Computerized Simulation of Meshing of Conventional Helical Involute Gears and Modification of Geometry

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COMPUTERIZED SIMULATION OF MESHING OF CONVENTIONAL HELICAL INVOLUTE GEARS AND MODIFICATION OF GEOMETRY

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ABSTRACT

An approach is proposed for computerized simulation of meshing of aligned and misaligned involute helical gears. Algorithms for TCA (Tooth Contact Analysis) computer programs were developed. Influence of misalignment on the shift of the bearing contact and transmission errors has been investigated. Numerical examples that illustrate the developed theory are provided.

NOMENCLATURE

\( \alpha_c \)  
parabola coefficient of function that represents the tooth profile of pinion rack-cutter (Fig. 10)

\( H_i \)  
lead

\( m_{21} \)  
gear ratio

\( M_{ij} \)  
coordinate transformation matrix (from \( S_j \) to \( S_i \))

\( n_f^{(i)} (i = 1, 2) \)  
unit normal vector to surface \( \Sigma_f \) represented in coordinate system \( S_f \)

\( n_i(\theta_j) (i = 1, 2) \)  
unit normal to surface \( \Sigma_i \)

\( N_i \)  
tooth number of the pinion \( (i = 1) \) or the gear \( (i = 2) \)

\( N_r \)  
normal vector to the rack-cutter surface \( \Sigma_r \) \( (r = c,t) \)

\( p_i (i = 1, 2) \)  
screw parameter

\( p_{ti} (i = 1, 2) \)  
circular pitch in transverse section

\( P_n \)  
normal diametral pitch

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\[ r_{bi} (i = 1, 2) \] radius of base cylinder (Figs. A.1.1 and A.1.2)

\[ r_{pi} (i = 1, 2) \] radius of pitch circle of the pinion (gear)

\[ \mathbf{f}(u, \theta_j) (i = 1, 2) \] position vector represented in \( S_i \)

\[ s_{i2} \] gear tooth thickness on the pitch cylinder

\[ S_i \] coordinate system \( i \)

\[ u_r, l_r (r = c, t) \] Surface parameters of \( S_r \) (Figs. 10 and A.5.1)

\[ \mathbf{v}(\theta) \] Relative velocity of surface \( \Sigma_i \) point with respect to surface \( \Sigma_j \) point

\[ w_{i1} \] Pinion space width on the pitch cylinder

\[ \alpha_{ni} \] Profile angle of the involute profile at the point of intersection of the involute profile with the pitch circle (Figs. A.1.1 and A.1.2)

\[ \beta \] helix angle on the pinion (gear) pitch cylinder (Figs. 9 and A.5.1)

\[ \theta_i, u_i (i = 1, 2) \] surface parameters (see \( \theta_i \) in Figs. A.1.1 and A.1.2)

\[ \mu_1 \] half of angular width on base cylinder (Fig. A.1.1)

\[ \Delta \alpha_n 1 \] error of profile angle of normal section of pinion rack-cutter

\[ \Delta \gamma \] error of shaft angle (Fig. A.1.3)

\[ \Delta \lambda_{p1} \] error of pinion lead angle \( \lambda_{p1} \)

\[ \Delta \Phi_2 \] gear transmission error (Fig. 3)

\[ \eta_2 \] half of tooth thickness on base cylinder (Fig. A.1.2)

\[ \lambda_{bi} (i = 1, 2) \] lead angle on the base cylinder of radius \( r_{bi} \)

\[ \lambda_{pi} (i = 1, 2) \] lead angle on the pitch cylinder of radius \( r_{pi} \)

\[ \Sigma_i (i = 1, 2) \] pinion and gear tooth surfaces

\[ \Sigma_r (r = c, t) \] rack-cutter surfaces

\[ \psi_j (i = 1, 2) \] rotation angles of pinion and gear being in mesh with the rack-cutter \( \Sigma_r (r = c, t) \), respectively (Fig. A.5.2)

1. INTRODUCTION

Computerized simulation of meshing and contact (Tooth Contact Analysis - TCA) was developed for spiral bevel and hypoid gear drives with tooth surfaces are in point contact. (User’s Manual, Litvin, F.L. and Gutman, Y., 1981, Litvin, F.L., et al., 1995, and Stadtfeld, H.J., 1993.) There is a great need to develop TCA computer programs for tooth surfaces that are initially in line contact that become point contact due to misalignment. A typical example of such gear drives is the conventional involute helical gear drive.

