THEORETICAL ANALYSIS OF THE LONGITUDINAL BEHAVIOR OF AN AUTOMATICALLY CONTROLLED SUPersonic INTERCEPTOR DURING THE ATTACK PHASE

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SUMMARY

A theoretical analysis has been made of the longitudinal behavior of an automatically controlled supersonic interceptor during the attack phase. The control system used to control the interceptor's flight path was one in which a pitching velocity was commanded in proportion to the longitudinal tracking error. Throughout the investigation the assumption is made that the target is flying on a straight-line path.

Factors considered in this investigation included effects of control-system parameters, effects of limitations on control deflection and rate of control deflection, effects of initial tracking errors, effects of nonlinear variations in drag and lift with angle of attack and Mach number, effects of nonlinear variations in pitching moment with angle of attack, effect of variations in interceptor forward velocity, and the effect of a normal acceleration limiter on the system performance.

The control system considered in this investigation was found to give acceptable control of the interceptor's flight path during attack runs against a nonmaneuvering target.

The inclusion of a nonlinear variation of drag and lift with angle of attack and Mach number resulted in relatively large variations in the interceptor forward velocity during the attack phase. However, the effects of velocity changes on the overall responses during the attack phase were considerably reduced when a signal proportional to the change in forward velocity was fed back to the elevator servo.

INTRODUCTION

In an interceptor research program engaged in at the Langley Aeronautical Laboratory of the National Advisory Committee for Aeronautics, one of the purposes was to evaluate the tracking performance of a supersonic interceptor equipped with various types of automatic control systems. The present paper is concerned with an analysis of that phase of the problem wherein the interceptor's radar locks on, with an initial vertical tracking error, to a bomber flying at a constant velocity; only maneuvers of the interceptor in the vertical plane are required to carry out the interception. The results obtained from analysis of this longitudinal phase of the general tracking problem are intended to provide information which will be useful in the synthesis of a satisfactory longitudinal control system for the interceptor being studied. The interceptor considered in this investigation is similar to that analyzed in reference 1, which has a notched delta wing of aspect ratio 3.2 and 55° sweepback of the leading edge.

For this investigation the interceptor is assumed to be flying initially in level flight at a Mach number of 2.2 at an altitude of 50,000 feet, and the target is flying in level flight toward the interceptor at a Mach number of 1.4, at various altitudes above 50,000 feet. No consideration
was given to the effects of altitude changes on the interception problem discussed in this paper. The guidance equations presented in this paper are for a lead-collision type of navigation. The results of this investigation are presented, for the most part, in the form of interceptor and kinematic responses subsequent to radar lock-on and were computed on the Reeves Electronic Analog Computer (REAC).

SYMBOLS

\[ C_D = \text{trim drag coefficient, } \frac{\text{Drag}}{qS} \]
\[ C_L = \text{trim lift coefficient, } \frac{\text{Lift}}{qS} \]
\[ C_{m} = \text{pitching-moment coefficient, } \frac{\text{Pitching moment}}{qS^2} \]
\[ C_{m,a} = \frac{\partial C_m}{\partial \alpha} \text{ per radian} \]
\[ C_{m,a} = \frac{\partial C_m}{\partial \beta} \text{ per radian} \]
\[ C_{m,a} = \frac{\partial C_m}{\partial \phi} \text{ per radian} \]
\[ C_{m,a} = \frac{\partial C_m}{\partial \theta} \text{ per radian} \]
\[ \bar{C}_m = \text{mean aerodynamic chord, ft} \]
\[ \bar{D} = \text{differential operator, } \frac{d}{dt} \]
\[ g = \text{acceleration due to gravity, ft/sec}^2 \]
\[ I_Y = \text{moment of inertia about } Y \text{ stability axis, slug-ft}^2 \]
\[ K = \text{tracking-loop gain constant, radians/sec/ \text{radian}} \]
\[ K_r = \text{rate of pitch-feedback gain, radians/sec/ \text{radian}} \]
\[ K_s = \text{elevator-servo gain constant, radians/radian /sec} \]
\[ M = \text{Mach number} \]
\[ \frac{M}{M} = \text{predicted miss distance, measured positive from interceptor to target, ft} \]
\[ M_{LS} = \text{component of } M \text{ along the instantaneous line of sight, positive when target is ahead of rockets at predicted time of impact, ft} \]
\[ M_{NLS} = \text{component of } M \text{ perpendicular to the instantaneous line of sight, positive when target is below rockets at predicted time of impact, ft} \]
\[ m = \text{mass of airplane, slugs} \]
\[ n = \text{normal acceleration, } g \text{ units} \]
\[ q = \text{dynamic pressure, lb/sq ft} \]
\[ R = \text{distance from interceptor to target along line of sight, measured positive from interceptor to target, ft} \]
\[ S = \text{wing area, sq ft} \]
\[ t = \text{time, sec} \]
\[ t_0 = \text{time of flight of interceptor from instantaneous position to firing point, sec} \]
\[ u = \text{change in forward velocity, ft/sec} \]
\[ u' = \text{relative change in forward velocity, } \frac{u}{V} \]
\[ V = \text{forward velocity, ft/sec} \]
\[ \alpha = \text{angle of attack, radians unless otherwise specified} \]
\[ \gamma = \text{flight-path angle } (\gamma = \theta - \alpha), \text{ radians unless otherwise specified} \]
\[ \delta_e = \text{elevator deflection, radians unless otherwise specified} \]
\[ \delta_r = -\delta_e \]
\[ \epsilon_f = \text{output of filter, radians} \]
\[ \epsilon_r = \text{error in interceptor's flight path at any given instant, } \epsilon_r = \frac{M_{NLS}}{-V'd_{\infty} + (V_r + V_\infty)t} \]
\[ \theta = \text{angle of pitch, radians unless otherwise specified} \]
\[ \sigma = \text{angle between interceptor } X \text{ body axis and radar line of sight, positive when line of sight is above body axis, radians unless otherwise specified} \]
\[ \tau = \text{time of flight of interceptor's rockets from firing point to predicted point of contact with target, sec} \]
\[ \tau_f = \text{filter time constant, sec} \]
\[ \tau_e = \text{elevator servosystem time constant, sec} \]
\[ \Omega = \theta \text{ angular velocity of line of sight, } (\Omega = \dot{\theta} + \theta), \text{ radians/sec; positive when line of sight is rotating upward} \]
Subscripts:
I  interceptor
i  input
L  limit
0  initial value
R  rocket
ss  steady state
T  target

