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NASA TN D-930



TECHNICAL NOTE

D-930

A LINEAR THEORY FOR INFLATABLE PLATES OF ARBITRARY SHAPE

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON

October 1961



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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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SUMMARY

A linear small-deflection theory is developed for the elastic behavior of inflatable plates of which Airmat is an example. Included in the theory are the effects of a small linear taper in the depth of the plate. Solutions are presented for some simple problems in the lateral deflection and vibration of constant-depth rectangular inflatable plates.

INTRODUCTION

For certain types of satellite and reentry vehicles it is desirable to have a structure which can be packaged in a compact form for launching and erected after injection into orbit or at the time of reentry. One possible way of meeting this requirement is to utilize an inflatable structure. Inflatable structures also have a variety of other applications where it is desirable to have a small package for transporting and simple erection capability at the destination. These applications range from inflatable airplanes to erectable living quarters. Where platelike structural components are needed in an inflatable structure, for example, for the lifting surfaces, fins, and control surfaces of a reentry glider, an inflatable plate such as Airmat (developed by Goodyear Aircraft Corporation) appears to be an efficient and useful scheme. This type of plate is illustrated schematically in figure 1. It consists of two woven covers having airtight coatings and held some distance apart by the combined action of drop cords and internal pressure. The drop cords are closely spaced, distributed throughout the plate, and may be of varying length to form a plate of variable thickness. In a reentry-glider application, for instance, metal wire would most likely be used to weave the covers and form the drop cords.

In order to make rational stress, deflection, and aeroelastic analyses of inflatable-plate structures, it is necessary to have a theory from which stresses and deflections in such plates can be calculated for various external loading conditions. In this report a linear theory for inflatable plates is derived, and some solutions to elementary static deflection and vibration problems are presented.

SYMBOLS

$A_{11}, A_{12}, A_{21}, A_{22}, A_{33}$	orthotropic elastic constants defined in equations (13)	
a, b	length of rectangular plate in x- and y-directions, respectively	
c	root chord length of triangular plate	
$E_w t$	extensional stiffness in warp direction of cover	L 9
$E_f t$	extensional stiffness in fill direction of cover	1 9
e	unit deformation through thickness of plate	
Gt	shear stiffness of cover	
h	depth of plate	
l	semispan of triangular plate	
M_x, M_y, M_{xy}	plate moment resultants	
$\bar{M}_x, \bar{M}_y, \bar{M}_{xy}$	external moments applied to edges of rectangular plate, positive when causing tension or positive shear in upper cover	
$\bar{M}_{N_x}, \bar{M}_{N_y}$	vector components of moments applied to edge of plate, positive in the positive x- and y-directions, respectively	
m, n	number of half waves in x- and y-directions, respectively	
N_x, N_y, N_{xy}	stress resultants associated with xyz coordinate system and dependent on displacements	
N_x^o, N_y^o, N_{xy}^o	stress resultants associated with xyz coordinate system and independent of displacements	
$N_x^* = N_x^o + N_x$		
$N_y^* = N_y^o + N_y$		
$N_{xy}^* = N_{xy}^o + N_{xy}$		

N'_x, N'_y, N'_{xy}	stresses in covers of inflatable plate associated with $x'y'z'$ coordinate systems
$\bar{N}_{Nx}, \bar{N}_{Ny}$	vector components of middle plane forces applied to edge of plate, positive in the positive x - and y -directions, respectively
p	internal pressure
q_x, q_y, q_z	x -, y -, and z -components, respectively, of external distributed loading per unit middle-plane area
S_x, S_y	transverse shear carried in covers due to taper (see eqs. (22))
s	coordinate along edge of plate (see fig. 1(a))
t	thickness of each cover
u, v, w	displacements in x -, y -, and z -directions, respectively
u', v', w'	displacements in x' -, y' -, and z' -directions, respectively
ΔV	change in volume of inflatable plate
\bar{V}_x, \bar{V}_y	external lateral loads applied to edges of rectangular plate
\bar{V}_N	external lateral load applied to edge of plate of arbitrary shape, positive in the positive z -direction
W	work of external loads and body forces
x, y, z	rectangular Cartesian coordinate system
x', y', z'	coordinate systems associated with covers
α, β	angles of rotation of the drop cords in the xz - and yz -planes, respectively, from their initial position normal to the xy -plane
$\epsilon'_x, \epsilon'_y, \gamma'_{xy}$	strains in covers of inflatable plate
δ	variational operator
ξ, η, ζ	rectangular Cartesian coordinates locating the final positions of points on the surfaces of the inflatable plate

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λ	length-width ratio of rectangular plate, b/a	
μ_{WF}	Poisson's ratio associated with a contraction in the fill direction caused by a tensile stress in the warp direction	
μ_{FW}	Poisson's ratio associated with a contraction in the warp direction caused by a tensile stress in the fill direction	
Π	total potential energy	
Π_C	strain energy in covers of inflatable plate	L 9
Π_I	work done against internal pressure	1 9
ρ	mass of plate per unit middle-plane area	
ω	circular frequency of vibration	

Subscripts:

F,W	pertain to fill and warp directions, respectively
m,n	integers
+, -	pertain to upper and lower covers, respectively

A comma followed by a subscript denotes partial differentiation with respect to the subscript. A dot over a symbol denotes partial differentiation with respect to time.

BASIC ASSUMPTIONS AND COORDINATE SYSTEMS

The inflatable plate shown in figure 1(a) is of arbitrary shape in planform and may have a small linear taper in depth in the x- and y-directions. The taper is assumed to be symmetric about the middle surface. The covers of the plate are assumed to be identical and are treated as orthotropic membranes. The closely spaced drop cords are assumed to be straight and inextensional, and they are conceptually spread continuously over the plate. The drop cords are assumed to be normal to the middle surface before deformation, but they are assumed to be hinged at the ends so that during deformation the angles between the drop cords and the middle surface or the covers may change. The internal pressure in the plate is assumed to be constant during deformation. Sidewalls are presumed to be present at the edge of the plate to contain the pressure, but their effects on plate behavior are not otherwise taken into account.

In this investigation the undeformed shape of the plate is assumed to be the shape existing after inflation and after inplane edge loads are applied. All displacement quantities in the derivation are to be interpreted as being measured from this undeformed state.

A rectangular xyz coordinate system is chosen so that the middle surface of the plate lies in the xy-plane as shown in figure 1(a). In addition to the basic xyz coordinate system, two other coordinate systems are used in the analysis. These additional systems are designated x'_+, y'_+, z'_+ and x'_-, y'_-, z'_- . The (+) subscript refers to the upper cover of the plate or the cover which lies on the positive-z side of the middle surface and the (-) subscript refers to the lower cover. This convention is used throughout the report.

The primed coordinate systems are obtained from the xyz system by small rotations equal to the taper angles of the covers. Thus, the $x'_+y'_+$ -plane is parallel to the upper cover and the $x'_-y'_-$ -plane is parallel to the lower cover. The coordinate transformations defining the primed systems are

$$\left. \begin{aligned} x &= x'_+ - z'_+ \frac{h_{,x}}{2} & y &= y'_+ - z'_+ \frac{h_{,y}}{2} & z &= z'_+ + x'_+ \frac{h_{,x}}{2} + y'_+ \frac{h_{,y}}{2} \\ x &= x'_- + z'_- \frac{h_{,x}}{2} & y &= y'_- + z'_- \frac{h_{,y}}{2} & z &= z'_- - x'_- \frac{h_{,x}}{2} - y'_- \frac{h_{,y}}{2} \end{aligned} \right\} (1)$$

where the quantities $h_{,x}/2$ and $h_{,y}/2$ represent the taper angles as shown in figure 1. These quantities are assumed to be small (i.e., $h_{,x}^2 \ll 1$, $h_{,y}^2 \ll 1$, and $h_{,x}h_{,y} \ll 1$) and constant throughout the plate.

DERIVATION OF DIFFERENTIAL EQUATIONS OF EQUILIBRIUM AND BOUNDARY CONDITIONS

Derivation of Stress-Displacement Relations

It is desirable to derive the theory in terms of stress and displacement quantities which are associated with the overall plate configuration rather than local quantities associated with the individual covers because these overall quantities can be identified with familiar quantities in plate theory. The relationships between the local and overall quantities are discussed in this section. In addition, there

are derived relations between the overall plate stress resultants and the overall plate displacements.

Strain-displacement relations.- Displacements in the individual covers of the inflatable plate in the primed coordinate systems are denoted by u_{\pm}' , v_{\pm}' , and w_{\pm}' . These quantities are related to u_{\pm} , v_{\pm} , and w_{\pm} , the displacements in the xyz coordinate system, by the transformations

$$\left. \begin{aligned} u_{+}' &= u_{+} + w_{+} \frac{h,x}{2} & u_{-}' &= u_{-} - w_{-} \frac{h,x}{2} \\ v_{+}' &= v_{+} + w_{+} \frac{h,y}{2} & v_{-}' &= v_{-} - w_{-} \frac{h,y}{2} \\ w_{+}' &= w_{+} - u_{+} \frac{h,x}{2} - v_{+} \frac{h,y}{2} & w_{-}' &= w_{-} + u_{-} \frac{h,x}{2} + v_{-} \frac{h,y}{2} \end{aligned} \right\} (2)$$

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The strains in the individual covers are defined in the primed coordinate systems in the usual manner:

$$\left. \begin{aligned} \epsilon_{x\pm}' &= \frac{\partial u_{\pm}'}{\partial x_{\pm}'} + \frac{1}{2} \left(\frac{\partial w_{\pm}'}{\partial x_{\pm}'} \right)^2 \\ \epsilon_{y\pm}' &= \frac{\partial v_{\pm}'}{\partial y_{\pm}'} + \frac{1}{2} \left(\frac{\partial w_{\pm}'}{\partial y_{\pm}'} \right)^2 \\ \gamma_{xy\pm}' &= \frac{\partial u_{\pm}'}{\partial y_{\pm}'} + \frac{\partial v_{\pm}'}{\partial x_{\pm}'} + \frac{\partial v_{\pm}'}{\partial x_{\pm}'} \frac{\partial w_{\pm}'}{\partial y_{\pm}'} \end{aligned} \right\} (3)$$

The following plate displacement quantities are now introduced:

$$\left. \begin{aligned} u &= \frac{u_{+} + u_{-}}{2} & v &= \frac{v_{+} + v_{-}}{2} & w &= \frac{w_{+} + w_{-}}{2} \\ \alpha &= \frac{u_{+} - u_{-}}{h} & \beta &= \frac{v_{+} - v_{-}}{h} & e &= \frac{w_{+} - w_{-}}{h} \end{aligned} \right\} (4)$$

The quantities u , v , and w are simply the averages of the displacements in the upper and lower covers in the x-, y-, and z-directions, respectively. For small displacements, the quantities α and β are

the angles between the drop cords and the z-axis in the xz- and yz-planes, respectively, or, in other words, α and β are the rotations of the drop cords during deformation. These displacements and rotations are shown in figure 2 in their positive senses. The quantity e represents the unit deformation or "strain" through the thickness of the plate. Since the drop cords are assumed inextensional, e is not an independent quantity for small deformations but is related to α and β by the equation

$$e = -\frac{\alpha^2}{2} - \frac{\beta^2}{2} \quad (5)$$

With the use of equations (1) to (4) the following sum-and-difference quantities necessary to the subsequent development may be calculated:

$$\begin{array}{ll} \epsilon'_{x+} + \epsilon'_{x-} & \epsilon'_{x+} - \epsilon'_{x-} \\ \epsilon'_{y+} + \epsilon'_{y-} & \epsilon'_{y+} - \epsilon'_{y-} \\ \gamma'_{xy+} + \gamma'_{xy-} & \gamma'_{xy+} - \gamma'_{xy-} \end{array}$$

In order to illustrate these calculations, consider the quantity

$$\epsilon'_{x+} + \epsilon'_{x-} = \frac{\partial u'_+}{\partial x'_+} + \frac{1}{2} \left(\frac{\partial w'_+}{\partial x'_+} \right)^2 + \frac{\partial u'_-}{\partial x'_-} + \frac{1}{2} \left(\frac{\partial w'_-}{\partial x'_-} \right)^2 \quad (6)$$

obtained from the first two of equations (3). From equations (1) and (2)

$$\begin{aligned} \frac{\partial u'_+}{\partial x'_+} &= \frac{\partial}{\partial x} \left[u_+(x,y) + \frac{h_{,x}}{2} w_+(x,y) \right] \frac{\partial x}{\partial x'_+} + \frac{\partial}{\partial y} \left[u_+(x,y) + \frac{h_{,x}}{2} w_+(x,y) \right] \frac{\partial y}{\partial x'_+} \\ &= \left(u_+ + \frac{h_{,x}}{2} w_+ \right)_{,x} \end{aligned} \quad (7)$$

Similarly,

$$\frac{\partial w'_+}{\partial x'_+} = \left(w_+ - \frac{h_{,x}}{2} u_+ - \frac{h_{,y}}{2} v_+ \right)_{,x} \quad (8)$$

and so forth. When equations (7) and (8) and their counterparts for the derivatives of u'_- and w'_- are used and when it is remembered that $h_{,x}$ and $h_{,y}$ are constants, equation (6) becomes

$$\begin{aligned}
\epsilon'_{x+} + \epsilon'_{x-} = & u_{+,x} + \frac{h_{,x}}{2} w_{+,x} + u_{-,x} - \frac{h_{,x}}{2} w_{-,x} + \frac{1}{2} \left(w_{+,x}^2 + \frac{h_{,x}^2}{4} u_{+,x}^2 \right. \\
& + \frac{h_{,y}^2}{4} v_{+,x}^2 - h_{,x} w_{+,x} u_{+,x} - h_{,y} w_{+,x} v_{+,x} + \frac{h_{,x} h_{,y}}{2} u_{+,x} v_{+,x} \\
& + w_{-,x}^2 + \frac{h_{,x}^2}{4} u_{-,x}^2 + \frac{h_{,y}^2}{4} v_{-,x}^2 + h_{,x} w_{-,x} u_{-,x} \\
& \left. + h_{,y} w_{-,x} v_{-,x} + \frac{h_{,x} h_{,y}}{2} u_{-,x} v_{-,x} \right) \quad (9)
\end{aligned}$$

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The use of equations (4) then leads to

$$\begin{aligned}
\epsilon'_{x+} + \epsilon'_{x-} = & 2u_{,x} + w_{,x}^2 + \frac{h_{,x}}{2} (h_{e,x} + h_{,x} e) + \frac{1}{4} (h_{e,x} + h_{,x} e)^2 + \frac{h_{,x}^2}{4} u_{,x}^2 \\
& + \frac{h_{,y}^2}{4} v_{,x}^2 + \frac{h_{,x}^2}{16} (h_{\alpha,x} + h_{,x} \alpha)^2 + \frac{h_{,y}^2}{16} (h_{\beta,x} + h_{,x} \beta)^2 \\
& - \frac{h_{,x}}{2} w_{,x} (h_{,x} \alpha + h_{\alpha,x}) - \frac{h_{,x}}{2} (h_{e,x} + h_{,x} e) u_{,x} \\
& - \frac{h_{,y}}{2} w_{,x} (h_{,x} \beta + h_{\beta,x}) - \frac{h_{,y}}{2} (h_{e,x} + h_{,x} e) v_{,x} \\
& + \frac{h_{,x} h_{,y}}{2} u_{,x} v_{,x} + \frac{h_{,x} h_{,y}}{8} (h_{,x} \alpha + h_{\alpha,x}) (h_{,x} \beta + h_{\beta,x}) \quad (10)
\end{aligned}$$

After substitution of equation (5) certain terms can be neglected in equation (10). In the first place, terms which contain u , v , w , α , and β and/or their derivatives to higher than second degree are assumed to be small. Such terms ultimately lead to nonlinear equilibrium equations and are neglected in this theory. Second-degree terms in equation (10) lead to linear terms in the equilibrium equations and are retained here. Secondly, terms containing $h_{,x}^2$, $h_{,y}^2$, or $h_{,x} h_{,y}$ are neglected because the taper is assumed to be small.