An approach is proposed that permits the investigation of the influence of gear misalignment on the shift of the bearing contact and transmission errors. Effective methods of crowning of gear tooth surfaces are proposed. The approach is complemented with a TCA computer program. Numerical examples that illustrate the developed approach are provided.
2. SIMULATION OF MESHING

General Considerations

For simulation of meshing, coordinate systems $S_1$ and $S_2$ that are rigidly connected to pinion 1 and gear 2, respectively (Fig. A.1.3), are applied. The meshing of the gear tooth surfaces is considered in the fixed coordinate system $S_f$ that is rigidly connected to the housing. Coordinate system $S_p$ is an auxiliary fixed coordinate system. Coordinate system $S_q$ is applied to simulate misalignment (Fig. A.1.3). Gear tooth surface $\Sigma_i$ ($i = 1, 2$) of a helical gear and the surface unit normal are represented in $S_i$ by vector functions $r_i(u_i, \theta_i)$ and $n_i(\theta_i)$, respectively, where $(u_i, \theta_i)$ are the surface parameters (see Appendix 1). Using coordinate transformation from $S_i$ to $S_p$, we represent the conditions of continuous tangency of gear tooth surfaces by following vector Eqs.(2-5).

\[ r_i'(u_1, \theta_1, \phi_i) - r_i'(u_2, \theta_2, \phi_2) = 0 \]  \hspace{1cm} (1)

\[ n_i'(\theta_1, \phi_i) - n_i'(\theta_2, \phi_2) = 0 \]  \hspace{1cm} (2)

Vector Eqs. (1) and (2) yield only five independent equations

\[ f_i(u_i, \theta_1, \phi_i, u_2, \theta_2, \phi_2) = 0 \quad f_i \in C^1 \quad (i = 1, 2, \ldots, 5) \]  \hspace{1cm} (3)

since

\[ |n_i'(\phi_i)| = |n_i'(\phi_2)| = 1 \]  \hspace{1cm} (4)

Equation system (3) contains six unknowns but one parameter, say $\phi_i$, may be chosen as input. The continuous solution of equation system (3) is an iterative process that is based on the following considerations:

1. Assume that system (3) of nonlinear equations is satisfied at a point

\[ p^{(0)} = (u_1^{(0)}, \theta_1^{(0)}, \phi_1^{(0)}, u_2^{(0)}, \theta_2^{(0)}, \phi_2^{(0)}) \]  \hspace{1cm} (5)

and the Jacobian of the fifth order satisfies the requirement

\[ \Delta_5 = \left| \begin{array}{cccccc} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \phi_1} & \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \phi_1} \\ \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial \theta_2} & \frac{\partial f_1}{\partial \phi_2} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial \theta_2} & \frac{\partial f_2}{\partial \phi_2} \end{array} \right| \neq 0 \quad (i = 1, 5) \]  \hspace{1cm} (6)

2. Then we can solve Eqs. (3) in the neighborhood of $p^{(0)}$ by functions

\[ \{u_i(\phi_i), \theta_1(\phi_i), u_2(\phi_i), \theta_2(\phi_i), \phi_2(\phi_i)\} \in C^1 \]  \hspace{1cm} (7)

3. Inequality (6) is observed when the gear tooth surfaces are in point contact. In this case, using functions (7) we are able to obtain: (i) the path of contact on surface $\Sigma_i$ ($i = 1, 2$) represented as

\[ r_i(u_i, \theta_1, \phi_i, u_2, \theta_2, \phi_2) \]  \hspace{1cm} (8)

and (ii) the transmission errors determined as

\[ \Delta \phi_1(\phi_i) = \phi_2(\phi_i) - \frac{N_1}{N_2} \phi_i \]  \hspace{1cm} (9)

The solution discussed above is found numerically. A subroutine for solution of nonlinear equations is represented in User’s Manual, 1989. The approach discussed is applied for simulation of meshing of misaligned helical gears. As a reminder that due to misalignment the line contact of tooth surfaces is turned out into point contact.
**Edge Contact**

Equations (1) and (2) describe continuous surface-to-surface tangency. However, it is not excluded that due to misalignment edge contact will occur that means curve-to-surface tangency when the edge of a surface of one gear (represented as a curve) will contact the surface of the mating gear. Considering the case when the edge of the pinion is in mesh with the surface of the gear, we will use the following equations

$$r_j^{(1)}(u_1(\theta_1), \theta_1, \phi_1) = r_j^{(2)}(u_2 \theta_2, \phi_2)$$  \hspace{1cm} (10)

$$\frac{\partial r_j^{(1)}}{\partial \theta_1} \cdot n_j^{(2)} = 0$$  \hspace{1cm} (11)

Equations (10) and (11) represent a system of four nonlinear equations in four unknowns ($f_j$ is considered as the input parameter). The solution to the system of Eqs. (10) and (11) allow to obtain functions

$$\theta_1(\phi_1), \theta_2(\phi_1), u_2(\phi_1), \phi_2(\phi_1)$$  \hspace{1cm} (12)

All three cases of meshing, such as surface-to-surface with instantaneous point contact, edge contact, and line contact (see below), may exist in meshing of involute helical gears when they are misaligned or aligned.

