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# TECHNICAL NOTE

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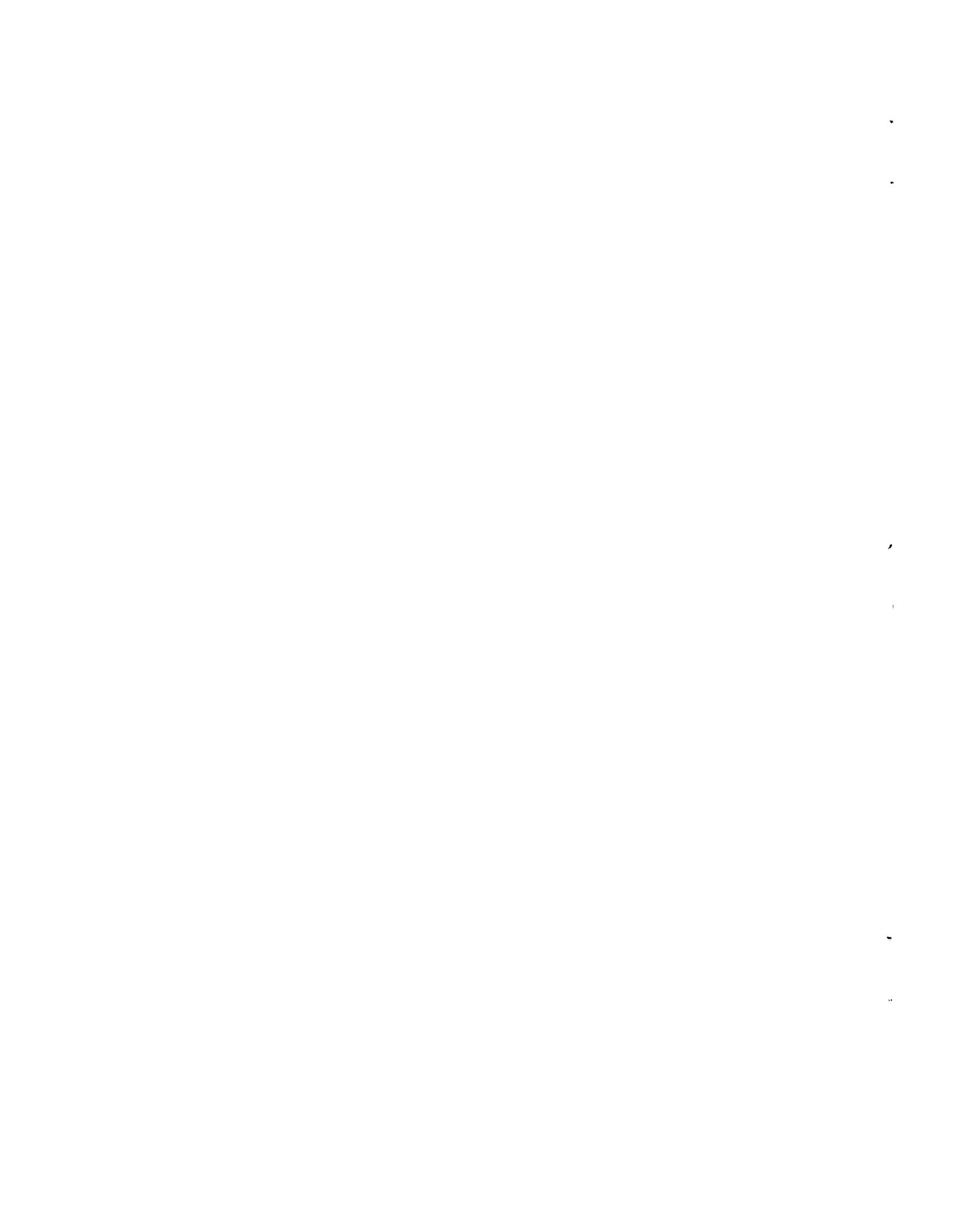
HEAT TRANSFER AT THE REATTACHMENT ZONE  
OF SEPARATED LAMINAR BOUNDARY LAYERS

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## SUMMARY

The flow and heat transfer are analyzed at the reattachment zone of two-dimensional separated laminar boundary layers. The fluid is considered to be flowing normal to the wall at reattachment. An approximate expression is derived for the heat transfer in the reattachment region and a calculated value is compared with an experimental measurement.

