1>}_{0\alpha}(y^p_A) + \gamma^{A<3>}_{0\alpha}(y^p_A) + \mathcal{O}(c^{-5}) =$$

$$= v^\alpha_{A_0}(y^0_A) + \frac{\partial}{\partial y^\alpha_A} K_A(y^0_A, y^\nu_A) +$$

$$+ \frac{\partial}{\partial y^\alpha_A} L_A(y^0_A, y^\nu_A) + \frac{\partial}{\partial y^\alpha_A} K_A(y^0_A, y^\nu_A) \frac{\partial}{\partial y^0_A} K_A(y^0_A, y^\nu_A) +$$

$$+ v_{A_0\nu} \frac{\partial}{\partial y^\alpha_A} Q^\nu_A(y^0_A, y^\nu_A) + \gamma_{\alpha\nu} \frac{\partial}{\partial y^0_A} Q^\nu_A(y^0_A, y^\nu_A) + \mathcal{O}(c^{-5}). \qquad (C4)$$

Expression $(C4)$ contains the terms of two orders of magnitude: c^{-1} and c^{-3}. However, as we discussed in Appendix B, in the post-Newtonian approximation for any arbitrary RF, one expects these components of the background metric tensor to be of the order $g_{0\alpha}(y^p_A) \sim \mathcal{O}(c^{-3})$. This gives the following condition for the function K_A:

$$v^\alpha_{A_0}(y^0_A) + \frac{\partial}{\partial y^\alpha_A} K_A(y^0_A, y^\nu_A) = \mathcal{O}(c^{-3}) \qquad (C5a)$$

Then, by formally integrating this last equation, we may find the following expression for the function K_A:

$$K_A(y^0_A, y^\nu_A) = P_A(y^0_A) - v_{A_0\mu} \cdot y^\mu_A + \mathcal{O}(c^{-4}) y^0_A. \qquad (C5b)$$

The result $(C5)$ considerably simplifies the calculation of the transformation rules between the different RFs. Thus, taking into account relation $(C3)$, one may obtain the following expression for the tensor $\gamma^A_{kl}(y^p_A)$:

$$\gamma^A_{00}(y^0_A, y^\nu_A) = 1 + 2\frac{\partial}{\partial y^0_A} K_A(y^0_A, y^\nu_A) + v_{A_0\beta} v^\beta_{A_0} +$$

$$+ 2\frac{\partial}{\partial y^0_A} L_A(y^0_A, y^\nu_A) + \Big(\frac{\partial}{\partial y^0_A} K_A(y^0_A, y^\nu_A)\Big)^2 +$$

$$+ 2 v_{A_0\beta} \frac{\partial}{\partial y^0_A} Q^\beta_A(y^0_A, y^\nu_A) + \mathcal{O}(c^{-6}), \qquad (C6a)$$

$$\gamma^A_{0\alpha}(y^0_A, y^\nu_A) = \frac{\partial}{\partial y^\alpha_A} L_A(y^0_A, y^\nu_A) - v_{A_0\alpha} \frac{\partial}{\partial y^0_A} K_A(y^0_A, y^\nu_A) +$$

$$+ v_{A_0\nu}(y^0_A) \frac{\partial}{\partial y^\alpha_A} Q^\nu_A(y^0_A, y^\nu_A) + \gamma_{\alpha\nu} \frac{\partial}{\partial y^0_A} Q^\nu_A(y^0_A, y^\nu_A) + \mathcal{O}(c^{-5}), \qquad (C6b)$$

$$\gamma^A_{\alpha\beta}(y^0_A, y^\nu_A) = \gamma_{\alpha\beta} + v_{A_0\alpha} v_{A_0\beta} +$$

$$+ \gamma_{\alpha\nu} \frac{\partial}{\partial y^\beta_A} Q^\nu_A(y^0_A, y^\nu_A) + \gamma_{\beta\nu} \frac{\partial}{\partial y^\alpha_A} Q^\nu_A(y^0_A, y^\nu_A) + \mathcal{O}(c^{-4}). \qquad (C6c)$$

Relations $(C6)$ are the KLQ parametrization of the metric γ^A_{mn}, which forms the background Riemann-flat space-time in the proper RF_A:

$$R_{klmn}(\gamma^A_{st}) = 0.$$

The functions $K_A, L_A,$ and Q^α_A will be chosen in order to separate the forces of inertia from the gravitational forces that are measured by the observer in this RF.

Relations $(C5)$ are a useful tool for simplifying the result $(C2)$ as well. Thus, for the determinant of the transformation matrix, we will get following expression:

$$det\Big[\lambda^{\overline{p}}_{Am}(y^p_A)\Big] = 1 + \frac{\partial}{\partial y^0_A} K_A(y^0_A, y^\nu_A) +$$

$$+ v_{A_0\lambda} v^\lambda_{A_0} + \frac{\partial}{\partial y^\mu_A} Q^\mu_A(y^0_A, y^\nu_A) + \mathcal{O}(c^{-4}). \qquad (C7)$$

C.3. Inverse Transformation of the Coordinate Base Vectors.

Using the transformation rule for the base vectors, $e^p_A = \partial/\partial y^p_A$, of the proper RF_A to those of the inertial barycentric RF_0, $e^p = \partial/\partial x^{\overline{p}}$, given by expressions (3.18), one easily obtains the inverse transformation matrix $\lambda^n_{A\overline{m}}(x^p) = \partial y^n_A/\partial x^m$ for this transition as well:

$$\frac{\partial y^0_A}{\partial x^0} = 1 - \frac{\partial}{\partial x^0} K_A\Big(x^0, x^\nu - y^\nu_{A_0}(x^0)\Big) -$$

$$- \frac{\partial}{\partial x^0} L_A\Big(x^0, x^\nu - y^\nu_{A_0}(x^0)\Big) + \frac{\partial}{\partial x^0}\Big[\frac{1}{2}\frac{\partial}{\partial x^0} K^2_A\Big(x^0, x^\nu - y^\nu_{A_0}(x^0)\Big) -$$

$$- v_{A_0\beta}(x^0) \cdot Q^\beta_A\Big(x^0, x^\nu - y^\nu_{A_0}(x^0)\Big)\Big] + \mathcal{O}(c^{-6}), \qquad (C8a)$$

$$\frac{\partial y^0_A}{\partial x^\alpha} = v_{A_0\alpha}(x^0) - \frac{\partial}{\partial x^\alpha} L_A\Big(x^0, x^\nu - y^\nu_{A_0}(x^0)\Big) +$$

$$+ \frac{\partial}{\partial x^\alpha}\Big[\frac{1}{2}\frac{\partial}{\partial x^0} K^2_A\Big(x^0, x^\nu - y^\nu_{A_0}(x^0)\Big) -$$

$$- v_{A_0\beta}(x^0) \cdot Q^\beta_A\Big(x^0, x^\nu - y^\nu_{A_0}(x^0)\Big)\Big] + \mathcal{O}(c^{-5}), \qquad (C8b)$$

$$\frac{\partial y^\alpha_A}{\partial x^0} = - v^\alpha_{A_0}(x^0) + \frac{\partial}{\partial x^0}\Big[- Q^\alpha_A\Big(x^0, x^\nu - y^\nu_{A_0}(x^0)\Big) +$$

$$+ v^\alpha_{A_0}(x^0) \cdot K_A\Big(x^0, x^\nu - y^\nu_{A_0}(x^0)\Big)\Big] + \mathcal{O}(c^{-5}), \qquad (C8c)$$

$$\frac{\partial y^\alpha_A}{\partial x^\beta} = \delta^\alpha_\beta + \frac{\partial}{\partial x^\beta}\Big[- Q^\alpha_A\Big(x^0, x^\nu - y^\nu_{A_0}(x^0)\Big) +$$

$$+ v^\alpha_{A_0}(x^0) \cdot K_A\Big(x^0, x^\nu - y^\nu_{A_0}(x^0)\Big)\Big] + \mathcal{O}(c^{-4}), \qquad (C8d)$$

where we have partially taken the result of $(C5b)$ into account in a form of the relation

$$\frac{\partial}{\partial x^\alpha} K_A\Big(x^0, x^\nu - y^\nu_{A_0}(x^0)\Big) = \frac{\partial}{\partial y^\alpha_A} K_A(y^0_A, y^\nu_A) + O(c^{-3}) = -v^\alpha_{A_0}(x^0) + \mathcal{O}(c^{-3}).$$

C.4. Mutual Transformation Between the Two Quasi-Inertial RFs.

The expressions for the transformation of the base vectors between two quasi-inertial RFs (e^p_B) and (e^p_A) may be obtained from relations (3.19). These transformations are given as

$$\frac{\partial y^0_B}{\partial y^0_A} = 1 + \frac{\partial}{\partial y^0_A} K_{BA}(y^0_A, y^\nu_A) + \frac{\partial}{\partial y^0_A} L_{BA}(y^0_A, y^\nu_A) + \mathcal{O}(c^{-6}), \tag{C9a}$$

$$\frac{\partial y^0_B}{\partial y^\alpha_A} = -v_{BA_0\alpha}(y^0_A) + \frac{\partial}{\partial y^\alpha_A} L_{BA}(y^0_A, y^\nu_A) + \mathcal{O}(c^{-5}), \tag{C9b}$$

$$\frac{\partial y^\alpha_B}{\partial y^0_A} = v^\alpha_{BA_0}(y^0_A) + \frac{\partial}{\partial y^0_A} Q^\alpha_{BA}(y^0_A, y^\nu_A) + \mathcal{O}(c^{-5}), \tag{C9c}$$

$$\frac{\partial y^\alpha_B}{\partial y^\beta_A} = \delta^\alpha_\beta + \frac{\partial}{\partial y^\beta_A} Q^\alpha_{BA}(y^0_A, y^\nu_A) + \mathcal{O}(c^{-4}), \tag{C9d}$$

where the functions K_{BA}, L_{BA}, and Q^α_{BA} are defined by expressions (3.20). From these expressions, $(C9)$, one may obtain the determinant of the transformation matrix $\lambda^{\overline{p}}_{BAm}(y^p_A)$ for the transformations between two different proper RFs as follows:

$$det\Big[\lambda^{\overline{p}}_{BAm}(y^p_A)\Big] = 1 + \frac{\partial}{\partial y^0_A} K_{BA}(y^0_A, y^\nu_A) +$$

$$+ v^\lambda_{BA_0}(y^0_A) v_{BA_0\lambda}(y^0_A) + \frac{\partial}{\partial y^\mu_A} Q^\mu_{BA}(y^0_A, y^\nu_A) + \mathcal{O}(c^{-4}). \tag{C10}$$

The condition $det\Big[\lambda^{\overline{p}}_{BAm}(y^p_A)\Big] = 0$ gives the boundary of validity of these transformations.

Appendix D: Transformations of Some Physical Quantities and Solutions.

In this appendix, we will present the transformation laws for the gauge conditions, the components of the Ricci tensor, and the components of an arbitrary energy-momentum tensor of matter of the matter distribution T^{mn} for the unperturbed solutions of the field equations $h^{(0)}_{mn}$ and for the interaction term h^{int}_{mn}, eqs.(3.1)–(3.4), under the general coordinate transformations discussed in Section 3 of this report.

D.1. Transformation of the Gauge Conditions.

With the help of eqs.($F1$) and the expansion of the metric tensor g_{mn} given by eqs.($B7$), we may obtain the relations for the gauge conditions expanded in a power series of the small parameter c^{-1}.

(i). In Cartesian coordinates of the inertial RF_0, the background space-time may be taken in a simple form of the Minkowski metric: $\gamma^{(0)}_{mn} = (1, -1, -1, -1)$. Then the power expansion of the gauge condition of eqs.($B7$) may be presented for $n = 0$ as follows:

$$\frac{1}{2}\frac{\partial}{\partial y^0_A}\Big(\gamma^{\epsilon\nu} g^{<2>}_{\epsilon\nu}(x^p) - g^{<2>}_{00}(x^p)\Big) - \gamma^{\epsilon\nu}\frac{\partial}{\partial y^\epsilon_A} g^{<3>}_{0\nu}(x^p) = \mathcal{O}(c^{-5}), \qquad (D1a)$$

and for $n = \alpha$ as follows:

$$\frac{1}{2}\gamma^{\alpha\lambda}\frac{\partial}{\partial y^\lambda_A}\Big(g^{<2>}_{00}(x^p) + \gamma^{\epsilon\nu} g^{<2>}_{\epsilon\nu}(x^p)\Big) -$$

$$-\gamma^{\epsilon\nu}\gamma^{\mu\alpha}\frac{\partial}{\partial y^\epsilon_A} g^{<3>}_{\nu\mu}(x^p) = \mathcal{O}(c^{-4}). \qquad (D1b)$$

(ii). In an analogous manner, one may obtain the expressions for the gauge conditions in coordinates (y^p_A) of the proper RF_A of body (A). For $n = 0$,

$$\frac{1}{2}\frac{\partial}{\partial y^0_A}\Big(\gamma^{\epsilon\nu} g^{<2>}_{\epsilon\nu}(y^p_A) - g^{<2>}_{00}(y^p_A)\Big) - \gamma^{\epsilon\nu}\frac{\partial}{\partial y^\epsilon_A} g^{<3>}_{0\nu}(y^p_A) +$$

$$+\frac{\partial^2 K_A}{\partial {y^0_A}^2} + \gamma^{\nu\lambda}\frac{\partial^2 L_A}{\partial y^\nu_A \partial^\lambda_A} + v_{A_0\mu}\Big(a^\mu_{A_0} + \gamma^{\nu\lambda}\frac{\partial^2 Q^\mu_A}{\partial y^\nu_A \partial^\lambda_A}\Big) = \mathcal{O}(c^{-5}), \qquad (D2a)$$

and for $n = \alpha$,

$$\frac{1}{2}\gamma^{\alpha\lambda}\frac{\partial}{\partial y^\lambda_A}\Big(g^{<2>}_{00}(y^p_A) + \gamma^{\epsilon\nu} g^{<2>}_{\epsilon\nu}(y^p_A)\Big) - \gamma^{\epsilon\nu}\gamma^{\mu\alpha}\frac{\partial}{\partial y^\epsilon_A} g^{<3>}_{\nu\mu}(y^p_A) +$$

$$+\Big(a^\alpha_{A_0} + \gamma^{\nu\lambda}\frac{\partial^2 Q^\alpha_A}{\partial y^\nu_A \partial^\lambda_A}\Big) = \mathcal{O}(c^{-4}). \qquad (D2b)$$

D.2. Transformation of the Ricci Tensor R_{mn}.