When the approach just described is extended to the other sum-and-difference quantities the results are

$$\begin{aligned}
 \epsilon'_{x+} + \epsilon'_{x-} &= 2u_{,x} + w_{,x}^2 - \frac{h}{2} [h_{,x}\alpha_{,x}(\alpha + w_{,x}) + h_{,x}\beta_{,x}\beta + h_{,y}\beta_{,x}w_{,x}] \\
 \epsilon'_{y+} + \epsilon'_{y-} &= 2v_{,y} + w_{,y}^2 - \frac{h}{2} [h_{,y}\beta_{,y}(\beta + w_{,y}) + h_{,y}\alpha_{,y}\alpha + h_{,x}\alpha_{,y}w_{,y}] \\
 \gamma'_{xy+} + \gamma'_{xy-} &= 2u_{,y} + 2v_{,x} + 2w_{,x}w_{,y} - \frac{h}{2} [h_{,x}\alpha_{,y}(\alpha + w_{,x}) \\
 &\quad + h_{,y}\beta_{,x}(\beta + w_{,y}) + h_{,x}\beta_{,y}\beta + h_{,x}\alpha_{,x}w_{,y} \\
 &\quad + h_{,y}\alpha_{,x}\alpha + h_{,y}\beta_{,y}w_{,x}] \\
 \epsilon'_{x+} - \epsilon'_{x-} &= h\alpha_{,x} + h_{,x}(\alpha + w_{,x}) - h_{,x}w_{,x}u_{,x} - h_{,y}w_{,x}v_{,x} \\
 \epsilon'_{y+} - \epsilon'_{y-} &= h\beta_{,y} + h_{,y}(\beta + w_{,y}) - h_{,x}w_{,y}u_{,y} - h_{,y}w_{,y}v_{,y} \\
 \gamma'_{xy+} - \gamma'_{xy-} &= h\alpha_{,y} + h\beta_{,x} + h_{,y}(\alpha + w_{,x}) + h_{,x}(\beta + w_{,y}) \\
 &\quad - h_{,x}(w_{,x}u_{,y} + w_{,y}u_{,x}) - h_{,y}(w_{,x}v_{,y} + w_{,y}v_{,x})
 \end{aligned}
 \tag{11}$$

Stress-strain relations.- The strains in the covers are related to stresses in the covers through the orthotropic stress-strain relations

$$\left. \begin{aligned}
 N'_{x\pm} &= A_{11}\epsilon'_{x\pm} + A_{12}\epsilon'_{y\pm} \\
 N'_{y\pm} &= A_{21}\epsilon'_{x\pm} + A_{22}\epsilon'_{y\pm} \\
 N'_{xy\pm} &= A_{33}\gamma'_{xy\pm}
 \end{aligned} \right\} \tag{12}$$

when the principal directions of the covers are aligned with x'_{\pm} and y'_{\pm} axes. Note that the N' quantities, the stresses in the covers, are expressed in terms of force per unit length rather than the conventional force per unit area. If the covers are simply woven fabric, for instance, with the warp and fill aligned with the x'_{\pm} and y'_{\pm} axes, respectively,

then the coefficients of the strains in equations (12) are related to the usual orthotropic elastic constants for the cover materials by

$$\left. \begin{aligned} A_{11} &= \frac{E_W t}{1 - \mu_{WF} \mu_{FW}} & A_{22} &= \frac{E_F t}{1 - \mu_{WF} \mu_{FW}} \\ A_{12} &= \frac{\mu_{FW} E_W t}{1 - \mu_{WF} \mu_{FW}} & A_{21} &= \frac{\mu_{WF} E_F t}{1 - \mu_{WF} \mu_{FW}} \\ A_{33} &= Gt \end{aligned} \right\} \quad (13)$$

The Poisson's ratios are related by $\mu_{FW} E_W t = \mu_{WF} E_F t$; therefore, $A_{12} = A_{21}$.

Stress-displacement relations.- It is appropriate, now, to introduce stress quantities $N_{x\pm}$, $N_{y\pm}$, and $N_{xy\pm}$ associated with the xyz coordinate system and to define the following overall plate stress and moment resultants:

$$\left. \begin{aligned} N_x &= N_{x+} + N_{x-} & M_x &= \frac{h}{2}(N_{x+} - N_{x-}) \\ N_y &= N_{y+} + N_{y-} & M_y &= \frac{h}{2}(N_{y+} - N_{y-}) \\ N_{xy} &= N_{xy+} + N_{xy-} & M_{xy} &= \frac{h}{2}(N_{xy+} - N_{xy-}) \end{aligned} \right\} \quad (14)$$

The assumption that the taper angles are small so that second-degree terms in h_x and h_y are negligible compared to unity justifies replacing $N_{x\pm}$ by $N'_{x\pm}$, $N_{y\pm}$ by $N'_{y\pm}$, and $N_{xy\pm}$ by $N'_{xy\pm}$. Thus, equations (14) become

$$\left. \begin{aligned} N'_{x+} + N'_{x-} &= N_x & N'_{x+} - N'_{x-} &= \frac{2M_x}{h} \\ N'_{y+} + N'_{y-} &= N_y & N'_{y+} - N'_{y-} &= \frac{2M_y}{h} \\ N'_{xy+} + N'_{xy-} &= N_{xy} & N'_{xy+} - N'_{xy-} &= \frac{2M_{xy}}{h} \end{aligned} \right\} \quad (15)$$

The sums of the stress resultants in the upper and lower covers become plate stress resultants and the differences are associated with plate moment and twist resultants. These plate quantities are shown in figure 2 in their positive senses.

When the three pairs of equations (12) are added and subtracted there results

$$\left. \begin{aligned}
 N'_{x+} + N'_{x-} &= A_{11}(\epsilon'_{x+} + \epsilon'_{x-}) + A_{12}(\epsilon'_{y+} + \epsilon'_{y-}) \\
 N'_{y+} + N'_{y-} &= A_{21}(\epsilon'_{x+} + \epsilon'_{x-}) + A_{22}(\epsilon'_{y+} + \epsilon'_{y-}) \\
 N'_{xy+} + N'_{xy-} &= A_{33}(\gamma'_{xy+} + \gamma'_{xy-}) \\
 N'_{x+} - N'_{x-} &= A_{11}(\epsilon'_{x+} - \epsilon'_{x-}) + A_{12}(\epsilon'_{y+} - \epsilon'_{y-}) \\
 N'_{y+} - N'_{y-} &= A_{21}(\epsilon'_{x+} - \epsilon'_{x-}) + A_{22}(\epsilon'_{y+} - \epsilon'_{y-}) \\
 N'_{xy+} - N'_{xy-} &= A_{33}(\gamma'_{xy+} - \gamma'_{xy-})
 \end{aligned} \right\} \quad (16)$$

Equations (11) and (15) are now substituted into equations (16) and all nonlinear terms are neglected to obtain the following linear stress displacement equations:

$$\left. \begin{aligned}
 N_x &= 2(A_{11}u_{,x} + A_{12}v_{,y}) \\
 N_y &= 2(A_{21}u_{,x} + A_{22}v_{,y}) \\
 N_{xy} &= 2A_{33}(u_{,y} + v_{,x}) \\
 M_x &= \frac{h^2}{2} \left\{ A_{11} \left[\alpha_{,x} + (\alpha + w_{,x}) \frac{h_{,x}}{h} \right] + A_{12} \left[\beta_{,y} + (\beta + w_{,y}) \frac{h_{,y}}{h} \right] \right\} \\
 M_y &= \frac{h^2}{2} \left\{ A_{21} \left[\alpha_{,x} + (\alpha + w_{,x}) \frac{h_{,x}}{h} \right] + A_{22} \left[\beta_{,y} + (\beta + w_{,y}) \frac{h_{,y}}{h} \right] \right\} \\
 M_{xy} &= \frac{h^2}{2} A_{33} \left[\alpha_{,y} + \beta_{,x} + (\alpha + w_{,x}) \frac{h_{,y}}{h} + (\beta + w_{,y}) \frac{h_{,x}}{h} \right]
 \end{aligned} \right\} \quad (17)$$

Use of the Principle of Minimum Potential Energy

The differential equations and boundary conditions are derived by using the principle of minimum potential energy. For the problem under consideration this principle may be stated as follows: When the plate is in equilibrium under external and body forces, the variation of the total potential energy with respect to variations in the five displacement quantities u , v , w , α , and β must be zero. The total potential energy is the sum of the strain energy in the covers and the potential energy of the internal pressurized gas minus the work done by the external and body forces. The principle can be written

$$\delta(\Pi_C + \Pi_I - W) = 0 \quad (18)$$

where the three quantities $\delta\Pi_C$, $\delta\Pi_I$, and δW are expressed in terms of plate stress resultants and plate displacements. The variational operations indicated by the symbol δ are performed in detail with respect to the displacements. When the coefficients of δu , δv , δw , $\delta\alpha$, and $\delta\beta$ are equated to zero there results a system of five linear partial differential equations and associated boundary conditions which govern the behavior of inflatable plates.

Strain energy in covers.- The variation of the strain energy in the covers corresponding to arbitrary variations δu , δv , δw , $\delta\alpha$, and $\delta\beta$ of the displacements may be written as follows for a slightly tapered plate of arbitrary shape:

$$\begin{aligned} \delta\Pi_C = & \iint \left[\left(N_{x+}^{\prime*} \delta\epsilon_{x+}^{\prime} + N_{y+}^{\prime*} \delta\epsilon_{y+}^{\prime} + N_{xy+}^{\prime*} \delta\gamma_{xy+}^{\prime} \right) \frac{\partial(x_+, y_+)}{\partial(x, y)} \right. \\ & \left. + \left(N_{x-}^{\prime*} \delta\epsilon_{x-}^{\prime} + N_{y-}^{\prime*} \delta\epsilon_{y-}^{\prime} + N_{xy-}^{\prime*} \delta\gamma_{xy-}^{\prime} \right) \frac{\partial(x_-, y_-)}{\partial(x, y)} \right] dx dy \quad (19) \end{aligned}$$

where

$$N_{x\pm}^{\prime*} = N_{x\pm}^{\prime 0} + N_{x\pm}^{\prime}$$

$$N_{y\pm}^{\prime*} = N_{y\pm}^{\prime 0} + N_{y\pm}^{\prime}$$

$$N_{xy\pm}^{\prime*} = N_{xy\pm}^{\prime 0} + N_{xy\pm}^{\prime}$$

or, since the Jacobians $\frac{\partial(x_{\pm}', y_{\pm}')}{\partial(x, y)}$ of the transformations (1) are both unity,

$$\begin{aligned} \delta\Pi_C = & \iint \left[\frac{1}{2}(N_{x+}'^* + N_{x-}'^*)\delta(\epsilon_{x+}' + \epsilon_{x-}') + \frac{1}{2}(N_{x+}'^* - N_{x-}'^*)\delta(\epsilon_{x+}' - \epsilon_{x-}') \right. \\ & + \frac{1}{2}(N_{y+}'^* + N_{y-}'^*)\delta(\epsilon_{y+}' + \epsilon_{y-}') + \frac{1}{2}(N_{y+}'^* - N_{y-}'^*)\delta(\epsilon_{y+}' - \epsilon_{y-}') \\ & \left. + \frac{1}{2}(N_{xy+}'^* + N_{xy-}'^*)\delta(\gamma_{xy+}' + \gamma_{xy-}') + \frac{1}{2}(N_{xy+}'^* - N_{xy-}'^*)\delta(\gamma_{xy+}' - \gamma_{xy-}') \right] dx dy \end{aligned} \quad (20)$$

The region of integration is the middle plane of the plate. Equation (20) can be expressed in terms of plate stress and displacement quantities by using equations (11) and the following equations analogous to equations (15):

$$\begin{aligned} N_{x+}'^* + N_{x-}'^* &= N_x^* & N_{x+}'^* - N_{x-}'^* &= \frac{2M_x}{h} \\ N_{y+}'^* + N_{y-}'^* &= N_y^* & N_{y+}'^* - N_{y-}'^* &= \frac{2M_y}{h} \\ N_{xy+}'^* + N_{xy-}'^* &= N_{xy}^* & N_{xy+}'^* - N_{xy-}'^* &= \frac{2M_{xy}}{h} \end{aligned}$$

When these substitutions are made, terms which consist of the product of inplane stress resultants (N_x^* , N_y^* , N_{xy}^*) and second-degree displacement quantities are retained because these stress resultants contain contributions which are independent of displacements (such as from internal pressure) and hence yield linear terms in the differential equations. Elsewhere, second-degree displacement quantities are neglected. With the use of equations (11) the expression for the variation of the strain energy in the covers can be written as follows with slight rearrangement:

$$\begin{aligned}
\delta\Pi_C = & \iint \left\{ N_x^* \delta \left[u_{,x} + \frac{1}{2} w_{,x}^2 - \frac{hh_{,x}}{4} \alpha_{,x} (\alpha + w_{,x}) - \frac{hh_{,x}}{4} \beta_{,x} \beta - \frac{hh_{,y}}{4} \beta_{,x} w_{,x} \right] \right. \\
& + N_y^* \delta \left[v_{,y} + \frac{1}{2} w_{,y}^2 - \frac{hh_{,y}}{4} \beta_{,y} (\beta + w_{,y}) - \frac{hh_{,y}}{4} \alpha_{,y} \alpha - \frac{hh_{,x}}{4} \alpha_{,y} w_{,y} \right] \\
& + N_{xy}^* \delta \left[u_{,y} + v_{,x} + w_{,x} w_{,y} - \frac{hh_{,x}}{4} \alpha_{,y} (\alpha + w_{,x}) - \frac{hh_{,y}}{4} \beta_{,x} (\beta + w_{,y}) \right. \\
& \left. - \frac{hh_{,x}}{4} \beta_{,y} \beta - \frac{hh_{,x}}{4} \alpha_{,x} w_{,y} - \frac{hh_{,y}}{4} \alpha_{,x} \alpha - \frac{hh_{,y}}{4} \beta_{,y} w_{,x} \right] + M_x \delta \alpha_{,x} \\
& \left. + M_y \delta \beta_{,y} + M_{xy} \delta (\alpha_{,y} + \beta_{,x}) + S_x \delta (\alpha + w_{,x}) + S_y \delta (\beta + w_{,y}) \right\} dx dy \quad (21)
\end{aligned}$$

where S_x and S_y are defined by

$$\left. \begin{aligned}
S_x &= \frac{1}{h} (h_{,x} M_x + h_{,y} M_{xy}) \\
S_y &= \frac{1}{h} (h_{,x} M_{xy} + h_{,y} M_y)
\end{aligned} \right\} \quad (22)$$

These quantities represent that portion of the transverse shear which is carried in the covers by virtue of the taper.

Potential energy of internal pressure.- The variation of the potential energy of the pressurized gas inside the inflatable plate is the product of the magnitude of the internal pressure and the change in volume due to virtual displacements δu , δv , δw , $\delta \alpha$, and $\delta \beta$. The increase in volume of the plate due to any set of small displacements u_{\pm} , v_{\pm} , and w_{\pm} in the covers and along the edge, denoted ΔV , is calculated from

$$\Delta V = - \iint_S \zeta(\xi, \eta) d\xi d\eta - \iint h dx dy \quad (23)$$

In the first integral ξ , η , and ζ are rectangular Cartesian coordinates locating the final positions of points on the surface of the plate and are functions of certain of the displacements u_{\pm} , v_{\pm} , and w_{\pm} .

The surface integration is performed over the entire outer surface of

the plate (over the upper and lower covers and along the edge) in such a manner that the surface is oriented with positive inward normal. (See ref. 1.) The second integral is performed over the middle surface of the plate at its initial position and represents the volume before deformation. The details of this calculation are presented in appendix A.

