TECHNICAL NOTE

D-492

A MODIFIED HANSEN'S THEORY AS APPLIED TO THE MOTION OF ARTIFICIAL SATELLITES

Peter Musen

Goddard Space Flight Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON November 1960

A MODIFIED HANSEN'S THEORY AS APPLIED TO THE MOTION OF ARTIFICIAL SATELLITES

by

Peter Musen

SUMMARY

This report presents a theory of oblateness perturbations of the orbits of artificial satellites based on Hansen's theory, with modification for adaptation to fast machine computation. The theory permits the easy inclusion of any gravitational terms and is suitable for the deduction of geophysical and geodetic data from orbit observations on artificial satellites. The computations can be carried out to any desired order compatible with the accuracy of the geodetic parameters.

CONTENTS

A MODIFIED HANSEN'S THEORY AS APPLIED TO THE MOTION OF ARTIFICIAL SATELLITES

INTRODUCTION

This work presents a systematic exposition of a theory of artificial satellites based on Hansen's treatment, starting from the basic principles of the perturbation theory.

Hansen's ideas influenced the development of the theories of celestial bodies for more than a century; but they are, generally speaking, difficult to understand — partly because of the unorthodox treatment of the perturbations and partly because of the way they are presented in his original papers. The basic idea consists in introducing a fictitious auxiliary satellite describing an auxiliary rotating ellipse of constant shape in accordance with Kepler's laws. The position of the real satellite is determined by its deviations in time and space from the position of this auxiliary satellite.

The perturbations *of* the orbit plane, which are small, are separated from the perturbations *in* the orbit plane, which are rather large. The perturbations *of* the orbit plane are determined by four interdependent parameters, two of which were introduced by Hansen himself. The perturbations *in* the orbit plane are the perturbations in the radius-vector and in the mean anomaly; they are determined by means of a single function W, for which a differential equation of the first order is formed. The special characteristics of Hansen's method consist in the addition of the angular perturbations to the mean anomaly and not to the true longitude, and also in the use of just one function to determine all the perturbations in the orbit plane.

In this modification of Hansen's theory the development is a numerical one designed to take full advantage of the speed of modern computing machines. Repetitive operations also are emphasized for the same reason. Finally, in several instances the development has been simplified and shortened by the application of vectorial operations in place of the notations of spherical astronomy that were popular in Hansen's time.

SOME FORMULAS FROM THE VECTORIAL THEORY OF PERTURBATIONS

It can be assumed in the first approximation that a central body consisting of homogeneous spherical layers is completely isolated and is rotating around an axis fixed in space. The center of the sphere is chosen as the origin of the inertial system of coordinates x, y, z. The x and the y axes are put in the equatorial plane, and the z axis has the positive direction of the axis of rotation. Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be the system of basic unit vectors connected with this coordinate system. If a satellite with negligible mass is placed in the gravitational field of this sphere, the differential equation of the satellite's motion has exactly the same form as in the two-body problem:

$$\ddot{\mathbf{r}} = -\frac{\mathbf{r}}{r^3}; \tag{1}$$

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$ (position vector of satellite);

$$\dot{\mathbf{r}} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$ (velocity);

$$\ddot{\mathbf{r}} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$$ (acceleration).

The mass of the central body, the radius, and the gravitational constant are put equal to unity. The orbit of the satellite is assumed to be an ellipse. In a more general case it is a conic section.

The following notations will be used:

$E_1 = a$ = semimajor axis of ellipse;

$E_2 = e$ = eccentricity;

$E_3 = \omega$ = argument of perigee;

$E_4 = \theta$ = longitude of ascending node in equatorial system;

$E_5 = i$ = inclination of orbit plane to equatorial plane;

$E_6 = g_0$ = mean anomaly at epoch;

$n = a^{-\frac{3}{2}}$ = mean motion;

$g = g_0 + n(t - t_0)$ = mean anomaly;

E = eccentric anomaly;

f = true anomaly;

$\mathbf{P} = P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}$ = unit vector directed from origin to perigee;

$\mathbf{R} = R_x\mathbf{i} + R_y\mathbf{j} + R_z\mathbf{k}$ = unit vector normal to orbit plane;

$\mathbf{Q} = \mathbf{R} \times \mathbf{P} = Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k}$;

$\mathbf{r}^\circ$ = unit vector in direction of $\mathbf{r}$;

$\mathbf{n}^\circ$ = unit vector normal to $\mathbf{r}$, lying in orbit plane.

The first six elements E_i $(i = 1, 2, 3, 4, 5, 6)$ represent the constants of integration. The complete solution is given by the following system of classical equations:

$$E - e \sin E = g ;$$

$$r \cos f = a(\cos E - e) \qquad r \sin f = a\sqrt{1 - e^2} \sin E; \tag{2}$$

$$r = a(1 - e \cos E) = \frac{a(1 - e^2)}{1 + e \cos f} ;$$

$$\begin{bmatrix} P_x & Q_x & R_x \\ P_y & Q_y & R_y \\ P_z & Q_z & R_z \end{bmatrix} = \begin{bmatrix} +\cos\theta & -\sin\theta & 0 \\ +\sin\theta & +\cos\theta & 0 \\ 0 & 0 & +1 \end{bmatrix} \cdot \begin{bmatrix} +1 & 0 & 0 \\ 0 & +\cos i & -\sin i \\ 0 & +\sin i & +\cos i \end{bmatrix} \cdot \begin{bmatrix} +\cos\omega & -\sin\omega & 0 \\ +\sin\omega & +\cos\omega & 0 \\ 0 & 0 & +1 \end{bmatrix} ; \tag{3}$$

$$\mathbf{r} = \mathbf{P}\,a(\cos E - e) + \mathbf{Q}\,a\sqrt{1 - e^2}\,\sin E\,; \tag{4}$$

$$\dot{\mathbf{r}} = \frac{1}{r\sqrt{a}}\left(\mathbf{Q}\,a\sqrt{1 - e^2}\,\cos E - \mathbf{P}\,a\sin E\right). \tag{5}$$

The position vector $\mathbf{r}_0$ and the velocity vector $\dot{\mathbf{r}}_0$ for the initial moment $t = t_0$ can be taken as the constants of integration instead of the elliptic elements E_i ($i = 1, 2, 3, 4, 5, 6$), which then must be determined by means of Equations 2 through 5.

