Radiative Heat Transfer in Finite Cylindrical Enclosures with Nonhomogeneous Participating Media

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Results of a numerical solution for radiative heat transfer in homogeneous and nonhomogeneous participating media are presented. The geometry of interest is a finite axisymmetric cylindrical enclosure. The integral formulation for radiative transport is solved by the YIX method. A three-dimensional solution scheme is applied to two-dimensional axisymmetric geometry to simplify kernel calculations and to avoid difficulties associated with treating boundary conditions. As part of the effort to improve modeling capabilities for turbulent jet diffusion flames, predicted distributions for flame temperature and soot volume fraction are used to calculate radiative heat transfer from soot particles in such flames. It is shown that the nonhomogeneity of radiative property has very significant effects. The peak value of the divergence of radiative heat flux could be underestimated by a factor of 7 if a mean homogeneous radiative property is used. Since recent studies have shown that scattering by soot agglomerates is significant in flames, the effect of magnitude of scattering is also investigated and found to be nonnegligible.

Nomenclature

\[ \begin{align*} 
A & = \text{area, } m^2 \\
a & = \text{absorption coefficient, } m^{-1} \\
a_n & = \text{coefficient of scattering phase function, Eq. (2)} \\
D & = \text{particle diameter, } \mu m \\
e & = \text{emissive power of the medium, } W/m^2 \\
i & = \text{radiation intensity, } W/m^2sr \\
K & = \text{kernel of integral equation, Eq. (8)} \\
L & = \text{distance between boundaries, m} \\
N_w & = \text{number of angular quadrature ordinates} \\
\alpha & = \text{scattering coefficient, } m^{-1} \\
\beta & = \text{scattering albedo; solid angle} \\
\delta & = \text{scattering coefficient, } m^{-1} \\
\epsilon & = \text{radiative heat flux vector, } W/m^2 \\
\Gamma & = \text{radiation quantity} \\
N_r & = \text{radiative heat flux vector} \\
R & = \text{distance from a point within the medium to the wall in the direction } \omega, \text{ Eq. (11)} \\
r & = \text{position vector} \\
r_o & = \text{cylinder radius, } m \\
\hat{r} & = \text{expansion function of Legendre series} \\
s & = \text{scattering coefficient, } m^{-1} \\
T & = \text{temperature, } K \\
V & = \text{volume, } m^3 \\
W & = \text{weights of discrete ordinates set} \\
\omega & = \text{scattering albedo; solid angle} \\
\omega_i & = \text{inward unit normal vector} \\
\Omega & = \text{Legendre function of order } n \\
\Phi & = \text{normalized scattering phase function} \\
\Theta & = \text{domain or boundary of integration, Eqs. (5-7)} \\
\omega & = \text{scattering albedo; solid angle} \\
\nu & = \text{incident direction} \\
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Subscripts

\[ \begin{align*} 
b & = \text{blackbody} \\
g & = \text{participating medium for radiative transfer} \\
r & = \text{radiation} \\
s & = \text{boundary surface} \end{align*} \]

Superscript

\[ \begin{align*} 
\nu & = \text{incident direction} \end{align*} \]

Introduction

Radiative heat transfer in a cylindrical enclosure with a participating medium is a problem of practical importance, e.g., in the design of industrial furnaces and many combustion devices. A solution method that is accurate, efficient of both computing time and storage, flexible in a complex geometry, and compatible with the energy equation is needed for the prediction of radiative performance. Few current methods can satisfy all or part of these requirements. The Monte Carlo method is flexible and requires little storage, but can be extremely time-consuming, and the results are subject to statistical error. The zonal method and the finite element method are accurate but require large amounts of computer time and storage. It is also difficult to handle nonhomogeneity or anisotropic scattering using the zonal method. The product integration method,\(^1\) while faster than the zonal and finite element methods, does not reduce the required storage. The discrete ordinates method, although accurate and less demanding of memory for large grid systems, suffers from ray effects\(^2\) and high computer time for multidimensional combined mode heat transfer problems. To deal with multidimensional nonhomogeneous media, adaptive grid and adaptive difference schemes must be used with the discrete-ordinates method to maintain the same order of local error. This will consume large amounts of CPU time and memory, and make calculation using the (Sn) method impractical. The spherical harmonics (\(P_\ell\)) method, which needs high-order approximation to achieve accurate results for an optically thin medium, is tedious in formulation and also requires large amounts of computer time and memory in dealing with non-