ANALYSIS

DERIVATION OF GUIDANCE EQUATIONS

The type of navigation or interception considered in this investigation is lead collision; that is, the interceptor endeavors to fly a constant flight path such that at only one point on the path the rockets of the interceptor may be fired and a hit obtained on the target. The rockets, subsequent to firing, fly a constant bearing course with the target to the predicted point of impact. The geometry of the attack problem is shown diagrammatically in figure 1. Generally, the vector equation which must be satisfied is:

\[ \overrightarrow{R} + V_T(t_o + \tau) = V_{i} + (V_I + V_R)\tau + \overrightarrow{M} \]  

The components of this vector equation along and normal to the instantaneous line of sight are:

\[ V_T(t_o + \tau) \cos (\sigma + \theta - \gamma_T) = [V_{i} + (V_I + V_R)\tau] \cos (\sigma + \alpha) + M_{LS} \]

\[ V_T(t_o + \tau) \sin (\sigma + \theta - \gamma_T) = [V_{i} + (V_I + V_R)\tau] \sin (\sigma + \alpha) + M_{NS} \]

The target flight-path angle \( \gamma_T \) is taken as zero when the target is in level flight going away from the interceptor and is taken as \( \pi \) when in level

![Figure 1. Geometry of lead collision navigation used in present investigation for \( M_{LS} = 0 \).](image-url)
flight coming toward the interceptor. The equations may be rewritten in terms of the range, rate of change of range, and angular velocity of the line of sight as:

$$\begin{align*}
R + R(t_o + \tau) - V_{RT} \cos (\sigma + \alpha) &= M_{LS} \\
-R\Omega(t_o + \tau) - V_{RT} \sin (\sigma + \alpha) &= M_{MLS}
\end{align*}$$

(3)

where

$$\begin{align*}
\dot{R} &= V_{T} \cos (\sigma + \theta - \gamma_T) - V_{T} \cos (\sigma + \alpha) \\
\dot{R}\Omega &= V_{T} \sin (\sigma + \alpha) - V_{T} \sin (\sigma + \theta - \gamma_T)
\end{align*}$$

The quantities $R$, $\sigma$, $\theta$, and $\alpha$ are defined as:

$$\begin{align*}
R &= R_o + \int \dot{R} dt \\
\sigma &= \sigma_o + \int \dot{\sigma} dt = \sigma_o + \int (\Omega - \theta) dt \\
\theta &= \theta_o + \int \dot{\theta} dt \\
\alpha &= \alpha_o + \int \dot{\alpha} dt
\end{align*}$$

In practice, $R$, $\sigma$, and $\dot{\sigma}$ would be available from the radar; however, in the analog solution of the problem these quantities were obtained from actual integrations.

Certain simplifying assumptions were made in this investigation. The angles $(\sigma + \theta)$ and $(\sigma + \alpha)$ were assumed to be small enough so that the cosines and sines of these angles are equal to unity and to the angle in radians, respectively. For these assumptions

$$\begin{align*}
\dot{R} &= V_{T} [\cos \gamma_T + (\sigma + \theta) \sin \gamma_T] - V_{T} \\
\dot{R}\Omega &= V_{T} (\sigma + \alpha) - V_{T} [(\sigma + \theta) \cos \gamma_T - \sin \gamma_T]
\end{align*}$$

and the guidance equations are

$$\begin{align*}
R + R(t_o + \tau) - V_{RT} \cos (\sigma + \alpha) &= M_{LS} \\
-R\Omega(t_o + \tau) - V_{RT} (\sigma + \alpha) &= M_{MLS}
\end{align*}$$

(4)

The solution of equations (4) is accomplished by computing continuously from the first of equations (4) the value of $(t_o + \tau)$ necessary to make $M_{MLS} = 0$ and then, from the second of equations (4), computing for this value of $(t_o + \tau)$ the value of $M_{LS}$ which will exist at $(t_o + \tau)$ seconds subsequent to the instantaneous time. The time at which $t_o$ is computed to be zero is the firing point for the interceptor's rockets. The time of flight of the interceptor's rockets is $\tau$, and throughout this investigation is assumed to be 1.5 seconds. The command to the control system is based on the error $\epsilon_r$ which exists at any time in the interceptor's flight path, which for this investigation is approximated by the expression

$$\epsilon_r = \frac{M_{MLS}}{V_T(t_o + (V_r + V_h)\tau)}$$

(5)

The validity of the foregoing assumptions in the guidance equations were checked by a digital solution on the Bell Telephone Laboratories X-6674 relay computer at Langley where the exact guidance equations were used. This comparison is discussed in a subsequent section.