With the help of the expansion for the components of the Ricci tensor, eqs.($B9$), one may obtain this tensor in coordinates of the different RFs.

(i). Thus, by making use of the relations for the covariant de Donder gauge conditions in coordinates (x^p) of inertial RF_0, eqs.$(D1)$, one may present the components for the Ricci tensor in the following form:

$$R_{00}(x^p) = -\frac{1}{2}\gamma^{\nu\lambda}\frac{\partial^2}{\partial x^\nu \partial x^\lambda} g_{00}^{<2>}(x^p) - \frac{1}{2}\gamma^{\nu\lambda}\frac{\partial^2}{\partial x^\nu \partial x^\lambda} g_{00}^{<4>}(x^p) -$$

$$-\frac{1}{2}\frac{\partial^2}{\partial x^{0\,2}} g_{00}^{<2>}(x^p) + \frac{1}{2}\gamma^{\lambda\mu}\gamma^{\nu\delta} g_{\lambda\nu}^{<2>}(x^p)\frac{\partial^2}{\partial x^\mu \partial x^\delta} g_{00}^{<2>}(x^p) +$$

$$+\frac{1}{2}\gamma^{\lambda\nu}\frac{\partial}{\partial x^\lambda} g_{00}^{<2>}(x^p)\frac{\partial}{\partial x^\nu} g_{00}^{<2>}(x^p) + \mathcal{O}(c^{-6}), \qquad (D3a)$$

$$R_{0\alpha}(x^p) = -\frac{1}{2}\gamma^{\nu\lambda}\frac{\partial^2}{\partial x^\nu \partial x^\lambda} g_{0\alpha}^{<3>}(x^p) + \mathcal{O}(c^{-5}), \qquad (D3b)$$

$$R_{\alpha\beta}(x^p) = -\frac{1}{2}\gamma^{\nu\lambda}\frac{\partial^2}{\partial x^\nu \partial x^\lambda} g_{\alpha\beta}^{<2>}(x^p) + \mathcal{O}(c^{-4}). \qquad (D3c)$$

(ii). From relations $(D2)$ and with the help of the expressions for the Ricci tensor given by eqs.$(B9)$, one may get this tensor in coordinates (y_A^p) of the proper RF_A as well:

$$R_{00}(y_A^p) = -\frac{1}{2}\gamma^{\nu\lambda}\frac{\partial^2}{\partial y_A^\nu \partial y_A^\lambda} g_{00}^{<2>}(y_A^p) - \frac{1}{2}\gamma^{\nu\lambda}\frac{\partial^2}{\partial y_A^\nu \partial y_A^\lambda} g_{00}^{<4>}(y_A^p) -$$

$$-\frac{1}{2}\frac{\partial^2}{\partial y_A^{0\,2}} g_{00}^{<2>}(y_A^p) + \frac{1}{2}\gamma^{\lambda\mu}\gamma^{\nu\delta} g_{\lambda\nu}^{<2>}(y_A^p)\frac{\partial^2}{\partial y_A^\mu \partial y_A^\delta} g_{00}^{<2>}(y_A^p) +$$

$$+\frac{1}{2}\gamma^{\lambda\nu}\frac{\partial}{\partial y_A^\lambda} g_{00}^{<2>}(y_A^p)\frac{\partial}{\partial y_A^\nu} g_{00}^{<2>}(y_A^p) +$$

$$+\frac{\partial}{\partial y_A^0}\left(\frac{\partial^2 K_A}{\partial y_A^{0\,2}} + \gamma^{\nu\lambda}\frac{\partial^2 L_A}{\partial y_A^\nu \partial y_A^\lambda} + v_{A_0\mu}\left(a_{A_0}^\mu + \gamma^{\nu\lambda}\frac{\partial^2 Q_A^\mu}{\partial y_A^\nu \partial y_A^\lambda}\right)\right) +$$

$$+\frac{1}{2}\left(a_{A_0}^\mu + \gamma^{\nu\lambda}\frac{\partial^2 Q_A^\mu}{\partial y_A^\nu \partial y_A^\lambda}\right)\frac{\partial}{\partial y_A^\mu} g_{00}^{<2>}(y_A^p) + \mathcal{O}(c^{-6}), \qquad (D4a)$$

$$R_{0\alpha}(y_A^p) = -\frac{1}{2}\gamma^{\nu\lambda}\frac{\partial^2}{\partial y_A^\nu \partial y_A^\lambda} g_{0\alpha}^{<3>}(y_A^p) + \frac{1}{2}\gamma_{\alpha\mu}\frac{\partial}{\partial y_A^0}\left(a_{A_0}^\mu + \gamma^{\nu\lambda}\frac{\partial^2 Q_A^\mu}{\partial y_A^\nu \partial y_A^\lambda}\right) +$$

$$+\frac{1}{2}\frac{\partial}{\partial y_A^\alpha}\left(\frac{\partial^2 K_A}{\partial y_A^{0\,2}} + \gamma^{\nu\lambda}\frac{\partial^2 L_A}{\partial y_A^\nu \partial y_A^\lambda} + v_{A_0\mu}\left(a_{A_0}^\mu + \gamma^{\nu\lambda}\frac{\partial^2 Q_A^\mu}{\partial y_A^\nu \partial y_A^\lambda}\right)\right) + \mathcal{O}(c^{-5}), \qquad (D4b)$$

$$R_{\alpha\beta}(y_A^p) = -\frac{1}{2}\gamma^{\nu\lambda}\frac{\partial^2}{\partial y_A^\nu \partial y_A^\lambda} g_{\alpha\beta}^{<2>}(y_A^p) +$$

$$+\frac{1}{2}\left(\gamma_{\mu\beta}\frac{\partial}{\partial y_A^\alpha} + \gamma_{\mu\alpha}\frac{\partial}{\partial y_A^\beta}\right)\left(a_{A_0}^\mu + \gamma^{\nu\lambda}\frac{\partial^2 Q_A^\mu}{\partial y_A^\nu \partial y_A^\lambda}\right) + \mathcal{O}(c^{-4}). \qquad (D4c)$$

D.3. Transformation Law for an Arbitrary Energy-Momentum Tensor T^{mn}.

In this subsection, we will present the power expansion for the components of $S_{mn} = T_{mn} - \frac{1}{2}g_{mn}T$ defined by equation $(B12)$.

(i). By assuming that each body (B) in the system may be described by the reduced energy-momentum tensor S^B_{mn}, one may easily obtain the total energy-momentum tensor S_{mn} for the entire system. Thus, in the coordinates of the inertial RF_0, this tensor may be presented as follows:

$$S_{mn}(x^p) = \sum_B \frac{\partial y^k_B}{\partial x^m}\frac{\partial y^l_B}{\partial x^n} S^B_{kl}(y^q_B(x^p)).$$

Then, with this relation above and from eqs.$(B10)$–$(B11)$, $(B13)$, and $(C6)$ for the coordinate transformations to the barycentric inertial RF_0, we will obtain the following result:

$$S_{00}(x^p) = \frac{1}{2}\sum_B \Bigg(T^{<0>00}_B(y^q_B(x^p)) + T^{<2>00}_B(y^q_B(x^p)) +$$

$$+2\, g^{<2>}_{00}(y^p_B(x^q)) \cdot T^{<0>00}_B(y^p_B(x^q)) - \gamma_{\epsilon\nu} T^{<2>\epsilon\nu}_B(y^p_B(x^q)) -$$

$$-\Big[2\frac{\partial}{\partial x^0} K_B\Big(x^0, x^\nu - y^\nu_{B_0}(x^0)\Big) + v_{B_0\nu}(x^0) v^\nu_{B_0}(x^0)\Big] \cdot T^{<0>00}_B(y^q_B(x^p)) -$$

$$-4\,\gamma_{\epsilon\mu} v^\epsilon_{B_0}(x^0)\, T^{<1>0\mu}_B(y^q_B(x^p))\Bigg) + \mathcal{O}(c^{-4}), \qquad (D5a)$$

$$S_{0\alpha}(x^p) = \sum_B \gamma_{\alpha\mu}\Bigg(T^{<1>0\mu}_B(y^q_B(x^p)) + v^\mu_{B_0}(x^0)\, T^{<0>00}_B(y^q_B(x^p))\Bigg) + \mathcal{O}(c^{-3}), \qquad (D5b)$$

$$S_{\alpha\beta}(x^p) = -\frac{1}{2}\gamma_{\alpha\beta}\sum_B T^{B<0>00}(y^q_B(x^p)) + \mathcal{O}(c^{-2}). \qquad (D5c)$$

(ii). One may obtain the relation for the energy-momentum tensor of the entire system in the coordinates of the proper RF_A as follows:

$$S_{mn}(y^p_A) = \sum_B \frac{\partial y^k_B}{\partial y^m_A}\frac{\partial y^l_B}{\partial y^n_A} S^B_{kl}(y^q_B(y^p_A)) = S^A_{mn}(y^p_A) + \sum_{B\neq A} \frac{\partial y^k_B}{\partial y^m_A}\frac{\partial y^l_B}{\partial y^n_A} S^B_{kl}(y^q_B(y^p_A)).$$

Then, making use of the formula above and from eqs.$(B10)$–$(B11)$ and $(B13)$, the expression for the quantity S_{mn} in coordinates (y^p_A) of the proper RF_A, with the help of eqs.$(C7)$, may be presented as follows:

$$S_{00}(y^p_A) = \frac{1}{2}\Bigg(T^{<0>00}_A(y^p_A) + T^{<2>00}_A(y^p_A) +$$

$$+2\, g^{<2>}_{00}(y^p_A)\, T^{<0>00}_A(y^p_A) - \gamma_{\epsilon\nu} T^{<2>\epsilon\nu}_A(y^p_A)\Bigg) +$$

$$+\frac{1}{2}\sum_{B\neq A}\Bigg(T^{<0>00}_B(y^q_B(y^p_A)) + T^{<2>00}_B(y^q_B(y^p_A)) +$$

$$+2g^{<2>}_{00}(y^q_B(y^p_A))\, T^{<0>00}_B(y^q_B(y^p_A)) - \gamma_{\epsilon\nu} T^{<2>\epsilon\nu}_B(y^q_B(y^p_A)) +$$

$$+\Big[2\frac{\partial}{\partial y_A^0}K_{BA}(y_A^0,y_A^\nu)-v_{BA_0}^\nu(y_A^0)v_{BA_0\nu}(y_A^0)\Big]\cdot T_B^{<0>00}(y_B^q(y_A^p))+$$

$$+4\gamma_{\epsilon\mu}v_{BA_0}^\epsilon(y_A^0)\,T_B^{<1>0\mu}(y_B^q(y_A^p))\Bigg)+\mathcal{O}(c^{-4}), \qquad (D6a)$$

$$S_{0\alpha}(y_A^p)=\gamma_{\alpha\mu}T_A^{<1>0\mu}(y_A^p)+$$

$$+\sum_{B\neq A}\gamma_{\alpha\mu}\Bigg(T_B^{<1>0\mu}(y_B^q(y_A^p))-v_{BA_0}^\mu(y_A^0)T_B^{<0>00}(y_B^q(y_A^p))\Bigg)+\mathcal{O}(c^{-3}), \qquad (D6b)$$

$$S_{\alpha\beta}(y_A^p)=-\frac{1}{2}\gamma_{\alpha\beta}\Bigg(T_A^{<0>00}(y_A^p)+\sum_{B\neq A}T_B^{<0>00}(y_B^q(y_A^p))\Bigg)+\mathcal{O}(c^{-2}). \qquad (D6c)$$

D.4. Transformation of the Unperturbed Solutions $h_{mn}^{(0)}$.

In this subsection, we will obtain the transformation rules for the unperturbed solutions.