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Mathematical Formulation

The radiative heat transfer equation is written as
\[ \frac{d(r, \omega)}{dr} = -\kappa(r, \omega) + a_{\text{id}}(r) + \frac{\pi}{4\pi} \int_{-\pi}^{\pi} i(r, \omega') \Phi(\omega', \omega') \, d\omega' \]  
(1)

where the phase function, based on the Mie theory, can be expressed as
\[ \Phi(\omega, \omega') = \sum_{n=0}^{N} (2n + 1)a_n P_n(\omega, \omega') \]  
(2)

For particles with the size parameter \((\pi D/\lambda)\) much less than unity, the scattering effect is usually neglected, e.g., Rayleigh particles. \(P_n(\omega, \omega')\) can be expanded using the addition theorem of the Legendre function, and
\[ \omega \cdot \omega' = \mu \mu' + \xi \xi' + \eta \eta' \]  
(3)

where \((\mu, \xi, \eta)\) are the direction cosines of \(\omega\) with respect to each coordinate axis. After the \(P_n(\omega, \omega')\) terms are expanded, the phase function can be expressed as
\[ \Phi(\omega, \omega') = \sum_{n=0}^{N} (2n + 1)a_n P_n(\omega, \omega') \]  
(4)

where \(a_0 = 1, b_1 = b_2 = b_3 = 3a_1, b_4 = 5a_2/4, b_5 = b_6 = 5a_3/12, \ldots\)

The integral formulation of radiative heat transfer in a general three-dimensional, gray, emitting, absorbing, and anisotropic scattering medium corresponding to Eq. (1) by Tan is used here. Crosbie and Farrell also developed similar integral expressions for intensity in three-dimensional cylindrical geometry. The present formulation is convenient to couple with the energy equation, since heat flux and its divergence are computed directly in addition to the computational efficiency for high-order scattering phase function
\[ 4e_i(r) - \frac{1}{\epsilon} \nabla \cdot q_i(r) \]  
(5)

\[ w_i(r) = \int_{\Omega} K(r, r') [\kappa e_i(r') - \frac{1}{\epsilon} \nabla \cdot q_i(r')] \, dV(r') \]  
(6)

\[ w_i(r) = \int_{\Omega} \int K(r, r') [\kappa e_i(r') - \frac{1}{\epsilon} \nabla \cdot q_i(r')] \, dV(r') \]  
(7)

In the above equations, \(e_i\) and \(q_i\) are the blackbody emissive powers of the medium and the boundary, and \(q_i\) is the net radiative heat flux at the wall. For isotropic scattering and nonscattering media, the second terms on the right sides of Eqs. (5-7) can be deleted. The kernel \(K\) is
\[ K(r, r') = \frac{1}{\pi |r - r'|^2} \left( -\int_{\Omega} K(r + \omega) \, d\omega \right) \]  
(8)

and the unit vector
\[ \omega = \frac{r - r'}{|r - r'|} \]  
(9)

When the medium is nongray, the integral equations are essentially the same, except that all radiative quantities are wavelength-dependent, and \(e_i\) and \(q_i\) are replaced by the spectral Planck function.

Numerical Method

Geometry

Unlike planar two-dimensional \((x-y)\) geometry, which extends infinitely in the third \((z)\) direction, axisymmetric cylindrical \((r-z)\) geometry does not extend infinitely in the third \((\theta)\) dimension. For the planar two-dimensional problem, by
integrating with respect to the infinite length direction in the volume integration in Eqs. (5-7), the Bickley functions of different degrees will be the kernels. However, if one performs the volume integrations with respect to $\theta$, as in the case of axisymmetry geometry (Fig. 1)

$$\int \int K(r, r') w(r') S_n(\omega) \, dV(r')$$

$$= \int_0^\infty \int_0^{2\pi} K(r, r') w(r') S_n(\omega) \, dr' \, d\theta' \, dz'$$

depending on $S_n$, the integration related to $\theta$ can be expressed as

$$\int_0^{2\pi} \exp \left[-\int_0^{\infty} \kappa(r + \omega') \, dt\right] \, d\theta'$$

$$= \int_0^{2\pi} \frac{\exp \left[-\kappa \sqrt{r^2 + r'^2 - 2rr' \cos \theta' + (z - z')^2}ight]}{\pi (r^2 + r'^2 - 2rr' \cos \theta' + (z - z')^2)^{1/2}} \, d\theta'$$

$$= \int_0^{2\pi} \frac{\exp \left[-\kappa (r + \omega') \, dt\right]}{\pi (r^2 + r'^2 - 2rr' \cos \theta' + (z - z')^2)^{1/2}} \, d\theta'$$