DISCUSSION OF FLIGHT-PATH CONTROL SYSTEM

The block diagram of the overall system is presented in figure 2. Briefly, the computed quantity $\epsilon_r$ is filtered, amplified, and used as the command to a pitch-rate command system. The dynamics of the filter and elevator servo are represented by simple first-order lag networks of the form $\frac{1}{1+\tau_1 D}$ and $\frac{1}{1+\tau_2 D}$ respectively. The transfer function $\frac{1}{1+\tau_1 D}$ is assumed to be representative of a low-pass filter which in practice would be necessary to attenuate the high-frequency radar noise present in the computed command signal $\gamma$. However, no attempt was made to include noise in the present investigation. The dynamics of the interceptor were obtained from the linearized equations of longitudinal motion. For certain cases, these equations were modified to include specific nonlinearities. All equations used in the analysis and the analog schematic diagram are presented in the appendix. The interceptor parameters and other constants used in the analysis are presented in table 1. The interceptor stability derivatives and mass parameters were obtained from unpublished data and the results of reference 1.
Theoretical Analysis of an Automatically Controlled Interceptor

Target motions

Interceptor motions

Radar

Computer

Filter

K

g-limiter

Block diagram of the longitudinal tracking system used in present investigation.

Table I

Stability Derivatives and Mass Characteristics of Interceptor and Other Constants Used in Investigation

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Value</th>
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<tr>
<td>Altitude, ft</td>
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<tr>
<td>Density ρ, slugs/ft³</td>
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</tr>
<tr>
<td>V₀, ft/sec</td>
<td>2.140</td>
</tr>
<tr>
<td>Mₐ</td>
<td>2.2</td>
</tr>
<tr>
<td>mₛ</td>
<td>776.4</td>
</tr>
<tr>
<td>Iₓₜ, slug-ft²</td>
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</tr>
<tr>
<td>qₗ, lb/sq ft</td>
<td>826</td>
</tr>
<tr>
<td>rₗ, ft</td>
<td>15</td>
</tr>
<tr>
<td>Sₔ, sq ft</td>
<td>401</td>
</tr>
<tr>
<td>Cₐₜ, per radian</td>
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</tr>
<tr>
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<tr>
<td>Cₐₜ, per radian</td>
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<tr>
<td>Cₐₔₜ, per radian</td>
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<td>0.033</td>
</tr>
<tr>
<td>αₙ, radians</td>
<td>0.033</td>
</tr>
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</table>

Results and Discussion

Selection of System Gain Constants and Effects of Various Aerodynamic Parameters on Attack Performance

System gain constants.—An investigation was first made to determine values of the gain constants \( K \) and \( K_r \), for which the attack performance of the interceptor would be reasonably satisfactory as a starting point for the general study. The servo gain constant \( K_r \) is taken as unity throughout this analysis. Examination of the block diagram (fig. 2) indicates that the forward-loop gain is \( KK_r \) and the feedback gain is \( K_r \); therefore, the assumption that \( K_r \) equals unity imposes no restrictions on the system gain constants or performance. From a theoretical analysis of the open-loop frequency response \( γ/τ_r \), and several preliminary runs on the REAC, acceptable values of \( K \) and \( K_r \) were found to be 3.0 and 0.375, respectively.

For these values of \( K \) and \( K_r \), tracking runs were computed for \( R₀=60,000 \) feet, \( γ_0=π \), and \( θ₀=7.5° \) and 15° and are presented in figure 3. For these runs the interceptor motions were computed from the linearized equations of longitudinal motions presented in the appendix. The responses shown in figure 3 include the predicted miss distance normal to the line of sight \( M_{NLS} \), the interceptor
normal acceleration \( n \), the elevator deflection \( \delta_e \), and the relative change in interceptor forward velocity \( u' = \frac{\Delta V}{V_{T,0}} \). For these runs the elevator deflection was limited to \( \pm 20^\circ \) and the rate of elevator deflection was limited to \( \pm 120^\circ/\text{sec} \). The miss distance \( M_{NLS} \) was started off-scale on the REAC recorder in order to bring out more clearly the characteristics of \( M_{NLS} \) in the vicinity of zero. For all time histories presented in this paper, the transient responses are plotted up to the time at which the interceptor's rockets are assumed to be fired \( (t_0 = 0) \). In view of the small angle assumptions made in deriving the guidance equations (eqs. (4)), the quantity \( (t_0 + \tau) \) is dependent only upon the initial range \( R_0 \), target velocity \( V_T \), rocket velocity \( V_R \), rocket time of flight \( \tau \), and interceptor forward velocity \( V_{I,0} (1 + u') \); also, for small values of \( u' \), the parameter \( (t_0 + \tau) \) varies linearly with time. For \( \sigma_0 = 7.5^\circ \) the change in forward velocity is \( 0.074 V_{I,0} \), the value of \( M_{NLS} \) at the assumed time of firing is \( -30 \) feet, and the peak normal acceleration is \( 7.6g \). For \( \sigma_0 = 15^\circ \), the change in forward velocity is \( 0.14 V_{I,0} \), \( M_{NLS} = -90 \) feet, and the peak acceleration is \( 7.8g \). For both values of \( \sigma_0 \), the maximum perturbation in \( \alpha \) was about 0.28 radian, but these transients are not presented.

The miss distance \( M_{NLS} \) for neither of these runs is zero at the end of the run, but this result is not due to the choice of gain constants. These non-zero values of \( M_{NLS} \) can be attributed to the decrease in the interceptor's forward velocity during the runs. The interceptor is unable to maintain a condition of steady tracking \( (n = 0, M_{NLS} = 0) \) as long as the forward velocity varies since, for the type of guidance considered, the interceptor must fly a constant flight path with constant velocity in order for the flight-path error \( e \), to be continuously zero. For these runs, \( u' \) never attains a constant value and consequently \( M_{NLS} \) is not zero at the assumed firing time.