(i). Using the following notation for the second term in expression (3.1):

$$H_{mn}^{(0)}(x^p)=\sum_B\frac{\partial y_B^k}{\partial x^m}\frac{\partial y_B^l}{\partial x^n}h_{kl}^{(0)B}(y_B^q(x^p)),$$

from eqs.($C6$), we will obtain the relations for components $H_{mn}^{(0)}(x^p)$ in the coordinates of the inertial RF_0 as follows:

$$H_{00}^{(0)}(x^p)=\sum_B\Bigg(h_{B00}^{(0)<2>}(y_B^q(x^p))+h_{B00}^{(0)<4>}(y_B^q(x^p))-$$

$$-2\frac{\partial}{\partial x^0}K_B\Big(x^0,x^\nu-y_{B_0}^\nu(x^0)\Big)\cdot h_{B00}^{(0)<2>}(y_B^q(x^p))-$$

$$-2v_{B_0}^\epsilon(x^0)\cdot h_{B0\epsilon}^{(0)<3>}(y_B^q(x^p))+v_{B_0}^\epsilon(x^0)v_{B_0}^\nu(x^0)\cdot h_{B\epsilon\nu}^{(0)<2>}(y_B^q(x^p))\Bigg)+\mathcal{O}(c^{-6}), \qquad (D7a)$$

$$H_{0\alpha}^{(0)}(x^p)=\sum_B\Bigg(h_{B0\alpha}^{(0)<3>}(y_B^q(x^p))+$$

$$+v_{B_0\alpha}(x^0)\cdot h_{B00}^{(0)<2>}(y_B^q(x^p))-v_{B_0}^\epsilon(x^0)\cdot h_{B\epsilon\alpha}^{(0)<2>}(y_B^q(x^p))\Bigg)+\mathcal{O}(c^{-5}), \qquad (D7b)$$

$$H_{\alpha\beta}^{(0)}(x^p)=\sum_B h_{B\alpha\beta}^{(0)<2>}(y_B^q(x^p))+\mathcal{O}(c^{-4}). \qquad (D7c)$$

(ii). The transformed components of $H^{(0)}_{mn}(y^p_A)$ in coordinates (y^p_A) of the proper RF$_A$ are defined as in eqn.(3.4):

$$H^{(0)}_{mn}(y^p_A) = \frac{\partial x^k}{\partial y^m_A}\frac{\partial x^l}{\partial y^n_A} H^{(0)}_{kl}(x^q(y^p_A)) = \sum_B \frac{\partial y^k_B}{\partial y^m_A}\frac{\partial y^l_B}{\partial y^n_A} h^{(0)B}_{kl}(y^q_B(y^p_A)) =$$

$$= h^{(0)B}_{mn}(y^p_A) + \sum_{B\neq A} \frac{\partial y^k_B}{\partial y^m_A}\frac{\partial y^l_B}{\partial y^n_A} h^{(0)B}_{kl}(y^q_B(y^p_A)).$$

Then, for these components, from the relations of $(C7)$, one may obtain the following result:

$$H^{(0)}_{00}(y^p_A) = h^{(0)<2>}_{A00}(y^p_A) + H^{(0)<4>}_{A00}(y^p_A) +$$

$$+ \sum_{B\neq A}\Bigg(h^{(0)<2>}_{B00}(y^q_B(y^p_A)) + h^{(0)<4>}_{B00}(y^q_B(y^p_A)) +$$

$$+2\frac{\partial}{\partial y^0_A} K_{BA}(y^0_A, y^\nu_A)\cdot h^{(0)<2>}_{B00}(y^q_B(x^p)) + 2v^\epsilon_{BA_0}(y^0_A)\cdot h^{(0)<3>}_{B0\epsilon}(y^q_B(y^p_A)) +$$

$$+v^\epsilon_{BA_0}(y^0_A) v^\nu_{BA_0}(y^0_A)\cdot h^{(0)<2>}_{B\epsilon\nu}(y^q_B(y^p_A))\Bigg) + \mathcal{O}(c^{-6}), \qquad (D8a)$$

$$H^{(0)}_{0\alpha}(y^p_A) = h^{(0)<3>}_{A0\alpha}(y^q_A) + \sum_{B\neq A}\Bigg(h^{(0)<3>}_{B0\alpha}(y^q_B(y^p_A)) -$$

$$-v_{BA_0\alpha}(y^0_A)\cdot h^{(0)<2>}_{B00}(y^q_B(y^p_A)) + v^\epsilon_{BA_0}(y^0_A)\cdot h^{(0)<2>}_{B\epsilon\alpha}(y^q_B(y^p_A))\Bigg) + \mathcal{O}(c^{-5}), \qquad (D8b)$$

$$H^{(0)}_{\alpha\beta}(y^p_A) = h^{(0)<2>}_{A\alpha\beta}(y^p_A) + \sum_{B\neq A} h^{(0)<2>}_{B\alpha\beta}(y^q_B(y^p_A)) + \mathcal{O}(c^{-4}). \qquad (D8c)$$

D.5. Transformation Rules for the Interaction Term h^{int}_{mn}.

The components of the interaction term $h^{int}_{mn}(x^s(y^p_A))$ in coordinates (y^p_A) of the proper RF$_A$ are given as follows:

$$h^{int}_{mn}(y^p_A) = \frac{\partial x^k}{\partial y^m_A}\frac{\partial x^l}{\partial y^n_A} h^{int}_{mn}(x^s(y^p_A)). \qquad (D9a)$$

By making use of expressions $(C6)$, the components of h^{int}_{mn} will take the following form:

$$h^{int}_{00}(y^p_A) = h^{int<4>}_{00}(x^s(y^p_A)) + \mathcal{O}(c^{-6}), \qquad (D9b)$$

$$h^{int}_{0\alpha}(y^p_A) = O(c^{-5}), \qquad h^{int}_{\alpha\beta}(y^p_A) = \mathcal{O}(c^{-4}). \qquad (D9c)$$

D.6. Transformation for the Energy-Momentum Tensor of a Perfect Fluid.

Let us define the model of matter distribution of a body (B) in its proper RF$_B$ by the tensor density $\hat{T}^{mn}_B$ given by

$$\hat{T}^{mn}_B(y^p_B) = \sqrt{-g}\Big(\Big[\rho_{B_0}(1+\Pi) + p\Big]u^m u^n - p g^{mn}\Big), \qquad (D10)$$

where all the quantities entering the formula above are calculated in the coordinates (y_B^p) of the non-inertial proper RF_B. Then, one may obtain the following post-Newtonian expansion of the tensor T_B^{mn} in coordinates (y_B^p) of the proper RF_B:

$$T^{00}(y_B^p) = \rho_{B_0}\Big[1 - v_\mu v^\mu + \Pi + 2\Big(\sum_C U_C - \frac{\partial K_B}{\partial y_B^0} - \frac{1}{2}v_{B_0\mu}v_{B_0}^\mu\Big) + \mathcal{O}(c^{-4})\Big], \qquad (D11a)$$

$$T^{0\alpha}(y_B^p) = \rho_{B_0}v^\alpha\Big[1 - v_\mu v^\mu + \Pi + 2\Big(\sum_C U_C - \frac{\partial K_B}{\partial y_B^0} - \frac{1}{2}v_{B_0\mu}v_{B_0}^\mu\Big) + \frac{p}{\rho_{B_0}} + \mathcal{O}(c^{-4})\Big], \qquad (D11b)$$

$$T^{\alpha\beta}(y_B^p) = \rho_{B_0}v^\alpha v^\beta - p\gamma^{\alpha\beta} + \rho\mathcal{O}(c^{-4}). \qquad (D11c)$$

Then, by using these relations, one may easily obtain the expressions for the right-hand side of the Hilbert–Einstein gravitational field equations in the form of the quantity S_{mn}, defined by eqs.(4.1), $(B13)$, and $(D5)$–$(D6)$.

(i). From eqs.$(D5)$ and with the help of the expressions of eqs.$(D10)$, we obtain components of the quantity S_{mn} in coordinates (x^p) of the barycentric inertial RF_0 as follows:

$$S_{00}(x^0, x^\nu) = \frac{1}{2}\sum_B \rho_{B_0}(y_B^q(x^p))\times$$

$$\times\Big[1 + \Pi - 2\sum_{B'} U_{B'} - 2v_\mu(x^p)v^\mu(x^p) + \frac{3p}{\rho} + \mathcal{O}(c^{-4})\Big], \qquad (D12a)$$

$$S_{0\alpha}(x^0, x^\nu) = \gamma_{\alpha\epsilon}\sum_B \rho_B(x^0, x^\nu - y_{B_0}^\nu(x^0))\Big[v^\epsilon(x^p) + \mathcal{O}(c^{-3})\Big], \qquad (D12b)$$

$$S_{\alpha\beta}(x^0, x^\nu) = -\frac{1}{2}\gamma_{\alpha\beta}\sum_B \rho_B(x^0, x^\nu - y_{B_0}^\nu(x^0))\Big[1 + \mathcal{O}(c^{-2})\Big], \qquad (D12c)$$

where the total mass density of the system is denoted as

$$\rho = \sum_B \rho_{B_0}.$$

(ii). In an analogous manner, but with the help of the expressions of eqs.$(D6)$, we may get the relations for tensor S_{mn} in the coordinates (y_A^p) of the proper RF_A:

$$S_{00}(y_A^0, y_A^\nu) = \frac{1}{2}\sum_B \rho_B(y_B^q(y_A^p))\Big[1 + \Pi - 2\sum_{B'} U_{B'} +$$

$$+2\frac{\partial K_A}{\partial y_A^0} - 2v_\mu(y_A^p)v^\mu(y_A^p) + v_{A_0\mu}v_{A_0}^\mu + \frac{3p}{\rho} + \mathcal{O}(c^{-4})\Big], \qquad (D13a)$$

$$S_{0\alpha}(y_A^0, y_A^\nu) = \gamma_{\alpha\epsilon}\sum_B \rho_B(y_A^0, y_A^\nu + y_{BA_0}^\nu(y_A^0))\Big[v^\epsilon(y_A^p) + \mathcal{O}(c^{-3})\Big], \qquad (D13b)$$

$$S_{\alpha\beta}(y_A^0, y_A^\nu) = -\frac{1}{2}\gamma_{\alpha\beta}\sum_B \rho_B(y_A^0, y_A^\nu + y_{BA_0}^\nu(y_A^0))\Big[1 + \mathcal{O}(c^{-2})\Big]. \qquad (D13c)$$

It should be noted that the functional dependence of the densities in expressions $(D12)$–$(D13)$ reflects the positions of all the bodies with respect to different RFs in the sense of Dirac's delta function.

Appendix E: Transformations of the Gravitational Potentials.

To establish the transformation properties of the unperturbed solutions for $h_{mn}^{(0)A}$ (given in Appendix A) for transitions from coordinates (y_A^p) of the proper RF_A to those of the barycentric RF_0 (and backwards), one should take into account that these solutions contain the integrals over the three-dimensional volumes of the bodies. For this reason, we should first derive the transformation laws for generalized gravitational potentials. The powerful technique for obtaining these rules was elaborated for some special cases of transformations earlier by Chandrasekhar & Contopulos (1967) (see also Brumberg & Kopejkin, 1988a; Will, 1993). It was noted that the transformation of the integrands should include the point transformation combined with the Lie transfer from one hypersurface to another. This transfer should be produced along the integral curves of the vector field of the body matter's four-velocity. The most sophisticated transformation at the post-Newtonian level is required for the Newtonian potential U_B. We will extend this technique to the general case of the coordinate transformations, which was discussed in Section 3 and in Appendix C.

For the transformation from the proper RF_A to the barycentric one, RF_0, with the help of expressions (3.18), one may establish the relationship between the observer's spatial coordinates and those of the integrating point as follows:

$$\frac{1}{|y_B^\nu - y_B'^\nu|} = \frac{1}{|x^\nu - x'^\nu|} \cdot \Bigg(1 - v_\beta(x^0, x'^\nu) v_{B_0\lambda}(x^0) \frac{(x^\beta - x'^\beta)(x^\lambda - x'^\lambda)}{|x^\nu - x'^\nu|^2} +$$

$$+ \Big[Q_B^\beta\Big(x^0, x^\nu - y_{B_0}^\nu(x^0)\Big) - Q_B^\beta\Big(x^0, x'^\nu - y_{B_0}^\nu(x^0)\Big)\Big] \cdot \frac{(x_\beta - x'_\beta)}{|x^\nu - x'^\nu|^2} + \mathcal{O}(c^{-4})\Bigg). \qquad (E1a)$$

By using the same procedure as above, from eqs.(3.19)–(3.20) we may obtain the expression for the observer's spatial coordinates and those of the integrating point while the transformation between two proper RFs (corresponding to the bodies (A) and (B)) is being performed:

$$\frac{1}{|y_B^\nu - y_B'^\nu|} = \frac{1}{|y_A^\nu - y_A'^\nu|} \cdot \Bigg[1 +$$

$$+ \Big(v_{A_0\beta}(y_A^0) + v_\beta(y_A^0, y_A'^\nu)\Big) v_{BA_0\lambda}(y_A^0) \cdot \frac{(y_A^\beta - y_A'^\beta)(y_A^\lambda - y_A'^\lambda)}{|y_A^\nu - y_A'^\nu|^2} +$$

$$+ \Bigg(\Big[Q_A^\beta(y_A^0, y_A^\nu) - Q_A^\beta(y_A^0, y_A'^\nu)\Big] - \Big[Q_B^\beta\Big(y_A^0 + y_{BA_0}^\nu(y_A^0)\Big) -$$

$$- Q_B^\beta\Big(y_A^0, y_A'^\nu + y_{BA_0}^\nu(y_A^0)\Big)\Big]\Bigg) \cdot \frac{(y_A^\beta - y_A'^\beta)}{|y_A^\nu - y_A'^\nu|^2} + \mathcal{O}(c^{-4})\Bigg]. \qquad (E1b)$$

For the transformation of the integrand, we should take into account the property of the invariant elementary volume (Kopejkin, 1988; Will, 1993):

$$d^3y_B' \cdot \sqrt{-g(y_B^p)}u^0(y_B^p) = d^3x' \cdot \sqrt{-g(x^p)}u^0(x^p), \qquad (E2)$$

where $\sqrt{-g}$ is the determinant of the metric tensor and u^0 is the temporal component of the invariant four-velocity.