**Simulation of Meshing of Aligned Helical Gears**

In this case, the gear tooth surfaces are in line contact and the Jacobian $D_5$ (Eq. (6)) becomes equal to zero. The TCA is based on the algorithm represented in Appendix 3. Two input parameters, say $f_1$ and $u_1$, must be applied and then we will get that the matrix

$$\begin{vmatrix} \frac{\partial f_i}{\partial \theta_1} & \frac{\partial f_i}{\partial \theta_2} & \frac{\partial f_i}{\partial \phi_1} & \frac{\partial f_i}{\partial \phi_2} \\ & & & \\ & & & \\ \end{vmatrix} (i = 1, 4)$$  \hspace{1cm} (13)

will be of rank 4. The instantaneous line of contact on surface $\Sigma_i (i = 1, 2)$ is the line of the vector function $r_i(u, v, q)$. This line is the tangent to the helix on the base cylinder of gear $i$ (Fig. 1). Computations confirm that the transmission function is a linear one represented as

$$\phi_2(\phi_1) = \frac{N_1}{N_2} \phi_1$$  \hspace{1cm} (14)

and the transmission errors are zero.

**Simulation of Meshing of Misaligned Involute Helical Gears**

The TCA computer program developed was applied to investigate the following errors: $D_9$—the shaft angle error (when the axes become crossed), $D_{a_1}$—normal profile angle error of the rack-cutters that generates the pinion, and $D_{l_{p1}}$—pinion lead angle error on the pitch cylinder.

The manufacturing of helical gears is based on application of two imaginary rack-cutters that like a molding complements a casting. The pinion and the gear are generated separately, but the imaginary rack-cutters have the same profile angle $a_n$ in the normal section. Error $D_{a_n}$ means that $a_n \neq a_{n2}$.

Parameter $a_n$ does not exist directly in the surface and surface unit normal equations (see Appendix 1). However, error $D_{a_n}$ affects the design parameters $a_{1l}, r_{b1}$, and $l_{b1}$ as follows (see Appendix 2)

$$\Delta a_{1l} = \frac{\sin 2\alpha_{1l}}{\sin 2\alpha_{n1}} \Delta a_{n1}$$  \hspace{1cm} (15)

$$\Delta \lambda_{b1} = \frac{\sin 2\lambda_{b1}}{2} \frac{\tan \alpha_{1l}}{\Delta a_{1l}}$$  \hspace{1cm} (16)
Contact lines

Figure 1.—Contact lines on tooth surfaces of a helical gear.

\[ \Delta r_{p1} = -r_{p1} \sin \alpha_{1} \Delta \alpha_{1} \tag{17} \]

In the real process of manufacturing, error \( D_{lpl} \) is caused due to lead error \( H_l \). Using the relationships between the design parameters (see Appendix 2), the following is obtained:

\[ \Delta \lambda_{p1} = \frac{\sin 2\lambda_{p1}}{2H_l} \Delta H_l \tag{18} \]

Equation (18) permits \( D_{lpl} \) to be obtained considering the lead error \( H_l \) is known.

Due to misalignment, the line contact of tooth surfaces is turned into point contact. The TCA for point contact is based on the application of equation system (3). The algorithm for initial guess for the solution of equation system (3) is represented in Appendix 4.

The influence of errors \( D_g \), \( D_{an1} \), and \( D_{lpl} \) was investigated in the following ways: (i) as the impact by the separate action of each of the errors, and (ii) as the impact by action of the following combination of errors: \( D_g \), \( D_{lpl} \), and \( D_{an1} \) and \( D_{lpl} \).

The separate action of errors \( D_{l} \) and \( D_{lpl} \) causes an edge contact (Fig. 2) and a discontinuous, almost linear function of transmission errors (Fig. 3). Error \( D_{an1} \) causes paths of contact shown in Fig. 4 but does not cause transmission errors. The drawings shown in Figs. 2 to 4 are based on computation for a gear drive with the following data: \( N_1 = 30 \), \( N_2 = 100 \), \( a_n = 20^\circ \), \( P_n = 5 \text{ (1/in.)} \), \( l_p = 60^\circ \), and \( L = 1.6 \text{ (in.)} \).

The combination of errors \( D_g \) and \( D_{lpl} \) when \( |\Delta \gamma| \neq |\Delta \lambda_{p1}| \) causes also an edge contact (Fig. 5). The function of transmission errors for one cycle of meshing is a combination of two almost linear functions with different values of slope (Fig. 6). The combination of errors \( D_{an1} \) and \( D_{lpl} \) causes the path of contact on the edge of the tooth length (Fig. 7) and an almost linear function of transmission errors (Fig. 8).

There is a particular case, when \( |\Delta \gamma| \neq |\Delta \lambda_{p1}| \). The meshing of gears may be considered in such a case as meshing of crossed involute helical gears which tooth surfaces are in point contact. The path of contact on the tooth surface is shown in Fig. 9, the transmission function is a linear one, and there is no transmission errors. However, the meshing of gear is not stable because even a small difference between \( |\Delta \gamma| \neq |\Delta \lambda_{p1}| \) will cause an edge contact and transmission errors.
Figure 2.—Edge contact caused by $\Delta \gamma$ or $\Delta \lambda_p = 3$ arc min: (a) pinion tooth surface, (b) gear tooth surface.