In this exposition, some classic results are presented in vectorial form. Equation 1 admits of two vectorial integrals. The area integral

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{c} = \frac{\mathbf{R}}{h}\,, \tag{6}$$

where

$$h = \frac{1}{\sqrt{a(1 - e^2)}}\,,$$

is obtained from Equation 1 by cross multiplying by $\mathbf{r}$ and integrating the result. The Laplacian integral

$$\mathbf{R} \times \mathbf{r} + h\,\mathbf{r}^\circ + \mathbf{g} = \mathbf{0}\,, \tag{7}$$

where

$$\mathbf{g} = h\,e\mathbf{P}\,, \tag{8}$$

is obtained from Equation 1 by cross multiplying by $\mathbf{R}$ and integrating the result, taking

$$\mathbf{R} \times \frac{\mathbf{r}}{r^3} = h(\mathbf{r} \times \dot{\mathbf{r}}) \times \frac{\mathbf{r}}{r^3} = \frac{h}{r^2}\left(r\dot{\mathbf{r}} - \mathbf{r}\,\frac{dr}{dt}\right) = h\,\frac{d\mathbf{r}^\circ}{dt}$$

into consideration. Forming the cross product of Equation 7 and $\mathbf{R}$, we have

$$\mathbf{R} \times (\mathbf{R} \times \dot{\mathbf{r}}) = -\mathbf{R} \times (h\mathbf{r}^\circ + \mathbf{g})\,,$$

or

$$\mathbf{R}\mathbf{R} \cdot \dot{\mathbf{r}} - \dot{\mathbf{r}}\mathbf{R} \cdot \mathbf{R} = -\mathbf{R} \times (h\mathbf{r}^\circ + \mathbf{g})\,.$$

But

$$\mathbf{R} \cdot \dot{\mathbf{r}} = 0 ,$$

and

$$\mathbf{R} \cdot \mathbf{R} = 1 .$$

Consequently,

$$\dot{\mathbf{r}} = \mathbf{R} \times (h\mathbf{r}^\circ + \mathbf{g}) . \tag{9}$$

This is the Hamiltonian integral in vectorial form. The vectors $\mathbf{c}$ and $\mathbf{g}$, taken as constants of integration, are not independent of each other. The condition

$$\mathbf{c} \cdot \mathbf{g} = \mathbf{0}$$

must be satisfied, and Equations 6 and 7 supply only five independent scalar integrals.

If the form of the central body is not exactly spherical and the distribution of densities is less regular, or if some other bodies — such as the sun and the moon — influence the motion of the satellite, then such a motion will become a disturbed one and the differential Equation 1 must be replaced by

$$\ddot{\mathbf{r}} = -\frac{\mathbf{r}}{r^3} + \mathbf{F} . \tag{10}$$

The disturbing force $\mathbf{F}$ is assumed to possess a force function Ω. In other words, we assume that $\mathbf{F}$ has the form

$$\mathbf{F} = \text{grad } \Omega = \frac{\partial\Omega}{\partial x}\mathbf{i} + \frac{\partial\Omega}{\partial y}\mathbf{j} + \frac{\partial\Omega}{\partial z}\mathbf{k} . \tag{11}$$

From the values of $\mathbf{r}$ and $\dot{\mathbf{r}}$ for any given moment t, we can deduce the system of instantaneous elliptic elements (the system of osculating elements) and obtain the position of the instantaneous orbit plane (the osculating orbit plane) as a plane passing through $\mathbf{r}$ and $\dot{\mathbf{r}}$:

$$\mathbf{r} = \mathbf{f}(t, a, e, \omega, i, \theta, g_0) = \mathbf{f}(t; E_1, E_2, \cdots, E_6) , \tag{12}$$

$$\dot{\mathbf{r}} = \boldsymbol{\phi}(t, a, e, \omega, i, \theta, g_0) = \boldsymbol{\phi}(t; E_1, E_2, \cdots, E_6) , \tag{13}$$

and the elements are functions of time themselves. From the way the osculating elements are defined, it can be concluded that

$$\frac{\partial \dot{\mathbf{r}}}{\partial t} = \dot{\mathbf{r}} \,, \tag{14}$$

$$\frac{\partial \dot{\mathbf{r}}}{\partial t} = - \frac{\mathbf{r}}{r^3} \,. \tag{15}$$

From the kinematical point of view, the osculating orbit plane can be considered as a rigid body (without mass) rotating around the origin. This plane, evidently, is tangent to the conic surface described by the radius vector of the satellite. Consequently, the instantaneous axis of rotation of the osculating orbit plane coincides with the radius vector. We have

$$\dot{\mathbf{r}} = \frac{\partial \mathbf{r}}{\partial t} + \sum_{i=1}^{6} \frac{dE_i}{dt} \cdot \frac{\partial \mathbf{r}}{\partial E_i} \,, \tag{16}$$

$$(i = 1, 2, 3, \cdots 6.)$$

$$\ddot{\mathbf{r}} = \frac{\partial \dot{\mathbf{r}}}{\partial t} + \sum_{i=1}^{6} \frac{dE_i}{dt} \cdot \frac{\partial \dot{\mathbf{r}}}{\partial E_i} \,. \tag{17}$$

Introducing the differential operator

$$\frac{\delta}{dt} = \sum_{i=1}^{6} \frac{dE_i}{dt} \cdot \frac{\partial}{\partial E_i} \,, \tag{18}$$

and taking Equations 14 and 15 into account, we obtain

$$\frac{\delta \mathbf{r}}{dt} = \mathbf{0} \tag{19}$$

and

$$\frac{\delta \dot{\mathbf{r}}}{dt} = \mathbf{F} \,. \tag{20}$$

Consequently,

$$\frac{\delta r}{dt} = \mathbf{0} \,, \qquad \frac{\delta \mathbf{r}^\circ}{dt} = \mathbf{0} \,. \tag{21}$$

These two well known equations serve as a basis for the geometrical theory of perturbations. For any osculating element E_i $(i = 1, 2, 3, \cdots, 6)$,

$$\frac{\delta E_i}{dt} = \frac{dE_i}{dt} .$$

In particular,

$$\frac{d\mathbf{c}}{dt} = \frac{\delta \mathbf{c}}{dt} , \qquad \frac{d\mathbf{g}}{dt} = \frac{\delta \mathbf{g}}{dt} . \tag{22}$$

From Equation 6, by applying the δ/dt operator, it can be deduced that

$$\mathbf{R} \frac{d}{dt} \frac{1}{h} + \frac{1}{h} \dot{\mathbf{R}} = \mathbf{r} \times \mathbf{F} . \tag{23}$$