## INTRODUCTION

The mechanism of heat transfer through separated regions is very complicated and little understood. Chapman in reference 1 first analyzed this problem and obtained an estimate of the average heat transfer in this region. A survey of the literature reveals that no local heat-transfer analysis has been done for a Chapman type separated region. Reference 2 indicates that the complete physical problem is too complex to be adequately described by a simple flow model. Existing experimental work (ref. 3) shows that the maximum heat transfer in a separated region occurs at the reattachment point. References 1 and 4 indicate that one may be able to analyze the flow and heat transfer near this point. This paper is concerned with heat transfer in the reattachment zone for normal reattachment of a two-dimensional separated laminar boundary layer. Figures 1 and 2 show the model considered.

The flow approaching the reattachment zone is considered inviscid but rotational. The viscous effect is assumed to be confined to the boundary layer which develops along the  $x$  axis. The partition into inviscid and viscous regions is justified because, as will be seen subsequently, the vorticity in the inviscid region is smaller than that in the boundary layer by an order of magnitude. In the following analysis the fluid in the inviscid region is assumed to be incompressible and a closed-form solution of the flow field is developed. This solution is then used to solve the boundary-layer equations for heat transfer along the  $x$  axis.

## NOMENCLATURE

h	total enthalpy
K	function defined by equation (15)
L	length of reattachment zone
l	length of separated mixing layer
Nu	Nusselt number, $\frac{q_w Pr l}{(h_e - h_w) \mu_e}$
P	pressure
Pr	Prandtl number
$q_w$	heat transfer to the wall per unit area per unit time
R	nose radius of a hypersonic blunt body
Re	Reynolds number, $\frac{\rho_e u_e l}{\mu_e}$
s	variable defined by equation (13)
u	x component of velocity
$u_e$	streamwise velocity at the outer edge of mixing layer
v	y component of velocity
$v_0$	$v(0, L)$
X	$\frac{x}{L}$
x	distance along the wall from reattachment point
$x_0$	particular value of x greater than L but still near the reattachment point
Y	$\frac{y}{L}$
y	distance in direction normal to wall
z	variable defined by equation (13) or function defined by equation (A7)

$\lambda$	parameter defined by equations (3) and (23)
$\mu$	dynamic viscosity
$\xi$	dummy variable
$\rho$	density
$\varphi, \chi$	Laplace transforms defined by equations (13)
$\psi$	stream function
$\Omega$	vorticity

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#### Subscripts

e	outer edge of mixing layer
r	average value for reattachment zone ( $0 < x < L$ )
t	reattachment point
w	wall

#### INVISCID FLOW REGION

The flow approaching the reattachment zone shall be considered inviscid, incompressible, but rotational. Figure 2 shows the flow model studied. For this case, the distribution of the streamlines on the x-y plane is given by the equation

$$\nabla^2 \psi = -\Omega(\psi) \quad (1)$$

where  $\psi$  is the usual stream function, defined as:

$$u = \frac{\partial \psi}{\partial y}; \quad v = -\frac{\partial \psi}{\partial x} \quad (2)$$

To obtain an expression for  $\Omega(\psi)$ , the velocity profile of the incoming stream at a distance  $y = L$  must be known. The distance  $L$  is defined such that the incoming stream is unaffected by the existence of the reattachment wall for  $y > L$ ; that is,  $u = 0$  for  $y \geq L$  and  $u$  is positive for  $y < L$ . Actually  $u$  will be slightly negative for  $y > L$ , as the fluid is being entrained into the mixing layer from the separated region, and  $u$  is positive for  $y < L$  because of the wall. The value

of  $L$  will be obtained a posteriori from the solution of the flow field. A study of the mixing layer solution of reference 1 shows that the stream-wise velocity distribution can be quite accurately represented by the expression

$$v(x, L) = -v_0 e^{-\lambda x} \quad (3)$$

From the definition of  $L$

$$u(x, L) = 0 \quad (4)$$

and

$$\frac{\partial u(x, L)}{\partial y} = 0 \quad (5)$$

The vorticity,  $\Omega(\psi)$ , may be evaluated at any boundary for it is constant along a streamline. In the present study, the vorticity is evaluated along the boundary at  $y = L$ . The vorticity distribution corresponding to the velocity profile of equation (3) and to condition (5) is derived in the following manner. From equation (3),

$$\Omega_{y=L} = \frac{dv}{dx} = \lambda v_0 e^{-\lambda x} \quad (6)$$

To express equation (6) in terms of  $\psi$ , we write equation (3) as

$$v_{y=L} = -\frac{d\psi}{dx} = -v_0 e^{-\lambda x}$$

When the above equation is integrated to satisfy the definition of the dividing streamline,  $\psi(0, L) = 0$ , (see ref. 1) there results the relationship,

$$\psi(x, L) = \frac{v_0}{\lambda} (1 - e^{-\lambda x}) \quad (7)$$

Equation (6) is now rewritten with the aid of equation (7) as

$$\Omega_{y=L}(\psi) = \lambda v_0 - \lambda^2 \psi \quad (8)$$

Equation (8) shows the relationship between the vorticity and the stream function at  $y = L$ . Vorticity, however, is constant along a streamline throughout the flow field. Equation (8), therefore, is applicable for the entire inviscid region.