Figure 3.—Function of transmission errors caused by $\Delta \gamma$ or $\Delta \lambda_p = 3$ arc min.

Figure 4.—Path of contact caused by $\Delta \alpha n_1 = 3$ arc min: (a) pinion tooth surface, (b) gear tooth surface.

Figure 5.—Edge contact caused by $\Delta \gamma = 3$ arc min and $\Delta \lambda_p = -4$ arc min: (a) pinion tooth surface, (b) gear tooth surface.
Figure 6.—Function of transmission errors caused by $\Delta \gamma = 3$ arc min and $\Delta \lambda_p = -4$ arc min.

Figure 7.—Edge contact caused by $\Delta \alpha_1 = 3$ arc min and $\Delta \lambda_p = -2$ arc min: (a) pinion tooth surface, (b) gear tooth surface.

Figure 8.—Function of transmission errors caused by $\Delta \alpha_1 = 3$ arc min and $\Delta \lambda_p = -2$ arc min.

Figure 9.—Path of contact caused by $\Delta \gamma = 3$ arc min and $\Delta \lambda_p = -3$ arc min: (a) pinion tooth surface, (b) gear tooth surface.
3. GENERATION AND MODIFICATION OF TOOTH SURFACES

The derivation of tooth surfaces is based on the imaginary process of generation of conjugate surfaces by application of two rack-cutters. The generating surfaces of the rack-cutters are represented respectively by plane \( \Sigma_c \) and cylindrical surface \( \Sigma_t \) that differs slightly from plane \( \Sigma_t \) (Fig. 10). The rack-cutter surfaces \( \Sigma_c \) and \( \Sigma_t \) are rigidly connected each to other in the process of the imaginary generation, and they are in tangency along straight line \( o_b z_b \) (Fig. 10). This line and axes of the gears form angle \( b \), that is equal to the helix angle on the pinion (gear) pitch cylinder. Figure 11 shows the normal sections of the rack-cutters. Rack-cutter surface \( \Sigma_c \) generates the pinion tooth surface \( \Sigma_1 \), and a rack-cutter surface \( \Sigma_t \) generates the gear tooth surface \( \Sigma_2 \).

**Applied Coordinate Systems**

Movable coordinate systems \( S_r (r = c, t) \), \( S_1 \) and \( S_2 \) are rigidly connected to the tools (rack-cutters), the pinion and the gear, respectively. The fixed coordinate systems \( S_m \) and \( S_n \) are rigidly connected to the frame of the cutting machine (Fig. A.5.2).

**Generating Surface**

The rack-cutter surface \( \Sigma_r (r = c, t) \) is represented in \( S_r \) by the equation

\[
r_r = r_r(u_r, l_r)
\]

where \( u_r, l_r \) are the surface parameters.

The normal to the rack-cutter surface is represented as

\[
N_r = \frac{\partial r_r}{\partial l_r} \times \frac{\partial r_r}{\partial u_r}
\]  

Equations (19) and (20) must be derived twice to represent the working surfaces of two rack-cutters that generate the pinion and the gear, respectively (see Appendix 5).
**Generated Surface**

The generated surface $\Sigma_i (i = 1, 2)$, the pinion or the gear surface, is determined as the envelope to the family of rack-cutter surfaces. Surface $\Sigma_i$ is determined in $S_i$ (see Appendix 5) by the Eqs. (3) and (4).

$$r_i(u_r, l_r, \psi_i) = M_{ir}(\psi_i)r_i(u_r, l_r) \quad (21)$$

$$N_r \cdot v^{(r)}_i = f(u_r, l_r, \psi_i) = 0 \quad (22)$$

Here: $(u_r, l_r)$ are the rack-cutter surface parameters; $\psi_i$ is the generalized parameter of motion in the process for generation; $N_r$ is the normal to the rack-cutter surface; $v^{(r)}_i$ is the relative velocity in meshing of the rack-cutter and the generated surface. Equation (21) represents in $S_i$ the family of rack-cutter surfaces. Equation (22) is the equation of meshing that is the necessary condition of envelope existence.

Equations (21) and (22) represent surface $\Sigma_i$ by three related parameters. Taking into account that $l_r$ is a linear parameter in equation of meshing, it can be eliminated and surface $\Sigma_i$ can be represented in two-parameter form, by parameters $u_r, \psi_i$ (see Appendix 5).