**Effect of nonlinear variation of drag and lift with angle of attack and Mach number.**—From unpublished wind-tunnel tests made for a model similar to the interceptor discussed in this paper, the variation of the drag coefficient \( C_D \) in the vicinity of the interceptor's trim angle of attack \( (\alpha_0 = 0.033 \text{ radian}) \) and initial Mach number \( (M_0 = 2.2) \) was found to be well approximated by the expression

\[
C_D = 0.014 + \frac{0.024}{M} + \left( \frac{5.92}{M} - 1.58 \right)^2
\]

and the variation of \( C_{L_{\alpha}} \) with Mach number in this range was given by

\[
C_{L_{\alpha}} = \frac{5.05}{M} \text{ per radian}
\]

If Mach number effects on \( C_D \) and \( C_{L_{\alpha}} \) are neglected, these expressions become, for \( \alpha_0 = 0.033 \text{ radian} \) and \( M_0 = 2.2 \),

\[
\begin{align*}
C_D &= 0.027 + 0.156 \Delta \alpha + 2.37(\Delta \alpha)^2 \\
C_{L_{\alpha}} &= 2.29 \text{ per radian}
\end{align*}
\]

The expressions for \( C_D \) and \( C_{L_{\alpha}} \), if only first-order changes in \( M \) are considered, become

\[
\begin{align*}
C_D &= 0.027 + 0.156 \Delta \alpha + 2.37(\Delta \alpha)^2 - [0.013 \\
&\quad + 0.134 \Delta \alpha + 2.03(\Delta \alpha)^2 \frac{\Delta M}{M_0}] \\
C_{L_{\alpha}} &= 2.29 \left( 1 - \frac{\Delta M}{M_0} \right)
\end{align*}
\]

**Figure 3.** Interceptor and kinematic time histories for \( K = 3.0 \) and \( K_r = 0.375 \).
Tracking responses were computed for $K=3.0, \; K_\epsilon=0.375, \; R_0=60,000$ feet, and $\sigma_0=7.5^\circ$ and $15^\circ$ for the cases where equations (6) and (7) were used for $C_D$ and $C_L\alpha$. The results are presented in figures 4(a) and 4(b). Also shown in these figures is the case for which $\Delta C_D$ is assumed to vary linearly with $\Delta \alpha$, that is, $\Delta C_D=0.156\Delta \alpha$. For $\sigma_0=7.5^\circ$ (fig. 4(a)), the change in interceptor forward velocity is seen to be $0.07\bar{V}_\gamma\_0$ for the linear case, $0.13\bar{V}_\gamma\_0$ when $\Delta C_D$ varies nonlinearly with $\Delta \alpha$, and $0.14\bar{V}_\gamma\_0$ when $\Delta C_D$ varies nonlinearly with $\Delta \alpha$ and $\Delta M$ and $C_L\alpha$ varies with Mach number. For the linear case $M_{NL\_S}=-30$ feet, and for the nonlinear cases $M_{NL\_S}=-50$ feet. As pointed out previously, the nonzero values of $M_{NL\_S}$ are due to the change in forward velocity. The same general trends are noted for $\sigma_0=15^\circ$ (fig. 4(b)), but the changes in forward velocity for this value of $\sigma_0$ are much greater for each drag condition investigated than were encountered for $\sigma_0=7.5^\circ$. Also, the values of $M_{NL\_S}$ at the end of the run are larger for this value of $\sigma_0$ than for $\sigma_0=7.5^\circ$.

Effect of feedback proportional to change in forward velocity. — A possible means of eliminating, or at least reducing, the value of $M_{NL\_S}$ at the assumed firing time is to feed back a signal to the elevator servo proportional to the change in forward velocity, such that a positive pitching moment is produced for a decrease in forward velocity. Time histories are presented in figures 5(a) and 5(b) for $\sigma_0=7.5^\circ$ and $15^\circ$. For these runs, $C_D$ varies nonlinearly with $\Delta \alpha$ and $\Delta M$ and a feedback gain of 0.12 is utilized. For each value of $\sigma_0$ the predicted value of $M_{NL\_S}$ is seen to be appreciably reduced. This type of feedback requires a bias error in the flight path in order for the interceptor to fly a constant flight path; but, on the basis of the predicted value of $M_{NL\_S}$ presented in figure 5, this bias appears to be small. For comparison, runs in which the change in forward velocity was assumed to be zero are also shown in these figures. Despite the fact that the velocity changed considerably during these runs, there appears to be no appreciable difference between the cases which included the feedback proportional to $\nu'$ in which forward velocity was allowed to vary and in which velocity changes

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Effect of nonlinear drag and lift on interceptor attack performance. $K=3.0; \; K_\epsilon=0.375; \; R_0=60,000$ feet.}
\end{figure}
were neglected. On the basis of these results, the remaining runs presented in this paper were computed with the assumption that velocity changes can be made to have a negligible effect on the attack performance of the interceptor being discussed.