The inviscid flow equation (1) and the complete boundary conditions can now be written for  $0 < x < \infty$  and  $0 < y < L$  as:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \lambda^2 \psi + v_0 \lambda = 0 \quad (9)$$

$$\psi(0, y) = 0 \quad (10)$$

$$\psi(x, 0) = 0 \quad (11)$$

$$\psi(x, L) = \frac{v_0}{\lambda} (1 - e^{-\lambda x}) \quad (7)$$

$$|\psi(x, y)| < M \quad (12)$$

where  $M$  is an arbitrarily large positive number and equation (12) expresses the bounded condition. The solution of the above boundary value problem is obtained by the Laplace transform method as follows.

The Laplace transforms of the stream function are first defined as

$$\left. \begin{aligned} \varphi(s, y) &= L_x \{ \psi(x, y) \} \\ \chi(s, z) &= L_y \{ \varphi(s, y) \} \end{aligned} \right\} \quad (13)$$

The transforms of the boundary conditions (11), (7), and (12) are also needed in the course of the solution. These are, respectively,

$$\varphi(s, 0) = 0 \quad (11a)$$

$$\varphi(s, L) = \frac{v_0}{s(s + \lambda)} \quad (7a)$$

and

$$|\varphi(s, y)| < N \quad \text{for all } s > 0 \quad (12a)$$

where  $N$  is another arbitrarily large positive number. When the Laplace transformation is performed twice in equation (9) and the boundary conditions (10) and (11a) are applied, the following equation results.

$$x(s, z) = \frac{A(z) + B(s) - (v_0\lambda/sz)}{K^2 + z^2} \quad (14)$$

where

$$K^2 = s^2 - \lambda^2 \quad (15)$$

and  $A(z)$  and  $B(s)$  are unknown but particular functions of  $z$  and  $s$ , respectively. An inverse Laplace transformation of equation (14) with the aid of boundary condition (7a) yields

$$\begin{aligned} \varphi(s, y) = & \frac{1}{\sin KL} \left\{ \frac{v_0 \sin Ky}{s(s + \lambda)} - \frac{1}{K} \sin Ky \int_0^L [\sin K(L - \xi)] A(\xi) d\xi \right. \\ & + \frac{v_0\lambda \sin Ky}{sK^2} (1 - \cos KL) + \frac{1}{K} \sin KL \int_0^y [\sin K(y - \xi)] A(\xi) d\xi \\ & \left. - \frac{v_0\lambda \sin KL}{sK^2} (1 - \cos Ky) \right\} \quad (16) \end{aligned}$$

Now, in the above equation, the integrals which include the unknown function  $A(\xi)$  in the integrand will be evaluated with the aid of the bounded condition (12a). Equation (16) shows that the first term on the right-hand side of the equation,  $1/\sin KL$ , becomes  $\infty$  when

$$K = \pm \frac{n\pi}{L} \quad (17)$$

where  $n = 0, 1, 2, \dots$ . From equations (15) and (17), this means that  $1/\sin KL$  becomes  $\infty$  when

$$s = \pm \sqrt{\left(\frac{n\pi}{L}\right)^2 + \lambda^2} \quad (18)$$

The boundary condition (12a) implies that the function  $\varphi(s, y)$  must be bounded for all values of  $s > 0$ . It is, therefore, necessary that the function in the braces in equation (16) be zero when  $s = \sqrt{(n\pi/L)^2 + \lambda^2}$  in order to satisfy the bounded condition (12a). The integral with the unknown integrand, therefore, becomes

$$\int_0^y \left[ \sin \frac{n\pi}{L} (L - \xi) \right] A(\xi) d\xi = \frac{n\pi v_0}{L} \left\{ \frac{\lambda [1 - (-1)^n]}{(n\pi/L)^2 \sqrt{(n\pi/L)^2 + \lambda^2}} + \frac{1}{\sqrt{(n\pi/L)^2 + \lambda^2} [\sqrt{(n\pi/L)^2 + \lambda^2} + \lambda]} \right\} \quad (19)$$