From expressions $(B2)$ and the components of the metric tensor of the order $\sim c^{-2}$ in the different RFs (given by (4.8) and (4.11)), we will get

$$\sqrt{-g(x^p)} = 1 + 2\sum_B U_B(y_B^q(x^p)) + \mathcal{O}(c^{-4}), \tag{E3a}$$

and

$$\sqrt{-g(y_A^p)} = 1 + \frac{\partial}{\partial y_A^0} K_A(y_A^p) + v_{A_0\beta}(y_A^0) v_{A_0}^\beta(y_A^0) +$$

$$+ \frac{\partial}{\partial y_A^\beta} Q_A^\beta(y_A^p) + 2\sum_B U_B(y_B^q(y_A^p)) + \mathcal{O}(c^{-4}). \tag{E3b}$$

The components of the invariant four-velocity are defined as follows:

$$u^k(z^p) = v^k(z^p)\Big[g_{00}(z^p) + 2g_{0\epsilon}(z^p)v^\epsilon(z^p) + g_{\nu\epsilon}(z^p)v^\epsilon(z^p)v^\nu(z^p)\Big]^{-1/2}, \tag{E4}$$

where $v^k(z^p) = dz_B^k/dz^0 = (1, \dot{z}_B^\epsilon)$. From this last expression and eqs.(4.8) and (4.11) one may obtain the relations for the component u^0 in the coordinates of the barycentric and the observer's proper RF as follows:

$$u^0(x^p) = 1 + \sum_B U_B(y_B^p(x^p)) - \frac{1}{2} v_\beta(x^p) v^\beta(x^p) + \mathcal{O}(c^{-4}), \tag{E5a}$$

and

$$u^0(y_A^p) = 1 + \sum_B U_B(y_B^q(y_A^p)) - \frac{1}{2} v_\beta(y_A^p) v^\beta(y_A^p) -$$

$$- \frac{1}{2} v_{A_0\beta}(y_A^0) v_{A_0}^\beta(y_A^0) - \frac{\partial}{\partial y_A^0} K_A(y_A^p) + \mathcal{O}(c^{-4}). \tag{E5b}$$

Then making use of the expression of eqs.$(E3) and (E5)$, we will have

$$\sqrt{-g(x^p)} u^0(x^p) = 1 + 3\sum_B U_B(y_B^p(x^p)) - \frac{1}{2} v_\beta(x^p) v^\beta(x^p) + \mathcal{O}(c^{-4}) \tag{E6a}$$

and

$$\sqrt{-g(y_A^p)} u^0(y_A^p) = 1 + 3\sum_B U_B(y_B^k(y_A^p)) - \frac{1}{2} v_\beta(y_A^p) v^\beta(y_A^p) +$$

$$+ \frac{\partial}{\partial y_A^\beta} Q_A^\beta(y_A^p) + \frac{1}{2} v_{A_0\beta}(y_A^0) v_{A_0}^\beta(y_A^0) + \mathcal{O}(c^{-4}). \tag{E6b}$$

From relation $(E2)$, the following transformation laws for the elementary volume may be established:

$$d^3 y_B' = d^3 x' \frac{\sqrt{-g(x^p)} u^0(x^p)}{\sqrt{-g(y_B^k(x^p))} u^0(y_B^k(x^p))} =$$

$$= d^3 x' \left(1 - v_{B_0\beta}(x^0) v^\beta(x^0, x'^\nu) - \frac{\partial}{\partial x'^\beta} Q_B^\beta\Big(x^0, x'^\nu - y_{B_0}^\nu(x^0)\Big) + \mathcal{O}(c^{-4})\right), \tag{E7a}$$

and

$$d^3y'_B = d^3y'_A \frac{\sqrt{-g(y^p_A)}u^0(y^p_A)}{\sqrt{-g(y^k_B(y^p_A))}u^0(y^k_B(y^p_A))} =$$

$$= d^3y'_A\Big(1 + v_{BA_0\beta}(y^0_A)\Big(v^\beta_{A_0}(y^p_{A_0}) + v^\beta(y^0_A, y'^\nu_A)\Big) +$$

$$+\frac{\partial}{\partial y'^\beta_A}\Big(Q^\beta_A(y^0_A, y'^\nu_A) - Q^\beta_B\Big(y^0_A, y'^\nu_A + y^\nu_{BA_0}(y^0_A)\Big)\Big) + \mathcal{O}(C^{-4})\Big). \qquad (E7b)$$

Since the quantities $\rho_B(z^0, z^\nu), \Pi(z^0, z^\nu)$, and $p(z^0, z^\nu)$ from the potentials defined in Appendix A are all measured in the co-moving *local* quasi-inertial frames, they are transformed as scalars, and for any given element of fluid, the following relations hold:

$$\rho_B(x^0, x^\nu) = \rho_B(y^k_B(x^p)), \qquad \Pi(x^0, x^\nu) = \Pi(y^k_B(x^p)), \qquad p(x^0, x^\nu) = p(y^k_B(x^p)). \qquad (E8)$$

Finally, the expressions $(E1) and (E7)$–$(E8)$ enable one to present the transformation law for the Newtonian potential as follows:

$$U_B(y^0_B, y^\nu_B) = U_B(x^0, x^\nu) + v^\beta_{B_0}(x^0)\cdot\frac{\partial^2}{\partial x^0 \partial x^\beta}\chi_B(x^0, x^\nu) +$$

$$+\int_B d^3x'\rho_B\Big(x^0, x'^\nu - y^\nu_{B_0}(x^0)\Big)\frac{\partial}{\partial x'^\lambda}\Big[\frac{Q^\lambda_B\Big(x^0, x^\nu - y^\nu_{B_0}(x^0)\Big) - Q^\lambda_B\Big(x^0, x'^\nu - y^\nu_{B_0}(x^0)\Big)}{|x^\nu - x'^\nu|}\Big] +$$

$$+\mathcal{O}(c^{-6}), \qquad (E9a)$$

and

$$U_B(y^0_B, y^\nu_B) = U_B(y^0_A, y^\nu_A) -$$

$$-v^\beta_{BA_0}(y^0_A)\left(\frac{\partial}{\partial y^0_A} - v^\mu_{A_0}(y^0_A)\frac{\partial}{\partial y^\mu_A}\right)\frac{\partial}{\partial y^\beta_A}\chi_B\Big(y^0_A, y^\nu_A\Big) -$$

$$-\int_B d^3y'_A\rho_B\Big(y^0_A, y'^\nu_A + y^\nu_{BA_0}(y^0_A)\Big)\frac{\partial}{\partial y'^\lambda_A}\left(\Big[\frac{Q^\lambda_A(y^0_A, y^\nu_A) - Q^\lambda_A(y^0_A, y'^\nu_A)}{|y^\nu_A - y'^\nu_A|}\Big] -\right.$$

$$\left.-\Big[\frac{Q^\lambda_B\Big(y^0_A, y^\nu_A + y^\nu_{BA_0}(y^0_A)\Big) - Q^\lambda_B\Big(y^0_A, y'^\nu_A + y^\nu_{BA_0}(y^0_A)\Big)}{|y^\nu_A - y'^\nu_A|}\Big]\right) + \mathcal{O}(c^{-6}). \qquad (E9b)$$

The Newtonian potential and the super-potential in the formulae above are given as follows:

$$U_B(z^0, z^\nu) = \int_B \frac{d^3z'}{|z^\nu - z'^\nu|}\rho_B(y'^q_B(z'^p)) + \mathcal{O}(c^{-6}) \qquad (E9c)$$

$$\chi_B(z^0, z^\nu) = -\int_B d^3z'\rho_B(y'^q_B(z'^p))\cdot|z^\nu - z'^\nu| + \mathcal{O}(c^{-4})L^2_B, \qquad (E9d)$$

where L_B is the proper dimensions of the body (B).

In order to establish the transformation properties for the potentials

$$V_B^\alpha(z^0, z^\nu), \qquad \Phi_{1B}(z^0, z^\nu) \quad \text{and} \quad \frac{\partial^2}{\partial z^{0\,2}}\chi_B(z^0, z^\nu),$$

one should find the transformation rules for the spatial components of the four-velocity $u^k(z^p)$ while transiting between RFs. Let $u^k(x^p)$ and $u^k(y_A^p)$ be four-velocities of matter measured in two different RFs under consideration. Since they are related by the usual tensorial law,

$$u^m(y_B^p) = u^k(x^p)\frac{\partial y_B^m}{\partial x^k} \qquad \Rightarrow \qquad \frac{dy_B^m}{ds} = \frac{\partial y_B^m}{\partial x^k}\frac{dx^k}{ds}, \tag{$E10$}$$

the following expression for the transformation of the invariant four-velocity may be obtained:

$$u^0(y_B^p) = \frac{dy_B^0}{ds} = \frac{\partial y_B^0}{\partial x^k}\frac{dx^k}{ds}, \tag{$E11a$}$$

$$u^\epsilon(y_B^p) = \frac{dy_B^\epsilon}{ds} = u^0(y_B^p)v^\epsilon(y_B^p) = \frac{\partial y_B^\epsilon}{\partial x^k}\frac{dx^k}{ds}. \tag{$E11b$}$$

The last two equations provide one with the result for transformation of the three-velocity, as follows:

$$v^\epsilon(y_B^p) = \frac{u^0(x^q)}{u^0(y_B^p(x^q))}\Big(\frac{\partial y_B^\epsilon}{\partial x^0} + v^\nu(x^p)\frac{\partial y_B^\epsilon}{\partial x^\nu}\Big). \tag{$E12$}$$

By collecting together expressions $(E5)$ and $(C8)$ and substituting them into eq.$(E12)$, one may get the relation between the components of the velocity while the transformation from the proper RF_A to the barycentric one, RF_0, is performed. This result may be written with the required accuracy as follows:

$$v^\alpha(y_B^q(x^p)) = v^\alpha(x^p) - v_{B_0}^\alpha(x^0) + \mathcal{O}(c^{-3}). \tag{$E13a$}$$

In an analogous manner, but with the help of the equation $(C9)$, one obtains the relations for velocities in two different proper RFs:

$$\begin{aligned} v^\alpha(y_B^q(y_A^p)) = {} & v^\alpha(y_A^p) + v_{BA_0}^\alpha(y_A^0) + \Big(\frac{\partial}{\partial y_A^0} + v^\lambda(y_A^p)\frac{\partial}{\partial y_A^\lambda}\Big)\Big[Q_A^\alpha(y_A^p) - Q_B^\alpha(y_A^p)\Big] - \\ & -\Big(v^\alpha(y_A^p) + v_{A_0}^\alpha(y_A^0)\Big)\Bigg(\frac{\partial}{\partial y_A^0}\Big[K_A(y_A^p) - K_B(y_A^p)\Big] - v^\lambda(y_A^p)v_{BA_0\lambda}(y_A^0)\Bigg) - \\ & -a_{B_0}^\alpha(y_A^0)\Big[K_A(y_A^p) - K_B(y_A^p)\Big] + \mathcal{O}(c^{-5}). \end{aligned} \tag{$E13b$}$$

Then, based on expressions $(E1)$ and $(E13)$, we may get the expression for the transformation law for the vector-potential V_B^ϵ:

$$V_B^\epsilon(y_B^0, y_B^\nu) = V_B^\epsilon(x^0, x^\nu) + v_{B_0}^\epsilon(x^0)\cdot U_B(x^0, x^\nu) + \mathcal{O}(c^{-5}), \tag{$E14a$}$$

and

$$V_B^\epsilon(y_B^0, y_B^\nu) = V_B^\epsilon(y_A^0, y_A^\nu) - v_{BA_0}^\epsilon(y_A^0)\cdot U_B(y_A^0, y_A^\nu) + \mathcal{O}(c^{-5}). \tag{$E14b$}$$

From expressions $(E1)$ and $(E13)$, we obtain the relation for the potential Φ_{1B}:

$$\Phi_{1B}(y_B^0, y_B^\nu) = \Phi_{1B}(x^0, x^\nu) - 2v_{B_0\epsilon}(x^0) \cdot V_B^\epsilon(x^0, x^\nu) -$$

$$-v_{B_0\epsilon}(x^0) v_{B_0}^\epsilon(x^0) \cdot U_B(x^0, x^\nu) + \mathcal{O}(c^{-6}), \qquad (E15a)$$

and

$$\Phi_{1B}(y_B^0, y_B^\nu) = \Phi_{1B}(y_A^0, y_A^\nu) + 2v_{BA_0\epsilon}(y_A^0) \cdot V_B^\epsilon(y_A^0, y_A^\nu) -$$

$$-v_{BA_0\epsilon}(y_A^0) v_{BA_0}^\epsilon(y_A^0) \cdot U_B(y_A^0, y_A^\nu) + \mathcal{O}(c^{-6}). \qquad (E15b)$$

Finally, for the transformation of the superpotential χ_B from $(E1), (E12)$, and $(C8)$–$(C9)$, one obtains

$$\frac{\partial^2}{\partial {y_B^0}^2} \chi_B(y_B^0, y_B^\nu) = \frac{\partial^2}{\partial {x^0}^2} \chi_B(x^0, x^\nu) + a_{B_0}^\epsilon(x^0) \frac{\partial}{\partial x^\epsilon} \chi_B(x^0, x^\nu) +$$

$$+2v_{B_0}^\epsilon(x^0) \frac{\partial^2}{\partial x^\epsilon \partial x^0} \chi_B(x^0, x^\nu) +$$

$$+v_{B_0}^\epsilon(x^0) v_{B_0}^\lambda(x^0) \frac{\partial^2}{\partial x^\epsilon \partial x^\lambda} \chi_B(x^0, x^\nu) + \mathcal{O}(c^{-6}), \qquad (E16a)$$

and

$$\frac{\partial^2}{\partial {y_B^0}^2} \chi_B(y_B^0, y_B^\nu) = \frac{\partial^2}{\partial {y_A^0}^2} \chi_B(y_A^0, y_A^\nu) - a_{BA_0}^\epsilon(y_A^0) \frac{\partial}{\partial y_A^\epsilon} \chi_B(y_A^0, y_A^\nu) -$$

$$-2v_{BA_0}^\epsilon(y_A^0) \frac{\partial^2}{\partial y_A^\epsilon \partial y_A^0} \chi_B(y_A^0, y_A^\nu) +$$

$$+v_{BA_0}^\epsilon(y_A^0) v_{BA_0}^\lambda(y_A^0) \frac{\partial^2}{\partial y_A^\epsilon \partial y_A^\lambda} \chi_B(y_A^0, y_A^\nu) + \mathcal{O}(c^{-6}). \qquad (E16b)$$

Appendix F: Christoffel Symbols in the Proper RF$_A$.

In this appendix, we will present some expressions that are in use in various parts of the present report.

F.1. Christoffel Symbols With Respect to the Background Metric γ^A_{mn}.