The variation of the potential energy of the internal gas can be written $\delta\Pi_I = -p\delta\Delta V$ and the results of the calculation in appendix A yield

$$\delta\Pi_I = -p\delta \left\{ \iint h \left[u_{,x} + v_{,y} + u_{,x}v_{,y} - u_{,y}v_{,x} + \frac{h^2}{12}(\alpha_{,x}\beta_{,y} - \alpha_{,y}\beta_{,x}) - \frac{\alpha^2}{2} - \frac{\beta^2}{2} - \alpha w_{,x} - \beta w_{,y} \right] dx dy \right\} \quad (24)$$

In this expression quantities which are higher than second degree in u , v , w , α , and β or their derivatives are neglected. The double integration is performed over the undeformed middle surface of the plate.

Work of external and body forces.- The plate may be loaded by distributed forces, inertia forces, and forces and moments acting on the boundaries. The variation of the work of these forces may be written as

$$\delta W = \iint \left(q_x \delta u + q_y \delta v + q_z \delta w - \rho \ddot{u} \delta u - \rho \ddot{v} \delta v - \rho \ddot{w} \delta w - \frac{h^2}{4} \rho \ddot{\alpha} \delta \alpha - \frac{h^2}{4} \rho \ddot{\beta} \delta \beta \right) dx dy + \oint \left(\bar{N}_{Nx} \delta u + \bar{N}_{Ny} \delta v + \bar{V}_N \delta w + \bar{M}_{Ny} \delta \alpha - \bar{M}_{Nx} \delta \beta \right) ds \quad (25)$$

As before, the double integral is performed over the middle surface of the undeformed plate. The line integral is around the boundary of the undeformed middle surface and positive in the clockwise sense looking in the positive z -direction. (See fig. 1(a).)

Equilibrium Equations and Boundary Conditions

The previously derived parts of the variation of the potential energy are now collected, and where required variational operations are performed in detail. Equation (18) thus becomes:

$$\begin{aligned}
\delta\Pi = \delta\Pi_C + \delta\Pi_I - \delta W = & \iint \left\{ N_x^* \left[\delta u_{,x} + w_{,x} \delta w_{,x} - \frac{hh_{,x}}{4} \alpha_{,x} (\delta\alpha + \delta w_{,x}) \right. \right. \\
& - \frac{hh_{,x}}{4} (\alpha + w_{,x}) \delta\alpha_{,x} - \frac{hh_{,x}}{4} \beta_{,x} \delta\beta - \frac{hh_{,x}}{4} \beta \delta\beta_{,x} - \frac{hh_{,y}}{4} \beta_{,x} \delta w_{,x} \\
& \left. \left. - \frac{hh_{,y}}{4} w_{,x} \delta\beta_{,x} \right] + N_y^* \left[\delta v_{,y} + w_{,y} \delta w_{,y} - \frac{hh_{,y}}{4} \beta_{,y} (\delta\beta + \delta w_{,y}) \right. \right. \\
& - \frac{hh_{,y}}{4} (\beta + w_{,y}) \delta\beta_{,y} - \frac{hh_{,y}}{4} \alpha_{,y} \delta\alpha - \frac{hh_{,y}}{4} \alpha \delta\alpha_{,y} - \frac{hh_{,x}}{4} \alpha_{,y} \delta w_{,y} \\
& \left. \left. - \frac{hh_{,x}}{4} w_{,y} \delta\alpha_{,y} \right] + N_{xy}^* \left[\delta u_{,y} + \delta v_{,x} + w_{,x} \delta w_{,y} + w_{,y} \delta w_{,x} \right. \right. \\
& - \frac{hh_{,x}}{4} \alpha_{,y} (\delta\alpha + \delta w_{,x}) - \frac{hh_{,x}}{4} (\alpha + w_{,x}) \delta\alpha_{,y} - \frac{hh_{,y}}{4} \beta_{,x} (\delta\beta + \delta w_{,y}) \\
& - \frac{hh_{,y}}{4} (\beta + w_{,y}) \delta\beta_{,x} - \frac{hh_{,x}}{4} \beta_{,y} \delta\beta - \frac{hh_{,x}}{4} \beta \delta\beta_{,y} - \frac{hh_{,x}}{4} \alpha_{,x} \delta w_{,y} \\
& - \frac{hh_{,x}}{4} w_{,y} \delta\alpha_{,x} - \frac{hh_{,y}}{4} \alpha_{,x} \delta\alpha - \frac{hh_{,y}}{4} \alpha \delta\alpha_{,x} - \frac{hh_{,y}}{4} \beta_{,y} \delta w_{,x} \\
& \left. \left. - \frac{hh_{,y}}{4} w_{,x} \delta\beta_{,y} \right] + M_x \delta\alpha_{,x} + M_y \delta\beta_{,y} + M_{xy} (\alpha_{,y} + \delta\beta_{,x}) \right. \\
& + S_x (\delta\alpha + \delta w_{,x}) + S_y (\delta\beta + \delta w_{,y}) - ph \left[\delta u_{,x} + \delta v_{,y} + u_{,x} \delta v_{,y} \right. \\
& + v_{,y} \delta u_{,x} - u_{,y} \delta v_{,x} - v_{,x} \delta u_{,y} + \frac{h^2}{12} (\alpha_{,x} \delta\beta_{,y} + \beta_{,y} \delta\alpha_{,x} - \alpha_{,y} \delta\beta_{,x} \\
& \left. - \beta_{,x} \delta\alpha_{,y}) - \alpha \delta\alpha - \beta \delta\beta - \alpha \delta w_{,x} - w_{,x} \delta\alpha - \beta \delta w_{,y} - w_{,y} \delta\beta \right] \\
& - q_x \delta u - q_y \delta v - q_z \delta w + \rho \ddot{u} \delta u + \rho \ddot{v} \delta v + \rho \ddot{w} \delta w + \frac{h^2}{4} \rho \ddot{\alpha} \delta \alpha + \frac{h^2}{4} \rho \ddot{\beta} \delta \beta \Big\} dx dy \\
& - \oint (\bar{N}_{Nx} \delta u + \bar{N}_{Ny} \delta v + \bar{V}_N \delta w + \bar{M}_{Ny} \delta \alpha - \bar{M}_{Nx} \delta \beta) ds = 0 \tag{26}
\end{aligned}$$

The terms with partial derivatives of varied quantities are integrated by using Gauss' theorem which can be written as follows (see ref. 1):

$$\iint (f_{,x} + g_{,y}) dx dy = - \oint \left(f \frac{\partial x}{\partial N} + g \frac{\partial y}{\partial N} \right) ds \quad (27)$$

The operator $\partial/\partial N$ denotes differentiation in the direction of the inward normal. For example, a term like $\iint N_x^* \delta u_{,x} dx dy$ in equation (26) is handled as follows:

$$\iint (N_x^* \delta u)_{,x} dx dy = - \oint N_x^* \delta u \frac{\partial x}{\partial N} ds \quad (28)$$

or

$$\iint N_x^* \delta u_{,x} dx dy = - \oint N_x^* \delta u \frac{\partial x}{\partial N} ds - \iint N_{x,x}^* \delta u dx dy \quad (29)$$

When Gauss' theorem is applied in this fashion to terms in the double integral of equation (26), there results

$$\begin{aligned} & - \iint \left((N_{x,x}^* + N_{xy,y}^* - ph_{,x} - ph_{,x}^v{}_{,y} + ph_{,y}^v{}_{,x} + q_x - \rho \ddot{u}) \delta u + (N_{y,y}^* \right. \\ & + N_{xy,x}^* - ph_{,y} - ph_{,y}^u{}_{,x} + ph_{,x}^u{}_{,y} + q_y - \rho \ddot{v}) \delta v + \left\{ (N_{x,x}^* + N_{xy,y}^*) w_{,x} \right. \\ & + (N_{y,y}^* + N_{xy,x}^*) w_{,y} + N_x^* w_{,xx} + 2N_{xy}^* w_{,xy} + N_y^* w_{,yy} + S_{x,x} + S_{y,y} + ph_{,x} \alpha \\ & + ph_{,y} \beta + ph(\alpha_{,x} + \beta_{,y}) - \frac{hh_{,x}}{4} \left[(N_{x,x}^* + N_{xy,y}^*) \alpha_{,x} + (N_{y,y}^* + N_{xy,x}^*) \alpha_{,y} \right. \\ & + N_x^* \alpha_{,xx} + 2N_{xy}^* \alpha_{,xy} + N_y^* \alpha_{,yy} \left. \right] - \frac{hh_{,y}}{4} \left[(N_{y,y}^* + N_{xy,x}^*) \beta_{,y} + (N_{x,x}^* \right. \\ & + N_{xy,y}^*) \beta_{,x} + N_x^* \beta_{,xx} + 2N_{xy}^* \beta_{,xy} + N_y^* \beta_{,yy} \left. \right] + q_z - \rho \ddot{w} \left. \right\} \delta w + \left\{ M_{x,x} + M_{xy,y} \right. \\ & - S_x - ph(\alpha + w_{,x}) - \frac{ph^2}{4} (h_{,x} \beta_{,y} - h_{,y} \beta_{,x}) - \frac{hh_{,x}}{4} \left[(N_{x,x}^* + N_{xy,y}^*) (\alpha + w_{,x}) \right. \end{aligned}$$

(Equation continued on next page)

$$\begin{aligned}
& + \left(N_{y,y}^* + N_{xy,x}^* \right) w_{,y} + N_x^* w_{,xx} + 2N_{xy}^* w_{,xy} + N_{y,w}^* \left. \right] - \frac{hh_{,y}}{4} (N_{y,y}^* + N_{xy,x}^*) \alpha \\
& - \frac{h^2}{4} \rho \ddot{\alpha} \left. \right\} \delta \alpha + \left\{ M_{y,y} + M_{xy,x} - S_y - ph(\beta + w_{,y}) - \frac{ph^2}{4} (h_{,y} \alpha_{,x} - h_{,x} \alpha_{,y}) \right. \\
& - \frac{hh_{,y}}{4} \left[(N_{y,y}^* + N_{xy,x}^*) (\beta + w_{,y}) + (N_{x,x}^* + N_{xy,y}^*) w_{,x} + N_x^* w_{,xx} + 2N_{xy}^* w_{,xy} \right. \\
& \left. \left. + N_{y,w}^* \right] - \frac{hh_{,x}}{4} (N_{x,x}^* + N_{xy,y}^*) \beta - \frac{h^2}{4} \rho \ddot{\beta} \right\} \delta \beta \Big) dx dy \\
& - \oint \left(\left(N_x^* \frac{\partial x}{\partial N} + N_{xy}^* \frac{\partial y}{\partial N} - ph \frac{\partial x}{\partial N} - phv_{,y} \frac{\partial x}{\partial N} + phv_{,x} \frac{\partial y}{\partial N} + \bar{N}_{Nx} \right) \delta u \right. \\
& \left. + \left(N_y^* \frac{\partial y}{\partial N} + N_{xy}^* \frac{\partial x}{\partial N} - ph \frac{\partial y}{\partial N} - phu_{,x} \frac{\partial y}{\partial N} + phu_{,y} \frac{\partial x}{\partial N} + \bar{N}_{Ny} \right) \delta v \right. \\
& + \left[N_x^* w_{,x} \frac{\partial x}{\partial N} + N_y^* w_{,y} \frac{\partial y}{\partial N} + N_{xy}^* w_{,x} \frac{\partial y}{\partial N} + N_{xy}^* w_{,y} \frac{\partial x}{\partial N} + S_x \frac{\partial x}{\partial N} + S_y \frac{\partial y}{\partial N} \right. \\
& + ph\alpha \frac{\partial x}{\partial N} + ph\beta \frac{\partial y}{\partial N} + \bar{V}_N - \frac{hh_{,x}}{4} \left(N_x^* \alpha_{,x} \frac{\partial x}{\partial N} + N_y^* \alpha_{,y} \frac{\partial y}{\partial N} + N_{xy}^* \alpha_{,y} \frac{\partial x}{\partial N} \right. \\
& \left. \left. + N_{xy}^* \alpha_{,x} \frac{\partial y}{\partial N} \right) - \frac{hh_{,y}}{4} \left(N_x^* \beta_{,x} \frac{\partial x}{\partial N} + N_y^* \beta_{,y} \frac{\partial y}{\partial N} + N_{xy}^* \beta_{,x} \frac{\partial y}{\partial N} + N_{xy}^* \beta_{,y} \frac{\partial x}{\partial N} \right) \right] \delta w \\
& + \left\{ M_x \frac{\partial x}{\partial N} + M_{xy} \frac{\partial y}{\partial N} + \bar{M}_{Ny} - \frac{hh_{,x}}{4} \left[N_x^* (\alpha + w_{,x}) \frac{\partial x}{\partial N} + N_y^* w_{,y} \frac{\partial y}{\partial N} + N_{xy}^* w_{,y} \frac{\partial x}{\partial N} \right. \right. \\
& \left. \left. + N_{xy}^* (\alpha + w_{,x}) \frac{\partial y}{\partial N} \right] - \frac{hh_{,y}}{4} \left(N_y^* \alpha \frac{\partial y}{\partial N} + N_{xy}^* \alpha \frac{\partial x}{\partial N} \right) - \frac{ph^3}{12} \left(\beta_{,y} \frac{\partial x}{\partial N} \right. \right. \\
& \left. \left. - \beta_{,x} \frac{\partial y}{\partial N} \right) \right\} \delta \alpha + \left\{ M_y \frac{\partial y}{\partial N} + M_{xy} \frac{\partial x}{\partial N} - \bar{M}_{Nx} - \frac{hh_{,y}}{4} \left[N_y^* (\beta + w_{,y}) \frac{\partial y}{\partial N} + N_x^* w_{,x} \frac{\partial x}{\partial N} \right. \right. \\
& \left. \left. + N_{xy}^* w_{,x} \frac{\partial y}{\partial N} + N_{xy}^* (\beta + w_{,y}) \frac{\partial x}{\partial N} \right] - \frac{hh_{,x}}{4} \left(N_x^* \beta \frac{\partial x}{\partial N} + N_{xy}^* \beta \frac{\partial y}{\partial N} \right) \right. \\
& \left. - \frac{ph^3}{12} \left(\alpha_{,x} \frac{\partial y}{\partial N} - \alpha_{,y} \frac{\partial x}{\partial N} \right) \right\} \delta \beta \Big) ds = 0 \tag{30}
\end{aligned}$$

In writing equation (30) second-degree terms in $h_{,x}$ and $h_{,y}$ are neglected, and use has been made of the fact that $h_{,x}$ and $h_{,y}$ are constants. By the usual arguments of the calculus of variations, the coefficients of the quantities δu , δv , δw , $\delta \alpha$, and $\delta \beta$ in the double integral can be equated to zero individually. This procedure gives a system of five partial differential equations of equilibrium. The first two equations describe the equilibrium conditions for the inplane forces, the third is the lateral-force equilibrium equation, and the final two are moment equilibrium equations.