**Simulation of Meshing**

Using the conditions of continuous tangency of gear tooth surfaces (1) and (2), and solving the equation systems (3), we can obtain the contact path on pinion and gear tooth surfaces, respectively. Since the pinion tooth surface is modified and generated by cylindrical surface $\Sigma_c$ of the rack-cutter, the pinion and gear tooth surfaces are in mesh in point contact. The contact path is directed along the tooth length. Although the misalignment $D_g$ occurs, the contact path is still close to the mean line. Figures 12 and 13 show the TCA results for the case of $D_g = 3$ arc min. Figure 12 shows that the contact path is in the middle of the pinion (gear) tooth surface and Fig. 13 shows that the function of transmission errors is an almost linear. The results of TCA confirm that the localization of contact is a necessary but not a sufficient condition of the improvement of meshing of helical gears. Due to the shape of transmission errors shown in Fig. 13, the transfer of meshing to the neighboring pair of teeth will be accompanied with high acceleration and vibration. This defect can be avoided by application of a predesigned parabolic function of transmission errors that is able to absorb the almost linear function of transmission errors (Litvin, F.L., 1994). Various methods of the predesign of a parabolic function of transmission errors were proposed in Litvin, F.L., et al., 1995.

**Figure 12.**—Path of contact caused by $\Delta\gamma = 3$ arc min: (a) modified pinion tooth surface, (b) gear tooth surface.

**Figure 13.**—Function of transmission errors for modified involute helical gear drive when $\Delta\gamma = 3$ arc min.
4. CONCLUSION

Based on the study contained herein the following conclusions can be drawn:

1. Equations of gear tooth surfaces with localized bearing contact were developed.
2. Algorithms for TCA (Tooth Contact Analysis) of aligned and misaligned involute helical gear drives have been developed.
3. Numerical examples that illustrate the developed theory and permit one to determine the influence of errors of alignment are presented. Lead correction may be accompanied with edge contact (Fig. 2).

REFERENCES

APPENDIX 1. EQUATIONS OF PINION AND GEAR TOOTH SURFACES

The equations presented in Litvin, F.L., 1994 for tooth surfaces of helical involute gears and the coordinate systems are used. The tooth side surface \( II (\Sigma_1) \) of a left-hand pinion (see the cross-section in Fig. A.1.10) and the unit normal \( n_1 \) to the surface are represented as follows:

\[
\begin{align*}
\mathbf{r}_1(u_1, \theta_1) &= \begin{bmatrix} r_1 \cos(\theta_1 + \theta_1) + u_1 \cos \lambda_{b1} \sin(\theta_1 + \theta_1) \\ -r_1 \sin(\theta_1 + \theta_1) + u_1 \cos \lambda_{b1} \cos(\theta_1 + \theta_1) \\ -u_1 \sin \lambda_{b1} + p_1 \theta_1 \end{bmatrix} \\
\mathbf{n}_1(q_1) &= \begin{bmatrix} -\sin \lambda_{b1} \sin(q_1 + m_1) \\ -\sin \lambda_{b1} \cos(q_1 + m_1) \\ -\cos \lambda_{b1} \end{bmatrix} \quad (A.1.1)
\end{align*}
\]

The surface tooth side \( II (\Sigma_2) \) of a right-hand gear (see the cross-section in Fig. A.1.2) and the unit normal \( n_2 \) to the surface are represented by the equations

\[
\begin{align*}
\mathbf{r}_2(u_2, \theta_2) &= \begin{bmatrix} r_2 \cos(\theta_2 - \theta_2) + u_2 \cos \lambda_{b2} \sin(\theta_2 - \theta_2) \\ -r_2 \sin(\theta_2 - \theta_2) + u_2 \cos \lambda_{b2} \cos(\theta_2 - \theta_2) \\ u_2 \sin \lambda_{b2} - p_1 \theta_2 \end{bmatrix} \\
\mathbf{n}_2(q_2) &= \begin{bmatrix} \sin \lambda_{b2} \sin(\theta_2 - \theta_2) \\ \sin \lambda_{b2} \cos(\theta_2 - \theta_2) \\ -\cos \lambda_{b2} \end{bmatrix} \quad (A.1.3)
\end{align*}
\]

Here: \((u_i, q_i) (i = 1, 2)\) are the surface parameters. The relationships between the design parameters are represented in Appendix 2 and their notations are given in the nomenclature.

For simulation of meshing coordinate systems \( S_1 \) and \( S_2 \) that are rigidly connected to pinion 1 and gear 2, respectively (Fig. A.1.3) are applied. The meshing of the gear tooth surfaces is considered in the fixed coordinate system \( S_f \) that is rigidly connected to the housing. Coordinate system \( S_p \) is an auxiliary fixed coordinate system. Coordinate system \( S_q \) is applied to simulate the errors of assembly (Fig. A.1.3).
Figure A.1.3.—Applied coordinate system.
APPENDIX 2.—RELATIONS BETWEEN THE DESIGN PARAMETERS