The connection components (or, so-called, Christoffel symbols) for the background metric $\gamma_{lm}(y^p_A)$ in the coordinates (y^p_A) of the proper RF$_A$ are defined as usual:

$$\gamma^{kA}_{nm}(y^p_A) = \frac{1}{2}\gamma^{kp}_A(y^p_A)\Big(\partial^A_n\gamma^A_{mp}(y^p_A) + \partial^A_m\gamma_{pn}(y^p_A) - \partial^A_p\gamma_{mn}(y^p_A)\Big),$$

where $\partial^A_p = \partial/\partial y^p_A$. Then from eqs.$(C6)$, one may obtain expressions for the Christoffel symbols for the metric tensor $\gamma_{lm}(y^p_A)$ as follows:

$$\gamma^{0A}_{00}(y^p_A) = \frac{\partial^2 K_A}{\partial {y^0_A}^2} + a_{A_0\epsilon}v^\epsilon_{A_0} + \frac{\partial^2 L_A}{\partial {y^0_A}^2} - a^\epsilon_{A_0}\frac{\partial L_A}{\partial y^\epsilon_A} -$$

$$-\Big(\frac{\partial^2 K_A}{\partial {y^0_A}^2} + a_{A_0\epsilon}v^\epsilon_{A_0}\Big)\Big(\frac{\partial K_A}{\partial y^0_A} + v_{A_0\epsilon}v^\epsilon_{A_0}\Big) + v_{A_0\epsilon}\Big(\frac{\partial^2 Q^\epsilon_A}{\partial {y^0_A}^2} - a_{A_0\epsilon}\frac{\partial Q^\epsilon_A}{\partial y^0_A}\Big) + \mathcal{O}(c^{-7}), \qquad (F1a)$$

$$\gamma^{0A}_{0\alpha}(y^p_A) = -a_{A_0\alpha} + \frac{\partial^2 L_A}{\partial y^0_A\partial y^\alpha_A} + v_{A_0\epsilon}\frac{\partial^2 Q^\epsilon_A}{\partial y^0_A\partial y^\alpha_A} + a_{A_0\alpha}\Big(\frac{\partial K_A}{\partial y^0_A} + v_{A_0\epsilon}v^\epsilon_{A_0}\Big) + \mathcal{O}(c^{-6}), \qquad (F1b)$$

$$\gamma^{0A}_{\alpha\beta}(y^p_A) = \frac{\partial^2}{\partial y^\alpha_A\partial y^\beta_A}\Big(L_A + v_{A_0\lambda}Q^\lambda_A\Big) + O(c^{-6}), \qquad (F1c)$$

$$\gamma^{\alpha A}_{00}(y^p_A) = a^\alpha_{A_0} + \Big(\frac{\partial^2 Q^\alpha_A}{\partial {y^0_A}^2} - a^\lambda_{A_0}\frac{\partial Q^\alpha_A}{\partial y^\lambda_A}\Big) - v^\alpha_{A_0}\Big(\frac{\partial^2 K_A}{\partial {y^0_A}^2} + a_{A_0\lambda}v^\lambda_{A_0}\Big) + \mathcal{O}(c^{-6}), \qquad (F1d)$$

$$\gamma^{\alpha A}_{0\beta}(y^p_A) = v^\alpha_{A_0}a_{A_0\beta} + \frac{\partial^2 Q^\alpha_A}{\partial y^0_A\partial y^\beta_A} + \mathcal{O}(c^{-5}), \qquad (F1e)$$

$$\gamma^{\alpha A}_{\beta\omega}(y^p_A) = \gamma^{\alpha<0>}_{\beta\omega} + \frac{\partial^2 Q^\alpha_A}{\partial y^\beta_A\partial y^\omega_A} + \mathcal{O}(c^{-4}), \qquad (F1f)$$

where $\gamma^{\alpha<0>}_{\beta\omega}$ is the Christoffel symbols in coordinates of the Galilean inertial RF (by choosing the quasi-Cartesian coordinates, we may make these symbols vanish: $\gamma^{\alpha<0>}_{\beta\omega} = 0$).

F.2. Christoffel Symbols With Respect to the Riemann Metric g^A_{mn}.

From the expressions of eqs.$(B8)$, one may also obtain the connection components $\Gamma^n_{kl}(y^p_A)$ with respect to the total Riemann metric $g_{lm}(y^p_A)$ in coordinates (y^p_A) of the RF$_A$:

$$\Gamma^0_{00}(y^p_A) = \frac{\partial}{\partial y^0_A}\left(\frac{\partial K_A}{\partial y^0_A} + \frac{1}{2}v_{A_0\epsilon}v^\epsilon_{A_0} - \sum_B U_B(y^0_A, y^\nu_A)\right) +$$

$$+\frac{\partial}{\partial y^0_A}\left(\frac{\partial L_A}{\partial y^0_A} + \frac{1}{2}\Big(\frac{\partial K_A}{\partial y^0_A}\Big)^2 + v_{A_0\epsilon}\frac{\partial Q^\epsilon_A}{\partial y^0_A} + \frac{1}{2}H^{<4>}_{00}(y^0_A, y^\nu_A)\right) -$$

$$-\frac{\partial}{\partial y^0_A}\left(\frac{\partial K_A}{\partial y^0_A} + \frac{1}{2}v_{A_0\epsilon}v^\epsilon_{A_0} - \sum_B U_B(y^0_A, y^\nu_A)\right)^2 -$$

$$-\left[\frac{\partial L_A}{\partial y_A^\epsilon} - v_{A_0\epsilon}\frac{\partial K_A}{\partial y_A^0} + \frac{\partial Q_{A\epsilon}}{\partial y_A^0} + v_{A_0\lambda}\frac{\partial Q_A^\lambda}{\partial y_A^\epsilon} + 4\,\gamma_{\epsilon\lambda}\sum_B V_B^\lambda(y_A^0, y_A^\nu)\right]\times$$

$$\times\Big(a_{A_0}^\epsilon + \gamma^{\epsilon\nu}\sum_B \frac{\partial}{\partial y_A^\nu}U_B(y_A^0, y_A^\nu)\Big) + \mathcal{O}(c^{-7}), \qquad (F2a)$$

$$\Gamma_{0\alpha}^0(y_A^p) = -a_{A_0\alpha} - \sum_B \frac{\partial}{\partial y_A^\alpha}U_B(y_A^0, y_A^\nu) +$$

$$+\frac{\partial}{\partial y_A^\alpha}\left(\frac{\partial L_A}{\partial y_A^0} + \frac{1}{2}\Big(\frac{\partial K_A}{\partial y_A^0}\Big)^2 + v_{A_0\epsilon}\frac{\partial Q_A^\epsilon}{\partial y_A^0} + \frac{1}{2}H_{00}^{<4>}(y_A^0, y_A^\nu)\right) +$$

$$+2\left(\frac{\partial K_A}{\partial y_A^0} + \frac{1}{2}v_{A_0\epsilon}v_{A_0}^\epsilon - \sum_B U_B(y_A^0, y_A^\nu)\right)\left(a_{A_0\alpha} + \sum_B \frac{\partial}{\partial y_A^\alpha}U_B(y_A^0, y_A^\nu)\right) + \mathcal{O}(c^{-6}), \qquad (F2b)$$

$$\Gamma_{\alpha\beta}^0(y_A^p) = \frac{\partial^2}{\partial y_A^\alpha \partial y_A^\beta}\Big(L_A + v_{A_0\lambda}Q_A^\lambda\Big) +$$

$$+\sum_B\left(2\gamma_{\beta\lambda}\frac{\partial}{\partial y_A^\alpha}V_B^\lambda(y_A^0, y_A^\nu) + 2\gamma_{\alpha\lambda}\frac{\partial}{\partial y_A^\beta}V_B^\lambda(y_A^0, y_A^\nu) - \gamma_{\alpha\beta}\frac{\partial}{\partial y_A^0}U_B(y_A^0, y_A^\nu)\right) + \mathcal{O}(c^{-5}), \qquad (F2c)$$

$$\Gamma_{00}^\alpha(y_A^p) = a_{A_0}^\alpha + \gamma^{\alpha\lambda}\sum_B \frac{\partial}{\partial y_A^\lambda}U_B(y_A^0, y_A^\nu) -$$

$$-\gamma^{\alpha\lambda}\frac{\partial}{\partial y_A^\lambda}\left(\frac{\partial L_A}{\partial y_A^0} + \frac{1}{2}\Big(\frac{\partial K_A}{\partial y_A^0}\Big)^2 + v_{A_0\epsilon}\frac{\partial Q_A^\epsilon}{\partial y_A^0} + \frac{1}{2}H_{00}^{<4>}(y_A^0, y_A^\nu)\right) +$$

$$+\frac{\partial}{\partial y_A^0}\left(\gamma^{\alpha\lambda}\frac{\partial}{\partial y_A^\lambda}\Big(L_A + v_{A_0\nu}Q_A^\nu\Big) - v_{A_0}^\alpha\frac{\partial K_A}{\partial y_A^0} + \frac{\partial Q_A^\alpha}{\partial y_A^0} + 4\sum_B V_B^\alpha(y_A^0, y_A^\nu)\right) -$$

$$-\left(v_{A_0}^\alpha v_{A_0}^\mu + \gamma^{\mu\lambda}\frac{\partial Q_A^\alpha}{\partial y_A^\lambda} + \gamma^{\alpha\lambda}\frac{\partial Q_A^\mu}{\partial y_A^\lambda} + 2\gamma^{\alpha\mu}\sum_B U_B(y_A^0, y_A^\nu)\right)\times$$

$$\times\Big(a_{A_0\mu} + \sum_B \frac{\partial}{\partial y_A^\mu}U_B(y_A^0, y_A^\nu)\Big) + \mathcal{O}(c^{-6}), \qquad (F2d)$$

$$\Gamma_{0\beta}^\alpha(y_A^p) = v_{A_0}^\alpha a_{A_0\beta} + \frac{\partial^2 Q_A^\alpha}{\partial y_A^0 \partial y_A^\beta} +$$

$$+\sum_B\left(2\frac{\partial}{\partial y_A^\beta}V_B^\alpha(y_A^0, y_A^\nu) - 2\gamma^{\alpha\mu}\gamma_{\beta\nu}\frac{\partial}{\partial y_A^\mu}V_B^\nu(y_A^0, y_A^\nu) + \delta_\beta^\alpha\frac{\partial}{\partial y_A^0}U_B(y_A^0, y_A^\nu)\right) + \mathcal{O}(c^{-5}), \qquad (F2e)$$

$$\Gamma_{\beta\omega}^\alpha(y_A^p) = \gamma_{\beta\omega}^{\alpha<0>} + \frac{\partial^2 Q_A^\alpha}{\partial y_A^\beta \partial y_A^\omega} +$$

$$+\sum_B\left(\delta_\beta^\alpha\frac{\partial}{\partial y_A^\omega}U_B(y_A^0, y_A^\nu) + \delta_\omega^\alpha\frac{\partial}{\partial y_A^\beta}U_B(y_A^0, y_A^\nu) - \gamma_{\beta\omega}\gamma^{\alpha\lambda}\frac{\partial}{\partial y_A^\lambda}U_B(y_A^0, y_A^\nu)\right) + \mathcal{O}(c^{-4}), \qquad (F2f)$$

where the quantity $H_{00}^{<4>}(y_A^0, y_A^\nu)$ comes from the relations for the metric tensor in coordinates of the proper RF_A, eqn.(4.11), and is given by relation (4.12).

Appendix G: The Component g_{00}^{A} and the Riemann Tensor.

In this appendix, we will present the expressions for the flat metric $\gamma_{00}^{A}(y_A^p)$, the "inertial friction," term and the interaction term $h_{00}^{int<4>}(x^p)$.

G.1. The Form of the Component γ_{00}^{A}.

By substituting in the relations of eqs.$(C6)$ the solutions for the transformation functions K_A, L_A, and Q_A^α that are given by the expressions of eqs.(5.11),(5.12),(5.23),(5.34), and (5.35), one obtains the following relations for the components of the metric $\gamma_{00}^{A}(y_A^p)$:

$$\gamma_{00}^{A<2>}(y_A^0, y_A^\nu) = \frac{\partial}{\partial y_A^0} K_A(y_A^0, y_A^\nu) + v_{A_0\beta}(y_A^0) v_{A_0}^\beta(y_A^0) =$$

$$= \sum_{B\neq A} \Big[\Big\langle U_B \Big\rangle_0 + y_A^\lambda \Big\langle \frac{\partial U_B}{\partial y_A^\lambda} \Big\rangle_0 \Big] + \zeta_1^A + \mathcal{O}(c^{-4}), \qquad (G1a)$$

$$\gamma_{00}^{A<4>}(y_A^0, y_A^\nu) = 2\frac{\partial}{\partial y_A^0} L_A(y^0, y^\nu) + \Big(\frac{\partial}{\partial y_A^0} K_A(y^0, y^\nu)\Big)^2 + 2v_{A_0\beta}(y_A^0) \frac{\partial}{\partial y_A^0} Q_A^\beta(y^0, y^\nu) =$$

$$= y_A^\mu y_A^\beta \cdot \Bigg(\gamma_{\mu\beta}\, a_{A_0\lambda} a_{A_0}^\lambda - a_{A_0\mu} a_{A_0\beta} + \sum_{B\neq A} \frac{\partial}{\partial y_A^0} \Big[\gamma_{\mu\beta} \Big\langle \frac{\partial U_B}{\partial y_A^0} \Big\rangle_0 - 4 \Big\langle \frac{\partial V_{B\beta}}{\partial y_A^\mu} \Big\rangle_0 \Big] \Bigg) +$$

$$- y_A^\mu \Big\langle \frac{\partial W_{00}^B}{\partial y_A^\mu} \Big\rangle_0 - \Big\langle W_{00}^B \Big\rangle_0 + 2\zeta_2^A +$$

$$+ 2\sum_{l\geq 3}^{k} \Bigg(\frac{\partial}{\partial y_A^0} L_{A\{L\}}(y_A^0) + v_{A_0\mu} \cdot \frac{\partial}{\partial y_A^0} Q_{A\{L\}}^\mu(y_A^0) \Bigg) \cdot y_A^{\{L\}} + \mathcal{O}(c^{-6}) + \mathcal{O}|y_A^\nu|^{k+1}), \qquad (G1b)$$

$$\gamma_{0\alpha}^{A<3>}(y_A^0, y_A^\nu) = \frac{\partial}{\partial y_A^\alpha} L_A(y^0, y^\nu) - v_{A_0\alpha}(y_A^0) \frac{\partial}{\partial y_A^0} K_A(y^0, y^\nu) +$$

$$+ v_{A_0\nu}(y_A^0) \frac{\partial}{\partial y_A^\alpha} Q_A^\nu(y^0, y^\nu) + \gamma_{\alpha\nu} \frac{\partial}{\partial y_A^0} Q_A^\nu(y^0, y^\nu) =$$