The last three partial differential equations, those associated with lateral forces and moments, can be simplified by some additional manipulation. The lateral-force equation is simplified by using the two inplane force equilibrium equations to substitute for the quantities $N_{x,x}^* + N_{xy,y}^*$ and $N_{y,y}^* + N_{xy,x}^*$. Terms in the resulting expression are neglected if they contain second-degree quantities in $h_{,x}$ and $h_{,y}$ or nonlinear quantities in the displacements or their derivatives. In addition, terms which contain products of an external loading and a displacement quantity are nonlinear and are dropped. The moment equilibrium equations are simplified by using the inplane force equations and also by substituting for the quantity $N_{xw,xx}^* + 2N_{xyw,xy}^* + N_{yw,yy}^*$ from the simplified lateral-force equilibrium equation. The same sorts of quantities are neglected as mentioned previously. In order to neglect all such quantities, the assumption must be made that the inplane stress resultants (N_x^* , N_y^* , N_{xy}^*) are not of a higher order of magnitude than the quantity ph . The simplified equations of equilibrium which result from these manipulations are

$$N_{x,x}^* + N_{xy,y}^* - ph_{,x}(1 + v_{,y}) + ph_{,y}v_{,x} + q_x = \rho \ddot{u} \quad (31a)$$

$$N_{y,y}^* + N_{xy,x}^* - ph_{,y}(1 + u_{,x}) + ph_{,x}u_{,y} + q_y = \rho \ddot{v} \quad (31b)$$

$$\begin{aligned} & N_x^0 \left[w - \frac{h}{4}(h_{,x}\alpha + h_{,y}\beta) \right]_{,xx} + N_y^0 \left[w - \frac{h}{4}(h_{,x}\alpha + h_{,y}\beta) \right]_{,yy} \\ & + 2N_{xy}^0 \left[w - \frac{h}{4}(h_{,x}\alpha + h_{,y}\beta) \right]_{,xy} + S_{x,x} + S_{y,y} + ph_{,x}(\alpha + w_{,x}) \\ & + ph_{,y}(\beta + w_{,y}) + ph(\alpha_{,x} + \beta_{,y}) + q_z = \rho \ddot{w} \end{aligned} \quad (31c)$$

$$M_{x,x} + M_{xy,y} - S_x - ph \left[\alpha + w_{,x} - \frac{h}{4}(h_{,x}\alpha + h_{,y}\beta) \right]_{,x} + \frac{hh_{,x}}{4}(q_z - \rho\ddot{w}) = \frac{h^2}{4} \rho\ddot{\alpha} \quad (31d)$$

$$M_{y,y} + M_{xy,x} - S_y - ph \left[\beta + w_{,y} - \frac{h}{4}(h_{,x}\alpha + h_{,y}\beta) \right]_{,y} + \frac{hh_{,y}}{4}(q_z - \rho\ddot{w}) = \frac{h^2}{4} \rho\ddot{\beta} \quad (31e)$$

Note that in equation (31c) the stress resultants N_x^* , N_y^* , and N_{xy}^* must be replaced by N_x^o , N_y^o , and N_{xy}^o in order to linearize the equation.

Nonlinear terms like $N_x \left[w - \frac{h}{4}(h_{,x}\alpha + h_{,y}\beta) \right]_{,xx}$ are assumed to be small compared with the linear terms and are neglected.

The appropriate boundary conditions are obtained by equating to zero the line integral in equation (30). The usual arguments of the calculus of variations lead to the following boundary conditions on the boundary of the plate middle surface:

$$\left. \begin{array}{l} u = 0 \\ \text{or} \\ - \left[N_x^* - ph(1 + v_{,y}) \right] \cos(N, x) - (N_{xy}^* + phv_{,x}) \cos(N, y) = \bar{N}_{Nx} \end{array} \right\} \quad (32a)$$

$$\left. \begin{array}{l} v = 0 \\ \text{or} \\ - \left[N_y^* - ph(1 + u_{,x}) \right] \cos(N, y) - (N_{xy}^* + phu_{,y}) \cos(N, x) = \bar{N}_{Ny} \end{array} \right\} \quad (32b)$$

$$\left. \begin{array}{l} w = 0 \\ \text{or} \\ - \left\{ N_x^o \left[w_{,x} - \frac{h}{4}(h_{,x}\alpha + h_{,y}\beta) \right]_{,x} + N_{xy}^o \left[w_{,y} - \frac{h}{4}(h_{,x}\alpha + h_{,y}\beta) \right]_{,y} + S_x \right. \\ \left. + ph\alpha \right\} \cos(N, x) - \left\{ N_y^o \left[w_{,y} - \frac{h}{4}(h_{,x}\alpha + h_{,y}\beta) \right]_{,y} \right. \\ \left. + N_{xy}^o \left[w_{,x} - \frac{h}{4}(h_{,x}\alpha + h_{,y}\beta) \right]_{,x} + S_y + ph\beta \right\} \cos(N, y) = \bar{V}_N \end{array} \right\} \quad (32c)$$

$$\alpha = 0$$

or

$$\left. \begin{aligned} & - \left\{ M_x - \frac{ph^3}{12} \beta_{,y} - \frac{h}{4} \left[N_x^0 h_{,x} (\alpha + w_{,x}) + N_{xy}^0 (h_{,y} \alpha + h_{,x} w_{,y}) \right] \right\} \cos(N,x) - \left\{ M_{xy} \right. \\ & + \left. \frac{ph^3}{12} \beta_{,x} - \frac{h}{4} \left[N_{xy}^0 h_{,x} (\alpha + w_{,x}) + N_y^0 (h_{,y} \alpha + h_{,x} w_{,y}) \right] \right\} \cos(N,y) = \bar{M}_{Ny} \end{aligned} \right\} \quad (32d)$$

$$\beta = 0$$

or

$$\left. \begin{aligned} & \left\{ M_y - \frac{ph^3}{12} \alpha_{,x} - \frac{h}{4} \left[N_y^0 h_{,x} (\beta + w_{,y}) + N_{xy}^0 (h_{,x} \beta + h_{,y} w_{,x}) \right] \right\} \cos(N,y) + \left\{ M_{xy} \right. \\ & + \left. \frac{ph^3}{12} \alpha_{,y} - \frac{h}{4} \left[N_{xy}^0 h_{,y} (\beta + w_{,y}) + N_x^0 (h_{,x} \beta + h_{,y} w_{,x}) \right] \right\} \cos(N,x) = \bar{M}_{Nx} \end{aligned} \right\} \quad (32e)$$

In these equations the relations

$$\left. \begin{aligned} \frac{\partial x}{\partial N} &= \cos(N,x) \\ \frac{\partial y}{\partial N} &= \cos(N,y) \end{aligned} \right\} \quad (33)$$

are used where (N,x) represents the angle between the inward normal and the positive x -axis and (N,y) represents the angle between the inward normal and the positive y -axis. The differential equations (31) and boundary conditions (32) govern the behavior of inflatable plates. In addition to these equations the linearized stress displacement relations, equations (17), are required to complete the system.

Limitation of Linear Theory

The range of validity of linear theory certainly depends on the specific problem under consideration. In some problems linear theory might be valid up to substantially larger deflections than in other problems. In order to obtain a rough indication of what this range of validity is, one simple problem is investigated in appendix B. The

lateral deflection of a long plate simply supported on its long edges and under uniform load is considered where the simply supported edges of the plate are prevented from moving together. Calculations made by a simplified nonlinear theory are compared with calculations made by the linear theory. The results show that the linear result is within 5 percent of the nonlinear result if the following condition holds:

$$(J + 1) \frac{w_1^2}{h^2} < 0.05$$

where

$$J = \frac{A_{11}}{2ph} \left(\frac{\pi h}{L} \right)^2$$

where w_1 is the lateral deflection at the center of the plate, and where L is the distance between the simply supported edges. Of course, this result is strictly applicable to this one problem only but it does give a rough indication of what might happen in other problems.

SOME SOLUTIONS FOR RECTANGULAR PLATES

Solutions are now presented to some specific problems in the deflection and vibration of constant-depth rectangular inflatable plates supported on all edges. Consideration is given to static deflection under uniformly distributed lateral load and vibration modes and frequencies of both simply supported and clamped plates. A comparison between results calculated from these solutions and some experiments conducted on a square inflatable plate is presented in reference 2.

Consider a rectangular plate of constant depth supported along edges $x = 0, a$ and $y = 0, b$. For the case where no edge loads are prescribed, the boundary conditions, equations (32), are as follows:

on $x = 0, a$,

$$\left. \begin{array}{l} \text{or} \\ u = 0 \\ N_x^* - ph(1 + \nu, y) = 0 \end{array} \right\} \quad (34a)$$

$$\left. \begin{array}{l} \text{or} \\ v = 0 \\ N_{xy}^* + ph\nu, y = 0 \end{array} \right\} \quad (34b)$$

$$\left. \begin{array}{l} \text{or} \\ w = 0 \\ N_x^0 w_{,x} + N_{xy}^0 w_{,y} + ph\alpha = 0 \end{array} \right\} \quad (34c)$$

$$\left. \begin{array}{l} \text{or} \\ \alpha = 0 \\ M_x - \frac{ph^3}{12} \beta_{,y} = 0 \end{array} \right\} \quad (34d)$$

$$\left. \begin{array}{l} \text{or} \\ \beta = 0 \\ M_{xy} + \frac{ph^3}{12} \alpha_{,y} = 0 \end{array} \right\} \quad (34e)$$

and on $y = 0, b,$

$$\left. \begin{array}{l} \text{or} \\ u = 0 \\ N_{xy}^* + phv_{,x} = 0 \end{array} \right\} \quad (35a)$$

$$\left. \begin{array}{l} \text{or} \\ v = 0 \\ N_y^* - ph(1 + u_{,x}) = 0 \end{array} \right\} \quad (35b)$$

$$\left. \begin{array}{l} \text{or} \\ w = 0 \\ N_y^0 w_{,y} + N_{xy}^0 w_{,x} + ph\beta = 0 \end{array} \right\} \quad (35c)$$

$$\left. \begin{array}{l} \text{or} \\ \alpha = 0 \\ M_{xy} + \frac{ph^3}{12} \beta_{,x} = 0 \end{array} \right\} \quad (35d)$$

$$\left. \begin{array}{l} \text{or} \\ \beta = 0 \\ M_y - \frac{ph^3}{12} \alpha_{,x} = 0 \end{array} \right\} \quad (35e)$$

If the \ddot{u} and \ddot{v} accelerations are neglected and only lateral loads are considered, the differential equations (31) become

$$\left. \begin{aligned}
 N_{x,x}^* + N_{xy,y}^* &= 0 \\
 N_{y,y}^* + N_{xy,x}^* &= 0 \\
 N_{x,w}^0 + N_{y,w}^0 + 2N_{xy,w}^0 + ph(\alpha_{,x} + \beta_{,y}) + q_z &= \rho \ddot{w} \\
 M_{x,x} + M_{xy,y} - ph(\alpha + w_{,x}) &= \frac{h^2}{4} \rho \ddot{\alpha} \\
 M_{y,y} + M_{xy,x} - ph(\beta + w_{,y}) &= \frac{h^2}{4} \rho \ddot{\beta}
 \end{aligned} \right\} \quad (36)$$

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The portions of the stress resultants which are independent of displacements must satisfy the following differential equations:

$$\begin{aligned}
 N_{x,x}^0 + N_{xy,y}^0 &= 0 \\
 N_{y,y}^0 + N_{xy,x}^0 &= 0
 \end{aligned}$$

and, if u and v are not prescribed on the boundaries, the following boundary conditions must be satisfied:

on $x = 0, a,$

$$\begin{aligned}
 N_x^0 - ph &= 0 \\
 N_{xy}^0 &= 0
 \end{aligned}$$

and on $y = 0, b,$

$$\begin{aligned}
 N_{xy}^0 &= 0 \\
 N_y^0 - ph &= 0
 \end{aligned}$$

The solution to this system is

$$\left. \begin{aligned}
 N_x^0 = N_y^0 = ph \\
 N_{xy}^0 = 0
 \end{aligned} \right\} \quad (37)$$

The first two of equations (36) contain only u and v and the remaining three contain only w , α , and β . Since for the particular problem under consideration the lateral deflection is desired, the fact that the u and v equations uncouple from the others means that it is necessary to consider only the last three of the differential equations (36) and the appropriate boundary conditions. These differential equations reduce to

$$\left. \begin{aligned} \text{ph}(\alpha + w, x)_{,x} + \text{ph}(\beta + w, y)_{,y} + q_z &= \rho \ddot{w} \\ M_{x,x} + M_{xy,y} - \text{ph}(\alpha + w, x) &= \frac{h^2}{4} \rho \ddot{\alpha} \\ M_{y,y} + M_{xy,x} - \text{ph}(\beta + w, y) &= \frac{h^2}{4} \rho \ddot{\beta} \end{aligned} \right\} \quad (38)$$

and the boundary conditions become:

on $x = 0, a$,

$$\left. \begin{aligned} w &= 0 \\ \text{or} \\ \text{ph}(\alpha + w, x) &= 0 \end{aligned} \right\} \quad (39a)$$

$$\left. \begin{aligned} \alpha &= 0 \\ \text{or} \\ M_x - \frac{\text{ph}^3}{12} \beta_{,y} &= 0 \end{aligned} \right\} \quad (39b)$$

$$\left. \begin{aligned} \beta &= 0 \\ \text{or} \\ M_{xy} + \frac{\text{ph}^3}{12} \alpha_{,y} &= 0 \end{aligned} \right\} \quad (39c)$$

and on $y = 0, b$,

$$\left. \begin{aligned} w &= 0 \\ \text{or} \\ \text{ph}(\beta + w, y) &= 0 \end{aligned} \right\} \quad (40a)$$

$$\left. \begin{array}{l} \alpha = 0 \\ M_{xy} + \frac{ph^3}{12} \beta_{,x} = 0 \end{array} \right\} \quad (40b)$$

$$\left. \begin{array}{l} \beta = 0 \\ M_y - \frac{ph^3}{12} \alpha_{,x} = 0 \end{array} \right\} \quad (40c)$$

Finally, from the last three of equations (17) the moments are written in terms of the rotations

$$\left. \begin{array}{l} M_x = \frac{h^2}{2} (A_{11} \alpha_{,x} + A_{12} \beta_{,y}) \\ M_y = \frac{h^2}{2} (A_{21} \alpha_{,x} + A_{22} \beta_{,y}) \\ M_{xy} = \frac{h^2}{2} A_{33} (\alpha_{,y} + \beta_{,x}) \end{array} \right\} \quad (41)$$

The differential equations (38) have the same form as the well-known equations for the lateral motion of a uniform plate with transverse shear flexibility and rotary inertia. (See, for example, ref. 3.) In this case the quantity ph plays the role of the transverse shear stiffness per unit width. The salient conclusion of this investigation, therefore, is that an inflatable plate can be considered as a particular type of sandwich plate where the pressurized gas acts as the core material. Note, however, that certain terms peculiar to inflatable plates appear in the boundary conditions (39) and (40). These terms are associated with the deformations of the edge walls in the presence of internal pressure. Consideration should be given to the importance of these terms before sandwich-plate solutions are applied to inflatable-plate problems.

When equations (41) are substituted into the differential equations (38) there results

$$\left. \begin{aligned}
 & ph(\alpha + w, x)_{,x} + ph(\beta + w, y)_{,y} + q_z = \rho \ddot{w} \\
 & \frac{h^2}{2}(A_{11}\alpha_{,xx} + A_{12}\beta_{,xy}) + \frac{h^2}{2} A_{33}(\alpha_{,yy} + \beta_{,xy}) - ph(\alpha + w, x) = \frac{\rho h^2}{4} \ddot{\alpha} \\
 & \frac{h^2}{2}(A_{21}\alpha_{,xy} + A_{22}\beta_{,yy}) + \frac{h^2}{2} A_{33}(\alpha_{,yx} + \beta_{,xx}) - ph(\beta + w, y) = \frac{\rho h^2}{4} \ddot{\beta}
 \end{aligned} \right\} (42)$$

The differential equations are now expressed entirely in terms of the plate lateral displacements w and plate rotations α and β .