Finally, the complete inverse transformation of equation (16) is obtained by finding the residues at all the single poles for  $s \leq 0$ . The solution of the boundary-value problem, equation (9) with its boundary conditions, is thus obtained and is

$$\frac{\psi(X, Y)}{v_0 L} = \frac{\coth \lambda L}{\lambda L} (\sinh \lambda Y) + \frac{1 - \cosh \lambda Y}{\lambda L} - \frac{Y}{\lambda L} \exp(-\lambda L X) - 2 \sum_{n=1}^{\infty} \frac{\lambda L (\sin n\pi Y)}{n\pi [(n\pi)^2 + (\lambda L)^2]} \exp\left[-\sqrt{(n\pi)^2 + (\lambda L)^2} X\right] \quad (20)^1$$

The value of  $\lambda L$  is found by satisfying condition (4) with the aid of the solution, equation (20). At  $y = L$ , from the continuity equation,  $u$ , if other than zero, is a monotonically increasing function of  $x$  with  $u(0, L) = 0$  and a maximum at  $x = \infty$ . Thus, condition (4) can be satisfied for all  $x$  if one sets  $u(\infty, L) = 0$ . Equation (20) can be used to show that

$$\frac{u(\infty, L)}{v_0} = \frac{1}{\sinh \lambda L} \quad (21)$$

The velocity components,  $u(x, y)$  and  $v(x, y)$ , were found to remain practically constant for  $|u(\infty, L)/v_0| \leq 0.01$ . Thus, condition (4) can be considered to be satisfied when  $u(\infty, L)/v_0 = 0.01$ , and from equation (21)  $\lambda L$  is found to be 5.3. In subsequent numerical work

$$\lambda L = 5.3 \quad (22)$$

is used.

A study of reference 1 shows that  $\lambda$  in equation (3) can be expressed quite accurately by

$$\lambda = \frac{1}{2.222} \frac{\sqrt{\text{Re}}}{l} \quad (23)$$

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<sup>1</sup>Equation (1) can also be solved by the method of separation of variables (see appendix).

Thus, for a given  $Re$  and  $\lambda$ ,  $L$  and  $\lambda$  can be readily determined from equations (22) and (23).

The velocity and pressure distributions along the wall, obtained from equation (20), are shown in figure 3.

#### BOUNDARY LAYER AND HEAT-TRANSFER ANALYSIS

To evaluate the heat transfer along the wall by conventional boundary-layer theory one must first investigate the effects of the vorticity and the enthalpy gradient of the inviscid region on the boundary-layer solution. The absolute magnitude of the vorticity interaction parameter (ratio of average vorticity in the inviscid region to that in the boundary layer) was estimated to be less than 0.1 for  $0 < X < 1$  and  $Re \geq 10^4$ . Classical boundary-layer theory can be applied with good engineering accuracy when the interaction parameter is of this order of magnitude (ref. 5). The effect of an enthalpy gradient in the inviscid region on the thermal boundary layer will be about the same as that of the inviscid vorticity on the momentum boundary layer.

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The present study concentrates on the reattachment zones that may exist on a hypersonic vehicle. The heat-transfer analysis (an approximation) is based on the theory of a highly cooled boundary layer for hypersonic blunt bodies, developed in reference 6, and the pressure and velocity distributions obtained in the preceding section.

A typical heat-transfer variation near the reattachment point is shown in figure 3. In view of the drastic variation of the local heat transfer within the small distance  $L$ , which is of the order of the mixing layer thickness, an average heat transfer in this zone is of greater engineering interest than the local heat transfer shown in figure 3. The reattachment zone is defined as that area of the wall along which flow readjustment takes place and is defined two-dimensionally by  $0 \leq X \leq 1$ . The average heat transfer at the reattachment zone was calculated for several pressures in the separated region and resulted in development of the following semiempirical expression

$$q_{w,r} \approx 2^{-3/2} Pr^{-2/3} \sqrt{\rho_t \mu_t} \sqrt{\frac{v_0}{L}} (h_t - h_w) \left( 0.76 + 1.411 \frac{Pe}{P_t} \right) \quad (24)$$

for  $0.1 \leq Pe/P_t \leq 0.5$ . To express the heat transfer in terms of the fluid properties at the outer edge of the mixing layer, the following approximations are made:

From reference 6:

$$\rho_t \mu_t \approx (\rho_e \mu_e) \frac{P_t}{P_e}$$

From reference 1 for  $Pr$  close to 1:

$$\frac{h_t - h_w}{h_e - h_w} \approx \frac{v_o}{u_e} = 0.587$$

Equation (24) now becomes, with the aid of equations (22) and (23),

$$Nu_r = \frac{q_{w,r} Pr l}{(h_e - h_w) \mu_e} \approx 0.0463 Pr^{1/3} Re^{3/4} \left( \frac{P_e}{P_t} \right)^{-1/2} \left( 0.76 + 1.411 \frac{P_e}{P_t} \right) \quad (25)$$

The pressure ratio  $P_e/P_t$  can be obtained from the assumption of an isentropic compression along the dividing streamline of the mixing layer as shown in reference 4. It should be remembered that the area of the reattachment zone varies with  $Re$ .