(10)

where $r$ and $r'$ are any two points within the cylinder, and $\theta'$ is the angle between $r$ and $r'$ projected on the $x$-$y$ plane. Equation (10) is a kernel function different from the Bickley function. The kernel function is a function of three variables ($r$, $r'$, and $z-z'$), which makes it difficult to tabulate or approximate by any computationally efficient function. Although direct numerical integration is possible, the amount of computation involved makes it essentially equivalent to a three-dimensional calculation. Another approach is to assume that the grid points and angular quadrature ordinates are predetermined, so that the new kernel function can be reduced to a single variable. The YIX integration points will also make the computation equivalent to a three-dimensional calculation.

Other difficulties in treating axisymmetric cylinder cases as a two-dimensional problem are 1) an artificial symmetry boundary condition must be imposed at $r = 0$ (Menguc and Viskanta,16 and Jamaluddin and Smith17), which could result in singularity caused by $1/r$ term; and 2) how to account for the curvature of the wall at $r = r_m$, rather than treating it as a flat wall.18 These difficulties are eliminated in three-dimensional schemes.

Numerical Quadratures

The integrations of Eqs. (5-7) are performed using the YIX method.3 The integral equations are first rewritten into the distance-angular form. To maintain the same order of accuracy in angular integration at volume and boundary elements in three-dimensional geometries, the fully symmetric discrete ordinates and weights sets were used.17 The use of discrete ordinates sets is discussed in Hsu et al.7 The volume and surface integrations on the right sides of Eqs. (5-7) are constructed as follows:

$$\int \int K(r, r') F(r') \, dV(r')$$

$$= \int_0^\infty \frac{d\omega}{\pi} \int_0^{2\pi} \exp \left[-\int_0^{\infty} \kappa(r + \omega') \, dt\right] F(r + \omega) \, d\omega$$

$$= \sum_i W_i \int_0^{2\pi} \exp \left[-\int_0^{\infty} \kappa(r + \omega') \, dt\right] F(r + \omega) \, d\omega$$

$$N_w = N_w$$

(11)

$$\int \int K(r, r') F(r') \cos (r - r', n) \, dA(r')$$

$$= \int_0^{\infty} \exp \left[-\int_0^{\infty} \kappa(r + \omega') \, dt\right] F(r + \omega R) \, d\omega$$

$$= \sum_i W_i \exp \left[-\int_0^{\infty} \kappa(r + \omega') \, dt\right] F(r + \omega R)$$

$$N_w = N_w$$

(12)

where the $R(r, \omega)$ is defined as

$$R(r, \omega) = \min (r + \omega)$$

where $N_w$ is the number of ordinates, which depends on the order of discrete ordinate sets used. For the $S_n$ discrete ordinate set, $N_w = n(n + 2)$. The distance integrals in Eq. (11) are evaluated using the YIX quadrature. It is interesting to note that the right sides of Eqs. (5) and (6) are essentially the same except for the additional $S_n$ term in Eq. (6). Therefore, the integrations in Eqs. (6) can be avoided and a significant reduction of computational time can be achieved. For calculations with a high-degree anisotropic scattering phase function, this causes only a minor increase in CPU time due to the second term on the right side of Eq. (5) compared with other methods. This is impossible for the two-dimensional formulation, where different kernels exist in Eqs. (5) and (6). Additionally, the current scheme is flexible enough to use multiple discrete ordinates sets in the formulation, which is very advantageous in dealing with ray effects.7 This flexibility does not exist in the conventional discrete ordinates method8,9 for the differential-integral formulation of the radiative transfer equation, where a consistent discrete ordinate set must be used even in the homogeneous case. By utilizing the axisymmetric geometry, only the first "wedge" of the boundary and volume elements is calculated (Fig. 1). To reduce computational time further, the angular quadrature is not carried out for all $N_w$ rays: in practice, only the rays with positive $y$ ordinates are calculated, due to the symmetry. The current scheme thus has the computational advantages of the three-
dimensional scheme, but at a cost approaching that of the two-dimensional scheme.