Simply Supported Rectangular Plates

For the case where all boundaries of the plate $x = 0, a$ and $y = 0, b$ are simply supported, a reasonable set of boundary conditions can be written from equations (39) and (40), with the help of equations (41), in the form

$$\left. \begin{aligned}
 & w(0, y) = w(a, y) = 0 \\
 & \beta(0, y) = \beta(a, y) = 0 \\
 & \alpha_{,x}(0, y) = \alpha_{,x}(a, y) = 0
 \end{aligned} \right\} (43)$$

and

$$\left. \begin{aligned}
 & w(x, 0) = w(x, b) = 0 \\
 & \alpha(x, 0) = \alpha(x, b) = 0 \\
 & \beta_{,y}(x, 0) = \beta_{,y}(x, b) = 0
 \end{aligned} \right\} (44)$$

Static deflection under uniform lateral load.- The response of the plate to a uniform lateral load is governed by equations (42) with terms \ddot{w} , $\ddot{\alpha}$, and $\ddot{\beta}$ set equal to zero and the term q_z regarded as constant. It is seen that the boundary conditions (43) and (44) are satisfied by displacements and rotations of the form

$$\left. \begin{aligned}
 w &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\
 \alpha &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\
 \beta &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \beta_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}
 \end{aligned} \right\} \quad (45)$$

When equations (45) are substituted into equations (42) reduced as just indicated, there results

$$\left. \begin{aligned}
 &\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{m\pi}{a} \left(\alpha_{mn} + w_{mn} \frac{m\pi}{a} \right) + \frac{n\pi}{b} \left(\beta_{mn} + w_{mn} \frac{n\pi}{b} \right) \right] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = \frac{q_z}{ph} \\
 &\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \left[A_{11} \left(\frac{m\pi}{a} \right)^2 + A_{33} \left(\frac{n\pi}{b} \right)^2 \right] \alpha_{mn} + (A_{12} + A_{33}) \frac{m\pi}{a} \frac{n\pi}{b} \beta_{mn} \right. \\
 &\quad \left. + \frac{2ph}{h^2} \left(\alpha_{mn} + \frac{m\pi}{a} w_{mn} \right) \right\} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = 0 \\
 &\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \left[A_{22} \left(\frac{n\pi}{b} \right)^2 + A_{33} \left(\frac{m\pi}{a} \right)^2 \right] \beta_{mn} + (A_{12} + A_{33}) \frac{m\pi}{a} \frac{n\pi}{b} \alpha_{mn} \right. \\
 &\quad \left. + \frac{2ph}{h^2} \left(\beta_{mn} + \frac{n\pi}{b} w_{mn} \right) \right\} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} = 0
 \end{aligned} \right\} \quad (46)$$

Equating to zero the coefficients of the Fourier series in the last two of equations (46) yields equations for α_{mn} and β_{mn} in terms of w_{mn} . The constant term q_z/ph in the first of equations (46) can be expanded in a double sine series in the interval $0 \leq x \leq a$, $0 \leq y \leq b$ to yield the equation

$$\left. \begin{aligned} w_{mn} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) + \frac{\alpha_{mn}}{\pi} \frac{m}{a} + \frac{\beta_{mn}}{\pi} \frac{n}{b} &= \frac{16}{\pi^4} \frac{q_z}{ph} \frac{1}{mn} & (m \text{ and } n \text{ odd}) \\ w_{mn} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) + \frac{\alpha_{mn}}{\pi} \frac{m}{a} + \frac{\beta_{mn}}{\pi} \frac{n}{b} &= 0 & (m \text{ or } n \text{ even}) \end{aligned} \right\} \quad (47)$$

When the expressions for α_{mn} and β_{mn} in terms of w_{mn} are substituted into equation (47), the resulting equation can be solved for w_{mn} . This procedure leads to the following expression for w_{mn} in dimensionless form (for m and n odd):

$$\frac{w_{mn}}{h} = \frac{R\bar{q}}{mn \left\{ \left(1 - \frac{R^2 \bar{p}^2}{4\Lambda_{mn}} \right) a_{mn} - \frac{R\bar{p}}{2\Lambda_{mn}} \left[e_{mn} + (\lambda^2 m^2 - n^2) \bar{A}_{33} \right] \right\}} \quad (48)$$

where

$$\lambda = \frac{b}{a}$$

$$\bar{q} = \frac{4q_z}{\pi^2 p}$$

$$R = \frac{4b^2}{\pi^2 h^2}$$

$$\bar{p} = \frac{ph}{A_{11}}$$

$$\Lambda_{mn} = \frac{1}{4} \left[(R\bar{p} + b_{mn})(R\bar{p} + c_{mn}) - d_{mn}^2 \right]$$

$$a_{mn} = \lambda^2 m^2 + n^2$$

$$b_{mn} = 2(\lambda^2 m^2 + \bar{A}_{33} n^2)$$

$$c_{mn} = 2(\bar{A}_{22}n^2 + \bar{A}_{33}\lambda_m^2)$$

$$d_{mn} = 2\lambda_{mn}(\bar{A}_{12} + \bar{A}_{33})$$

$$e_{mn} = \lambda_m^2 n^2 (1 + \bar{A}_{22} - 2\bar{A}_{12})$$

and where

$$\bar{A}_{12} = \frac{A_{12}}{A_{11}}$$

$$\bar{A}_{22} = \frac{A_{22}}{A_{11}}$$

$$\bar{A}_{33} = \frac{A_{33}}{A_{11}}$$

For either m or n even, all coefficients w_{mn} , α_{mn} , and β_{mn} vanish.

Natural frequencies of vibration.- The free vibration of a simply supported rectangular plate is governed by differential equations (42) with q_z set equal to zero and boundary conditions (eqs. (43) and (44)) assumed to hold for all time. The functions (for $m = 1, 2, \dots$ and $n = 1, 2, \dots$)

$$\left. \begin{aligned} w &= \bar{w}_{mn} e^{i\omega t} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ \alpha &= \bar{\alpha}_{mn} e^{i\omega t} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ \beta &= \bar{\beta}_{mn} e^{i\omega t} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \end{aligned} \right\} \quad (49)$$

are seen to satisfy the boundary conditions, equations (43) and (44). When equations (49) are substituted into equations (42) (with $q_z = 0$) there results (for $m = 1, 2, \dots$ and $n = 1, 2, \dots$)

$$\left. \begin{aligned}
& \left\{ \text{ph} \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right] - \rho\omega^2 \right\} \bar{w}_{mn} + \text{ph} \frac{m\pi}{a} \bar{\alpha}_{mn} + \text{ph} \frac{n\pi}{b} \bar{\beta}_{mn} = 0 \\
& \text{ph} \frac{m\pi}{a} \bar{w}_{mn} + \left\{ \text{ph} + \frac{h^2}{2} \left[A_{11} \left(\frac{m\pi}{a} \right)^2 + A_{33} \left(\frac{n\pi}{b} \right)^2 \right] - \frac{\rho h^2}{4} \omega^2 \right\} \bar{\alpha}_{mn} \\
& + \frac{h^2}{2} \frac{m\pi}{a} \frac{n\pi}{b} (A_{12} + A_{33}) \bar{\beta}_{mn} = 0 \\
& \text{ph} \frac{n\pi}{b} \bar{w}_{mn} + \frac{h^2}{2} \frac{m\pi}{a} \frac{n\pi}{b} (A_{12} + A_{33}) \bar{\alpha}_{mn} + \left\{ \text{ph} + \frac{h^2}{2} \left[A_{22} \left(\frac{n\pi}{b} \right)^2 + A_{33} \left(\frac{m\pi}{a} \right)^2 \right] \right. \\
& \left. - \frac{\rho h^2}{4} \omega^2 \right\} \bar{\beta}_{mn} = 0
\end{aligned} \right\} (50)$$

or in dimensionless form

$$\begin{vmatrix}
\frac{h}{b} \left[\bar{p} (\lambda^2 m^2 + n^2) - k^2 \right] & \bar{p} \lambda m & \bar{p} n & \left| \frac{\bar{w}_{mn}}{h} \right| \\
\bar{p} \lambda m & \frac{\bar{p} b}{h} + \frac{h \pi^2}{2b} (\lambda^2 m^2 + \bar{A}_{33} n^2 - \frac{k^2}{2}) & \frac{h \pi^2}{2b} \lambda m (\bar{A}_{12} + \bar{A}_{33}) & \left| \frac{\bar{\alpha}_{mn}}{\pi} \right| \\
\bar{p} n & \frac{h \pi^2}{2b} \lambda m (\bar{A}_{12} + \bar{A}_{33}) & \frac{\bar{p} b}{h} + \frac{h \pi^2}{2b} (\bar{A}_{22} n^2 + \bar{A}_{33} \lambda^2 m^2 - \frac{k^2}{2}) & \left| \frac{\bar{\beta}_{mn}}{\pi} \right|
\end{vmatrix} = 0 \quad (51)$$

where $k^2 = \frac{\rho \omega^2 b^2}{A_{11} \pi^2}$ is the frequency parameter. When the determinant of the coefficients of \bar{w}_{mn} , $\bar{\alpha}_{mn}$, and $\bar{\beta}_{mn}$ is equated to zero and after some algebraic manipulation, the following equation results:

$$k^6 - \left[\bar{p} (2R + a_{mn}) + b_{mn} + c_{mn} \right] k^4 + \left[\bar{p}^2 R (R + a_{mn}) + \bar{p} (R + a_{mn}) (b_{mn} + c_{mn}) + (b_{mn} c_{mn} - d_{mn}^2) \right] k^2 - \bar{p}^2 R f_{mn} - \bar{p} a_{mn} (b_{mn} c_{mn} - d_{mn}^2) = 0 \quad (52)$$

where the only symbol not previously defined is

$$f_{mn} = \lambda^2 m^2 b_{mn} + n^2 c_{mn} + 2 \lambda m n d_{mn}$$

The solutions to equations (52) define the natural vibration frequencies of the simply supported plate corresponding to the mode shapes given by equations (49).

Clamped Rectangular Plates

Suppose, now, that instead of boundary conditions as given by equations (43) and (44), the following boundary conditions are prescribed for a rectangular plate:

$$\left. \begin{aligned} w(0,y) = w(a,y) = 0 \\ \alpha(0,y) = \alpha(a,y) = 0 \\ \beta(0,y) = \beta(a,y) = 0 \end{aligned} \right\} \quad (53)$$

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and

$$\left. \begin{aligned} w(x,0) = w(x,b) = 0 \\ \alpha(x,0) = \alpha(x,b) = 0 \\ \beta(x,0) = \beta(x,b) = 0 \end{aligned} \right\} \quad (54)$$

These boundary conditions are appropriate for a plate with all edges clamped. The differential equations (42) must now be solved in conjunction with conditions (53) and (54).

Static deflection under uniform lateral load.- In equations (42) set \ddot{w} , $\ddot{\alpha}$, and $\ddot{\beta}$ equal to zero and regard q_z as constant. The following deflection and rotation functions satisfy the boundary conditions (53) and (54):

$$\left. \begin{aligned} w &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ \alpha &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \alpha_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ \beta &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \beta_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \end{aligned} \right\} \quad (55)$$

that is, w , α , and β are zero on all boundaries. Substituting equations (55) in the reduced differential equations (42) yields

$$\left. \begin{aligned}
 & \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ -w_{mn} \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} + \alpha_{mn} \frac{m\pi}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right. \\
 & \left. + \beta_{mn} \frac{n\pi}{b} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \right\} = - \frac{q_z}{ph} \\
 & \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ -w_{mn} ph \frac{m\pi}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} - \alpha_{mn} \left[ph + \left(A_{11} \frac{m^2 \pi^2}{a^2} + A_{33} \frac{n^2 \pi^2}{b^2} \right) \frac{h^2}{2} \right] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right. \\
 & \left. + \beta_{mn} \frac{m\pi}{a} \frac{n\pi}{b} \frac{h^2}{2} (A_{12} + A_{33}) \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \right\} = 0 \\
 & \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ -w_{mn} ph \frac{n\pi}{b} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} + \alpha_{mn} \frac{m\pi}{a} \frac{n\pi}{b} \frac{h^2}{2} (A_{12} + A_{33}) \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \right. \\
 & \left. - \beta_{mn} \left[ph + \left(A_{22} \frac{n^2 \pi^2}{b^2} + A_{33} \frac{m^2 \pi^2}{a^2} \right) \frac{h^2}{2} \right] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right\} = 0
 \end{aligned} \right\} (56)$$

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The Galerkin method is now applied by multiplying through all of equations (56) by $\sin \frac{i\pi x}{a} \sin \frac{j\pi y}{b}$ (where i and j are integers) and integrating over x from 0 to a and over y from 0 to b . The resulting equations are written in dimensionless form as follows:

$$\left. \begin{aligned}
 & \frac{w_{ij}}{h} \left(\frac{h}{b} \right) \pi^2 (i^2 \lambda^2 + j^2) + \sum_{m=1}^{\infty} m \lambda \alpha_{mj} \gamma_{im} + \sum_{n=1}^{\infty} n \beta_{in} \gamma_{jn} = \frac{4q_z b}{ph \pi^2} \frac{[(-1)^i - 1][(-1)^j - 1]}{ij} \\
 & \bar{p} \sum_{m=1}^{\infty} m \lambda \frac{w_{mj}}{h} \gamma_{im} - \left[\frac{\bar{p}b}{h} + \frac{h\pi^2}{2b} (i^2 \lambda^2 + \bar{A}_{33} j^2) \right] \alpha_{ij} + \frac{h}{2b} (\bar{A}_{12} + \bar{A}_{33}) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m \lambda n \beta_{mn} \gamma_{im} \gamma_{jn} = 0 \\
 & \bar{p} \sum_{n=1}^{\infty} n \frac{w_{in}}{h} \gamma_{jn} + \frac{h}{2b} (\bar{A}_{12} + \bar{A}_{33}) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m \lambda n \alpha_{mn} \gamma_{im} \gamma_{jn} - \left[\frac{\bar{p}b}{h} + \frac{h\pi^2}{2b} (\bar{A}_{22} j^2 + \bar{A}_{33} i^2 \lambda^2) \right] \beta_{ij} = 0
 \end{aligned} \right\} (57)$$

where (using r and s as integers)

$$\gamma_{rs} = \frac{(-1)^{r-s} - 1}{r-s} + \frac{(-1)^{r+s} - 1}{r+s} \quad (r \neq s)$$

$$\gamma_{rs} = 0 \quad (r = s)$$

Examination of equations (57) shows that the unknowns w_{ij} are nonzero only for odd values of i and j , the unknowns α_{ij} are nonzero only for even values of i and odd values of j , and the unknowns β_{ij} are nonzero only for odd values of i and even values of j . For a first approximation, truncate the system (57) by taking $i = 1, 2$ and $j = 1, 2$ and summing over $m = 1, 2$ and $n = 1, 2$. Equations (57) reduce to

$$\begin{vmatrix} \frac{h}{b} \pi^2 (\lambda^2 + 1) & \frac{8}{3} \lambda & \frac{8}{3} \\ -\frac{8}{3} \bar{p} \lambda & -\left[\frac{\bar{p}b}{h} + \frac{h\pi^2}{2b} (4\lambda^2 + \bar{A}_{33}) \right] & -\frac{32}{9} \frac{h}{b} (\bar{A}_{12} + \bar{A}_{33}) \lambda \\ -\frac{8}{3} \bar{p} & -\frac{32}{9} \frac{h}{b} (\bar{A}_{12} + \bar{A}_{33}) \lambda & -\left[\frac{\bar{p}b}{h} + \frac{h\pi^2}{2b} (4\bar{A}_{22} + \lambda^2 \bar{A}_{33}) \right] \end{vmatrix} \begin{vmatrix} \frac{w_{11}}{h} \\ \alpha_{21} \\ \beta_{12} \end{vmatrix} = \begin{vmatrix} \frac{16q_2 b}{ph\pi^2} \\ 0 \\ 0 \end{vmatrix} \quad (58)$$

and three additional sets of equations which are homogeneous by virtue of the symmetry of the loading and, hence, yield the trivial results $w_{12} = w_{21} = w_{22} = \alpha_{11} = \alpha_{12} = \alpha_{22} = \beta_{11} = \beta_{21} = \beta_{22} = 0$. The last two of equations (58) can be solved for α_{21} and β_{12} in terms of w_{11}/h . Substituting these expressions into the first of equations (58) and solving for w_{11}/h results in

$$\frac{w_{11}}{h} = \frac{R\bar{q}}{\left(1 - \frac{16}{9\pi^2} \frac{R^2 \bar{p}^2}{\Lambda}\right) a_{11} - \frac{16}{9\pi^2} \frac{R\bar{p}}{\Lambda} \left(\lambda^2 c_{12} + b_{21} - \frac{32}{9\pi^2} \lambda d_{22}\right)} \quad (59)$$

where

$$\Lambda = \frac{1}{4} \left[(R\bar{p} + b_{21})(R\bar{p} + c_{12}) + \left(\frac{16d_{22}}{9\pi^2}\right)^2 \right]$$

More accurate approximations can be obtained by taking more terms in the series for w , α , and β , but the number of simultaneous equations which must be solved increases rapidly.