Taking

$$\mathbf{R} \cdot \mathbf{R} = 1, \qquad \mathbf{R} \cdot \dot{\mathbf{R}} = 0, \qquad \mathbf{R} \times \mathbf{R} = 0, \qquad \mathbf{R} \cdot \mathbf{r} = 0$$

into account, we deduce from Equation 23 that

$$\frac{d}{dt} \frac{1}{h} = \mathbf{R} \cdot \mathbf{r} \times \mathbf{F} ;$$

and

$$\mathbf{R} \times \dot{\mathbf{R}} = h \mathbf{R} \times (\mathbf{r} \times \mathbf{F}) ,$$

$$\mathbf{R} \times (\mathbf{r} \times \mathbf{F}) = \mathbf{r} \mathbf{R} \cdot \mathbf{F} - \mathbf{F} \mathbf{R} \cdot \mathbf{r} .$$

Finally,

$$\mathbf{R} \times \dot{\mathbf{R}} = h \mathbf{r} \mathbf{R} \cdot \mathbf{F} ,$$

and, with

$$\mathbf{R} \times (\mathbf{R} \times \dot{\mathbf{R}}) = -\dot{\mathbf{R}}$$

taken into account,

$$\dot{\mathbf{R}} = h(\mathbf{r} \times \mathbf{R}) (\mathbf{R} \cdot \mathbf{F}) .$$

This last equation shows that the vector

$$\boldsymbol{\omega} = h\mathbf{r}(\mathbf{R} \cdot \mathbf{F}) \tag{23a}$$

represents the angular velocity of rotation of vector $\mathbf{R}$ and, consequently, is also the angular velocity of rotation of the osculating plane around the instantaneous radius vector of the satellite.

We now introduce a system of coordinates X, Y, Z rigidly connected with the osculating orbit plane, with the X and Y axes lying in that plane and with the Z-axis normal to it. The intersection of the X-axis with the celestial sphere is called the departure point. The angle between the X-axis and the line of nodes is designated by σ. Three Eulerian angles θ, σ, i determine the position of the XYZ system with respect to the xyz system. The satellite is moving in the XY plane.

In order to investigate the motion of the satellite with respect to the XYZ system we must use the formulas giving the connection between the absolute and the relative velocities and the relative accelerations. If $\mathbf{r}$ is written in the form

$$\mathbf{r} = X\mathbf{i}' + Y\mathbf{j}' + Z\mathbf{k}' ,$$

where $\mathbf{i}', \mathbf{j}', \mathbf{k}' = \mathbf{R}$ are the basic unit vectors in the XYZ system, then

$$\frac{d\mathbf{r}}{dt} = \frac{dX}{dt}\mathbf{i}' + \frac{dY}{dt}\mathbf{j}' + \frac{dZ}{dt}\mathbf{k}'$$

is the relative velocity, and

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d^2X}{dt^2}\mathbf{i}' + \frac{d^2Y}{dt^2}\mathbf{j}' + \frac{d^2Z}{dt^2}\mathbf{k}'$$

is the relative acceleration.

The basic formulas giving the connection between the absolute and the relative motions are

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} + \omega \times \mathbf{r} , \tag{24}$$

$$\ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2} + 2\omega \times \frac{d\mathbf{r}}{dt} + \omega \times (\omega \times \mathbf{r}) + \frac{d\omega}{dt} \times \mathbf{r} = -\frac{\mathbf{r}}{r^3} + \mathbf{F} ; \tag{25}$$

or, taking Equation 23 into account,

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} ,$$

$$\ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2} + h\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right)(\mathbf{R} \cdot \mathbf{F})$$

$$= \frac{d^2\mathbf{r}}{dt^2} + \mathbf{R}(\mathbf{R} \cdot \mathbf{F}) = -\frac{\mathbf{r}}{r^3} + \mathbf{F} .$$

By writing $\mathbf{F}$ in the form

$$\mathbf{F} = (\mathbf{F}) + \frac{\partial \Omega}{\partial Z} \mathbf{k}' ,$$

where $(\mathbf{F})$ is the projection of the disturbing force on the XY plane,

$$(\mathbf{F}) = \frac{\partial \Omega}{\partial X} \mathbf{i}' + \frac{\partial \Omega}{\partial Y} \mathbf{j}' , \tag{26}$$

we obtain

$$\mathbf{R} \cdot \mathbf{F} = \frac{\partial \Omega}{\partial Z} \tag{27}$$

and

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{\mathbf{r}}{r^3} + (\mathbf{F}) . \tag{28}$$

An important conclusion (Reference 1) is noted here: *The differential equation of the motion of the satellite relative to the system rigidly connected with the osculating orbit plane has the same form as in the inertial system.* The scalar equations of the motion are, consequently:

In the rectangular coordinates,

$$\frac{d^2X}{dt^2} + \frac{X}{r^3} = \frac{\partial \Omega}{\partial X} ,$$

$$\frac{d^2Y}{dt^2} + \frac{Y}{r^3} = \frac{\partial \Omega}{\partial Y} ;$$

and in the polar coordinates,

$$(\mathbf{F}) = \frac{\partial \Omega}{\partial r} \mathbf{r}^\circ + \frac{1}{r} \frac{\partial \Omega}{\partial v} \mathbf{R} \times \mathbf{r}^\circ , \qquad v = \angle (X, \mathbf{r}) \tag{28a}$$

$$\frac{d^2 r}{dt^2} - r\left(\frac{dv}{dt}\right)^2 = -\frac{1}{r^2} + \frac{\partial\Omega}{\partial r}\,, \tag{28b}$$

$$\frac{d}{dt}\left(r^2 \frac{dv}{dt}\right) = \frac{\partial\Omega}{\partial v}\,. \tag{28c}$$

Thus, Hansen's classic results about the relative motion can be obtained in a more direct way without appealing to the equations for the variation of constants (Reference 2).

Because of the form of Equation 28, which is similar to Equation 1, the idea of osculation can be extended to the moving system of coordinates; and, introducing the operator δ/dt, we can claim that in the moving system

$$\frac{\delta \mathbf{r}}{dt} = \mathbf{0}\,, \tag{29}$$

$$\frac{\delta}{dt}\,\frac{d\mathbf{r}}{dt} = (\mathbf{F})\,. \tag{30}$$

The quasi-integrals, Equations 6, 7, and 9 retain their form. In particular the area "integral" may be written in the form

$$r^2 \frac{dv}{dt} = \frac{1}{h}\,. \tag{31}$$

The vector $\mathbf{R}$ is a constant vector in the XYZ system. By substituting

$$\mathbf{R}\cdot\mathbf{F} = \frac{\partial\Omega}{\partial Z}$$

into Equation 28a we have

$$\omega = h\,\mathbf{r}\,\frac{\partial\Omega}{\partial Z}\,, \tag{32}$$

and by substituting Equation 31 into 28c, another classic equation is easily deduced:

$$\frac{d}{dt}\,\frac{1}{h} = \frac{\partial\Omega}{\partial v}\,. \tag{33}$$

The equation for $d\mathbf{g}/dt$ is deduced by applying the operator δ/dt to the Laplacian "integral":

$$\mathbf{R} \times \frac{d\mathbf{r}}{dt} + h\mathbf{r}^\circ + \mathbf{g} = \mathbf{0} . \tag{34}$$