Solution Procedure

The discretized integrals of Eqs. (5-7) are solved by iteration. The steps are 1) give an initial guess for \( V \cdot q_r \), \( w_s(r) \), and \( q_s(r) \), with \( e_r(r) \) and \( e_s(r) \) known; 2) calculate integrals on the right side of Eqs. (5-7) by the YIX quadrature; 3) calculate new \( V \cdot q_r \), \( w_s(r) \), and \( q_s(r) \); and 4) go to step 2 unless the convergence criterion is satisfied.

For the case with \( e_r(r) \) and \( e_s(r) \) as the unknowns, \( V \cdot q_r \) is obtained from the given heat source term of the energy equation and \( q_s(r) \) from the given flux boundary conditions. The iterative procedure is the same as above.

Results

The computation was performed on a Sun 4/690 workstation. In all calculations, the first YIX integration point is 0.01, except in the optically thin case, with overall optical thickness less than 0.1, where a first integration point of 0.001 is used. Typical run times can be several seconds to several hundred seconds, depending on the solution accuracy required and the optical thickness of the problem. Run time will be discussed later for the examples shown in Figs. 2 and 3.

In Fig. 2, the YIX method is applied to a benchmark problem: a uniform temperature \( (T_a) \) non-scattering medium enclosed in a black cold cylinder with unity radius and height equivalent to one diameter. The net surface heat flux \( q_r \) (or, in this particular case, heat flux \( q_s \) at \( r = 1 \)), by three different solutions is shown in the figure. The YIX results are very close to the "exact" solution by Dua and Cheng. The maximum differences between YIX results and the exact solution are 3.2, 1.9, and 9.6% for \( \tau = 5.0, 1.0, \) and 0.1, respectively. The P3 solution by Sun,\(^2\) which is similar to that of Menguc and Viskanta,\(^3\) is reasonably accurate for \( \tau = 1 \), except at large \( z \), where it overpredicts. At low optical thickness, P3 has a large deviation from the exact solution. The finite volume method\(^4\) results are nearly the same as the exact solutions and are not shown in the figure. The calculation time for the YIX method increases as the optical thickness increases. The CPU times for \( \tau = 5.0, 1.0, \) and 0.1 are 335, 195, and 32 min respectively, with 0.001 as the first integration point and S16 as the angular quadrature.

Experimental cylindrical furnace data were available for comparison with several numerical solutions. The 0.9-m diam and 5-m length water-cooled Delft furnace data were obtained from Jamaluddin and Smith\(^1\) and Wu and Fricker.\(^2\) The radiative medium for this experiment was modeled as gray, non-scattering with constant extinction coefficient of 0.3 m\(^{-1}\). The boundaries are at 425 K with an emissivity of 0.8, except at the inlet and exit plenum, where the boundaries are treated as black surfaces at 300 K. Figure 3 shows the measured wall heat flux and calculations by various methods. All the numerical solutions correctly predict the location of maximum heat flux. However, the P3 method\(^3\) seriously underpredicts the value of maximum heat flux, while other methods are reasonably close to each other. In the legend in Fig. 3, YIX/Sn represents the YIX quadrature using 0.001 as the first integration point and an angular quadrature using the Sn discrete ordinates set. It is interesting to note that the YIX/S4 and finite volume\(^4\) methods produce nearly identical results. The higher-order quadrature (YIX/S16) predicts a higher peak flux than all other methods at an axial distance 1 m from the furnace base. Compared with results from the YIX and finite volume methods, the S4 result\(^5\) is lower for almost the entire length of the furnace. It is acknowledged\(^6,7\) that the measured values cannot be reproduced even with more accurate numerical methods. One of the reasons for the discrepancy is believed to be the assumption of a uniform extinction coefficient. This shows the necessity for treating the medium as non-homogeneous in radiation transfer calculation for combustion systems.

For the experimental furnace problem, the YIX run times and maximum errors for using different integration points and angular quadrature orders are shown in Table 1. Note that the S16/0.001 case is used as the baseline for error comparison. As expected, with the higher-order discrete ordinates set, the CPU time increases in proportion to the corresponding Ny. While the smaller first integration point used in the YIX quadrature reduces the integration error, it also increases the CPU time.