The input design parameters are: \( N_1, N_2, P_n, \alpha_n, \) and \( \lambda_n. \) For ideal gearing we have: \( \alpha_{n1} = \alpha_{n2}, \lambda_{p1} = \lambda_{p2}, \lambda_{b1} = \lambda_{b2}. \) The remaining design parameters (see the nomenclature) are determined as follows

\[
\tan \alpha_{ti} = \frac{\tan \alpha_{ni}}{\sin \lambda_{pi}}, \quad (i = 1, 2) \tag{A.2.1}
\]

\[
P_{ti} = \frac{\pi}{P_n \sin \lambda_{pi}} \tag{A.2.2}
\]

\[
w_{t1} = \frac{P_n}{2} \quad \text{(assuming that the backlash is zero)} \tag{A.2.3}
\]

\[
s_{t2} = \frac{N_2}{2} \tag{A.2.4}
\]

\[
r_{pi} = \frac{N_i}{2P_n \sin \lambda_{pi}} \tag{A.2.5}
\]

\[
n_{bi} = r_{pi} \cos \alpha_{ti} \tag{A.2.6}
\]

\[
\tan \lambda_{bi} = \frac{\tan \lambda_{pi}}{\cos \alpha_{ti}} \tag{A.2.7}
\]

\[
p_i = n_{bi} \tan \lambda_{bi} = r_{pi} \tan \lambda_{pi} \tag{A.2.8}
\]

\[
H_i = 2\pi_{pi} \tag{A.2.9}
\]

\[
\frac{P_1}{P_2} = \frac{r_{b1}}{r_{b2}} = \frac{r_{p1}}{r_{p2}} = \frac{N_1}{N_2} = m_{21} \tag{A.2.10}
\]
APPENDIX 3.—ALGORITHM OF TCA FOR ALIGNED GEARS

The TCA computer program requires the solution of the system of Eq. (3). The algorithm developed is based on computation by the following steps:

**Step 1:** In the case of aligned gears we have

\[ \lambda_{b1} = \lambda_{b2} = \lambda_b \]  

(A.3.1)

**Step 2:** Considering \( n_{xf}^{(1)} = n_{xf}^{(2)} \) and \( n_{xf}^{(1)} = n_{xf}^{(2)} \), we obtain the equation

\[ \theta_1 + \mu_1 - \phi_1 = \theta_2 - \eta_2 + \phi_2 \]  

(A.3.2)

where \( f_i \) is the input parameter.

**Step 3:** Considering \( x_f^{(1)} = x_f^{(2)} \) and Eqs. (A.3.1) to (A.3.2), we obtain

\[ (\eta_{b1} + \eta_{b2}) \cos(\theta_1 + \mu_1 - \phi_1) \times \cos \lambda_b \sin(\theta_1 + \mu_1 - \phi_1) = E \]  

(A.3.3)

**Step 4:** Considering \( y_f^{(1)} = y_f^{(2)} \) and Eqs. (A.3.1) to (A.3.2), we obtain

\[ (\eta_{b1} + \eta_{b2}) \sin(\theta_1 + \mu_1 - \phi_1) \times \cos \lambda_b \cos(\theta_1 + \mu_1 - \phi_1) = 0 \]  

(A.3.4)

**Step 5:** Considering \( z_f^{(1)} = z_f^{(2)} \) and Eq. (A.3.1), we obtain

\[ (\eta_{b1} + \eta_{b2}) \sin \lambda_b - p_1 \theta_1 - p_2 \theta_2 = 0 \]  

(A.3.5)

**Step 6:** Equations (A.3.3) and (A.3.4) yield

\[ \theta_1 = \alpha_r + \phi_1 - \mu_1 \]  

(A.3.6)

**Step 7:** Equations (A.3.4) and (A.3.5) yield

\[ \theta_2 = \left( \frac{N_1}{N_2} + 1 \right) \tan \alpha_r - \frac{N_1}{N_2} \theta_1 \]  

(A.3.7)

**Step 8:** Equations (A.3.2) and (A.3.6) yield

\[ \phi_2 = \eta_2 - \theta_2 + \alpha_r \]  

(A.3.8)

**Step 9:** Equation (A.3.5) yields

\[ u_2 = \frac{n_{b1} \theta_1 + n_{b2} \theta_2}{\cos \lambda_b} - \mu_1 \]  

(A.3.9)

Note: parameter \( u_1 \) is the second input parameter, in addition to \( \phi_1 \), that is required for the TCA for aligned gears.
APPENDIX 4.—TCA ALGORITHM FOR MISALIGNED GEAR DRIVES

An initial guess is required for the solution of equation system (3) in the case of gear misalignment. The required set of parameters, \( p^{(0)} = (u^{(0)}, \phi^{(0)}, \phi_1^{(0)}, u_2^{(0)}, \theta_2^{(0)}, \phi_2^{(0)}) \), can be obtained by the following derivations.

**Step 1:** Using equation \( n_{sf}^{(1)} = n_{sf}^{(2)} \), we obtain

\[
\cos \Phi_2 = \frac{\cos \lambda_{b1} - \cos \lambda_{b2} \cos \Delta \gamma}{\sin \lambda_{b2} \sin \Delta \gamma}, \quad (\Phi_2 = \theta_2 - \eta_2 + \phi_2) \quad (A.4.1)
\]

We consider that when pinion lead angle error, \( D_{p1} \), occurs, we have \( \lambda_{b1} \neq \lambda_{b2} \) and \( \lambda_{p1} \neq \lambda_{p2} \) (see Appendix 2).