Sometimes, it may be desired to calculate the average heat transfer to an area which is a bit larger than the reattachment zones, but includes it. From figure 2 and the definition of  $L$  one can see that the pressure along the walls is essentially constant for  $x > L$ . Hence, the average heat transfer for  $L < x \leq x_o$  is readily found to be

$$Nu_{(L < x \leq x_o)} \approx \frac{0.0926 Pr^{1/3} Re^{3/4}}{0.2922(x_o/l)^{1/2} Re^{1/4} + 1} \quad (26)$$

A weighted average of equations (25) and (26) would give the average heat transfer near the reattachment point for  $x_o > L$ .

#### DISCUSSION AND CONCLUDING REMARKS

There are no experimental data available, to the authors' present knowledge, which could be used directly to compare the above theory. However, a rough comparison can be made with the experimental results of reference 3 in which an average heat transfer for  $x_o/l = 0.08$  was measured for a separated laminar boundary layer reattaching at an angle of about  $45^\circ$ . The experimental conditions of reference 3 were used to calculate the average heat transfer which was found to be within 10 percent of the measured value.

The present analysis is for the normal reattachment only and therefore its comparison with the experimental result for the  $45^\circ$  reattachment angle lends only rough support. It is doubtful, however, that the variation of the reattachment angle between  $45^\circ$  and  $90^\circ$  would change the heat-transfer result drastically.

It is interesting to investigate the dependence of the average laminar heat transfer on Reynolds number. Equations (25) and (26) indicate that the Nusselt number varies with  $Re^{3/4}$  in the reattachment zone and with a slightly smaller power of  $Re$  outside this zone. Thus, the average heat transfer for a given  $x_0 > L$  varies between  $Re^{1/2}$  and  $Re^{3/4}$ , the exact value depending on  $x_0$  and  $Re$ . It is worth noting that the length of the reattachment zone,  $L$ , varies inversely with  $Re^{1/2}$ ; therefore, as  $Re$  increases, for a given  $x_0 > L$ , the dependence of the average heat transfer on  $Re$  decreases.

Finally, one can make an approximate comparison of the heat transfer at the stagnation point of a hypersonic vehicle with that at a probable reattachment zone. This comparison showed that the heat transfer at the reattachment zone could be as much as two or more times that at the stagnation point when  $l/R \leq 1$ .

In an actual separated region, the boundary-layer thickness at the separation point will not be zero as assumed in reference 1. For this case, the present analysis remains unchanged through equation (22). The heat-transfer equations (25) and (26), however, should be modified with the proper values of  $\lambda$  and  $v_0$  at  $y = L$  which may be obtained from an analysis similar to that in reference 7.

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Moffett Field, Calif., July 10, 1961

## APPENDIX

## SOLUTION OF EQUATION (1) BY SEPARATION OF VARIABLES

In a region defined by  $0 < x < L$  and  $0 < y < L$  the streamwise velocity distribution of reference 1 can be closely approximated by

$$v(x, L) = -v_0 \frac{\sinh \lambda(L - x)}{\sinh \lambda L} \quad (A1)$$

Following the method outlined earlier the inviscid flow equation (1) and its accompanying boundary conditions can be written for  $0 < x < L$  and  $0 < y < L$  as

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \lambda^2 \psi - \lambda v_0 \coth \lambda L \quad (A2)$$

$$\psi(x, 0) = 0 \quad (A3)$$

$$\psi(0, y) = 0 \quad (A4)$$

$$\frac{\partial \psi}{\partial x}(L, y) = 0 \quad (A5)$$

$$\psi(x, L) = \frac{v_0}{\lambda} \left[ \coth \lambda L - \frac{\cosh \lambda(L - x)}{\sinh \lambda L} \right] \quad (A6)$$

Condition (12) has been replaced by the requirement that the fluid along the wall at  $x = L$  be moving parallel to the  $x$  axis. That this is the case is apparent from the symmetry of equation (1) and condition (4).

