On modeling pressure diffusion in non-homogeneous shear flows

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New models are proposed for the "slow" and "rapid" parts of the pressure diffusive transport based on the examination of DNS databases for plane mixing layers and wakes. The model for the "slow" part is non-local, but requires the distribution of the triple-velocity correlation as a local source. The latter can be computed accurately for the normal component from standard gradient diffusion models, but such models are inadequate for the cross component. More work is required to remedy this situation.

1. Introduction

In higher-order turbulence models 'pressure diffusion' is usually neglected, or at best added to 'turbulent diffusion' (Launader 1984) and the two modeled in aggregate. Pressure diffusion refers to the term $\partial_i u_j p$ in the Reynolds stress budget; turbulent diffusion refers to $\partial_k u_i u_j u_k$. The latter represents the ensemble averaged effect of random convection and can often be modeled as a diffusion process; the former, however, is harder to explain as diffusion. Turbulent diffusion is usually considered to be the dominant diffusion mechanism, and pressure diffusion is considered to be negligible. However, Lumley (1975a) showed that for homogeneous turbulence the application of symmetry and incompressibility constraints to the exact equation for the "slow" or non-linear part of the pressure diffusion led to the result that its magnitude is 20% that of the triple velocity correlation. In addition, it is of opposite sign, so that if turbulent diffusion could be modeled as a gradient transport, pressure diffusion would represent counter-gradient transport. Demuren et al. (1994) examined DNS databases for several shear flows, namely: the mixing layer simulation of Rogers and Moser (1994); the wake simulation of Moser et al. (1996); the boundary layer simulation of Spalart (1988); and the backward facing step simulation of Le et al. (1993). These confirm for the $\bar{q}^2$-equation that in simple shear regions pressure diffusion is roughly 20-30% of turbulent diffusion, and it appears to be mostly counter-gradient transport, so that it merely reduces the effect of turbulent diffusion, which is mostly gradient transport. Thus, the current practice of absorbing pressure diffusion and turbulent diffusion into a single model term appears reasonable, as far as the main shear regions are concerned. But the DNS data show that near the edges of the shear layers turbulent diffusion decreases

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rapidly to zero, while pressure diffusion decreases only very gradually, so the latter then becomes dominant. Thus, the budgets show that near the free stream edge the balance is between pressure transport and mean convection, or temporal drift, rather than between turbulent transport and the latter. Further, where shear layer interactions occur, as near the middle of a wake, pressure diffusion no longer follows counter-gradient transport (see Fig. 1), and Lumley’s model becomes inadequate. It in fact becomes additive very close to the center, leading to an overall increase in total transport, in contrast to the effect in the simple shear layers on either side. Both effects cannot be captured by a mere change in model coefficient. Therefore, in order to build a model for total diffusive transport in general shear flows one must look beyond the homogeneous model of Lumley. The pressure diffusion should also be modeled separately. Inhomogeneous effects can be introduced via a non-local model based on the elliptic relaxation concept of Durbin (1991, 1993) as implemented in the previous study of Demuren et al. (1994). The local model, required in this formulation, will be based on the “slow” or non-linear part of the pressure diffusion. A splitting of the pressure diffusion into “slow” and “rapid” parts is therefore necessary.

DNS databases for the mixing layer simulation of Rogers and Moser (1994) and the wake simulation of Moser et al. (1996) are post-processed to split the velocity-pressure gradient and pressure diffusion terms into “slow” and “rapid” parts. Separate models are then proposed for these terms.
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Figure 2. Reynolds stress budget of $u_z^2$ from the DNS of plane wake. (See Fig. 1 for legend.)

Figure 3. Reynolds stress budget of $u_1u_2$ from the DNS of plane wake. (See Fig. 1 for legend.)
2. Governing equations

The Reynolds stress equations can be written for time-developing plane shear flows as:

\[ D_t \overline{u_i u_j} = -(\overline{u_i u_k \partial_k U_j} + \overline{u_j u_k \partial_k U_i}) - \partial_k \overline{u_i u_j u_k} - (\overline{u_i \partial_j (p/\rho)} + \overline{u_j \partial_i (p/\rho)}) \]
\[ -2\nu \overline{\partial_k u_i \partial_k u_j} + \nu \nabla^2 \overline{u_i u_j} \]  

\( (D_t \) represents the total derivative and \( \partial_k \) the partial derivative in the \( x_k \) coordinate.) Thus, the time-derivative is balanced by the production, turbulent transport, velocity-pressure gradient correlation, dissipation, and viscous diffusion, respectively. For the cases under consideration in this study, only normal (\( x_2 \)) derivatives of turbulent statistics are non-zero. There is also only one non-zero mean-velocity gradient, \( \partial_2 U_1 \).