Taking Equations 34, 33, and 28a into consideration, we obtain

$$\frac{d\mathbf{g}}{dt} = \frac{\partial\Omega}{\partial r} \mathbf{r}^\circ \times \mathbf{R} + \left(\frac{1}{r} + h^2\right)\frac{\partial\Omega}{\partial v} \mathbf{r}^\circ . \tag{35}$$

In the inertial system of coordinates,

$$\dot{\mathbf{g}} = \frac{\partial\Omega}{\partial r} \mathbf{r}^\circ \times \mathbf{R} + \left(\frac{1}{r} + h^2\right) \frac{\partial\Omega}{\partial v} \mathbf{r}^\circ + \omega \times \mathbf{g} . \tag{35a}$$

This equation can be used for the computation of the special perturbations in the elements. If χ is the angular distance of the osculating perigee from the departure point,

$$\mathbf{g} = he\mathbf{P} = he\,(\mathbf{i}' \cos \chi + \mathbf{j}' \sin \chi) . \tag{36}$$

By substituting Equation 36 and

$$\mathbf{r}^\circ = \mathbf{i}' \cos v + \mathbf{j}' \sin v ,$$

$$\mathbf{r}^\circ \times \mathbf{R} = \mathbf{i}' \sin v - \mathbf{j}' \cos v$$

into Equation 35, two classic formulas

$$\frac{d(he \cos \chi)}{dt} = + \frac{\partial\Omega}{\partial r} \sin v + \left(\frac{1}{r} + h^2\right)\frac{\partial\Omega}{\partial v} \cos v ,$$

$$\frac{d(he \sin \chi)}{dt} = - \frac{\partial\Omega}{\partial r} \cos v + \left(\frac{1}{r} + h^2\right)\frac{\partial\Omega}{\partial v} \sin v$$

are obtained. These, however, will not be used in this exposition.

Introduction of the Auxiliary Satellite

In any satellite theory some first approximation to the real orbit and the real motion is used as a starting point. It is customary to call such a first approximation an "intermediary orbit."

The choice of an "intermediary" is not unique. In the theory of Laplace, as well as in the theory of Hansen, it is a rotating ellipse. In the theory of

Hill it is a so-called "variational curve," which in the lunar theory is obtained by neglecting the eccentricities of the satellite and of the sun and by neglecting the solar parallax. In the case of the artificial satellite the "variational inequalities" depend only on the mean argument of the latitude (Reference 3). We imagine a fictitious auxiliary satellite describing the intermediary orbit in accordance with the prescribed law. The choice of the intermediary and of the law of motion on it must be made in such a way that the difference between the positions of the real and auxiliary satellites is small.

In the Hansen-type theory which is presented here, the intermediary orbit is an ellipse of constant shape lying in the osculating orbit plane, with a_0, e_0, and $n_0 = a_0^{-3/2}$ fixed. The auxiliary satellite is describing this ellipse in accordance with Kepler's law. The ellipse is rotating uniformly with respect to the eccentric anomaly of the auxiliary satellite around the axis normal to the osculating plane. The directions and absolute values of the radius vectors of the real and auxiliary satellites do not coincide; but the difference is small, being of the order of perturbations. Let $\mathbf{r}^\circ$ be the unit vector along the radius vector $\mathbf{r}$ of the real satellite at the time t. The radius vector $\bar{\mathbf{r}}$ of the auxiliary satellite will have the same direction at some other moment say $\mathfrak{z}$. Then,

$$\mathbf{r}^\circ(t) = \bar{\mathbf{r}}^\circ(\mathfrak{z}) . \tag{37}$$

The time t and the "pseudo time" $\mathfrak{z}$ differ from each by the order of perturbations, and their difference is small. The absolute values of the radius vectors $\mathbf{r}$ and $\bar{\mathbf{r}}$ also differ from each other by the order of perturbations, and we can put

$$r(t) = (1 + \nu)\, \bar{\mathbf{r}}(\mathfrak{z}) , \tag{38}$$

where ν is small and the factor $1 + \nu$ defines how the vector $\bar{\mathbf{r}}$ must be "stretched out" or "contracted" in order to become equal with $\mathbf{r}$. Using the accepted terminology, call $\mathfrak{z}$ the "disturbed time," and the difference $\mathfrak{z} - t = \delta\mathfrak{z}$ can be understood as the "perturbation of time."

These two simple relations, Equations 37 and 38, serve as a basis for Hansen's development. The motion of the auxiliary satellite in the basic ellipse is governed by the usual equations familiar from the two-body problem. If the true anomaly of the auxiliary satellite is $\bar{f}$ and the eccentric anomaly is E, then

$$\bar{r} \cos \bar{f} = a_0(\cos E - e_0) , \tag{39}$$

$$\bar{r} \sin \bar{f} = a_0\sqrt{1 - e_0^2}\, \sin E , \tag{39a}$$

$$\bar{r} = a_0(1 - e_0 \cos E) , \tag{39b}$$

$$E - e_0 \sin E = g_0 + n_0 \mathfrak{z} = g_0 + n_0(t - t_0) + n_0 \delta \mathfrak{z} . \tag{40}$$

The area "integral" for the auxiliary satellite takes the form

$$\bar{r}^2 \frac{d\bar{f}}{d\mathfrak{z}} = \frac{1}{h_0} ; \quad h_0 = \frac{1}{\sqrt{a_0(1 - e_0^2)}} .$$