Effect of variations in the pitching moment due to angle of attack.—Recent results of wind-tunnel tests of complete models have often indicated a nonlinear variation of pitching moment with angle of attack. In order to check, at least qualitatively, the effect of a nonlinear pitching moment, runs were made for the assumed pitching variation presented in figure 6(a). This variation of $C_w$ with angle of attack is generally similar to the type of variation obtained from wind-tunnel tests, but it should be pointed out that the range of $\alpha$ for which the pitching moment is nonlinear was arbitrarily selected on the basis that this range of $\alpha$ is in the range likely to be encountered in this particular problem. The results for this variation are presented in figure 6(b) and afford a comparison between the linear and nonlinear cases. For these runs $K=3.36$, $K_r=1.0$, $R_0=60,000$ ft, and $\alpha_0=7.5^\circ$. The altitude and Mach number of the interceptor and target are the same as before. The most significant effects of the assumed nonlinearity appeared to be reflected in the $M_{NLS}$, $n$, and $\delta$ transients and hence these are the variables presented in figure 6(b). The maximum normal acceleration is about 9g for the nonlinear case as compared with approximately 7g for the linear case. The nonlinear case also shows that there is considerable overshoot in the $M_{NLS}$ transient, and the $\delta$ motion is rather irregular. It should be noted, however, that this assumed nonlinearity does not prevent the predicted $M_{NLS}$ from being approximately zero at the assumed time of firing. The results obtained for this assumed pitching-moment variation are in agreement with those which would have been intuitively expected. The slope of the pitching-moment curve (fig. 6(a)) is seen to decrease in magnitude as the angle of attack increases, and finally reverses its sign. The general effect of reducing $C_w$ is to reduce the system spring constant in pitch and hence to
In this discussion, the rise time is defined as the time required for \( M_{NLS} \) to reach initially the point of zero miss and the response time is defined as the time for \( M_{NLS} \) to reach and remain less than 30 feet.

**Effect of the tracking loop gain \( K \).**—The effect of the gain constant \( K \) on the system performance is shown in figure 7 for values of \( \alpha_0 \) equal to 7.5° and 15°. The results are presented for \( K=3,5, \) and 9. For these cases the initial range was 60,000 feet and the feedback gain \( K_r \) was equal to 0.375. The effect of \( K \) is seen to be essentially the same for both values of \( \alpha_0 \). As the gain is increased the system becomes more and more oscillatory and for \( K=9.0 \) the \( M_{NLS} \) response is probably unsatisfactory for both values of \( \alpha_0 \). The effects of \( K \) with respect to the rise time, the response time and overshoot of the \( M_{NLS} \) transient, and the maximum normal acceleration encountered are summarized in figure 7(c) for \( \alpha_0=7.5^\circ \) and 15°. As \( K \) is increased, there is only a slight variation in rise time and peak acceleration, which is due, to a large extent, to the fact that the elevator reaches its maximum deflection of \(-20^\circ \) very quickly and remains at that deflection for a time dependent upon \( K \) and \( \alpha_0 \). As \( K \) is increased, the response time tends first to decrease and then to become large as \( K \) is further increased. The response time for \( K=9 \) and \( \alpha_0=15^\circ \) is not shown, since for this combination of \( K \) and \( \alpha_0 \), \( M_{NLS} \) never reaches the condition where it remains less than 30 feet. For both values of \( \alpha_0 \), the overshoot in \( M_{NLS} \) increases progressively with increases in \( K \).

**Effect of rate feedback gain \( K_r \).** For \( K=3.0, \alpha_0=7.5^\circ, \) and \( R_b=60,000 \) feet, results are presented in figure 8(a) for values of \( K_r=0,0.20,0.375,0.60,\) and 1.0. For \( K_r=0 \) the responses are seen to be rather oscillatory. As \( K_r \) is increased the system becomes more stable, but the overshoot in \( M_{NLS} \) is seen to increase with increases in \( K_r \). These effects are due to the fact that, as the interceptor is stabilized, its response to control inputs becomes slower and results in an increase in the \( M_{NLS} \) overshoot. A summary of the effects of \( K_r \) on the \( M_{NLS} \) transient is presented in figure 8(b). Inclusion of the pitch-rate feedback tends to reduce the response time and overshoot of \( M_{NLS} \) and, in addition, to eliminate the oscillatory condition which exists for \( K_r=0 \); but for values of \( K_r \) greater than 0.375, the overshoot is larger than...
for $K_0 = 0$. There is seen to be a relatively negligible effect of $K_0$ on the rise time because, as mentioned previously, $\delta_0$ is at its limit of $-20^\circ$ at the beginning of the run and the rate feedback is ineffective during the early portion of the run.

**Effect of initial error in $\sigma$.**—For $K = 3.0$, $K_0 = 0.375$, and $R_0 = 60,000$ feet, several runs were made to evaluate the ability of the system to score a hit as the initial error in $\sigma$ is increased. The value of $\sigma$ actually reflects the altitude difference and the horizontal distance between the interceptor and target; at the beginning of the run the altitude difference is given by the expression $R_0 \sin(\sigma_0 + \theta_0)$. Results are presented in figure 9 for $\sigma_0 = 2^\circ$, $7.5^\circ$, $10^\circ$, $15^\circ$, and $20^\circ$. As $\sigma_0$ is increased, the time which elapses between reaching the correct flight path ($M_{SLA} = 0$) and the firing point becomes less and less, until for $\sigma_0 = 20^\circ$ the interceptor is unable to reduce $M_{SLA}$ to zero. The maximum allowable initial value of $\sigma_0$ is a function of the range and the maximum acceleration that the airplane can pull. In the present system for $\delta_{\gamma,t} = -20^\circ$, $\sigma_0$ is approximately $5g$.