**Step 2:** Using equation \( n_{sf}^{(1)} = n_{sf}^{(2)} \), we obtain

\[
\sin \Phi_1 = \frac{\sin \lambda_{b2}}{\sin \lambda_{b1}} \sin \Phi_2, \quad (\Phi_1 = \theta_1 + \mu_1 - \phi_1) \quad (A.4.2)
\]

**Step 3:** Using equation \( x_f^{(1)} = x_f^{(2)} \), we obtain

\[
u_2 = \frac{1}{\cos \lambda_{b2} \sin \Phi_2} \left( E - n_{b2} \cos \Phi_2 - x_f^{(1)} \right) \quad (A.4.3)
\]

**Step 4:** Using equation \( y_f^{(1)} = y_f^{(2)} \), we obtain

\[
\theta_2 = \frac{1}{p_2 \sin \Delta \gamma} \left[ y_f^{(1)} + u_2 \sin \lambda_{b2} \sin \Delta \gamma - \left( n_{b2} \sin \Phi_2 - u_2 \cos \lambda_{b2} \cos \Phi_2 \right) \cos \Delta \gamma \right] \quad (A.4.4)
\]

**Step 5:** Using equation \( z_f^{(1)} = z_f^{(2)} \), we obtain

\[
\theta_1 = \frac{1}{p_1} \left( z_f^{(2)} + u_1 \sin \lambda_{b1} \right) \quad (A.4.5)
\]

**Step 6:** Equation (A.4.2) yields

\[
\phi_1 = \theta_1 + \mu_1 - \Phi_1 \quad (A.4.6)
\]

**Step 7:** Equations (A.4.1) and (A.4.2) yield

\[
\phi_2 = \Phi_2 - \theta_2 + \eta_2 \quad (A.4.7)
\]

When \( u_1 \) is chosen, the other five parameters may be determined by Eqs. (A.4.3) to (A.4.7). Parameter \( u_1 \) is related with the magnitude \( r_1 \) of the position vector of the pinion cross-section by the equation...
\[ u_1 = \frac{\sqrt{r_1^2 - r_b^2}}{\cos \beta_b} \]  

(A.4.8)

where \( r_d \leq r_1 \leq r_a \).

Equations (A.4.3) to (A.4.5) and (A.4.7) to (A.4.8) permit one to determine parameters \( P^{(0)} \) as the initial guess for the solution of the system of Eq. (3) considering that \( f_1 \) is given.
APPENDIX 5.—SURFACES OF TOOTH RACK-CUTTERS

As a reminder, that two rack-cutters are applied for the separate generation of the gear and the pinion (Fig. 11(a)). The gear rack-cutter is a conventional one that is used for generation of involute gears and its normal section is represented in $S_{c2}$ (Fig. 11(c)) as

$$r_{c2}(u_t) = M_{c2b} \hat{r}_b(u_t) = \begin{bmatrix} u_t \cos \alpha_n \\ a_m - u_t \sin \alpha_n \\ 0 \\ 1 \end{bmatrix}$$  \hspace{1cm} (A.5.1)

where (Fig. 11(c))

$$a_m = \frac{\pi}{4R_n} \cdot \hat{r}_b(u_t) = [-u_t \quad 0 \quad 1]^T$$  \hspace{1cm} (A.5.2)

The equations of the surface and the unit normal to the surface of gear rack-cutter $\Sigma_2$ are represented in $S_i$ as

$$r_i(u_t, l_i) = M_{ic2} r_{c2}(u_t)$$ \hspace{1cm} (A.5.3)

$$n_i = \left\| \frac{\partial r_i}{\partial l_i} \times \frac{\partial r_i}{\partial u_t} \right\|$$ \hspace{1cm} (A.5.4)

where (Fig. A.5.1(b))

$$M_{ic2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \beta & \sin \beta & l_i \sin \beta \\ 0 & -\sin \beta & \cos \beta & l_i \cos \beta \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ \hspace{1cm} (A.5.5)

For the purpose of localization of contact, the normal section of pinion rack-cutter deviates from the normal section of the gear rack-cutter and is represented in $S_{c1}$ as (Fig. 11(b))

$$r_{c1}(u_c) = M_{clb} \hat{r}_b(u_c) = \begin{bmatrix} -u_c \cos \alpha_n - a_c u_c^2 \sin \alpha_n \\ u_c \sin \alpha_n - a_c u_c^2 \cos \alpha_n - \alpha_m \\ 0 \\ 1 \end{bmatrix}$$ \hspace{1cm} (A.5.6)

Figure A.5.1.—To derivation of rack-cutter surface.
where

\[ r_b(u_c) = \begin{bmatrix} -u_c & -a_cu_c^2 & 0 & 1 \end{bmatrix}^T \]  \hfill (A.5.7)