$$= -4\gamma_{\alpha\mu} \sum_{B\neq A} \Big[y_A^\lambda \Big\langle \frac{\partial V_B^\mu}{\partial y_A^\lambda} \Big\rangle_0 + \Big\langle V_B^\mu \Big\rangle_0 \Big] + \sigma_\alpha^A -$$

$$- \frac{1}{2} (\gamma_{\alpha\epsilon}\delta_\lambda^\beta + \gamma_{\alpha\lambda}\delta_\epsilon^\beta - \gamma_{\epsilon\lambda}\delta_\alpha^\beta)\, y_A^\epsilon y_A^\lambda \cdot \sum_{B\neq A} \frac{\partial}{\partial y_A^0} \Big\langle \frac{\partial U_B}{\partial y_A^\beta} \Big\rangle_0 +$$

$$+ \sum_{l\geq 3}^{k} \Bigg(\frac{\partial}{\partial y_A^0} Q_{A\{L\}}^\alpha(y_A^0) \cdot y_A^{\{L\}} + \Big(L_{A\{L\}}(y_A^0) + v_{A_0\nu} \cdot Q_{A\{L\}}^\nu(y_A^0) \Big) \cdot \frac{\partial}{\partial y_A^\alpha} \Big(y_A^{\{L\}} \Big) \Bigg) +$$

$$+ \mathcal{O}(c^{-5}) + \mathcal{O}(|y_A^\nu|^{k+1}), \qquad (G1c)$$

$$\gamma^{A<2>}_{\alpha\beta}(y^0_A, y^\nu_A) = v_{A_0\alpha}(y^0_A)v_{A_0\beta}(y^0_A) + \gamma_{\alpha\nu}\frac{\partial}{\partial y^\beta_A}Q^\nu_A(y^0_A, y^\nu_A) + \gamma_{\beta\nu}\frac{\partial}{\partial y^\alpha_A}Q^\nu_A(y^0_A, y^\nu_A) =$$

$$= -2\gamma_{\alpha\beta}\sum_{B\neq A}\Big[y^\lambda_A\Big\langle\frac{\partial U_B}{\partial y^\lambda_A}\Big\rangle_0 + \Big\langle U_B\Big\rangle_0\Big] + \sigma^A_{\alpha\beta}+$$

$$+\sum_{l\geq 3}^{k}\Big(\gamma_{\alpha\nu}Q^\nu_{A\{L\}}(y^0_A)\frac{\partial}{\partial y^\beta_A} + \gamma_{\beta\nu}Q^\nu_{A\{L\}}(y^0_A)\frac{\partial}{\partial y^\alpha_A}\Big)\cdot y^{\{L\}}_A + \mathcal{O}(c^{-4}) + \mathcal{O}(|y^\nu_A|^{k+1}). \qquad (G1d)$$

G.2. Lemma. The following relation holds for any values of k:

$$a^{\{K\}} - b^{\{K\}} = \sum_{s=1}^{k}(-)^{k-s}\frac{P^{k-s+1}_k}{(s-1)!(k-s+1)!}a^{\{S-1\}}\Big(a^\nu - b^\nu\Big)^{\{K-S+1\}}, \qquad (G2)$$

where $a^{\{K\}} = a^{\nu_1}a^{\nu_2}...a^{\nu_k}$, and P^p_n is the operation of all the possible arrangements of p different objects from n ones.

Formula $(G2)$ may be proved by direct verification of several arbitrary values of s. Thus, for $s = 1$ and $s = 2$, this formula is trivial. For $s = 3$ and $s = 4$ from the right-hand side of equation $(G2)$, one may check that these relations hold as well.

Indeed, by straightforward calculation, we have the following result for $s = 3$:

$$(a^{\nu_1} - b^{\nu_1})(a^{\nu_2} - b^{\nu_2})(a^{\nu_3} - b^{\nu_3}) - a^{\nu_1}(a^{\nu_2} - b^{\nu_2})(a^{\nu_3} - b^{\nu_3})-$$

$$-a^{\nu_2}(a^{\nu_3} - b^{\nu_3})(a^{\nu_1} - b^{\nu_1}) - a^{\nu_3}(a^{\nu_1} - b^{\nu_1})(a^{\nu_2} - b^{\nu_2})+$$

$$+a^{\nu_1}a^{\nu_2}(a^{\nu_3} - b^{\nu_3}) + a^{\nu_2}a^{\nu_3}(a^{\nu_1} - b^{\nu_1}) + a^{\nu_3}a^{\nu_1}(a^{\nu_2} - b^{\nu_2}) =$$

$$= a^{\nu_1}a^{\nu_2}a^{\nu_3} - b^{\nu_1}b^{\nu_2}b^{\nu_3} = a^{\{3\}} - b^{\{3\}}.$$

And for $s = 4$,

$$-(a^{\nu_1} - b^{\nu_1})(a^{\nu_2} - b^{\nu_2})(a^{\nu_3} - b^{\nu_3})(a^{\nu_4} - b^{\nu_4}) + a^{\nu_1}(a^{\nu_2} - b^{\nu_2})(a^{\nu_3} - b^{\nu_3})(a^{\nu_4} - b^{\nu_4})+$$

$$+a^{\nu_2}(a^{\nu_3} - b^{\nu_3})(a^{\nu_4} - b^{\nu_4})(a^{\nu_1} - b^{\nu_1}) + a^{\nu_3}(a^{\nu_4} - b^{\nu_4})(a^{\nu_1} - b^{\nu_1})(a^{\nu_2} - b^{\nu_2})(a^{\nu_3} - b^{\nu_3})+$$

$$+a^{\nu_4}(a^{\nu_1} - b^{\nu_1})(a^{\nu_2} - b^{\nu_2})(a^{\nu_3} - b^{\nu_3}) - a^{\nu_1}a^{\nu_2}(a^{\nu_3} - b^{\nu_3})(a^{\nu_4} - b^{\nu_4}) - a^{\nu_1}a^{\nu_3}(a^{\nu_2} - b^{\nu_2})(a^{\nu_4} - b^{\nu_4})-$$

$$-a^{\nu_1}a^{\nu_4}(a^{\nu_2} - b^{\nu_2})(a^{\nu_3} - b^{\nu_3}) - a^{\nu_2}a^{\nu_3}(a^{\nu_1} - b^{\nu_1})(a^{\nu_4} - b^{\nu_4}) - a^{\nu_2}a^{\nu_4}(a^{\nu_1} - b^{\nu_1})(a^{\nu_3} - b^{\nu_3})-$$

$$-a^{\nu_3}a^{\nu_4}(a^{\nu_1} - b^{\nu_1})(a^{\nu_2} - b^{\nu_2}) + a^{\nu_1}a^{\nu_2}a^{\nu_3}(a^{\nu_4} - b^{\nu_4}) + a^{\nu_2}a^{\nu_3}a^{\nu_4}(a^{\nu_1} - b^{\nu_1}) + a^{\nu_3}a^{\nu_4}a^{\nu_1}(a^{\nu_2} - b^{\nu_2})+$$

$$+a^{\nu_4}a^{\nu_1}a^{\nu_2}(a^{\nu_3} - b^{\nu_3}) = a^{\nu_1}a^{\nu_2}a^{\nu_3}a^{\nu_4} - b^{\nu_1}b^{\nu_2}b^{\nu_3}b^{\nu_4} = a^{\{4\}} - b^{\{4\}}.$$

Then by induction, one may extrapolate the validity of expression $(G2)$ for any $s > 4$.

Making use of the relation $(G2)$, we will simplify the form of some expressions for the metric tensor in the proper RF_A and the interaction term in the coordinates (x^p) of the barycentric inertial RF_0. Let us present two expressions that will be necessary for the future analysis. The following integral is easy to calculate in the form

$$\int_B d^3y'_A \rho_B\Big(y^0_A, y'^\nu_A + y^\nu_{BA_0}(y^0_A)\Big)\frac{\partial}{\partial y'^\lambda_A}\Big[\frac{y^{\{K\}}_A - y'^{\{K\}}_A}{|y^\nu_A - y'^\nu_A|}\Big] =$$

$$= \sum_{s=1}^{k} \frac{(-)^{k-s+1}P^{k-s+1}_k}{(s-1)!(k-s+1)!}\cdot y^{\{S-1\}}_A \times$$

$$\times \int_B d^3y'_A \cdot \rho_B\Big(y^0_A, y'^\nu_A + y^\nu_{BA_0}(y^0_A)\Big)\frac{\partial}{\partial y'^\lambda_A}\Big[\frac{\Big(y^\nu_A - y'^\nu_A\Big)^{\{K-S+1\}}}{|y^\nu_A - y'^\nu_A|}\Big] =$$

$$= \sum_{s=1}^{k} \frac{(-)^{k-s+1}P^{k-s+1}_k}{(s-1)!(k-s+1)!}\cdot y^{\{S-1\}}_A \cdot \frac{\partial}{\partial y^\lambda_A} Z(y^p_A)^{(K-S+1)}_B. \qquad (G3)$$

The same quantity will have the following form in the coordinates (x^p) of the inertial RF_0:

$$\int_B d^3x' \rho_B\Big(x^0, x'^\nu - y^\nu_{B_0}(x^0)\Big)\times$$

$$\times \frac{\partial}{\partial x'^\lambda}\Big[\frac{\Big(x^\nu - y^\nu_{B_0}(x^0)\Big)^{\{K\}} - \Big(x'^\nu - y^\nu_{B_0}(x^0)\Big)^{\{K\}}}{|x^\nu - x'^\nu|}\Big] =$$

$$= \sum_{s=1}^{k} \frac{(-)^{k-s+1}P^{k-s+1}_k}{(s-1)!(k-s+1)!}\Big(x^\nu - y^\nu_{B_0}(x^0)\Big)^{\{S-1\}} \times$$

$$\times \int_B d^3x' \cdot \rho_B\Big(x^0, x'^\nu - y^\nu_{B_0}(x^0)\Big)\cdot \frac{\partial}{\partial x'^\lambda}\Big[\frac{(x^\nu - x'^\nu)^{\{K-S+1\}}}{|x^\nu - x'^\nu|}\Big] =$$

$$= \sum_{s=1}^{k} \frac{(-)^{k-s+1}P^{k-s+1}_k}{(s-1)!(k-s+1)!}\Big(x^\nu - y^\nu_{B_0}(x^0)\Big)^{\{S-1\}} \cdot \frac{\partial}{\partial x^\lambda} Z(x^p)^{(K-S+1)}_B, \qquad (G4)$$

where potential $Z(x^p)^{(S)}_B$ was defined as

$$Z(z^p)^{(S)}_B = \int_B \frac{d^3z'}{|z^\nu - z'^\nu|}\cdot \rho_B\Big(z^0, z'^\nu - z^\nu_{B_0}(z^0)\Big)\cdot (z^\nu - z'^\nu)^{\{S\}}. \qquad (G5)$$

G.3. The Form of the 'Inertial Friction' Term.

The following term in the temporal component of the metric tensor $g^A_{00}(y^0_A, y^\nu_A)$, eq.(4.16b), has the meaning of the gravitational inertial friction:

$$\int_B d^3y'_A \rho_B\Big(y^0_A, y'^\nu_A + y^\nu_{BA_0}(y^0_A)\Big)\frac{\partial}{\partial y'^\lambda_A}\Big[\frac{Q^\lambda_A(y^0_A, y^\nu_A) - Q^\lambda_A(y^0_A, y'^\nu_A)}{|y^\nu_A - y'^\nu_A|}\Big].$$

Substitution in this relation of the obtained function Q_A^α will enable us to present the 'inertial friction' term as follows:

$$\int_B d^3y'_A\rho_B\Big(y_A^0, y_A'^\nu + y_{BA_0}^\nu(y_A^0)\Big)\frac{\partial}{\partial y_A'^\lambda}\Big[\frac{Q_A^\lambda(y_A^0, y_A^\nu) - Q_A^\lambda(y_A^0, y_A'^\nu)}{|y_A^\nu - y_A'^\nu|}\Big] =$$

$$= 2U_B(y_A^0, y_A^\nu)\sum_{B'\neq A}\Big[\Big\langle U_{B'}\Big\rangle_0 + y_A^\mu\Big\langle\frac{\partial U_{B'}}{\partial y_A^\mu}\Big\rangle_0\Big] +$$

$$+\frac{1}{2}v_{A_0}^\lambda(y_A^0)v_{A_0}^\beta(y_A^0)\cdot\frac{\partial^2}{\partial y_A^\lambda\partial y_A^\beta}\chi_B(y_A^0, y_A^\nu) +$$

$$+\frac{5}{2}a_{A_0}^\lambda(y_A^0)\cdot\frac{\partial}{\partial y_A^\lambda}\chi_B(y_A^0, y_A^\nu) - f_A^{\lambda\beta}\cdot\frac{\partial^2}{\partial y_A^\lambda\partial y_A^\beta}\chi_B(y_A^0, y_A^\nu) +$$

$$+\sum_{l\geq 3}^{k} Q_{A\{L\}}^\lambda(y_A^0)\cdot\sum_{s=1}^{l}\frac{(-)^{l-s+1}P_l^{l-s+1}}{(s-1)!(l-s+1)!}\cdot y_A^{\{S-1\}}\cdot\frac{\partial}{\partial y_A^\lambda}Z(y_A^q)_B^{(L-S+1)} +$$

$$+\mathcal{O}(c^{-4}) + \mathcal{O}(|y_A^q|^{k+1}). \qquad (G6)$$

G.4. The Form of the Interaction Term.