It is assumed that the auxiliary ellipse is rotating uniformly with respect to E and that the position of its perigee is determined by the equation

$$\pi = \pi_0 + \gamma \Delta E ,$$

where

$$\Delta E = E - E_0 .$$

The "angular speed" of rotation γ cannot be taken arbitrarily, but must be determined in such a way that the development of the perturbations in the mean anomaly $n_0 \delta \mathfrak{z}$ does not contain any secular term. The polar angle of the auxiliary satellite at the moment $\mathfrak{z}$ with respect to the X-axis is $\bar{f} + \pi_0 + \gamma \Delta E$. Let the polar angle of the real satellite at the moment t be designated by v. The condition (Equation 37) of equality of the unit vectors evidently can be replaced by the condition of the equality of these two polar angles:

$$v = \bar{f} + \pi_0 + \gamma \Delta E . \tag{41}$$

The node θ and the argument of the departure point contain two types of terms: the secular and the periodic terms. Consequently, the constants θ_0, σ_0, α, and η may be determined in such a way that the expressions

$$2N = \sigma_0 + \theta_0 - \sigma - \theta - 2\alpha \Delta E , \tag{42}$$

$$2K = \sigma_0 - \theta_0 - \sigma + \theta + 2\eta \Delta E \tag{43}$$

contain neither constant nor secular parts. Such a determination, together with the proper determination of γ, which is discussed later, leads to the development of x, y, z containing periodic terms only. Now

$$\sigma = \sigma_0 - (\alpha - \eta) \Delta E - N - K , \tag{44}$$

$$\theta = \theta_0 - (\alpha + \eta) \Delta E - N + K . \tag{45}$$

For the argument of latitude of the real satellite,

$$v - \sigma = \bar{f} + (\pi_0 - \sigma_0) + (\gamma + \alpha - \eta)\,\Delta E + N + K \,. \tag{46}$$

The expressions

$$(\sigma) = \sigma_0 - (\alpha - \eta)\,\Delta E \,, \tag{47}$$

$$(\theta) = \theta_0 - (\alpha + \eta)\,\Delta E \,, \tag{48}$$

$$(\omega) = (\pi_0 - \sigma_0) + (\gamma + \alpha - \eta)\,\Delta E \,, \tag{49}$$

containing only the constant and secular parts, are called the mean values of the corresponding elements.

Disturbing Function and its Derivatives

The disturbing function is defined as the negative of the difference between the gravitational potential and the potential of a spherical earth of the same mass. In the present theory, Ω is taken in the form of an expansion in zonal harmonics:

$$\begin{aligned} \Omega &= \frac{k_2}{r^3}(1 - 3\psi^2) + \frac{k_3}{r^4}(3\psi - 5\psi^3) + \frac{k_4}{r^5}(3 - 30\psi^2 + 35\psi^4) + \cdots \\ &= \Omega_2 + \Omega_3 + \Omega_4 + \cdots \,, \end{aligned} \tag{50}$$

where

$$\psi = \sin i \sin (v - \sigma) \tag{51}$$

is the sine of the latitude. Taking

$$v = \bar{f} + \pi_0 + \gamma\,\Delta E \tag{52}$$

and

$$\sigma = \sigma_0 - (\alpha - \eta)\,\Delta E - N - K \,, \tag{53}$$

$$(\omega) = (\pi_0 - \sigma_0) + (\gamma + \alpha - \eta)\,\Delta E \tag{54}$$

into account, we obtain

$$\psi = \sin i \sin [\bar{f} + (\omega) + N + K] . \tag{55}$$

The expressions for $\sin i$, N, and K have the form of trigonometric series in E and (ω). It is, however, inconvenient to keep these series in the argument in ψ.

Introducing the parameters

$$\begin{aligned} \lambda_1 &= \sin \frac{i}{2} \cos N , & \lambda_3 &= \cos \frac{i}{2} \sin K , \\ \lambda_2 &= \sin \frac{i}{2} \sin N , & \lambda_4 &= \cos \frac{i}{2} \cos K , \end{aligned} \tag{56}$$

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1 , \tag{57}$$

yields a different form for ψ:

$$\psi = 2(\lambda_1\lambda_4 - \lambda_2\lambda_3) \sin [\bar{f} + (\omega)] + 2(\lambda_2\lambda_4 + \lambda_1\lambda_3) \cos [\bar{f} + (\omega)] , \tag{58}$$

which leaves only $\bar{f} + (\omega)$ in the arguments. The parameters λ_1 and λ_2 were introduced by Hansen. The idea of introducing also λ_3 and λ_4 is a natural one. In the first approximation,

$$\lambda_1 = \sin \frac{i_0}{2} , \qquad \lambda_3 = 0 , \tag{59}$$

$$\lambda_2 = 0 , \qquad \lambda_4 = \cos \frac{i_0}{2} . \tag{60}$$

The next approximations add only small periodic oscillations about the mean values of λ_i, and those mean values will differ from Equations 59 and 60 only slightly.

Thus, the introduction of λ parameters permits an easy separation of the secular motion of the orbit plane from the periodic oscillation of this plane about its mean position. (The mean position of the plane is understood to be the position affected only by the secular motion of the node.) The components of the rotation matrix representing these small oscillations about the mean position of the orbit plane are polynomials in λ_i. The price paid for such a simple representation of the rotation matrix is the condition in Equation 57, that only three parameters, in fact, are independent. The relation (Equation 57) is, however, not a great inconvenience and it can be used also to check the accuracy of the development. If an introduction of only three independent parameters is necessary, then probably a satisfactory choice would be

$$p = +\frac{\lambda_1}{\lambda_4} = +\frac{\cos N}{\cos K}\tan\frac{i}{2}, \tag{61}$$

$$q = -\frac{\lambda_2}{\lambda_4} = -\frac{\sin N}{\cos K}\tan\frac{i}{2}, \tag{62}$$

$$s = +\frac{\lambda_3}{\lambda_4} = +\tan K . \tag{63}$$

The elements of the matrix of rotation of the orbit plane about its mean position are the rational fractions in p, q, s. The λ parameters are analogous to the parameters of Euler in the theory of the rotation of a rigid body (Reference 4), and the identity of p, q, s with the components of Gibbs' rotation vector (Reference 5) is also easily recognizable. The form of ψ which is convenient to use in developing the disturbing function is

$$\psi = 2\,\frac{a_0}{\bar{r}}\left[(\lambda_1\lambda_4 - \lambda_2\lambda_3)\,m + (\lambda_2\lambda_4 + \lambda_1\lambda_3)\,\ell\right], \tag{64}$$

where

$$\begin{aligned}\ell = \frac{\bar{r}}{a_0}\cos[\bar{f} + (\omega)] &= \frac{1}{2}\left(1 + \sqrt{1 - e_0^2}\right)\cos[E + (\omega)] \\ &+ \frac{1}{2}\left(1 - \sqrt{1 - e_0^2}\right)\cos[E - (\omega)] \\ &- e_0\cos(\omega) ;\end{aligned} \tag{65}$$

$$\begin{aligned}m = \frac{\bar{r}}{a_0}\sin[\bar{f} + (\omega)] &= \frac{1}{2}\left(1 + \sqrt{1 - e_0^2}\right)\sin[E + (\omega)] \\ &- \frac{1}{2}\left(1 - \sqrt{1 - e_0^2}\right)\sin[E - (\omega)] \\ &- e_0\sin(\omega) .\end{aligned} \tag{66}$$

For $a_0/\bar{r}$ the classic formula (Reference 6) is:

$$\frac{a_0}{\bar{r}} = \frac{2}{\sqrt{1 - e_0^2}}\left(\frac{1}{2} + \beta\cos E - \beta^2\cos 2E + \cdots\right).$$

It is necessary to point out that the eccentric anomaly E plays two roles. In the motion of the auxiliary satellite in its ellipse, E has the usual geometrical meaning; but in the expressions for perturbations it is the independent variable replacing the time. These two types of E must be distinguished from each other because the partial derivative $\partial\Omega/\partial E$, which appears in this theory, is taken with respect to the "elliptic" E. The best way to keep these two eccentric anomalies separated from each other is to use a temporary notation F for the elliptic E in the development of the disturbing function.