Presented in figure 10 is a plot of the initial range $h_i$ against the maximum initial $\sigma_0$ for which the present system could score a hit. This curve was obtained from a simultaneous solution of the equations

$$V_f \int_0^{t_d} \cos \gamma(t) dt + (V_f + V_r) \tau \cos |\gamma(t_d)|$$

$$- V_r \cos \gamma(t_d + \tau) - R_0 \cos (\sigma_0 + \theta_0)$$

$$V_f \int_0^{t_d} \sin \gamma(t) dt + (V_f + V_r) \tau \sin |\gamma(t_d)|$$

$$- V_r \sin \gamma(t_d + \tau) - R_0 \sin (\sigma_0 + \theta_0)$$

for the case where $\gamma_0 = \pi$. Equations (8) relate the horizontal and vertical distances traveled by the interceptor, rockets, and target to the horizontal and vertical distances which exist between the interceptor and target at $t=0$. If these equations were satisfied, a hit would be obtained.
The function $\gamma(t)$ was calculated from the longitudinal equations of motion for a step input on $\delta_e = -20^\circ$, which is taken as the maximum value of $\delta_e$ throughout these calculations. Also presented in Figure 10 is the variation of $R_e$ with $\sigma_0$ which was calculated on the assumption that $\gamma(t) = \gamma(0) + \dot{\gamma}t$. The value of $\dot{\gamma}$ used in this expression is the steady-state $\dot{\gamma}$ due to an elevator deflection of $-20^\circ$, and it was assumed that the airframe attained this output immediately upon
The results obtained for this simplified approach to the problem are seen to be substantially the same as those obtained when the more exact approach is used. The curve indicates that for $R_0 = 60,000$ feet the largest $\sigma_0$ which can be used is approximately $22^\circ$. As a check, runs were made for a large value of $K$ (in order to keep $\delta_s = \delta_{s0}$ for the entire run) and the results indicated that $\sigma_0 = 20^\circ$ was about the largest value which could be tolerated.

In order to obtain a hit for $R_0 = 60,000$ feet and $\sigma_0 = 22^\circ$, the required time of flight of the interceptor from its initial position to the assumed firing point is seen from figure 10 to be 16.8 seconds. This explains the difference between the maximum allowable $\sigma_0$ indicated by figure 10 and the value obtained from the REAC results. For the REAC runs, the simplifications made in the guidance equations eliminated the dependence of $(t_0 + \tau)$ on $\sigma_0$; and for the initial range $R_0 = 60,000$ feet, $t_0$ is equal to 14.8 seconds. Actually, $t_0$ for the REAC runs was slightly less than 14.8 seconds, since $R_0$ was closer to 59,000 feet due to voltage limitations on the REAC.

A check was also made for $R_0 = 30,000$ feet and the result also agreed well with the curve of figure 10.

**Effect of filter time constant $\tau_f$.**—The effect of $\tau_f$ is illustrated in figures 11(a) and 11(b) for $\tau_f = 0$, 3.3, 0.6, and 1.2 seconds. For these runs $K = 3.0$, $K_r = 0.375$, $\sigma_0 = 2^\circ$ and 7.5°, and $R_0 = 60,000$ feet. As $\tau_f$ is increased from 0 to 1.20 seconds, the initial response in $M_{NLS}$ is seen to become progressively slower, which is due to the increased lag between the initial command.
Effect of initial values of radar elevation angle \( \sigma \) on interceptor attack performance. \( K = 3.0; K_r = 0.375; R_0 = 60,000 \) feet.

The values of \( \sigma_0 \) chosen to illustrate the effect of \( \tau_f \) have no special significance, but were used only on the basis that the results obtained for these values of \( \sigma_0 \) were typical of the results obtained for all values of \( \sigma_0 \) up to the maximum allowable value for \( R_0 = 60,000 \) feet. The same statement may be made concerning the values of \( \sigma_0 \) used in the subsequent sections.

Effect of servo time constant \( \tau_s \). The effect of the servo time constant \( \tau_s \) is demonstrated in figure 12 for \( \tau_s = 0.03, 0.10, 0.20, \) and 0.30 second. The cases presented are for \( K = 3.0, K_r = 0.375, \sigma_0 = 2^\circ, \) and \( R_0 = 60,000 \) feet. As \( \tau_s \) is increased from 0.03 second to 0.30 second, the most important effect is that the overshoot in the \( M_{NLS} \) response is seen to become larger, but at the assumed firing point the miss distance is zero in either case. The effects of \( \tau_s \) on the \( M_{NLS} \) response are summarized in figure 12(b).

Effect of control-surface limitations. The effects of limitations on the rate of control-surface deflection and the magnitude of the surface deflection were investigated briefly for \( K = 3.0, K_r = 0.375, \) and \( R_0 = 60,000 \) feet. All of the results presented in this paper up to this point were obtained for the condition where \( \delta_0 \) was limited to \( \pm 120^\circ/\text{sec} \) and \( \delta_{\infty} \) to \( \pm 20^\circ \). The effect of reducing the maximum value of \( \delta_0 \) to \( \pm 60^\circ/\text{sec} \) and \( \pm 30^\circ/\text{sec} \) for this tracking run may be seen from figure 13(a). For these runs \( \sigma_0 = 10^\circ \). The limiting control deflection was kept at \( \pm 20^\circ \). The effect on this run of reducing \( \delta_{\infty} \) was to reduce slightly the peak normal acceleration as \( \delta_{\infty} \) is reduced from \( \pm 120^\circ/\text{sec} \) to \( \pm 30^\circ/\text{sec} \). This reduction in acceleration causes a slight increase in the rise and response time of \( M_{NLS} \), but it may be concluded that, at least for this case, the effects of limitations on the rate of control deflection were small. However, for cases where higher values of \( K \) would be required (for example, a maneuvering target) the effects of rate limitation would probably be much more important and should be investigated thoroughly. The effects of limiting \( \delta_{\infty} \) on the \( M_{NLS} \) responses are summarized in figure 13(b).