Making use of the solutions for the functions K_A, L_A, and Q_A^α of the coordinate transformation, one may also obtain the form of the interaction term $h_{00}^{int<4>}$ in any coordinate system. Thus, for example, in the coordinates of the inertial RF_0 from eq.(4.6), we have the following expression:

$$h_{00}^{int<4>}(x^0, x^\nu) = 4\int_B\frac{d^3x'}{|x^\nu - x'^\nu|}\sum_B\rho_B\Big(x^0, x'^\nu - y_{B_0}^\nu(x^0)\Big)\sum_C U_C(x'^p) +$$

$$+2\Big(\sum_B U_B(x^p)\Big)^2 + 2\sum_B\Big[2a_{B_0}^\lambda(x^0)\cdot\frac{\partial}{\partial x^\lambda}\chi_B(x^0, x^\nu) -$$

$$-2a_{B_0}^\lambda(x^0)\cdot\Big(x_\lambda - y_{B_0\lambda}(x^0)\Big)\cdot U_B(x^0, x^\nu) -$$

$$+\sum_{l\geq 3}^{k} Q_{B\{L\}}^\lambda(x^0)\sum_{s=1}^{l}\frac{(-)^{l-s+1}P_l^{l-s+1}}{(s-1)!(l-s+1)!}\Big(x^\nu - y_{B_0}^\nu(x^0)\Big)^{\{S-1\}}\cdot\frac{\partial}{\partial x^\lambda}Z(x^q)_B^{(P-S+1)} -$$

$$-2\zeta_1^B\cdot U_B(x^0, x^\nu) - f^{\lambda\beta}\cdot\frac{\partial^2}{\partial x^\lambda\partial x^\beta}\chi_B(x^0, x^\nu)\Big] + \mathcal{O}(c^{-4}) + \mathcal{O}(|x'^\nu - y_{B_0}^\nu(x^0)|^{k+1}). \qquad (G7)$$

G.5. The Form of the Riemann Tensor in the Proper RF_A.

We are using the following notation for the components of the Riemann tensor:

$$R_{mnp}^k = \partial_p\Gamma_{mn}^k - \partial_n\Gamma_{mp}^k + \Gamma_{mn}^l\Gamma_{lp}^k - \Gamma_{mp}^l\Gamma_{ln}^k. \qquad (G8)$$

(i). By making use of the expressions for the metric tensor $g^A_{mn}(y^p_A)$ given by eqs.(6.7), one will obtain the following post-Newtonian expansions of the components of this tensor in an arbitrary RF (z^p):

$$R_{0\alpha0\beta}(z^p) = \frac{\partial}{\partial z^\beta}\Gamma^{0<2>}_{\alpha0} + g^{<2>}_{00}\frac{\partial}{\partial z^\beta}\Gamma^{0<2>}_{\alpha0} + \frac{\partial}{\partial z^\beta}\Gamma^{0<4>}_{\alpha0} - \frac{\partial}{\partial z^0}\Gamma^{0<3>}_{\alpha\beta} +$$

$$+\Gamma^{0<2>}_{\alpha0}\Gamma^{0<2>}_{\beta0} - \Gamma^{\nu<2>}_{\alpha\beta}\Gamma^{0<2>}_{\nu0} + \mathcal{O}(c^{-6}), \qquad (G9a)$$

$$R_{0\nu\alpha\beta}(z^p) = \frac{\partial}{\partial z^\beta}\Gamma^{0<3>}_{\nu\alpha} - \frac{\partial}{\partial z^\alpha}\Gamma^{0<3>}_{\nu\beta} + \mathcal{O}(c^{-5}), \qquad (G9b)$$

$$R_{\alpha\mu\beta\sigma}(z^p) = \gamma_{\alpha\lambda}\left(\frac{\partial}{\partial z^\sigma}\Gamma^{\lambda<2>}_{\mu\beta} - \frac{\partial}{\partial z^\beta}\Gamma^{\lambda<2>}_{\mu\sigma}\right) + \mathcal{O}(c^{-4}). \qquad (G9c)$$

(ii). By making use of expressions $(G9)$, one may obtain the components of the Riemann tensor R_{mnpk} in the coordinates (x^p) of the barycentric inertial RF_0 as follows:

$$R_{0\alpha0\beta}(x^0, x^\nu) = -\sum_B \frac{\partial^2 U_B}{\partial x^\alpha \partial x^\beta} + \frac{1}{2}\frac{\partial^2 H^{<4>}_{00}}{\partial x^\alpha \partial x^\beta} + \sum_B\left(\frac{\partial U_B}{\partial x^\alpha}\frac{\partial U_B}{\partial x^\beta} - \gamma_{\alpha\beta}\gamma^{\mu\nu}\frac{\partial U_B}{\partial x^\mu}\frac{\partial U_B}{\partial x^\nu}\right) -$$

$$-\sum_B\left(2\gamma_{\beta\lambda}\frac{\partial^2 V^\lambda_B}{\partial x^0 \partial x^\alpha} + 2\gamma_{\alpha\lambda}\frac{\partial^2 V^\lambda_B}{\partial x^0 \partial x^\beta} - \gamma_{\alpha\beta}\frac{\partial^2 U_B}{\partial x^{02}}\right) + \mathcal{O}(c^{-6}), \qquad (G10a)$$

$$R_{0\mu\alpha\beta}(x^0, x^\nu) = 2\sum_B\left(\gamma_{\alpha\lambda}\frac{\partial^2 V^\lambda_B}{\partial x^\mu \partial x^\beta} - \gamma_{\beta\lambda}\frac{\partial^2 V^\lambda_B}{\partial x^\mu \partial y^\alpha_A}\right) +$$

$$+\sum_B\left(\gamma_{\mu\beta}\frac{\partial^2 U_B}{\partial x^0 \partial x^\alpha} - \gamma_{\mu\alpha}\frac{\partial^2 U_B}{\partial x^0 \partial x^\beta}\right) + \mathcal{O}(c^{-5}), \qquad (G10b)$$

$$R_{\alpha\mu\beta\sigma}(x^0, x^\nu) = \sum_B\left(\gamma_{\alpha\beta}\frac{\partial^2 U_B}{\partial x^\mu \partial x^\sigma} + \gamma_{\mu\sigma}\frac{\partial^2 U_B}{\partial x^\alpha \partial x^\beta} -\right.$$

$$\left.-\gamma_{\beta\mu}\frac{\partial^2 U_B}{\partial x^\alpha \partial x^\sigma} - \gamma_{\alpha\sigma}\frac{\partial^2 U_B}{\partial x^\mu \partial x^\beta}\right) + \mathcal{O}(c^{-4}). \qquad (G10c)$$

(iii). By making use of the expressions for the connection components Γ^k_{mp} presented by the relations of eqs.$(F2)$, one may obtain the components of the Riemann tensor R_{mnpk} in the coordinates (y^p_A) of the proper RF_A as follows:

$$\left\langle R_{0\alpha0\beta}\right\rangle_0 = -\sum_B\left\langle\frac{\partial^2 U_B}{\partial y^\alpha_A \partial y^\beta_A}\right\rangle_0 + \frac{1}{2}\left\langle\frac{\partial^2 H^{<4>}_{00}}{\partial y^\alpha_A \partial y^\beta_A}\right\rangle_0 +$$

$$+\left\langle\frac{\partial U_A}{\partial y^\alpha_A}\frac{\partial U_A}{\partial y^\beta_A}\right\rangle_0 - \gamma_{\alpha\beta}\gamma^{\mu\nu}\left\langle\frac{\partial U_A}{\partial y^\mu_A}\frac{\partial U_A}{\partial y^\nu_A}\right\rangle_0 + \gamma_{\alpha\beta}a^\lambda_{A_0}a_{A_0\lambda} - a_{A_0\alpha}a_{A_0\beta} -$$

$$-\sum_B\frac{\partial}{\partial y^0_A}\left(2\gamma_{\beta\lambda}\left\langle\frac{\partial V^\lambda_B}{\partial y^\alpha_A}\right\rangle_0 + 2\gamma_{\alpha\lambda}\left\langle\frac{\partial V^\lambda_B}{\partial y^\beta_A}\right\rangle_0 - \gamma_{\alpha\beta}\left\langle\frac{\partial U_B}{\partial y^0_A}\right\rangle_0\right) + \mathcal{O}(c^{-6}), \qquad (G11a)$$

$$\Big\langle R_{0\mu\alpha\beta}\Big\rangle_0 = \sum_B \Bigg(2\gamma_{\alpha\lambda}\Big\langle\frac{\partial^2 V_B^\lambda}{\partial y_A^\mu \partial y_A^\beta}\Big\rangle_0 - 2\gamma_{\beta\lambda}\Big\langle\frac{\partial^2 V_B^\lambda}{\partial y_A^\mu \partial y_A^\alpha}\Big\rangle_0 +$$

$$+\gamma_{\mu\beta}\Big\langle\frac{\partial^2 U_B}{\partial y_A^0 \partial y_A^\alpha}\Big\rangle_0 - \gamma_{\mu\alpha}\Big\langle\frac{\partial^2 U_B}{\partial y_A^0 \partial y_A^\beta}\Big\rangle_0\Bigg) + \mathcal{O}(c^{-5}), \qquad (G11b)$$

$$\Big\langle R_{\alpha\mu\beta\sigma}\Big\rangle_0 = \sum_B \Bigg(\gamma_{\alpha\beta}\Big\langle\frac{\partial^2 U_B}{\partial y_A^\mu \partial y_A^\sigma}\Big\rangle_0 + \gamma_{\mu\sigma}\Big\langle\frac{\partial^2 U_B}{\partial y_A^\alpha \partial y_A^\beta}\Big\rangle_0 -$$

$$-\gamma_{\beta\mu}\Big\langle\frac{\partial^2 U_B}{\partial y_A^\alpha \partial y_A^\sigma}\Big\rangle_0 - \gamma_{\alpha\sigma}\Big\langle\frac{\partial^2 U_B}{\partial y_A^\mu \partial y_A^\beta}\Big\rangle_0\Bigg) + \mathcal{O}(c^{-4}). \qquad (G11c)$$

It is interesting to note that in the case when the *local* gravity produced by the body (A) under consideration may be neglected, the Riemann curvature tensor ($G10$) is formed only by the gravitational field of the other bodies in the system. This suggests that one may extend the generalized Fermi conditions in the *local* region of body (A) (or at the immediate vicinity of its world-line γ_A, given by relations (5.2)), as follows:

$$g_{mn}(y_A^p) = \; g_{mn}^{(loc)}(y_A^p) + \delta g_{mn}^{(ext)}(\sim |y_A^\alpha|^2) + \mathcal{O}(|y_A^\alpha|^3), \qquad (G12a)$$

$$\Gamma_{mn}^k(y_A^p) = \; \Gamma_{mn}^{k(loc)}(y_A^p) + \delta\Gamma_{mn}^{k(ext)}(\sim |y_A^\alpha|) + \mathcal{O}(|y_A^\alpha|^2), \qquad (G12b)$$

$$R_{mnkl}(y_A^p) = \; R_{mnkl}(y_A^0)\Big|_{\gamma_A} + \mathcal{O}(|y_A^\alpha|), \qquad (G12c)$$

where superscript *ext* denotes the external sources of gravity. Relations ($G12$) summarize our expectations based on the Equivalence Principle about the *local* gravitational environment of the self-gravitating and arbitrarily shaped extended bodies.

Appendix H: Some Important Identities.

In this appendix, we will present some identities necessary to reduce the expressions in Section 6. We will use the definition for the total mass density of the system $\overline{\rho}$ in the coordinates (y_A^p) as given by (6.4); for the total Newtonian potential $\overline{U}$ as given by (6.17); and for the total vector potential of the system $\overline{V}^\alpha$ as given by expression (6.20). Then, one may obtain the required identities simply by using the eq.m. (6.6), the Poisson equations for the potentials $\overline{U}$ and $\overline{V}^\alpha$ (6.18) and (6.21), respectively, and with the help of expression (6.22):

$$\overline{\rho}\frac{\partial\overline{V}^\alpha}{\partial y_A^0}+\overline{\rho}v^\beta\Big[\partial_\beta\overline{V}^\alpha-\partial^\alpha\overline{V}_\beta\Big]=$$

$$=\frac{1}{4\pi}\frac{\partial}{\partial y_A^0}\left(-\partial^\alpha\overline{U}\frac{\partial\overline{U}}{\partial y_A^0}+\partial_\nu\overline{U}\Big[\partial^\alpha\overline{V}^\nu-\partial^\nu\overline{V}^\alpha\Big]\right)+$$

$$+\frac{1}{4\pi}\frac{\partial}{\partial y_A^\beta}\left(\partial^\alpha\overline{U}\frac{\partial\overline{V}^\beta}{\partial y_A^0}+\partial^\beta\overline{U}\frac{\partial\overline{V}^\alpha}{\partial y_A^0}-\gamma^{\alpha\beta}\partial_\nu\overline{U}\frac{\partial\overline{V}^\nu}{\partial y_A^0}+\Big[\partial^\nu\overline{V}^\alpha-\partial^\alpha\overline{V}^\nu\Big]\Big[\partial_\nu\overline{V}^\beta-\partial^\beta\overline{V}_\nu\Big]-\right.$$

$$\left.-\gamma^{\alpha\beta}\Big[\partial^\mu\overline{V}^\nu\partial_\mu\overline{V}^\nu-\partial^\mu\overline{V}^\nu\partial_\nu\overline{V}_\mu\Big]+\frac{1}{2}\gamma^{\alpha\beta}\left(\frac{\partial\overline{U}}{\partial y_A^0}\right)^2\right), \qquad (H1)$$

$$\overline{\rho}v^\alpha\Big(\frac{\partial\overline{U}}{\partial y_A^0}+v^\mu\partial_\mu\overline{U}\Big)=\frac{\partial}{\partial y_A^0}\Big(\overline{\rho}v^\alpha\overline{U}\Big)+\frac{\partial}{\partial y_A^\beta}\Big[\overline{\rho}v^\alpha v^\beta\overline{U}-p\overline{U}\gamma^{\alpha\beta}\Big]+p\partial^\alpha\overline{U}+\overline{\rho}\overline{U}\partial^\alpha\overline{U}, \qquad (H2)$$

$$\overline{\rho}\partial^\alpha\overline{V}^\beta=\frac{1}{4\pi}\partial_\nu\Big[\partial^\nu\overline{U}\partial^\alpha\overline{V}^\beta+\partial^\alpha\overline{U}\partial^\nu\overline{V}^\beta-\gamma^{\alpha\nu}\partial^\mu\overline{U}\partial^\mu\overline{V}^\beta\Big]+\overline{\rho}v^\beta\partial^\alpha\overline{U}, \qquad (H3)$$

$$\overline{\rho}_0\partial^\alpha f=\frac{1}{2\pi}\partial_\beta\Gamma^{\alpha\beta}(f)-\frac{1}{4\pi}\partial^\alpha\overline{U}\partial_\mu\partial^\mu f, \qquad (H4)$$

where $\Gamma^{\alpha\beta}$ is defined as follows:

$$\Gamma^{\alpha\beta}(f)=\frac{1}{2}\Big[\partial^\alpha f\partial^\beta\overline{U}+\partial^\beta f\partial^\alpha\overline{U}-\gamma^{\alpha\beta}\partial^\nu f\partial^\nu\overline{U}\Big]. \qquad (H5)$$