In terms of a new function  $z$ , where

$$\psi(x, y) = z(x, y) + \frac{v_0}{\lambda} \coth \lambda L \quad (A7)$$

the differential equation and its boundary conditions become

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \lambda^2 z \quad (A8)$$

$$z(x, 0) = -\frac{v_0}{\lambda} \coth \lambda L \quad (A9)$$

$$z(0, y) = -\frac{v_0}{\lambda} \coth \lambda L \quad (A10)$$

$$\frac{\partial z}{\partial x}(L, y) = 0 \quad (\text{A11})$$

$$z(x, L) = -\frac{v_0}{\lambda} \frac{\cosh \lambda(L-x)}{\sinh \lambda L} \quad (\text{A12})$$

This new function can be broken into the sum of three functions such that

$$z(x, y) = z_1(x, y) + z_2(x, y) + z_3(x, y) \quad (\text{A13})$$

The superposition principle can be used to break equations (A8) through (A12) into three simpler problems.

For  $z_1$ :

$$\nabla^2 z_1 = \lambda^2 z_1$$

$$z_1(x, 0) = -\frac{v_0}{\lambda} \coth \lambda L$$

$$z_1(0, y) = 0$$

$$\frac{\partial z_1}{\partial x}(L, y) = 0$$

$$z_1(x, L) = 0$$

For  $z_2$ :

$$\nabla^2 z_2 = \lambda^2 z_2$$

$$z_2(x, 0) = 0$$

$$z_2(0, y) = -\frac{v_0}{\lambda} \coth \lambda L$$

$$\frac{\partial z_2}{\partial x}(L, y) = 0$$

$$z_2(x, L) = 0$$

For  $z_3$ :

$$\nabla^2 z_3 = \lambda^2 z_3$$

$$z_3(x, 0) = 0$$

$$z_3(0, y) = 0$$

$$\frac{\partial z_3}{\partial x}(L, y) = 0$$

$$z_3(x, L) = -\frac{v_0}{\lambda} \frac{\cosh \lambda(L-x)}{\sinh \lambda L}$$

Each of these problems is readily solved by the method of separation of variables. Combining their solutions with the aid of equations (A13) and (A7) will give a solution to equation (1). For sufficiently large values of  $\lambda L$  ( $\lambda L \geq 5.3$ ), this solution is equivalent to equation (20). That this should be so can also be seen by examining equations (A1), (A2), and (A6) which for large  $\lambda L$  reduce to equations (3), (9), and (7).

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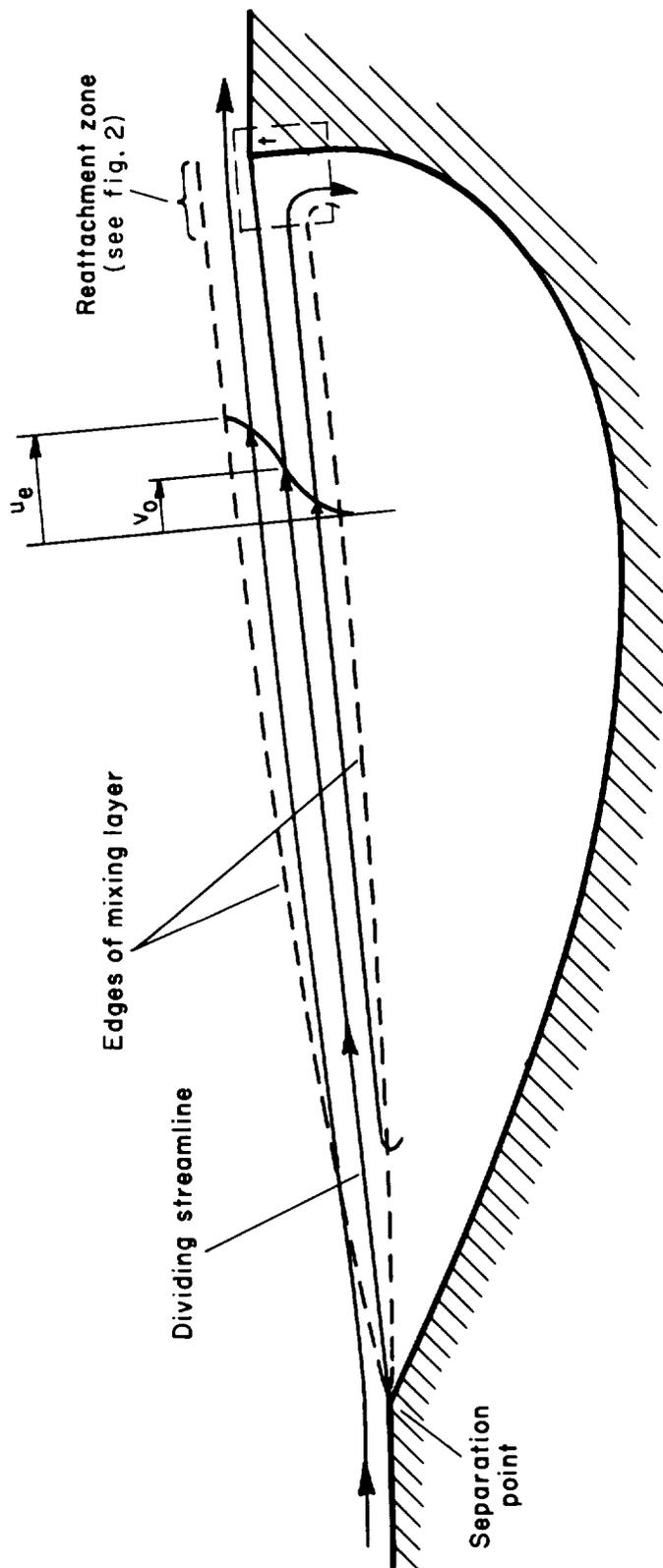