Budgets for the \( \overline{u_2^2} \) normal component of the Reynolds stress and the shear stress \( \overline{u_1 u_2} \), obtained from the DNS database for the plane wake of Moser et al. (1996), are presented in Figs. 2 and 3, respectively. (In all these figures, \( y_c \) is the center-line, \( \delta \) is the wake half-width, and \( \delta_m \) is the mixing layer momentum thickness.) In both cases, turbulent and pressure transports are of comparable magnitude. Therefore, it would be inconsistent to model one and not the other. And, if one tries to model them together, then the comment in the introduction with respect to the \( q^2 \) equation also applies to the \( u_2^2 \)-equation. On the other hand, in the \( \overline{u_1 u_2} \)-equation, these terms virtually cancel each other out, except near the edges of the shear layers. Hence, it appears unlikely that a single composite model could reproduce all these features.

The velocity-pressure gradient correlation can be split into a pressure-strain correlation and a pressure diffusive transport as:

\[ -(\overline{u_i \partial_j (p/\rho)} + \overline{u_j \partial_i (p/\rho)}) = (p/\rho)(\overline{\partial_j u_i} + \overline{\partial_i u_j}) - (1/\rho)(\delta_{ij} \partial_2 \overline{p u_i} + \delta_{ij} \partial_2 \overline{p u_j}) \]  

Further, each of these terms can be split into “slow” and “rapid” parts by splitting the pressure \( p \), used in the correlations, into \( p_s \) and \( p_r \), respectively. In the present study, \( p_s \) and \( p_r \) are obtained from solution of the equations:

\[ \nabla^2 p_s = -\partial_q u_p \partial_p u_q + \partial_{22} \overline{u_2^2} \]
\[ \nabla^2 p_r = -2\partial_2 U_1 \partial_1 u_2 \]  

Figure 4 shows results of the splittings of the velocity-pressure gradient correlations into “slow” and “rapid” parts for the four non-zero components of the Reynolds stresses. We note that for the diagonal components all energy is produced in the streamwise component \( \overline{u_1^2} \) and then transferred to the normal \( \overline{u_2^2} \) and transverse \( \overline{u_3^2} \) components via pressure scrambling. The transfer mechanism appears to be quite different for these components; transfer to the normal component is solely
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3. Proposed transport models

It is proposed to model the transport terms in the Reynolds stress equations in three parts, namely, turbulent diffusion, “slow” pressure diffusion, and “rapid” pressure diffusion.

3.1 Turbulent diffusion model

Turbulent diffusive transport (TDIFF) is modeled following the proposal of Mellor and Herring (1973), (hereafter denoted MH) as:

\[ TDIFF_{\bar{u}_i \bar{u}_j} = -\left[ \bar{u}_i \bar{u}_j \bar{u}_k \right]_{,k} = c_3 \left[ (k^3 / \epsilon) \left( \bar{u}_j \bar{u}_k \right)_{,i} + \left( \bar{u}_k \bar{u}_i \right)_{,j} + \left( \bar{u}_i \bar{u}_j \right)_{,k} \right] \]  

(“\( ,k \)” represents derivative with respect to \( x_k \).) This model is derived from the isotropization of coefficients in the more complex Hanjalic and Launder (1972) diffusion model. It does preserve the symmetry of the indices in the triple-velocity correlation. It was found by Demuren and Sarkar (1993) to yield the correct anisotropy of the Reynolds stress in the wake region of channel flows, and a variant by Mellor and Yamada (1986) is widely used in geophysical flows. The model will also be used to calculate triple-velocity correlations which are required for the modeling of “slow” pressure diffusion in the next section.