The disturbing function can be written in the form

$$\Omega = \frac{k_2}{a_0^3} (1 + \nu)^{-3} (1 - 3\psi^2) \frac{a_0^3}{\bar{r}^3} + \frac{k_3}{a_0^4} (1 + \nu)^{-4} (3\psi - 5\psi^2) \frac{a_0^4}{\bar{r}^4}$$

$$+ \frac{k_4}{a_0^5} (1 + \nu)^{-5} (3 - 30\psi^2 + 35\psi^4) \frac{a_0^5}{\bar{r}^5} \cdots . \qquad (67)$$

The elliptic E is replaced by F, and Ω by

$$\Omega^* = \frac{k_2}{a_0^3} (1 + \nu)^{-3} \left(\frac{a_0}{\bar{\rho}}\right)^3 (1 - 3\psi^{*2})$$

$$+ \frac{k_3}{a_0^4} (1 + \nu)^{-4} \left(\frac{a_0}{\bar{\rho}}\right)^4 (3\psi^* - 5\psi^{*2})$$

$$+ \frac{k_4}{a_0^5} (1 + \nu)^{-5} \left(\frac{a_0}{\bar{\rho}}\right)^5 (3 - 30\psi^{*2} + 35\psi^{*4}) + \cdots , \qquad (68)$$

where

$$\psi^* = 2 \frac{a_0}{\bar{\rho}} \left[(\lambda_1\lambda_4 - \lambda_2\lambda_3) m^* + (\lambda_2\lambda_4 + \lambda_1\lambda_3) \ell^*\right], \qquad (69)$$

$$\ell^* = \frac{\bar{\rho}}{a_0} \cos [\bar{\varphi} + (\omega)] = \frac{1}{2} (1 + \sqrt{1 - e_0^2}) \cos [F + (\omega)]$$

$$- \frac{1}{2} (1 - \sqrt{1 - e_0^2}) \cos [F - (\omega)] - e_0 \cos (\omega), \qquad (70)$$

$$m^{*} = \frac{\bar{\rho}}{a_0} \sin [\bar{\varphi} + (\omega)] = \frac{1}{2} (1 + \sqrt{1 - e_0^2}) \sin [F + (\omega)]$$

$$- \frac{1}{2} (1 - \sqrt{1 - e_0^2} \sin [F - (\omega)] - e_0 \sin (\omega), \quad (71)$$

and

$$\frac{a_0}{\bar{\rho}} = \frac{2}{\sqrt{1 - e_0^2}} \left(\frac{1}{2} + \beta \cos F + \beta^2 \cos^2 F + \cdots \right) . \quad (72)$$

No replacement of E by F is done in $1 + \nu$, λ_1, λ_2, λ_3, λ_4. Consequently, the introduction of the "temporary" eccentric anomaly F and Ω^* instead of Ω permits us to distinguish between two types of E and, if necessary, to keep track of changes that occur in Ω (or Ω^*) from one iteration to another. Each iteration leads to a development of the form

$$\Omega^* = \Sigma C \cos (i E + 2 j \omega + k F) + \Sigma S \sin [i E + (2 j + 1) \omega + k F]$$

and

$$\Omega = \Sigma C \cos (i E + 2 j \omega) + \Sigma S \sin [i E + (2 j + 1) \omega] .$$

The last equation will not be used. In this exposition there appears the partial derivative $\partial\Omega/\partial E$, which in fact must be understood as

$$\frac{\partial\Omega}{\partial E} = \overline{\frac{\partial\Omega^*}{\partial F}} . \quad (73)$$

The "bar" operation means in Hansen's notation the replacement of F by E again. Each iteration step leads to an expression of the form

$$\frac{\partial\Omega}{\partial E} = \Sigma S \sin (i E + 2 j \omega) + \Sigma C \cos [i E + (2 j + 1) \omega] .$$

Taking

$$\frac{\partial\Omega}{\partial F} = \frac{\partial\Omega}{\partial \bar{r}} \frac{\partial \bar{r}}{\partial E} + \frac{\partial\Omega}{\partial \bar{f}} \frac{\partial \bar{f}}{\partial E} ,$$

$$\frac{\partial \bar{f}}{\partial E} = \frac{a_0 \sqrt{1 - e_0^2}}{\bar{r}} ,$$

$$\frac{\partial \bar{r}}{\partial E} = a_0 e_0 \sin E ,$$

and

$$\frac{\partial \Omega}{\partial v} = \frac{\partial \Omega}{\partial \bar{f}} ,$$

$$r \frac{\partial \Omega}{\partial r} = \bar{r} \frac{\partial \Omega}{\partial \bar{r}} ,$$

we obtain

$$\frac{\partial \Omega}{\partial v} = \frac{\bar{r}}{a_0 \sqrt{1 - e_0^2}} \frac{\partial \Omega}{\partial E} - \frac{e_0 \sin E}{\sqrt{1 - e_0^2}} r \frac{\partial \Omega}{\partial r} . \tag{74}$$

The development of the derivative $r(\partial\Omega/\partial r)$ presents no difficulties:

$$\begin{aligned} r \frac{\partial \Omega}{\partial r} = &- \frac{3k_2}{a_0^3} (1 + \nu)^{-3} \left(\frac{a_0}{\bar{r}}\right)^3 (1 - 3\psi^2) \\ &- \frac{4k_3}{a_0^4} (1 + \nu)^{-4} \left(\frac{a_0}{\bar{r}}\right)^4 (3\psi - 5\psi^3) \\ &- \frac{5k_4}{a_0^5} (1 + \nu)^{-5} \left(\frac{a_0}{\bar{r}}\right)^5 (3 - 30\psi^2 + 35\psi^4) + \cdots \\ = &- 3\,\Omega_2 - 4\,\Omega_3 - 5\,\Omega_4 - \cdots \\ = &- 3\,\bar{\Omega}_2^* - 4\,\bar{\Omega}_3^* - 5\,\bar{\Omega}_4^* . \end{aligned} \tag{75}$$

Perturbations *in* the Orbit Plane

The perturbations in the orbit plane are the perturbations of the radius vector, $1 + \nu$, and of the mean anomaly, $n_0 \delta z$. Thus, the perturbations in the mean anomaly and in the argument of perigee (from the standpoint of Lagrange's variation of elements) are combined into a single angle $n_0 \delta z$, representing the perturbations of the mean anomaly from the standpoint of the Hansen theory.