Figure 1.- Separated region studied.

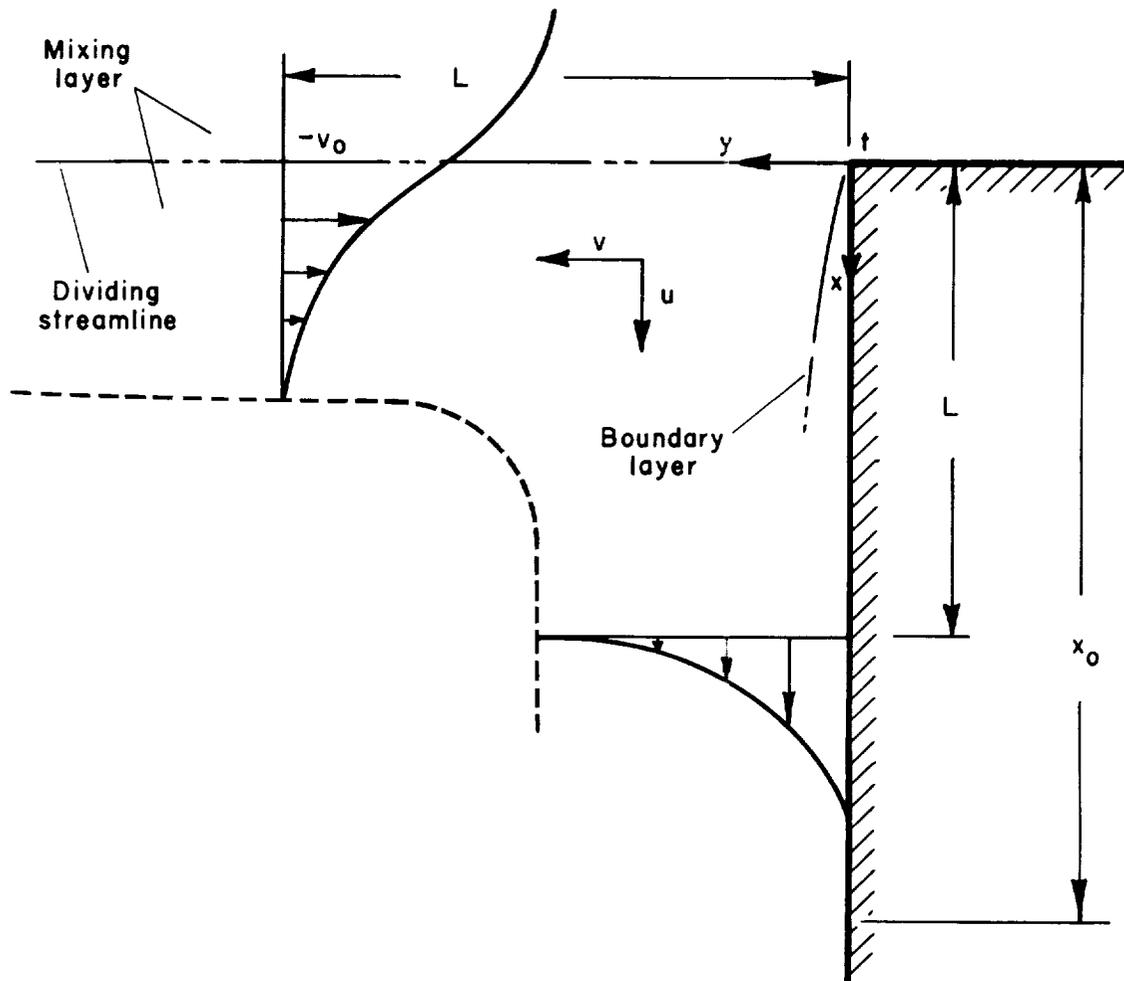


Figure 2.- Reattachment zone flow model.