Natural frequencies of vibration.- For the free vibrations of a clamped rectangular plate assume deflection and rotation functions as follows:

$$\left. \begin{aligned} w &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \bar{w}_{mn} e^{i\omega t} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ \alpha &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \bar{\alpha}_{mn} e^{i\omega t} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ \beta &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \bar{\beta}_{mn} e^{i\omega t} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \end{aligned} \right\} (60)$$

When equations (60) are substituted into differential equations (42) with q_z set equal to zero there results

$$\left. \begin{aligned} &\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ \left[\frac{\rho\omega^2}{ph} - \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} \right) \right] \bar{w}_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right. \\ &+ \bar{\alpha}_{mn} \frac{m\pi}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} + \bar{\beta}_{mn} \frac{n\pi}{b} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \left. \right\} = 0 \\ &\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ -\bar{w}_{mn} ph \frac{m\pi}{a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} + \bar{\alpha}_{mn} \left[\frac{\rho h^2}{4} \omega^2 - ph - \frac{h^2}{2} \left(A_{11} \frac{m^2\pi^2}{a^2} + A_{33} \frac{n^2\pi^2}{b^2} \right) \right] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \right. \\ &+ \bar{\beta}_{mn} \frac{m\pi}{a} \frac{n\pi}{b} \frac{h^2}{2} (A_{12} + A_{33}) \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \left. \right\} = 0 \\ &\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ -\bar{w}_{mn} ph \frac{n\pi}{b} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} + \bar{\alpha}_{mn} \frac{m\pi}{a} \frac{n\pi}{b} \frac{h^2}{2} (A_{12} + A_{33}) \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \right. \\ &+ \bar{\beta}_{mn} \left[\frac{\rho h^2}{4} \omega^2 - ph - \frac{h^2}{2} \left(A_{22} \frac{n^2\pi^2}{b^2} + A_{33} \frac{m^2\pi^2}{a^2} \right) \right] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \left. \right\} = 0 \end{aligned} \right\} (61)$$

Application of the Galerkin method as used in the previous section leads to the following system of equations in dimensionless form:

$$\left. \begin{aligned}
 & \frac{\pi^2 h}{b} \left[k^2 - \bar{p} (i^2 \lambda^2 + j^2) \right] \frac{\bar{w}_{ij}}{h} - \bar{p} \sum_{m=1}^{\infty} m \lambda \bar{\alpha}_{mj} \gamma_{im} - \bar{p} \sum_{n=1}^{\infty} n \bar{\beta}_{in} \gamma_{jn} = 0 \\
 & \bar{p} \sum_{m=1}^{\infty} m \lambda \frac{\bar{w}_{mj}}{h} \gamma_{im} + \left[\frac{\pi^2 h}{4b} k^2 - \frac{\bar{p}b}{h} - \frac{\pi^2 h}{2b} (i^2 \lambda^2 + \bar{A}_{33} j^2) \right] \bar{\alpha}_{ij} \\
 & + \frac{h}{2b} (\bar{A}_{12} + \bar{A}_{33}) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m \lambda n \bar{\beta}_{mn} \gamma_{im} \gamma_{jn} = 0 \\
 & \bar{p} \sum_{n=1}^{\infty} n \frac{\bar{w}_{in}}{h} \gamma_{jn} + \frac{h}{2b} (\bar{A}_{12} + \bar{A}_{33}) \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m \lambda n \bar{\alpha}_{m1} \gamma_{im} \gamma_{jn} \\
 & + \left[\frac{\pi^2 h}{4b} k^2 - \frac{\bar{p}b}{h} - \frac{\pi^2 h}{2b} (\bar{A}_{22} j^2 + \bar{A}_{33} i^2 \lambda^2) \right] \bar{\beta}_{ij} = 0
 \end{aligned} \right\} (62)$$

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Limit consideration to the first four terms of each of the series in equations (60); thus, the integers i and j in equations (62) are taken to be 1 and 2 and the summations are carried out over $m = 1, 2$ and $n = 1, 2$. It is found that only certain terms couple. The \bar{w}_{11} coefficient couples with $\bar{\alpha}_{21}$ and $\bar{\beta}_{12}$, \bar{w}_{12} couples with $\bar{\alpha}_{22}$ and $\bar{\beta}_{11}$, \bar{w}_{21} couples with $\bar{\alpha}_{11}$ and $\bar{\beta}_{22}$, and finally \bar{w}_{22} couples with $\bar{\alpha}_{12}$ and $\bar{\beta}_{21}$. The frequencies for these various modes within the framework of the 1, 2 approximation are given by the following equations:

For the \bar{w}_{11} mode,

$$\left| \begin{array}{ccc}
 \frac{\pi^2 h}{b} [k^2 - \bar{p}(\lambda^2 + 1)] & -\frac{8}{3} \bar{p} \lambda & -\frac{8}{3} \bar{p} \\
 -\frac{8}{3} \bar{p} \lambda & \frac{\pi^2 h}{4b} k^2 - \frac{\bar{p}b}{h} - \frac{\pi^2 h}{2b} (4\lambda^2 + \bar{A}_{33}) & -\frac{32}{9} \frac{h}{b} \lambda (\bar{A}_{12} + \bar{A}_{33}) \\
 -\frac{8}{3} \bar{p} & -\frac{32}{9} \frac{h}{b} \lambda (\bar{A}_{12} + \bar{A}_{33}) & \frac{\pi^2 h}{4b} k^2 - \frac{\bar{p}b}{h} - \frac{\pi^2 h}{2b} (\lambda^2 \bar{A}_{33} + 4\bar{A}_{22})
 \end{array} \right| = 0 \quad (63)$$

for the \bar{w}_{12} mode,

$$\begin{vmatrix} \frac{\pi^2 h}{b} [k^2 - \bar{p}(\lambda^2 + 4)] & -\frac{8}{3} \bar{p} \lambda & \frac{8}{3} \bar{p} \\ -\frac{8}{3} \bar{p} \lambda & \frac{\pi^2 h}{4b} k^2 - \frac{\bar{p}b}{h} - \frac{\pi^2 h}{2b} (4\lambda^2 + 4\bar{A}_{33}) & \frac{32}{9} \frac{h}{b} \lambda (\bar{A}_{12} + \bar{A}_{33}) \\ \frac{8}{3} \bar{p} & \frac{32}{9} \frac{h}{b} \lambda (\bar{A}_{12} + \bar{A}_{33}) & \frac{\pi^2 h}{4b} k^2 - \frac{\bar{p}b}{h} - \frac{\pi^2 h}{2b} (\lambda^2 \bar{A}_{33} + \bar{A}_{22}) \end{vmatrix} = 0 \quad (64)$$

for the \bar{w}_{21} mode,

$$\begin{vmatrix} \frac{\pi^2 h}{b} [k^2 - \bar{p}(4\lambda^2 + 1)] & \frac{8}{3} \bar{p} \lambda & -\frac{8}{3} \bar{p} \\ \frac{8}{3} \bar{p} \lambda & \frac{\pi^2 h}{4b} k^2 - \frac{\bar{p}b}{h} - \frac{\pi^2 h}{2b} (\lambda^2 + \bar{A}_{33}) & \frac{32}{9} \frac{h}{b} \lambda (\bar{A}_{12} + \bar{A}_{33}) \\ -\frac{8}{3} \bar{p} & \frac{32}{9} \frac{h}{b} \lambda (\bar{A}_{12} + \bar{A}_{33}) & \frac{\pi^2 h}{4b} k^2 - \frac{\bar{p}b}{h} - \frac{\pi^2 h}{2b} (4\lambda^2 \bar{A}_{33} + 4\bar{A}_{22}) \end{vmatrix} = 0 \quad (65)$$

and for the \bar{w}_{22} mode,

$$\begin{vmatrix} \frac{\pi^2 h}{b} [k^2 - \bar{p}(4\lambda^2 + 4)] & \frac{8}{3} \bar{p} \lambda & \frac{8}{3} \bar{p} \\ \frac{8}{3} \bar{p} \lambda & \frac{\pi^2 h}{4b} k^2 - \frac{\bar{p}b}{h} - \frac{\pi^2 h}{2b} (\lambda^2 + 4\bar{A}_{33}) & -\frac{32}{9} \frac{h}{b} \lambda (\bar{A}_{12} + \bar{A}_{33}) \\ \frac{8}{3} \bar{p} & -\frac{32}{9} \frac{h}{b} \lambda (\bar{A}_{12} + \bar{A}_{33}) & \frac{\pi^2 h}{4b} k^2 - \frac{\bar{p}b}{h} - \frac{\pi^2 h}{2b} (4\lambda^2 \bar{A}_{33} + \bar{A}_{22}) \end{vmatrix} = 0 \quad (66)$$

In each case the smallest real positive value of k which makes the determinant vanish corresponds to the desired natural frequency.

TOTAL POTENTIAL ENERGY EXPRESSION

The derivation of the theory has been carried out using the potential energy in varied form. It is of interest to write the total potential energy expression because such an expression is useful in obtaining

approximate solutions. The potential energy is assumed to be measured from the condition of the plate after it has been inflated and after inplane edge loads \bar{N}_{Nx} and \bar{N}_{Ny} have been applied. In this state the displacements and rotations u , v , w , α , and β are assumed to be zero and the quantities N_x^o , N_y^o , and N_{xy}^o are assumed to be known and independent of u , v , w , α , and β . Then, additional loads q_x , q_y , q_z , \bar{V}_N , \bar{M}_{Nx} , and \bar{M}_{Ny} are applied. For this situation, the total potential energy for the small deflections of a tapered inflatable plate of arbitrary planform shape can be written in terms of displacements and rotations as

$$\begin{aligned}
 \Pi = & \iint \left(N_x^o u_{,x} + N_y^o v_{,y} + N_{xy}^o (u_{,y} + v_{,x}) + \frac{1}{2} \left\{ N_x^o \left[w_{,x} - \frac{h}{4} (h_{,x} \alpha + h_{,y} \beta) \right]_{,x}^2 \right. \right. \\
 & + N_y^o \left[w_{,y} - \frac{h}{4} (h_{,x} \alpha + h_{,y} \beta) \right]_{,y}^2 + 2N_{xy}^o \left[w_{,x} - \frac{h}{4} (h_{,x} \alpha + h_{,y} \beta) \right]_{,x} \left[w_{,y} \right. \\
 & - \left. \frac{h}{4} (h_{,x} \alpha + h_{,y} \beta) \right]_{,y} - \frac{hh_{,x}}{2} \left[N_x^o \left(\frac{\alpha^2}{2} + \frac{\beta^2}{2} \right)_{,x} + N_{xy}^o \left(\frac{\alpha^2}{2} + \frac{\beta^2}{2} \right)_{,y} \right] \\
 & \left. \left. - \frac{hh_{,y}}{2} \left[N_y^o \left(\frac{\alpha^2}{2} + \frac{\beta^2}{2} \right)_{,y} + N_{xy}^o \left(\frac{\alpha^2}{2} + \frac{\beta^2}{2} \right)_{,x} \right] \right\} + A_{11} u_{,x}^2 + A_{22} v_{,y}^2 \right. \\
 & + A_{33} (u_{,y} + v_{,x})^2 + 2A_{12} u_{,x} v_{,y} + \frac{h^2}{4} \left\{ A_{11} \left[\alpha_{,x} + (\alpha + w_{,x}) \frac{h_{,x}}{h} \right]^2 \right. \\
 & + A_{22} \left[\beta_{,y} + (\beta + w_{,y}) \frac{h_{,y}}{h} \right]^2 + 2A_{12} \left[\alpha_{,x} + (\alpha + w_{,x}) \frac{h_{,x}}{h} \right] \left[\beta_{,y} \right. \\
 & \left. \left. + (\beta + w_{,y}) \frac{h_{,y}}{h} \right] + A_{33} \left[\alpha_{,y} + \beta_{,x} + (\alpha + w_{,x}) \frac{h_{,y}}{h} + (\beta + w_{,y}) \frac{h_{,x}}{h} \right]^2 \right\} \\
 & - ph \left[u_{,x} + v_{,y} + u_{,x} v_{,y} - u_{,y} v_{,x} - \frac{\alpha^2}{2} - \frac{\beta^2}{2} - \alpha w_{,x} - \beta w_{,y} \right. \\
 & \left. + \frac{h^2}{12} (\alpha_{,x} \beta_{,y} - \alpha_{,y} \beta_{,x}) \right] - q_x u - q_y v - q_z w \Big) dx dy \\
 & - \oint (\bar{N}_{Nx} u + \bar{N}_{Ny} v + \bar{V}_N w + \bar{M}_{Ny} \alpha - \bar{M}_{Nx} \beta) ds \tag{67}
 \end{aligned}$$

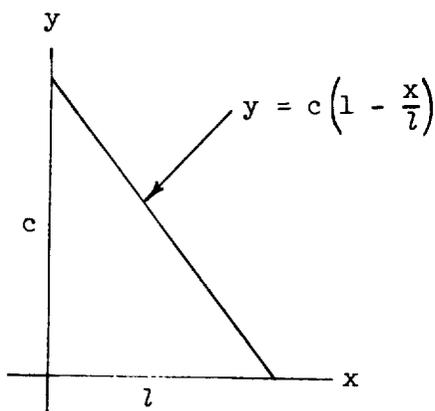
In writing equation (67) terms of higher than second degree in displacements and rotations or their derivatives have been neglected. In addition, inertia effects are neglected. Variation of equation (67) with respect to u , v , w , α , and β yields with a little manipulation the differential equations (31) (with the inertia terms set equal to zero) and the boundary conditions (32).

In many problems only lateral deflections are desired and inplane displacements are not of primary concern. Then equation (67) can be simplified for use in obtaining approximate solutions by dropping all terms containing u or v or their derivatives.