Differentiating

$$v = \bar{f} + \bar{r}_0 + \gamma \Delta E ,$$

gives

$$\frac{dv}{dt} = \frac{d\bar{f}}{d\mathfrak{z}} \frac{d\mathfrak{z}}{dt} + \gamma \frac{dE}{dt} .$$

Then, eliminating dv/dt and $d\bar{f}/d\mathfrak{z}$ by means of

$$\frac{dv}{dt} = \frac{1}{hr^2}$$

and

$$\frac{d\bar{f}}{d\mathfrak{z}} = \frac{1}{h_0 \bar{r}^2} ,$$

we have

$$\frac{d\mathfrak{z}}{dt} = \frac{h_0}{h}\left(\frac{\bar{r}}{r}\right)^2 - \gamma h_0 \bar{r}^2 \frac{dE}{dt} ; \tag{76}$$

or, taking into consideration

$$\frac{\bar{r}}{r} = \frac{1}{1 + \nu} ,$$

$$h_0 = \frac{1}{a_0^2 n_0 \sqrt{1 - e_0^2}} ,$$

we obtain from Equation 76,

$$\frac{d\mathfrak{z}}{dt} = \frac{h_0}{h(1 + \nu)^2} - \frac{\gamma}{n_0\sqrt{1 - e_0^2}} \left(\frac{\bar{r}}{a_0}\right)^2 \frac{dE}{dt} . \tag{77}$$

Equation 76 can be written in the form

$$\frac{d\delta\mathfrak{z}}{dt} = \bar{W} + \frac{h_0}{h} \cdot \frac{\nu^2}{(1 + \nu)^2} - \frac{\gamma}{n_0\sqrt{1 - e_0^2}} \left(\frac{\bar{r}}{a_0}\right)^2 \frac{dE}{dt} , \tag{78}$$

where

$$\overline{W} = -1 - \frac{h_0}{h} + 2\,\frac{h_0}{h}\,\frac{1}{1+\nu}. \tag{79}$$

The first and the third terms in Equation 78 are of the first order, and the second term is of the second order with respect to the disturbing forces. In order to obtain the main part of δz, we must find $\overline{W}$; that is the next problem.

The equation of the orbit,

$$r = \frac{a(1-e^2)}{1+e\cos f},$$

can be written in the form

$$\frac{1}{h(1+\nu)} = h\bar{r} + \bar{\mathbf{r}}\cdot\mathbf{g};$$

and, as a consequence, Equation 79 becomes

$$\overline{W} = -1 - \frac{h_0}{h} + 2h_0 h\bar{r} + 2h_0\bar{\mathbf{r}}\cdot\mathbf{g}. \tag{80}$$

By taking

$$h_0^2 a_0 (1 - e_0^2) = 1$$

into account, the last equation can be rewritten in the form

$$\overline{W} = -1 - \frac{h_0}{h} + 2\,\frac{h}{h_0}\,\frac{\bar{r}}{a_0(1-e_0^2)} + 2\,\frac{\bar{\mathbf{r}}\cdot\mathbf{g}}{h_0 a_0(1-e_0^2)}. \tag{81}$$

This is a classic equation written in terms of vectors. In forming the differential equation determining $\overline{W}$, it would be preferable to keep the perturbations separated from the elliptic motion of the auxiliary satellite. For this reason it will be more convenient to introduce another function W instead of $\overline{W}$, by replacing $\bar{\mathbf{r}}$ with $\bar{\boldsymbol{\rho}}$. The vector $\bar{\boldsymbol{\rho}}$ is defined as a function of the temporary eccentric anomaly F which replaces E and which is considered as a temporary constant; or $\bar{\boldsymbol{\rho}}$ may also be considered as a function of the true anomaly $\bar{\varphi}$, which temporarily replaces $\bar{f}$. After the integration is performed, F is replaced by E again. Instead of Equations 39 through 39b, we then have

$$\bar{\rho}\cos\bar{\varphi} = a_0(\cos F - e_0), \tag{82}$$

$$\bar{\rho}\sin\bar{\varphi} = a_0\sqrt{1-e_0^2}\,\sin F, \tag{82a}$$

$$\bar{\rho} = a_0 (1 - e_0 \cos F), \tag{82b}$$

$$\bar{\boldsymbol{\rho}} = \mathbf{i}' \bar{\rho} \cos (\bar{\varphi} + \pi_0 + \gamma \Delta E) + \mathbf{j}' \bar{\rho} \sin (\bar{\varphi} + \pi_0 + \gamma \Delta E) . \tag{82c}$$

In view of the foregoing remarks, W takes the form

$$W = -1 - \frac{h_0}{h} + 2 \frac{h}{h_0} \frac{\bar{\rho}}{a_0 (1 - e_0^2)} + \frac{2 \bar{\boldsymbol{\rho}} \cdot \mathbf{g}}{h_0 a_0 (1 - e_0^2)} . \tag{83}$$

Substituting Equations 82c and 36 into Equation 83 results in a classic formula:

$$W = -1 - \frac{h_0}{h} + 2 \frac{h}{h_0} \frac{\bar{\rho}}{a_0} \frac{1 + e \cos (\bar{\varphi} + \pi_0 + \gamma \Delta E - \chi)}{1 - e_0^2} . \tag{84}$$

Eliminating $\bar{\varphi}$ in favor of F by means of Equations 82 through 82b, we obtain another formula:

$$W = \Xi + \Upsilon \cos F + \Psi \sin F , \tag{85}$$

where

$$\Xi = -1 - \frac{h_0}{h} + 2 \frac{h}{h_0} \frac{1}{1 - e_0^2} \left[1 - e e_0 \cos (\pi_0 + \gamma \Delta E - \chi) \right] , \tag{86}$$

$$\Upsilon = 2 \frac{h}{h_0} \frac{e \cos (\pi_0 + \gamma \Delta E - \chi) - e_0}{1 - e_0^2} ; \tag{86a}$$

$$\Psi = -2 \frac{h}{h_0} \frac{e \sin (\pi_0 + \gamma \Delta E - \chi)}{\sqrt{1 - e_0^2}} . \tag{86b}$$

It follows from Equations 86 and 86a that

$$e_0 \Upsilon + \Xi = -1 - \frac{h_0}{h} + 2 \frac{h}{h_0} . \tag{87}$$

The formulas 85 and 87 are used in the following exposition in the problem of determining the constants of integration and determining the perturbations in h/h_0.