The following identities are also easy to verify:

$$\overline{\rho}\delta\ddot{w}^\alpha_{A_0}=\overline{\rho}\partial^\alpha\Big(y_{A\mu}\delta\ddot{w}^\mu_{A_0}\Big)=\frac{1}{2\pi}\partial_\beta\Gamma^{\alpha\beta}\Big(y_{A\mu}\delta\ddot{w}^\mu_{A_0}\Big), \qquad (H6)$$

$$\overline{\rho}\left(a^\alpha_{A_0}a_{A_0\lambda}-\delta^\alpha_\lambda\cdot a^\mu_{A_0}a_{A_0\mu}\right)y^\lambda_A=\frac{1}{2\pi}\partial^\alpha\overline{U}\cdot a_{A_0\lambda}a^\lambda_{A_0}+$$

$$+\frac{1}{4\pi}\partial_\beta\Gamma^{\alpha\beta}\Big[\Big(y^\lambda_Ay^\mu_A-\gamma^{\lambda\mu}y_{A\nu}y^\nu_A\Big)a_{A_0\lambda}a_{A_0\mu}\Big], \qquad (H7)$$

$$+\overline{\rho}\left(\sum_{B\neq A}\frac{\partial}{\partial y_A^0}\Big[2\Big\langle\frac{\partial V_B^{\alpha)}}{\partial y_A^{(\lambda}}\Big\rangle_A+2\Big\langle v^{(\alpha}\frac{\partial U_B}{\partial y_A^{\lambda)}}\Big\rangle_A-\delta_\lambda^\alpha\frac{\partial}{\partial y_A^0}\Big\langle U_B\Big\rangle_A\Big]\right)y_A^\lambda=$$

$$=\frac{1}{4\pi}\partial_\beta\Gamma^{\alpha\beta}\Bigg[\sum_{B\neq A}\frac{\partial}{\partial y_A^0}\Big(4y_A^\lambda y_A^\epsilon\Big[\Big\langle\frac{\partial V_{B\epsilon}}{\partial y_A^\lambda}\Big\rangle_A+\Big\langle v_\epsilon\frac{\partial U_B}{\partial y_A^\lambda}\Big\rangle_A\Big]-$$

$$-y_{A\mu}y_A^\mu\frac{\partial}{\partial y_A^0}\Big\langle U_B\Big\rangle_A\Big)\Bigg]-\frac{1}{4\pi}\partial^\alpha\overline{U}\cdot\frac{\partial^2}{\partial {y_A^0}^2}\Big\langle U_B\Big\rangle_A, \qquad (H8)$$

$$\overline{\rho}\ddot{a}_{A_0\lambda}\Big(y_A^\alpha y_A^\lambda+\frac{1}{2}\gamma^{\alpha\lambda}y_A^\mu y_{A\mu}\Big)=-\frac{5}{4\pi}\partial^\alpha\overline{U}\cdot\ddot{a}_{A_0\lambda}y_A^\lambda+$$

$$+\frac{1}{4\pi}\frac{\partial}{\partial y_A^\beta}\Big[\Big(\gamma^{\alpha\beta}\gamma_{\lambda\mu}-\delta_\lambda^\alpha\delta_\mu^\beta-\delta_\mu^\alpha\delta_\lambda^\beta\Big)\Big(y_A^\mu y_A^\nu+\frac{1}{2}\gamma^{\mu\nu}y_{A\epsilon}y_A^\epsilon\Big)\ddot{a}_{A_0\nu}\partial^\lambda\overline{U}\Big]. \qquad (H9)$$

From the equation for the potential $\overline{W}$ (6.23a) and with the help of ($H4$), one obtains

$$\overline{\rho}\partial^\alpha\overline{W}=\frac{1}{2\pi}\partial_\beta\Gamma^{\alpha\beta}(\overline{W})+\frac{1}{2\pi}\partial^\alpha\overline{U}\Big[4\pi\sum_B\overline{\rho}_B\Big(\Pi-2v_\mu v^\mu+\frac{3p}{\overline{\rho}}\Big)-$$

$$-\sum_B\partial_{00}^2U_B-2\sum_B\partial_\lambda U_B\Big(2a_{A_0}^\lambda+\sum_{B'}\partial^\lambda U_{B'}\Big)+$$

$$+\sum_{l\geq 3}Q_{A\{L\}}^\alpha(y_A^0)\Big(2\sum_B\partial_{\epsilon\lambda}^2U_B\partial^\lambda+\sum_B\partial_\epsilon U_B\partial_\mu\partial^\mu\Big)y_A^{\{L\}}\Big]+\overline{\rho}\mathcal{O}(|y_A^\nu|^{k+1})+\overline{\rho}\mathcal{O}(c^{-6}). \qquad (H10)$$

The following identity may be written in two different ways. In order to reflect this ambiguity, we present it as follows:

$$\overline{\rho}\frac{\partial\overline{U}}{\partial y_A^0}+2\overline{\rho}v^\mu\frac{\partial\overline{U}}{\partial y_A^\mu}=\frac{\partial}{\partial y_A^0}\Big(a_1\overline{\rho}\overline{U}+\frac{2a_1-3}{8\pi}\partial_\mu\overline{U}\partial^\mu\overline{U}\Big)+$$

$$+\frac{\partial}{\partial y_A^\mu}\Big(\frac{1-a_1}{4\pi}\partial^\mu\overline{U}\frac{\partial\overline{U}}{\partial y_A^0}+\frac{a_2}{4\pi}\overline{U}\partial^\mu\frac{\partial\overline{U}}{\partial y_A^0}+(a_1+a_2)\overline{\rho}\overline{U}v^\mu+$$

$$+\frac{2-a_1-a_2}{4\pi}\partial_\nu\overline{U}[\partial^\nu\overline{V}^\mu-\partial^\mu\overline{V}^\nu]\Big), \qquad (H11)$$

where a_1 and a_2 are arbitrary numbers.

One can verify the correctness of the following identities necessary to reduce the terms in equation (6.32) that contain the functions $Q_{A\{L\}}^\alpha$ with $l\geq 3$:

$$\overline{\rho}a_{A_0\nu}\sum_{l\geq 3}^k\Big[Q_{A\{L\}}^\alpha(y_A^0)\partial^\alpha y_A^{\{L\}}\Big]=$$

$$=\frac{1}{2\pi}\partial_\beta\Gamma^{\alpha\beta}\Big(a_{A_0\nu}\sum_{l\geq 3}^k\Big[Q_{A\{L\}}^\nu(y_A^0)y_A^{\{L\}}\Big]\Big)-\frac{1}{4\pi}\partial^\alpha\overline{U}\Big[a_{A_0\nu}\sum_{l\geq 3}^kQ_{A\{L\}}^\nu(y_A^0)\cdot\partial_\lambda\partial^\lambda y_A^{\{L\}}\Big], \qquad (H12)$$

$$\overline{\rho}\sum_{l\geq 3}^{k}\partial_{00}^{2}Q^{\alpha}_{A\{L\}}(y^0_A)\cdot y^{\{L\}}_A+2\overline{\rho}v_\mu\sum_{l\geq 3}^{k}\partial_0 Q^{\alpha}_{A\{L\}}(y^0_A)\cdot\partial^\mu y^{\{L\}}_A+$$

$$+\partial_0(\overline{\rho}v^\lambda)\sum_{l\geq 3}^{k}Q^{\alpha}_{A\{L\}}(y^0_A)\cdot\partial_\lambda y^{\{L\}}_A=\frac{\partial}{\partial y^\lambda_A}\Big[\overline{\rho}v^\lambda\sum_{l\geq 3}^{k}\partial_0 Q^{\alpha}_{A\{L\}}(y^0_A)\cdot y^{\{L\}}_A\Big]+$$

$$+\frac{\partial}{\partial y^0_A}\Big[\overline{\rho}\sum_{l\geq 3}^{k}\partial_0 Q^{\alpha}_{A\{L\}}(y^0_A)\cdot y^{\{L\}}_A+\overline{\rho}v^\lambda\sum_{l\geq 3}^{k}Q^{\alpha}_{A\{L\}}(y^0_A)\cdot\partial_\lambda y^{\{L\}}_A\Big], \qquad (H13)$$

$$\overline{\rho}\partial^\alpha\overline{U}\sum_{l\geq 3}^{k}Q^{\mu}_{A\{L\}}(y^0_A)\cdot\partial_\mu y^{\{L\}}_A-\overline{\rho}\partial^\mu\overline{U}\sum_{l\geq 3}^{k}Q^{\mu}_{A\{L\}}(y^0_A)\cdot\partial^\alpha y^{\{L\}}_A-$$

$$-\frac{1}{4\pi}\partial^\alpha\overline{U}\sum_{l\geq 3}^{k}Q^{\nu}_{A\{L\}}(y^0_A)\Big[2\sum_B\partial^2_{\nu\lambda}U_B\cdot\partial^\lambda y^{\{L\}}_A+\partial_\nu U_B\cdot\partial_\mu\partial^\mu y^{\{L\}}_A\Big]=$$

$$=\frac{1}{4\pi}\frac{\partial}{\partial y^\beta_A}\Bigg(\sum_{l\geq 3}^{k}Q^{\lambda}_{A\{L\}}(y^0_A)\Big[\partial^\beta\overline{U}\partial^\alpha\overline{U}\cdot\partial^\lambda y^{\{L\}}_A-\Big(\partial^\beta\overline{U}\cdot\partial^\alpha y^{\{L\}}_A+\partial^\alpha\overline{U}\cdot\partial^\beta y^{\{L\}}_A\Big)\partial_\lambda\overline{U}\Big]+$$

$$+\gamma^{\alpha\beta}\partial_\lambda\overline{U}\sum_{l\geq 3}^{k}\partial_\mu y^{\{L\}}_A\Big(\partial^\mu\overline{U}Q^{\lambda}_{A\{L\}}(y^0_A)-\frac{1}{2}\partial^\lambda\overline{U}Q^{\mu}_{A\{L\}}(y^0_A)\Big)+$$

$$+\partial_\lambda\overline{U}\sum_{l\geq 3}^{k}Q^{\beta}_{A\{L\}}(y^0_A)\Big(\frac{1}{2}\partial^\lambda\overline{U}\partial^\alpha y^{\{L\}}_A-\partial^\alpha\overline{U}\partial^\lambda y^{\{L\}}_A\Big)\Bigg), \qquad (H14)$$

$$\frac{\partial}{\partial y^0_A}\Big[\overline{\rho}\sum_{l\geq 3}^{k}\partial_0 Q^{\alpha}_{A\{L\}}(y^0_A)\cdot y^{\{L\}}_A+\overline{\rho}v^\lambda\sum_{l\geq 3}^{k}Q^{\alpha}_{A\{L\}}(y^0_A)\cdot\partial_\lambda y^{\{L\}}_A\Big]+$$

$$+\frac{\partial}{\partial y^\lambda_A}\Big[\overline{\rho}v^\lambda\sum_{l\geq 3}^{k}\partial_0 Q^{\alpha}_{A\{L\}}(y^0_A)\cdot y^{\{L\}}_A+(\overline{\rho}v^\lambda v^\mu-\gamma^{\lambda\mu}p)\sum_{l\geq 3}^{k}Q^{\alpha}_{A\{L\}}(y^0_A)\cdot\partial_\lambda y^{\{L\}}_A\Big]=$$

$$=\overline{\rho}\frac{d^2}{dy^{0\,2}_A}\Big[\sum_{l\geq 3}^{k}Q^{\alpha}_{A\{L\}}(y^0_A)y^{\{L\}}_A\Big]-\frac{\partial}{\partial y^\beta_A}\Big[p\sum_{l\geq 3}^{k}Q^{\alpha}_{A\{L\}}(y^0_A)\cdot\partial^\beta y^{\{L\}}_A\Big]. \qquad (H15)$$

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Appendix I: Astrophysical Parameters Used in the Report.

In this appendix, we present the astrophysical parameters used in the calculations of the gravitational effects for the *Mercury Orbiter* mission in Section 7 of this report:

$$\text{Solar radius} : R_\odot = 695\,980 \text{ km},$$

$$\text{Solar gravitational constant} : \mu_\odot = \frac{GM_\odot}{c^2} = 1.4766 \text{ km},$$

$$\text{Solar quadrupole coefficient (Brown } et\ al.\text{, 1989)} : J_{2\odot} = (1.7 \pm 0.17) \times 10^{-7},$$

$$\text{Solar rotation period} : \tau_\odot = 25.36 \text{ days},$$

$$\text{Mercury's mean distance} : a_M = 0.3870984 \text{ AU} = 5.791 \times 10^7 \text{ km},$$

$$\text{Mercury's radius} : R_M = 2\,439 \text{ km},$$

$$\text{Mercury's gravitational constant} : \mu_M = \frac{GM_M}{c^2} = 1.695 \times 10^{-7} \mu_\odot,$$

$$\text{Mercury's sidereal period} : T_M = 0.241 \text{ yr} = 87.96 \text{ days},$$

$$\text{Mercury's rotational period} : \tau_M = 59.7 \text{ days},$$

$$\text{Eccentricity of Mercury's orbit} : e_M = 0.20561421,$$

$$\text{Jupiter's gravitational constant} : \mu_J = 9.547 \times 10^{-4} \mu_\odot,$$

$$\text{Jupiter's sidereal period} : T_J = 11.865 \text{ yr},$$

$$\text{Astronomical Unit} : AU = 1.49597892(1) \times 10^8 \text{ km}.$$

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