The surface equation of pinion rack-cutter \( \Sigma_c \) is represented in \( S_c \)

\[ r_c(u_c, l_c) = M_{c1}r_c(u_c) \]  \hfill (A.5.8)

where (Fig. A.5.1(a))

\[
M_{c1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \beta & -\sin \beta & -l_c \sin \beta \\ 0 & -\sin \beta & \cos \beta & l_c \cos \beta \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]  \hfill (A.5.9)

The unit normal to \( \Sigma_c \) is represented in \( S_c \) by equations

\[ n_c = \frac{N_c}{|N_c|}, N_c = \frac{\partial r_c}{\partial l_c} \times \frac{\partial r_c}{\partial u_c} \]  \hfill (A.5.10)

that yield

\[ n_c = \frac{1}{\sqrt{1 + 4a_c^2u_c^2}} \begin{bmatrix} -\sin \alpha_n + 2a_cu_c \cos \alpha_n \\ -\cos \alpha_n + 2a_cu_c \cos \alpha_n \cos \beta \\ -(\cos \alpha_n + 2a_cu_c \cos \alpha_n) \cos \beta \end{bmatrix} \]  \hfill (A.5.11)

**Equation of Meshing**

In the process for the generation, the two rigidly connected rack-cutters perform translational motion, while the pinion and gear perform rotational motions as shown in Fig. A.5.2.

The equation of meshing between surface \( \Sigma_r (r = c, t) \) of the rack-cutter and the pinion (gear) tooth surface \( \Sigma_i (i = 1, 2) \) can be represented as

![Figure A.5.2.—Generation of pinion and gear by rack-cutters.](image-url)
where $\psi_i$ is the angle of rotation of the gear in the process for generation. The derivation of equation of meshing is based on the theorem that the common normal to $\Sigma_i$ and $\Sigma_i$ must pass through the instantaneous axis of rotation. Thus, we have

$$\frac{X_r - x_r}{n_x} = \frac{Y_r - y_r}{n_y} = \frac{Z_r - z_r}{n_z}$$  \hspace{1cm} (A.5.13)$$

Here (Fig. A.5.2) we have

(i) $(X_r = 0, Y_i = r_p y_2)$ in the case for gear generation.
(ii) $(X_r = 0, Y_i = -r_p y_1)$ in the case for pinion generation.

After transformations, we obtain the following equations of meshing between $\Sigma_1$ and $\Sigma_c$, and $\Sigma_2$ and $\Sigma_i$, respectively

$$f(u_c, l_c, \psi_1) = r_p \psi_1 - l_c \sin \beta - a_m \cos \beta$$

$$+ \frac{u_c(1 + 2a_c^2u_c^2) \cos \beta}{\sin \alpha_n - 2a_c u_c \cos \alpha_n} = 0$$ \hspace{1cm} (A.5.14)

$$f(u_i, l_i, \psi_2) = r_p \psi_2 - l_i \sin \beta - a_m \cos \beta$$

$$+ \frac{u_i \cos \beta}{\sin \alpha_n} = 0$$ \hspace{1cm} (A.5.15)

where $\psi_i (i = 1, 2)$ is the angle of pinion and gear rotation.

**Tooth Surfaces**

Pinion tooth surface $\Sigma_1$ is generated by rack-cutter tooth surface $\Sigma_c$ and is represented in $S_1$ by the equations

$$r_1(u_c, l_c, \psi_1) = M_{1n} M_{nc} r_c(u_c, l_c), f(u_c, l_c, \psi_1) = 0$$  \hspace{1cm} (A.5.16)

where (Fig. A.5.2(a))

$$M_{1n} = \begin{bmatrix} \cos \psi_1 & \sin \psi_1 & 0 & 0 \\ -\sin \psi_1 & \cos \psi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ \hspace{1cm} (A.5.17)

$$M_{nc} = \begin{bmatrix} 1 & 0 & 0 & r_{pl} \\ 0 & 1 & 0 & r_{pl} \psi_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ \hspace{1cm} (A.5.18)

Gear tooth surface $\Sigma_2$ is represented in $S_2$ by the following equation

$$r_2(u_c, l_c, \psi_2) = M_{2n} M_{nc} r_c(u_c, l_c), f(u_c, l_c, \psi_2) = 0$$  \hspace{1cm} (A.5.19)
where (Fig. A.5.2(b))

\[
M_{2m} = \begin{bmatrix}
\cos \psi_2 & -\sin \psi_2 & 0 & 0 \\
\sin \psi_2 & \cos \psi_2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(A.5.20)

\[
M_{mt} = \begin{bmatrix}
1 & 0 & 0 & r_{p2} \\
0 & 1 & 0 & -r_{p2} \psi_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(A.5.21)
# Computerized Simulation of Meshing of Conventional Helical Involute Gears and Modification of Geometry

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## Abstract
An approach is proposed for computerized simulation of meshing of aligned and misaligned involute helical gears. Algorithms for TCA (Tooth Contact Analysis) computer programs were developed. Influence of misalignment on the shift of the bearing contact and transmission errors has been investigated. Numerical examples that illustrate the developed theory are provided.