For a rectangular plate with edges $x = 0, a$ and $y = 0, b$ the double integration in equation (67) is, of course, performed over x from 0 to a and over y from 0 to b . By noting that on the edge $x = a$, $\bar{N}_{Nx} = \bar{N}_x$, $\bar{N}_{Ny} = \bar{N}_{xy}$, $\bar{V}_N = \bar{V}_x$, $\bar{M}_{Ny} = \bar{M}_x$, and $\bar{M}_{Nx} = -\bar{M}_{xy}$, on the edge $x = 0$, $\bar{N}_{Nx} = -\bar{N}_x$, $\bar{N}_{Ny} = -\bar{N}_{xy}$, $\bar{V}_N = -\bar{V}_x$, $\bar{M}_{Ny} = -\bar{M}_x$, $\bar{M}_{Nx} = \bar{M}_{xy}$, and so forth, the line integral in equation (67) becomes

$$\begin{aligned}
 & - \oint (\bar{N}_{Nx}u + \bar{N}_{Ny}v + \bar{V}_Nw + \bar{M}_{Ny}\alpha - \bar{M}_{Nx}\beta) ds \\
 & = - \int_0^b (\bar{N}_xu + \bar{N}_{xy}v + \bar{V}_xw + \bar{M}_x\alpha + \bar{M}_{xy}\beta) \Big|_0^a dy \\
 & \quad - \int_0^a (\bar{N}_yv + \bar{N}_{xy}u + \bar{V}_yw + \bar{M}_{xy}\alpha + \bar{M}_y\beta) \Big|_0^b dx \quad (68)
 \end{aligned}$$

For a triangular plate oriented as shown in the following sketch



the double integration in equation (67) can be performed first over y from 0 to $c\left(1 - \frac{x}{l}\right)$ and then over x from 0 to l . The line integral becomes

$$\begin{aligned}
 & - \oint (\bar{N}_{Nx}u + \bar{N}_{Ny}v + \bar{V}_Nw + \bar{M}_{Ny}\alpha - \bar{M}_{Nx}\beta) ds \\
 & = - \int_0^l (\bar{N}_{Nx}u + \bar{N}_{Ny}v + \bar{V}_Nw + \bar{M}_{Ny}\alpha - \bar{M}_{Nx}\beta) \left| \sqrt{1 + \frac{c^2}{l^2}} \right|_{y=c\left(1-\frac{x}{l}\right)} dx \\
 & \quad + \int_0^c (\bar{N}_xu + \bar{N}_{xy}v + \bar{V}_xw + \bar{M}_x\alpha + \bar{M}_{xy}\beta) \Big|_{x=0} dy \\
 & \quad + \int_0^l (\bar{N}_{xy}u + \bar{N}_yv + \bar{V}_yw + \bar{M}_{xy}\alpha + \bar{M}_y\beta) \Big|_{y=0} dx \qquad (69)
 \end{aligned}$$

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CONCLUDING REMARKS

A linear theory has been developed which describes the behavior of inflatable plates such as Airmat. A rough indication of the range of applicability of the linear theory is obtained in an appendix by carrying out a nonlinear analysis of a simple problem and comparing the results with the linear solution. The theory turns out to be essentially the same as sandwich-plate theory in which transverse shear deformations are taken into account. The internal pressure in the inflatable plate is analogous to the transverse shear modulus in the sandwich plate. Some simple static lateral deflection and vibration problems are solved for rectangular inflatable plates of constant depth.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Field, Va., June 21, 1961.

APPENDIX A

CHANGE IN VOLUME OF INFLATABLE PLATE

Analytical Approach

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In order to calculate the work done against the internal pressure in an inflatable plate it is necessary to know how the volume of the plate changes due to displacements in the covers and along the edges. The derivation of the change in volume in terms of u , v , w , α , and β is presented in this appendix.

The derivation is carried out using the formula from calculus for determining the volume of a three-dimensional body by integrating over the surface enclosing the body. (See ref. 1.) Thus, the volume of the deformed plate is given by

$$V = - \iint_S \zeta \, d\xi \, d\eta \quad (A1)$$

where ξ , η , and ζ denote rectangular Cartesian coordinates (along the x-, y-, and z-axes, respectively) locating points in the surface of the deformed plate. The double integral in equation (A1) is a surface integration carried out over all external surfaces of the plate in the deformed state. The negative sign in equation (A1) arises because the surface of integration is assumed to be oriented so that an inward normal is positive.

The initial volume of the plate can be represented by the integral

$$\iint h \, dx \, dy \quad (A2)$$

which is simply a double integration over the undeformed middle plane of the plate. The change in volume ΔV is given by the difference between V and the quantity (A2) as

$$\Delta V = - \iint_S \zeta \, d\xi \, d\eta - \iint h \, dx \, dy \quad (A3)$$

Evaluation of Surface Integral

Upper cover.- The contribution of the surface integral over the upper cover of the plate can be written by substituting

$$\left. \begin{aligned} \xi &= x + u_+(x, y) \\ \eta &= y + v_+(x, y) \\ \zeta &= \frac{h}{2} + w_+(x, y) \end{aligned} \right\} \quad (A4)$$

into the integral in equation (A1). Over the upper cover, then, this integral is

$$- \iint_{S_+} \zeta \, d\xi \, d\eta = \iint \zeta \frac{\partial(\xi, \eta)}{\partial(x, y)} \, dx \, dy \quad (A5)$$

where the deformation of the upper surface is considered to be a transformation of coordinates from the xyz system to the $\xi\eta\zeta$ system as given by equations (A4). The double integral on the right-hand side of equation (A5) is performed over the undeformed middle plane of the plate, and the quantity $\frac{\partial(\xi, \eta)}{\partial(x, y)}$ is the Jacobian of the transformation from the $\xi\eta$ -plane to the xy -plane. The upper cover projects onto the xy -plane in a negatively oriented region; thus, the minus sign of equation (A1) is canceled out on the right-hand side of equation (A5). The Jacobian is

$$\begin{aligned} \frac{\partial(\xi, \eta)}{\partial(x, y)} &= \begin{vmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} \end{vmatrix} \\ &= (1 + u_{+,x})(1 + v_{+,y}) - u_{+,y}v_{+,x} \end{aligned} \quad (A6)$$

On substitution of equations (A4) and (A6) into the integral (A5), there results for the contribution of the surface integral over the upper cover to the volume of the deformed plate

$$\iint \zeta \frac{\partial(\xi, \eta)}{\partial(x, y)} dx dy = \iint \left[\frac{h}{2} (1 + u_{+,x} + v_{+,y} + u_{+,x} v_{+,y} - u_{+,y} v_{+,x}) \right. \\ \left. + w_+ + w_+ u_{+,x} + w_+ v_{+,y} + w_+ u_{+,x} v_{+,y} - w_+ u_{+,y} v_{+,x} \right] dx dy \quad (A7)$$

Lower cover.- For the lower cover of the plate the quantities ξ , η , and ζ are expressed in terms of displacements and the xyz coordinates as follows:

$$\left. \begin{aligned} \xi &= x + u_-(x, y) \\ \eta &= y + v_-(x, y) \\ \zeta &= -\frac{h}{2} + w_-(x, y) \end{aligned} \right\} \quad (A8)$$

The surface integral over the lower cover becomes

$$- \iint_{S_-} \zeta d\xi d\eta = - \iint \zeta \frac{\partial(\xi, \eta)}{\partial(x, y)} dx dy \quad (A9)$$

where the minus sign must be retained because the lower cover projects onto the xy-plane in a positively oriented region. The Jacobian is now written

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} \\ = (1 + u_{-,x})(1 + v_{-,y}) - u_{-,y} v_{-,x} \quad (A10)$$

and the contribution of the integral over the lower cover to the volume of the deformed plate is

$$\begin{aligned}
-\iint \zeta \frac{\partial(\xi, \eta)}{\partial(x, y)} dx dy &= \iint \left[\frac{h}{2}(1 + u_{-,x} + v_{-,y} + u_{-,x}v_{-,y} - u_{-,y}v_{-,x}) \right. \\
&\quad - w_- - w_-u_{-,x} - w_-v_{-,y} - w_-u_{-,x}v_{-,y} \\
&\quad \left. + w_-u_{-,y}v_{-,x} \right] dx dy \tag{A11}
\end{aligned}$$

Edge. - It is assumed that the drop cords along the edge of the plate remain straight as they are displaced. Thus, the final coordinates of points along the edge of the plate are given by the following functions:

$$\left. \begin{aligned}
\xi &= x + \frac{u_+(x, y) - u_-(x, y)}{h} z + \frac{u_+(x, y) + u_-(x, y)}{2} \\
\eta &= y + \frac{v_+(x, y) - v_-(x, y)}{h} z + \frac{v_+(x, y) + v_-(x, y)}{2} \\
\zeta &= z + \frac{w_+(x, y) - w_-(x, y)}{h} z + \frac{w_+(x, y) + w_-(x, y)}{2}
\end{aligned} \right\} \tag{A12}$$

In these equations x and y are on the boundary of the middle plane of the plate and can be thought of as functions of s , the coordinate along this boundary. The surface integral over the plate edge can be written

$$-\iint_{\text{Edge}} \zeta d\xi d\eta = -\oint \int_{-h/2}^{h/2} \zeta \frac{\partial(\xi, \eta)}{\partial(z, s)} dz ds \tag{A13}$$

The normal vector to the plate edge projects onto the z, s surface in a positive region; hence, the minus sign is retained in equation (A13). The Jacobian in equation (A13) is

$$\begin{aligned}
\frac{\partial(\xi, \eta)}{\partial(z, s)} &= \begin{vmatrix} \frac{\partial \xi}{\partial z} & \frac{\partial \xi}{\partial s} \\ \frac{\partial \eta}{\partial z} & \frac{\partial \eta}{\partial s} \end{vmatrix} \\
&= \begin{vmatrix} \frac{u_+ - u_-}{h} & \frac{\partial \xi}{\partial x} \frac{dx}{ds} + \frac{\partial \xi}{\partial y} \frac{dy}{ds} \\ \frac{v_+ - v_-}{h} & \frac{\partial \eta}{\partial x} \frac{dx}{ds} + \frac{\partial \eta}{\partial y} \frac{dy}{ds} \end{vmatrix} \\
&= \frac{u_+ - u_-}{h} \left[\left(\frac{v_+ - v_-}{h} \right)_{,x} z + \frac{v_{+,x} + v_{-,x}}{2} \right] \frac{dx}{ds} + \frac{u_+ - u_-}{h} \left[1 + \left(\frac{v_+ - v_-}{h} \right)_{,y} z \right. \\
&\quad \left. + \frac{v_{+,y} + v_{-,y}}{2} \right] \frac{dy}{ds} - \frac{v_+ - v_-}{h} \left[1 + \left(\frac{u_+ - u_-}{h} \right)_{,x} z + \frac{u_{+,x} + u_{-,x}}{2} \right] \frac{dx}{ds} \\
&\quad - \frac{v_+ - v_-}{h} \left[\left(\frac{u_+ - u_-}{h} \right)_{,y} z + \frac{u_{+,y} + u_{-,y}}{2} \right] \frac{dy}{ds} \tag{A14}
\end{aligned}$$

When the last of equations (A12) and equation (A14) are substituted into equation (A13), the contribution to the volume of the surface integral over the plate edges becomes

$$\begin{aligned}
& - \oint \left\{ \left(1 + \frac{w_+ - w_-}{h} \right) \left(\frac{u_+ - u_-}{h} \right) \frac{h^3}{12} \left[\left(\frac{v_+ - v_-}{h} \right)_{,x} \frac{dx}{ds} + \left(\frac{v_+ - v_-}{h} \right)_{,y} \frac{dy}{ds} \right] \right. \\
& - \left(1 + \frac{w_+ - w_-}{h} \right) \left(\frac{v_+ - v_-}{h} \right) \frac{h^3}{12} \left[\left(\frac{u_+ - u_-}{h} \right)_{,x} \frac{dx}{ds} + \left(\frac{u_+ - u_-}{h} \right)_{,y} \frac{dy}{ds} \right] \\
& + \frac{w_+ + w_-}{2} \left[\left(\frac{u_+ - u_-}{h} \right) \left(\frac{v_{+,x} + v_{-,x}}{2} \right) h \frac{dx}{ds} + \left(1 + \frac{v_{+,y} + v_{-,y}}{2} \right) \left(\frac{u_+ - u_-}{h} \right) h \frac{dy}{ds} \right. \\
& \left. - \left(1 + \frac{u_{+,x} + u_{-,x}}{2} \right) \left(\frac{v_+ - v_-}{h} \right) h \frac{dx}{ds} - \left(\frac{v_+ - v_-}{h} \right) \left(\frac{u_{+,y} + u_{-,y}}{2} \right) h \frac{dy}{ds} \right] \Big\} ds \tag{A15}
\end{aligned}$$

Total volume.- The total volume in the deformed plate is, of course, simply the sum of equations (A7), (A11), and (A15). When equations (4) are used to express the deformed volume in terms of the overall plate displacement and rotation quantities, there is obtained

$$\begin{aligned}
 V = & \iint \left\{ (1 + e)h \left[1 + u_{,x} + v_{,y} + u_{,x}v_{,y} - u_{,y}v_{,x} + \frac{1}{4}(h\alpha)_{,x}(h\beta)_{,y} \right. \right. \\
 & \left. \left. - \frac{1}{4}(h\alpha)_{,y}(h\beta)_{,x} \right] + w \left[(h\alpha)_{,x} + (h\beta)_{,y} + (h\beta)_{,y}u_{,x} - (h\beta)_{,x}u_{,y} \right. \right. \\
 & \left. \left. + (h\alpha)_{,x}v_{,y} - (h\alpha)_{,y}v_{,x} \right] \right\} dx dy - \oint \left\{ \left[(1 + e) \frac{h^3}{12} (\alpha\beta)_{,x} - \beta\alpha_{,x} \right] \right. \\
 & \left. + wh(\alpha v_{,x} - \beta u_{,x}) - wh\beta \right] \frac{dx}{ds} + \left[(1 + e) \frac{h^3}{12} (\alpha\beta)_{,y} - \beta\alpha_{,y} \right] \\
 & \left. + wh(\alpha v_{,y} - \beta u_{,y}) + wh\alpha \right] \frac{dy}{ds} \Big\} ds \tag{A16}
 \end{aligned}$$

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Simplified Expression for Change in Volume

Since a linear theory is sought, any terms in equation (A16) which are of a degree higher than two in the plate displacements and rotations or their derivatives may be neglected. Higher degree terms lead to nonlinearities in the differential equations which result from the application of the variational technique. With this simplification the change in volume due to displacements in the inflatable plate becomes

$$\begin{aligned}
 \Delta V = & \iint \left[hu_{,x} + hv_{,y} + hu_{,x}v_{,y} - hu_{,y}v_{,x} + \frac{h}{4}(h\alpha)_{,x}(h\beta)_{,y} - \frac{h}{4}(h\alpha)_{,y}(h\beta)_{,x} \right. \\
 & \left. + he + w(h\alpha)_{,x} + w(h\beta)_{,y} \right] dx dy - \oint \left\{ \left[\frac{h^3}{12} (\alpha\beta)_{,x} - \beta\alpha_{,x} \right] - wh\beta \right] \frac{dx}{ds} \\
 & \left. + \left[\frac{h^3}{12} (\alpha\beta)_{,y} - \beta\alpha_{,y} \right] + wh\alpha \right] \frac{dy}{ds} \Big\} ds \tag{A17}
 \end{aligned}$$

When the relations

$$\left. \begin{aligned} \frac{dx}{ds} &= \frac{\partial y}{\partial N} \\ \frac{dy}{ds} &= - \frac{\partial x}{\partial N} \end{aligned} \right\} \quad (A18)$$

are utilized, Gauss' theorem in the form given in equation (27) can be applied to the line integral in equation (A17). The final result for ΔV , after substitution for e in terms of α and β , is

$$\begin{aligned} \Delta V = \iint h \left[u_{,x} + v_{,y} + u_{,x}v_{,y} - u_{,y}v_{,x} + \frac{h^2}{12}(\alpha_{,x}\beta_{,y} - \alpha_{,y}\beta_{,x}) \right. \\ \left. - \frac{\alpha^2}{2} - \frac{\beta^2}{2} - \alpha w_{,x} - \beta w_{,y} \right] dx dy \end{aligned} \quad (A19)$$

APPENDIX B

ESTIMATION OF VALIDITY OF LINEAR THEORY

A simple problem is discussed in this appendix in order to provide some idea of the range of parameters for which linear theory is valid. The results of a nonlinear analysis of the problem are compared with the results of the linear analysis. The problem considered is a long plate of constant depth which, after inflation, is simply supported on its long edges in such a manner that these edges cannot move. For simplicity the plate is assumed to be loaded by a uniform lateral load equally divided between the upper and lower covers. For the linear solution, the manner in which the loading is divided between the covers is immaterial. For a nonlinear solution, on the other hand, if the loading is not equally divided between the covers, an additional nonlinear term arises in the theory. A cross section of the plate under consideration is shown in figure 3. The x-axis is assumed to be normal to the long, simply supported edges, and the distance between these edges is L.