Several runs were made for the case of \( \delta_0 = 120^\circ/\text{sec} \) and \( \delta_{\infty} \) reduced from \( \pm 20^\circ \) to \( \pm 10^\circ \); the results are presented in figure 14. For these cases \( \sigma_0 \) is equal to \( 5^\circ \). The general effect of reducing the control deflection limits from \( 20^\circ \) to \( 10^\circ \) is to reduce the peak normal acceleration of the interceptor, and hence the rise and response time of the \( M_{NLS} \) response.
The maximum $\sigma_0$ for which $M_{N\text{LM}}$ can be reduced to zero is directly dependent on the limit imposed on $\delta$.

**Effect on tracking of limiting the command $\theta_i$.**

Since the result obtained from filtering and amplifying the error signal $e_i$ is used as the command to a pitch-rate command system, this command should be limited if it is desired to limit the interceptor's normal acceleration from aerodynamic, pilot comfort, structural, or other considerations. The normal acceleration response to $\delta_i$ is given by

$$n \left( \frac{\dot{\theta}}{\dot{\theta}} \right)_i \left( \frac{\theta}{\dot{\theta}} \right)_i = \frac{K_i}{(\delta \dot{\theta})_i + K_i, K_i = \frac{1}{4.85 + K_i}}$$

Hence the steady-state normal acceleration can be limited to any desired value by limiting the input command $\theta_i$. For the interceptor being considered

$$\left( \frac{\dot{\theta}}{\dot{\theta}} \right)_s = \frac{K_i}{(\delta \dot{\theta})_s + K_i, K_i = \frac{1}{4.85 + K_i}}$$

and, in general,

$$\left( \frac{\dot{\theta}}{\dot{\theta}} \right)_s = \frac{V_i}{g}$$

Therefore,

$$\left( \frac{\dot{\theta}}{\dot{\theta}} \right)_s = \left( \frac{\dot{\theta}}{\dot{\theta}} \right)_s \left( \frac{\theta}{\dot{\theta}} \right)_s$$

If $\theta_i$ is limited by this expression, it should be pointed out that only the steady-state $n$ is being limited by use of this expression. The effectiveness of limiting the output transient acceleration by this method depends primarily on the response characteristics of $(\theta_i \dot{\theta})$. If the system gains are chosen to give a fast response of $n$ to $\theta_i$ with little or no overshoot, this means of limiting $n$ should be satisfactory. The results presented in figure 15 afford a comparison of the cases for which there is no $g$-limiter and for the case where $g$ is limited by equation (10). For these cases $K = 3.0$, $K_i = 0.375$, $R_0 = 60,000$ feet, $\sigma_0 = 15^\circ$ and $(\theta_i \dot{\theta}) = 5g$. The unframed steady-state normal-acceleration
(e) Summary plots.
Figure 11.—Concluded.
response to the limiting value of $\delta$ is approximately $4.9g$; hence, the unlimited $g$ case and the case of $(n)_{L} = 5g$ should oscillate about the same value when $\delta = (\delta)_{L}$. As can be seen, the peak $n$ for the unlimited case is roughly $8g$. When $(\theta)_{L}$ is limited by equation (10), the peak $g$ response is reduced roughly to $6.5g$. The rise time of $M_{NL}$ is seen to increase slightly when $\theta_{L}$ is limited, but the response time is less than that for the unlimited case.

**Digital check on validity of simplified guidance equations.** In order to check the validity of the small-angle assumptions made in the guidance equations (see previous section entitled "Derivation of Guidance Equations"), a solution, using the exact guidance of kinematic equations, was obtained from the Bell Computer for comparison with the REAC$^3$ tracking solutions which utilized the simplified equations. This comparison is presented in figure 16 for the case of $\sigma_{n} = 7.5g$, $R_{e} = 60,000$ feet, $K = 3.0$, and $K_{v} = 0.375$, and the agreement is seen to be excellent.

---

**Figure 12.**—Effect of servo time constant $\tau_{s}$ on interceptor attack performance. $K = 3.0$; $K_{v} = 0.375$; $R_{e} = 60,000$ feet.

---

**Figure 12.**—Concluded.
CONCLUSIONS

The following conclusions were reached from a theoretical investigation of the longitudinal tracking behavior of an automatically controlled interceptor against a nonmaneuvering target:

1. The control system considered in this investigation (i.e., command on rate of pitch proportional to longitudinal tracking error) was found to give acceptable control of the interceptor's flight path during attack runs against a nonmaneuvering target.

2. The inclusion of a nonlinear variation of drag and lift with angle of attack and Mach number resulted in relatively large variations in the interceptor forward velocity during the attack runs.

3. The changes in forward velocity computed for the runs in this investigation, although rather large, had a relatively small effect on the overall responses when a signal proportional to the change in forward velocity was fed back to the elevator servo.

4. Consideration of a nonlinear variation of pitching moment with angle of attack which tended toward static instability at high angles of attack indicated that its primary effect was to increase the magnitudes of the interceptor's motions during the tracking runs.

5. The general effect of increasing the tracking gain $K$ was to destabilize the tracking loop.

6. Increases in the rate-feedback gain $K_r$ tended to stabilize the interceptor's longitudinal short-period oscillation, but had a destabilizing effect on the tracking loop.

7. The maximum initial angularity between the interceptor's flight path and radar line of sight for
which a hit can be obtained can be well approximated from the initial range and normal acceleration capabilities of the interceptor.

8. Increases in either the filter time constant $\tau_f$ or the servo time constant $\tau_s$ had an adverse effect on the attack performance because of the increased lag between the input command and the elevator motions.

9. The effects of limitations on the rate of control deflection did not appear to be large in this investigation for the limiting rates considered.

10. The general effect of limitations on the magnitude of the control deflection was to slow down the interceptor's responses, and hence the maximum initial tracking error which can be tolerated decreases as the limits on elevator deflection are reduced.