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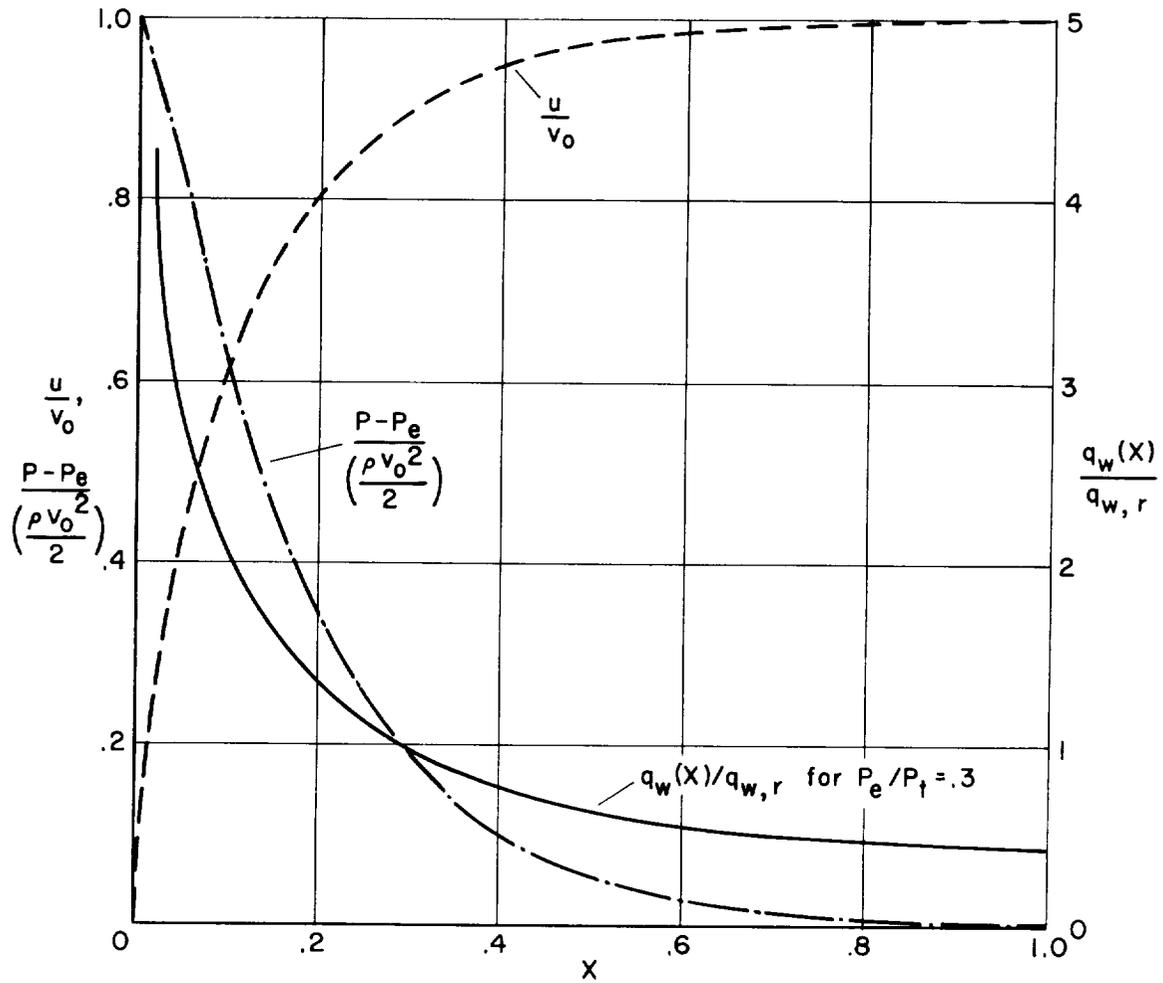


Figure 3.- Inviscid velocity, pressure, and heat-transfer distributions along the wall.