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For this problem the potential energy in varied form can be written as follows when nonlinear terms are included:

$$\delta \Pi = \int_0^L \left[N_x^* \delta \left(u_{,x} + \frac{1}{2} w_{,x}^2 + \frac{h^2}{8} e_{,x}^2 \right) + M_x \delta (\alpha_{,x} + e_{,x} w_{,x}) - ph \delta (u_{,x} + e - \alpha w_{,x} + e u_{,x}) - q_z \delta w \right] dx \quad (B1)$$

where

$$\left. \begin{aligned} N_x^* &= ph + N_x \\ N_x &= 2A_{11} \left(u_{,x} + \frac{1}{2} w_{,x}^2 + \frac{h^2}{8} e_{,x}^2 \right) \\ M_x &= \frac{h^2}{2} A_{11} (\alpha_{,x} + e_{,x} w_{,x}) \\ e &= - \frac{\alpha^2}{2} \end{aligned} \right\} \quad (B2)$$

When the variation of equation (B1) is performed and appropriate integrations by parts are carried out, there results the following system of nonlinear differential equations:

$$\left. \begin{aligned}
 N_{x,x} + ph\alpha\alpha_{,x} &= 0 \\
 N_x w_{,xx} + N_{x,x} w_{,x} + ph(\alpha + w_{,x})_{,x} - (M_x \alpha\alpha_{,x})_{,x} + q_z &= 0 \\
 M_{x,x} - N_x \frac{h^2}{4} \alpha\alpha_{,x}^2 - \frac{ph^3}{4} \alpha\alpha_{,x}^2 + \left(N_x \frac{h^2}{4} \alpha_{,x} \alpha^2 \right)_{,x} + \frac{ph^3}{4} (\alpha_{,x} \alpha^2)_{,x} \\
 - (M_x w_{,x} \alpha)_{,x} + M_x w_{,x} \alpha_{,x} - ph(\alpha + w_{,x}) - ph u_{,x} \alpha &= 0
 \end{aligned} \right\} \quad (B3)$$

and the following boundary conditions at $x = 0$ and L :

$$\left. \begin{aligned}
 u = w = 0 \\
 M_x - M_x w_{,x} \alpha + N_x \frac{h^2}{4} \alpha_{,x} \alpha^2 + \frac{ph^3}{4} \alpha_{,x} \alpha^2 &= 0
 \end{aligned} \right\} \quad (B4)$$

This system of equations can be simplified (partially linearized) by assuming that α and its derivatives are small compared with w and its derivatives and by neglecting terms of second degree or higher in α and terms which contain products of α times w or u or their derivatives. For practical inflatable-plate proportions, this assumption appears justified in this problem. When the equations are simplified in this manner there results the differential equations

$$\left. \begin{aligned}
 N_{x,x} &= 0 \\
 N_x w_{,xx} + ph(\alpha + w_{,x})_{,x} + q_z &= 0 \\
 M_{x,x} - ph(\alpha + w_{,x}) &= 0
 \end{aligned} \right\} \quad (B5)$$

and the boundary conditions at $x = 0$ and L

$$u = w = M_x = 0 \quad (B6)$$

where

$$\left. \begin{aligned} N_x &= 2A_{11} \left(u_{,x} + \frac{1}{2} w_{,x}^2 \right) \\ M_x &= \frac{h^2}{2} A_{11} \alpha_{,x} \end{aligned} \right\} \quad (B7)$$

An exact solution to this system of equations can be obtained. The result is in terms of rather involved transcendental expressions, however, and it is difficult to find a relationship between the nonlinear and linear solutions. For the purposes of this appendix a simple approximate solution using the Galerkin method suffices.

Equations (B5) can be written

$$u_{,x} + \frac{1}{2} w_{,x}^2 = C_1 \quad (B8a)$$

$$2A_{11} C_1 w_{,xx} + \text{ph}(\alpha + w_{,x})_{,x} = -q_z \quad (B8b)$$

$$\frac{h^2}{2} A_{11} \alpha_{,xx} - \text{ph}(\alpha + w_{,x}) = 0 \quad (B8c)$$

where C_1 is a constant and the boundary conditions (B6) at $x = 0$ and L are

$$u = w = \alpha_{,x} = 0 \quad (B9)$$

The following assumptions for w and α satisfy the last two of equations (B9) exactly:

$$w = w_1 \sin \frac{\pi x}{L} \quad (B10a)$$

$$\alpha = \alpha_1 \cos \frac{\pi x}{L} \quad (B10b)$$

When equations (B10) are substituted into equation (B8b) and the resulting equation is multiplied by $\sin \frac{\pi x}{L}$ and integrated over x from 0 to L , there results

$$A_{11}C_1w_1\left(\frac{\pi}{L}\right)^2 + \frac{ph}{2}\alpha_1\frac{\pi}{L} + \frac{ph}{2}w_1\left(\frac{\pi}{L}\right)^2 = q_z\frac{2}{\pi} \quad (B11)$$

From equation (B8c) there is obtained

$$\alpha_1 = \frac{-w_1\frac{\pi}{L}}{\frac{h^2}{2}\frac{A_{11}}{ph}\left(\frac{\pi}{L}\right)^2 + 1} \quad (B12)$$

The constant C_1 is obtained by substituting equation (B10a) into equation (B8a), integrating to find u , and using the boundary conditions $u = 0$ at $x = 0$ and $x = L$. The result is

$$C_1 = \frac{w_1^2}{4}\left(\frac{\pi}{L}\right)^2 \quad (B13)$$

When equations (B12) and (B13) are substituted into equation (B11), some algebraic manipulations lead to

$$\frac{w_1}{h}\left[(J+1)\left(\frac{w_1}{h}\right)^2 + 1\right] = \frac{4}{\pi}\frac{q_z}{p}\frac{L^2}{\pi^2h^2}\frac{J+1}{J} \quad (B14)$$

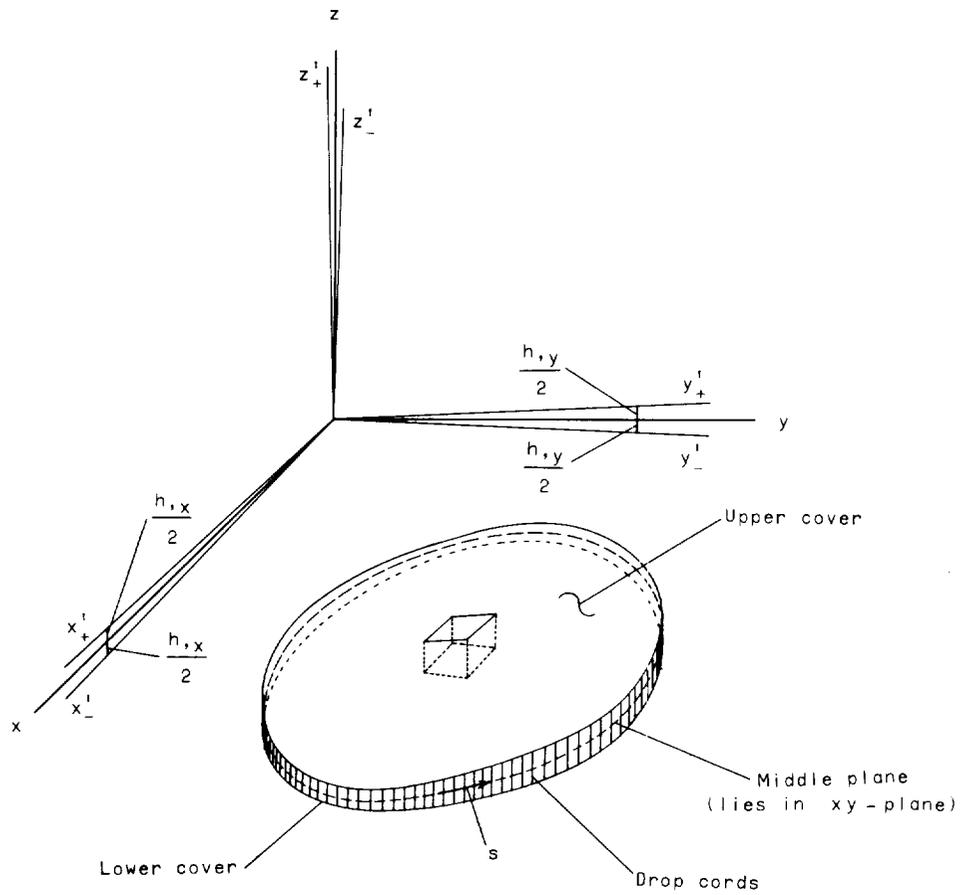
where $J = \frac{A_{11}}{2ph}\left(\frac{\pi h}{L}\right)^2$. Calculations showed very little difference between this approximate result and the exact solution. The linear solution is obtained when $(J+1)\left(\frac{w_1}{h}\right)^2$ is neglected compared with unity. It is easy to see, for instance, that the linear solution is within 5 percent of the nonlinear solution provided that

$$(J+1)\left(\frac{w_1}{h}\right)^2 < 0.05 \quad (B15)$$

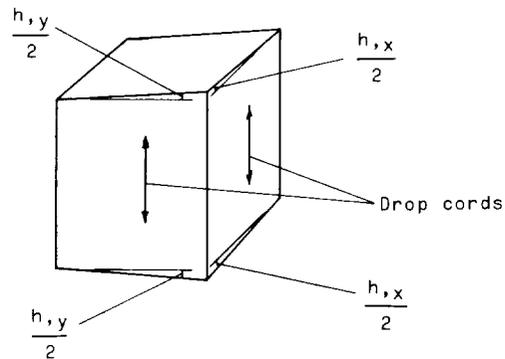
This result is strictly applicable only to the problem under consideration in this appendix but does give a rough indication of what might happen in other problems.

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(a) Plate having linear taper in depth and arbitrary shape in planform.



(b) Enlarged view of plate element showing taper angles.

Figure 1.- Inflatable plate with basic coordinate systems.

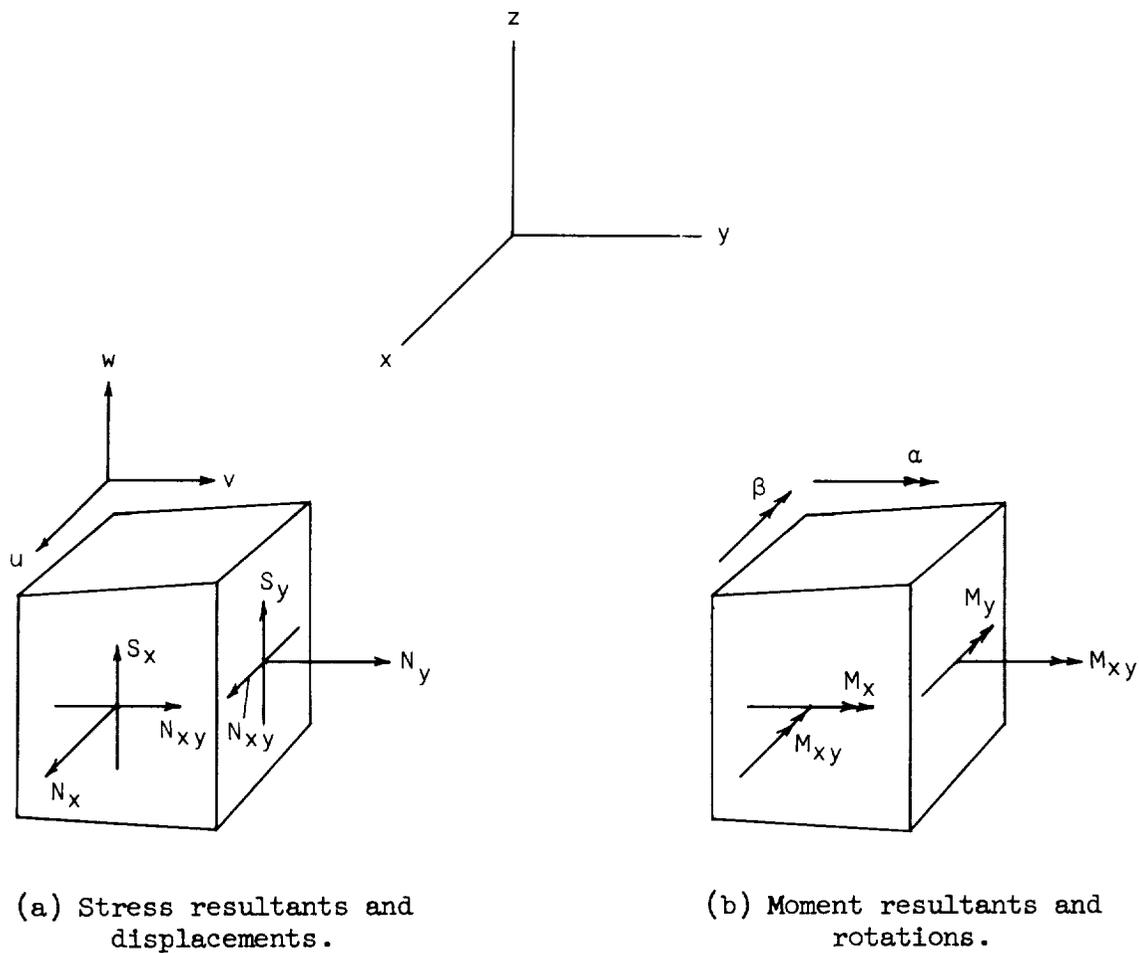


Figure 2.- Plate stress resultants, moments, and twists; plate displacements and rotations applied to an element in their positive senses.

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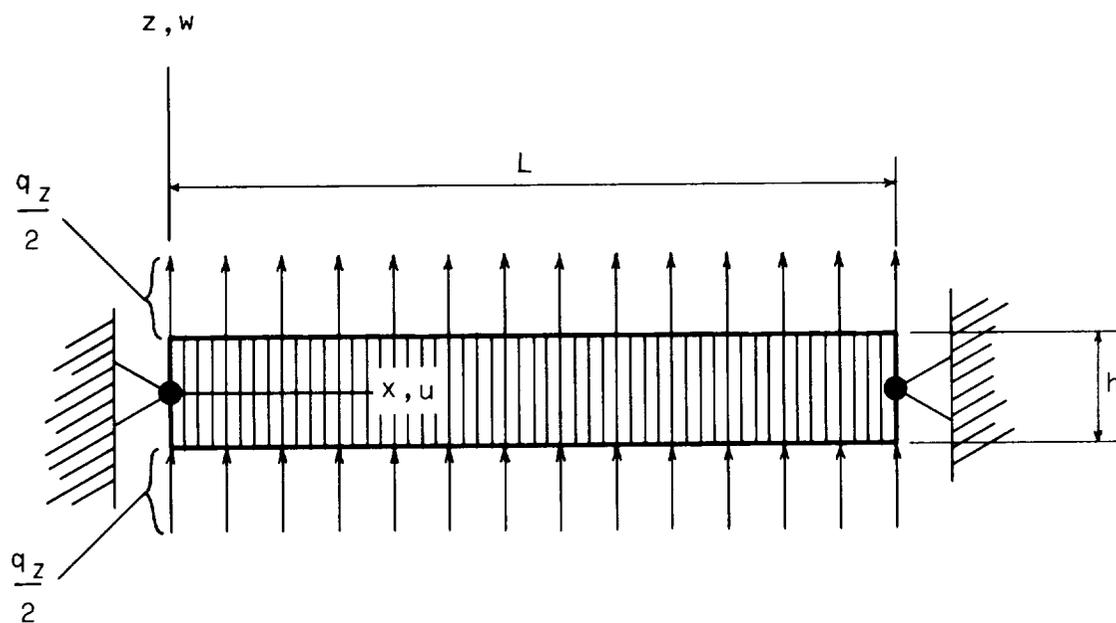


Figure 3.- Cross section of long uniformly loaded inflatable plate simply supported on long edges.