11. Limiting of the interceptor normal acceleration was achieved by limiting the input command on pitching velocity, which effectively limits the interceptor's steady-state normal acceleration.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Field, Va., October 22, 1954.
All equations used in this investigation are presented in this appendix. The equations are presented in both general form and with numerical substitutions for the parameters in table 1. It will be noted that certain of the equations presented in numerical form have been multiplied by constants. These constants were used in order to adjust the REAC voltages to satisfactory levels in the analog procedure.

Linearized airframe equations \((\gamma_{L,0}=0)\):

\[
\begin{align*}
\ddot{\phi} &= -\tau_{v_{L}} \frac{qS}{I_{Y}} - \tau_{v_{\alpha}} \frac{qS}{I_{Y}} \phi + \tau_{v_{\alpha}} \frac{qS}{I_{Y}} \phi \\
\dot{\alpha} &= -\tau_{L_{\alpha}} \frac{qS}{mV_{L,0}} \alpha - 2\tau_{L_{\alpha}} \frac{qS}{mV_{L,0}} \dot{\theta} \\
\dot{u} &= -2\tau_{\alpha} \frac{qS}{mV_{L,0}} u - \tau_{\alpha} \frac{qS}{mV_{L,0}} \dot{\theta}
\end{align*}
\]

It will be noted that \(C_{m_{\alpha}}, C_{L_{\alpha}},\) and \(\delta_{e}\) appear in these equations rather than \(C_{m_{\alpha}}, C_{L_{\alpha}},\) and \(\delta_{e}\). The relations exist that

\[
\begin{align*}
\delta_{e}' &= -\delta_{e} \\
C_{m_{\alpha}} &= -C_{m_{\alpha}} \\
C_{L_{\alpha}} &= -C_{L_{\alpha}}
\end{align*}
\]

This convention is adopted in order that the true physical phase relationships of the pitch-rate command loop be obtained for positive gain constants.

Control equations:

\[
\begin{align*}
\dot{\epsilon}_{f} &= \frac{\epsilon_{f}}{\tau_{f}} \\
\dot{\theta} &= K_{\epsilon} \frac{\theta}{\tau_{\theta}} \\
\dot{\delta}_{e}' &= K_{\delta} \left( \frac{\delta_{e}}{\tau_{\delta}} - \frac{\delta_{e}'}{\tau_{\delta}} \right)
\end{align*}
\]

Kinematic equations (simplified, \(\gamma_{r} = \pi\)):

\[
\begin{align*}
t_{o} + \tau &= -V_{\alpha} \frac{\theta}{R} \\
\dot{R} &= -V_{r} - V_{r} \\
R \Omega &= V_{r}(\sigma + \alpha) + V_{r}(\sigma + \theta) \\
M_{NLS} &= -R \Omega \left( t_{o} + \tau \right) - V_{\alpha} \left( \sigma + \alpha \right) \\
\epsilon_{s} &= -V_{r}(\sigma + \tau) + V_{\alpha} \\
V_{r} &= V_{l,0} + V_{l,0} u \\
R &= R_{0} + \int \dot{R} dt \\
\alpha &= \alpha_{0} + \int \dot{\alpha} dt \\
\dot{\theta} &= \dot{\theta}_{o} + \int \dot{\theta} dt \\
\sigma &= \sigma_{o} + \int (\Omega - \theta) dt \\
\Omega &= \frac{\Omega}{R}
\end{align*}
\]

For the parameters presented in table 1, equations (A1), (A2), and (A3) take the following form:

Airframe equations:

\[
\begin{align*}
\ddot{\phi} &= -0.1845 \dot{\phi} - 0.01819 \alpha - 10.382 \alpha + 5.4695 \delta_{e}' \\
\ddot{\alpha} &= -0.4565 \alpha - 0.0303 \dot{u}' + 0.0329 \delta_{e}'
\end{align*}
\]
Control equations:
\[-20\dot{u}' - 0.215 u' + 0.303\dot{\theta} + 0.579\alpha - 0.303\alpha\]

Kinematic equations:
\[\dot{l}_o + \tau = \frac{3000 - R}{R}\]
\[12\dot{R} = -16,320 - 12V_1\]
\[20R\omega = 42,800 (\sigma + \alpha) + 27,200 (\sigma + \theta)\]
\[-4M_{yls} = 4R\omega (t_o + \tau) + 12,000 (\sigma + \alpha)\]
\[-2\dot{\epsilon}_1 = \frac{-2M_{yls}}{V_1 (t_o + \tau) - 3000}\]
\[\dot{\epsilon}_1'' = 42,800\omega - 42,800u'\]
\[-R' = 60,000 - \int \dot{R} dt\]
\[-2\alpha = 0.036 - 2\int \dot{\alpha} dt\]

\[-\dot{\theta} = -0.033 - \int \dot{\theta} dt\]
\[\sigma = \sigma_o + \int (\Omega - \dot{\theta}) dt\]
\[-10\Omega = -10R\Omega\]

The \(\alpha\) and \(\theta\) appearing in equations (A3) are total \(\alpha\) and \(\theta\), whereas the \(\alpha\) and \(\theta\) in equations (A1) are perturbations away from the trimmed condition. The solution was slowed down to the extent that 2 seconds of machine time was equivalent to 1 second of problem time. The scale factors used were: 100 volts = 100 seconds, 100 volts = 1 radian, and 100 volts = 60,000 feet.

The analog schematics of equations (A1), (A2), and (A3) are presented in figures 17(a), 17(b), and 17(c), respectively.

REFERENCE

(b) Control equations (A2). (See fig. 17(a) for key.)

Figure 17.—Continued.
(e) Kinematic equations (A3). (See fig. 17(a) for key.)
Figure 17. Concluded.