As part of the effort to model a turbulent jet diffusion sooting flame, the radiation heat transfer within such a system is treated rigorously. Flame temperature (Fig. 4) and soot volume fraction data are obtained from numerical results for a turbulent ethylene jet diffusion flame in quiescent air with a nozzle diameter of \( D = 0.58 \) mm and a fuel flow rate of \( 3.96 \) cm\(^3\)/s.\(^8\) Since the extinction coefficient of soot aggregates can be approximated by that of Rayleigh spheres,\(^9\) it can be calculated from volume fraction with known soot refractive index and wavelength. We chose \( m = 1.7 - i0.7 \) and 0.5-\( \mu \)m wavelength for this study. We are not claiming that the medium should be assumed gray. Rather, we do this so that
Table 1: Comparison of CPU time and maximum error for the example problem in Fig. 3

<table>
<thead>
<tr>
<th></th>
<th>CPU time, s</th>
<th>Maximum error, %</th>
<th>CPU time, s</th>
<th>Maximum error, %</th>
</tr>
</thead>
<tbody>
<tr>
<td>The first Y1X integration point</td>
<td>S16</td>
<td>S4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.001</td>
<td>27.0</td>
<td>0 (base case)</td>
<td>24.8</td>
<td>4.2</td>
</tr>
<tr>
<td>0.01</td>
<td>37.4</td>
<td>4.7</td>
<td>3.8</td>
<td>5.9</td>
</tr>
</tbody>
</table>

A 3 × 17 grid system is used in all calculations on a Sun workstation.

Fig. 4: Temperature (K) distribution of a turbulent jet diffusion flame.

Fig. 5: Extinction coefficient (m⁻¹) distribution of a turbulent jet diffusion flame.

we can focus on effects of nonhomogeneity and scattering and be more computationally efficient. The local extinction coefficient, shown in Fig. 5, varies from as low as 10⁻¹⁵ m⁻¹ to about 10 m⁻¹ within the whole computational domain, which is much larger than the region covered in the figures. The computational domain is between 0 ≤ r/D ≤ 130 and 0 ≤ z/D ≤ 500 using a 65 × 50 grid. The boundaries are assumed to be black at 300 K. The radiative heat flux is normalized with respect to the blackbody emissive power at 1000 K. In Fig. 7 the same temperature data are used, but the extinction coefficient is treated as a constant. An equivalent mean extinction coefficient is obtained by averaging the local extinction coefficients of all the volume elements whose temperatures are higher than 300 K. As shown in Fig. 7, the peak flux divergence is still at the flame center, but its value is reduced to about 49 from 370, by a factor of more than 7. In the right parts of Figs. 6–10, the corresponding qₙ at r/D = 31 for different conditions is plotted. The homogeneous case (Fig. 7) has smooth qₙ distribution due to its constant extinction coefficient. On the other hand, nonhomogeneous cases have a sharp
peak $q_\tau$. It is evident that the effect of neglecting radiative property nonhomogeneity is very significant. To predict the flame behavior accurately with strong radiation effect and nonuniform soot distribution, it is necessary to model the radiation heat transfer by considering the local radiative property variation.

Figures 8–10 show the results under the same conditions as Figs. 4 and 5, but with isotropic scattering added where the scattering albedo equals 0.2, 0.5, and 0.8, respectively. Recent analysis and measurements both show that the scattering effect can be very significant, since soot particles are agglomerated in flames. Figure 8 shows a nearly identical flux divergence distribution as a nonscattering case (Fig. 6), except that the peak value is reduced to about 330 from 370 and the radial heat flux also decreases. As the scattering albedo increases, both radiative flux and its divergence decrease. Note that the scattering albedo may be a function of the position in the flame, since at later stages of the combustion (or at a higher position above the jet nozzle), the soot aggregates may grow larger, which increases the scattering albedo. In the current numerical scheme using the YIX method, it is very easy to incorporate the distribution of the scattering albedo. However, since this data is not yet available, the calculation will be left for future study.

Conclusions

Numerical solutions of radiative heat transfer within a finite axisymmetric cylindrical enclosure involving homogeneous and nonhomogeneous media are treated rigorously. The use of the YIX quadrature and discrete ordinates is shown to be able to solve complicated multidimensional homogeneous as well as nonhomogeneous problems accurately. Numerical results for soot radiation from a turbulent jet diffusion flame show that neglecting the nonhomogeneity in radiative property can cause significant errors. They also show that the magnitude of scattering albedo has a significant effect on the radiative heat flux. It should be pointed out that the current numerical solution can also easily include nonuniform distribution of any other radiative properties, such as the scattering albedo.

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References