3.2 “Slow” pressure diffusion model

The “slow” part of the pressure transport (SPDIFF) is modeled using a non-local elliptic relaxation approach as:

\[ SPDIFF_{\bar{u}_i \bar{u}_j} = -\left[ \delta_{jk} \bar{p} \bar{u}_i + \delta_{ik} \bar{p} \bar{u}_j \right]_{,k} = \left[ \delta_{jk} f_i + \delta_{ik} f_j \right]_{,k} \]  

where

\[ L^2 \partial_x^2 f_i - f_i = -f_i^L \]

The local source term \( f_i^L \) is given by Lumley’s (1975a) model as:

\[ f_i^L = \bar{p} \bar{u}_i^L = -0.2q^2 u_i \]
It is assumed that the same length scale which governs the non-locality in the pressure redistribution would also govern the non-locality in the pressure transport. For the $q^2 u_2$ and $u_2^2$ equations, respectively, $q^2 u_1$ and $q^2 u_2$ are obtained from the MH model. This treatment represents a generalization of the previous study by Demuren et al. (1994) in which the $k - \varepsilon$ turbulence model was used.

The principal effect of the non-local model is to "elliptically" spread the influence of the local source over the length scale $L$. Figures 6 and 7 present comparisons of the pressure diffusion, based on the local model, to DNS data for mixing layer. In each case, two model computations are made; one assumes that the triple-velocity correlations are known from DNS, and the other computes them from the MH model. For $u_2^2$, shown in Fig. 6, both computations yield similar results, with peaks that are somewhat higher than in the DNS. Hence, full application of the non-local model
would produce quite good agreement. For $\overline{u_1u_2}$, shown in Fig. 7, only the first approach, which assumes a pre-knowledge of the triple-velocity correlation, gives results in agreement with DNS. The MH model grossly underpredicts the $q^2u_1$, and hence its derivative. This appears to be a general flaw of gradient-diffusion models for the triple-velocity correlation. They are usually calibrated to reproduce the normal component of the turbulent diffusion in simple shear flows. They fail to reproduce other components, if these are present, as in the 3D boundary layer study of Schwarz and Bradshaw (1994) or the shearless mixing layer study of Briggs et al. (1996). This problem will have to be addressed before a reliable, self-contained model for the pressure diffusion of $\overline{u_1u_2}$ can be produced.

### 3.3 "Rapid" pressure diffusion model

The "rapid" part of the pressure transport (RPDIFF) is modeled in terms of the Reynolds stresses and mean velocity gradients. The simplest such model has the form:

$$\text{RPDIFF}_{\overline{u_iu_j}} = -\left[\delta_{jk}\overline{p_ru_i} + \delta_{ik}\overline{p_ru_j}\right] + \epsilon = \left[\delta_{jk}g_i + \delta_{ik}g_j\right]$$  \hspace{1cm} (8)

where

$$g_{i,k} = c_r\overline{u_iu_l}U_{l,k}$$  \hspace{1cm} (9)

Equation (8) is similar in form to the "rapid" part of some pressure-strain models. This is consistent with the suggestion of Lumley (1975b) that the traditional separation of velocity-pressure gradient correlation into a pressure-strain correlation and a pressure transport is not unique. Preliminary tests show that $c_r$ should have a value between 0.1 and 0.3. It has been suggested that this part should also be modeled with non-local effects, consistent with the modeling of the slow part.
Figure 6. Model predictions of pressure diffusion of $u_2^2$ in the plane mixing layer: ---, DNS; ----, model with triple-correlations from DNS; -----, model with triple-correlations from MH.

Figure 7. Model predictions of pressure diffusion of $u_1u_2$ in the plane mixing layer. (See Fig. 6 for legend.)
and the pressure-strain correlation, but evidence for such behavior could not be discerned from the DNS data. Further testing is desirable.

4. Conclusions

The "slow" and "rapid" parts of the velocity-pressure gradient correlations and the pressure transport have been calculated from DNS databases of plane mixing layers and wakes. These show that, in agreement with homogeneous shear flow simulation, the mechanism for the transfer of energy from the streamwise component of the Reynolds stress to the normal component is via the "slow" part, whereas for the transverse component it is through the "rapid" part. But pressure transport is distributed significantly into both "slow" and "rapid" parts, the former being mostly counter-gradient transport, and the latter closer to gradient transport. Models are proposed for both parts, which show qualitative agreement with DNS data for the normal component of the Reynolds stress but have shortcomings when applied to the Reynolds shear stress. The main flaw is the inability of gradient diffusion models to predict other than the normal component of the triple-velocity correlation for which they have been calibrated. Further development and testing is required.

REFERENCES