It can easily be seen that W is changing under the influence of the perturbations only. The "elliptic" motion is present in Equation 85 in the form of F and is considered as "frozen" for the time being. This leads to the conclusion that

$$\frac{\delta W}{dt} = \frac{dW}{dt} . \tag{88}$$

The perturbations in $\bar{\boldsymbol{\rho}}$ are present only through $\bar{\pi}_0 + \gamma \Delta E$ in Equation 82c in the argument. The only influence of the perturbative force on $\bar{\boldsymbol{\rho}}$ will consist of the rotation of $\bar{\boldsymbol{\rho}}$ around the Z-axis with the angular velocity $\gamma\,(dE/dt)$. Consequently,

$$\frac{\delta \bar{\boldsymbol{\rho}}}{dt} = (\mathbf{R} \times \bar{\boldsymbol{\rho}})\, \gamma \, \frac{dE}{dt} , \tag{89}$$

where $\bar{\rho}$ is completely uninfluenced by perturbations; then

$$\frac{\delta \bar{\rho}}{dt} = 0 . \tag{90}$$

There is obtained from Equation 83, by applying the operator δ/dt and taking Equations 88, 89, 90, 35, and 33 into consideration,

$$\frac{dW}{dt} = h_0 \frac{\partial \Omega}{\partial v} \left[2 \frac{\bar{\boldsymbol{\rho}} \cdot \mathbf{r}^\circ}{r} - 1 + 2 \frac{h^2}{h_0^2} \cdot \frac{\bar{\boldsymbol{\rho}} \cdot \mathbf{r}^\circ - \bar{\rho}}{a_0 (1 - e_0^2)} \right]$$

$$+ 2h_0 \left(\mathbf{R} \cdot \bar{\boldsymbol{\rho}} \times \frac{\mathbf{r}^\circ}{r} \right) r \frac{\partial \Omega}{\partial r} + \frac{2\mathbf{g} \cdot \mathbf{R} \times \bar{\boldsymbol{\rho}}}{h_0 a_0 (1 - e_0^2)} \gamma \frac{dE}{dt} . \tag{91}$$

Designating by $\boldsymbol{\rho}^\circ$ the unit vector in the direction of $\bar{\boldsymbol{\rho}}$, we have

$$\bar{\boldsymbol{\rho}} = \bar{\rho} \cdot \boldsymbol{\rho}^\circ .$$

It follows from Equations 82 through 82c that

$$\frac{d\bar{\boldsymbol{\rho}}}{dF} = \frac{d\bar{\rho}}{dF} \boldsymbol{\rho}^\circ + \bar{\rho} \frac{d\boldsymbol{\rho}^\circ}{d\bar{\varphi}} \frac{d\bar{\varphi}}{dF} , \tag{92}$$

where

$$\frac{d\bar{\rho}}{dF} = a_0 e_0 \sin F ,$$

$$\frac{d\bar{\varphi}}{dF} = \frac{a_0}{\bar{\rho}} \sqrt{1 - e_0^2} . \tag{92a}$$

Evidently

$$\frac{d\boldsymbol{\rho}^\circ}{d\bar{\varphi}} = \mathbf{R} \times \boldsymbol{\rho}^\circ . \tag{92b}$$

It follows from 92 to 92b that

$$\frac{d\bar{\boldsymbol{\rho}}}{dF} = \boldsymbol{\rho}^\circ \, a_0 e_0 \sin F + (\mathbf{R} \times \bar{\boldsymbol{\rho}}) \frac{a_0}{\bar{\rho}} \sqrt{1 - e_0^2} .$$

Differentiating Equation 83 with respect to F gives

$$\frac{\partial W}{\partial F} = 2 \frac{h}{h_0} \cdot \frac{e_0 \sin F}{1 - e_0^2} + \frac{2\mathbf{g} \cdot}{h_0 a_0 (1 - e_0^2)} \left[\bar{\boldsymbol{\rho}}^\circ a_0 e_0 \sin F + (\mathbf{R} \times \bar{\boldsymbol{\rho}}) \cdot \frac{a_0}{\bar{\rho}} \sqrt{1 - e_0^2} \right] ,$$

or

$$\frac{2\mathbf{g} \cdot \mathbf{R} \times \bar{\boldsymbol{\rho}}}{h_0 a_0 (1 - e_0^2)} = \left[\frac{\bar{\rho}}{a_0} \frac{\partial W}{\partial F} - \left(W + \frac{h_0}{h} + 1 \right) e_0 \sin F \right] \frac{1}{\sqrt{1 - e_0^2}} .$$

Substituting this last result into Equation 91 gives us

$$\frac{dW}{dt} = h_0 \frac{\partial \Omega}{\partial v} \left[2 \frac{\bar{\boldsymbol{\rho}} \cdot \mathbf{r}^\circ}{r} - 1 + 2 \frac{h^2}{h_0^2} \frac{\bar{\boldsymbol{\rho}} \cdot \mathbf{r}^\circ - \bar{\rho}}{a_0 (1 - e_0^2)} \right] + 2h_0 \left(\mathbf{R} \cdot \bar{\boldsymbol{\rho}} \times \frac{\mathbf{r}^\circ}{r} \right) r \frac{\partial \Omega}{\partial r}$$

$$+ \frac{\mathfrak{z}}{\sqrt{1 - e_0^2}} \left[\frac{\bar{\rho}}{a_0} \frac{\partial W}{\partial F} - \left(W + \frac{h_0}{h} + 1 \right) e_0 \sin F \right] \frac{dE}{dt} \cdot \tag{93}$$

From Equation 79,

$$\frac{h_0}{h} = \frac{1 + \nu}{1 - \nu} (1 + \bar{W}) ,$$

and substituting into Equation 78 gives us

$$\frac{dn_0 \, \delta \mathfrak{z}}{dt} = n_0 \frac{\bar{W} + \nu^2}{1 - \nu^2} - \frac{\mathfrak{z}}{\sqrt{1 - e_0^2}} \left(\frac{\bar{r}}{a_0} \right)^2 \frac{dE}{dt} . \tag{94}$$

The last equation represents a generalized formula of Hill (Reference 7) in the theory of planetary perturbations. Differentiating Kepler's equation

$$E - e_0 \sin E = g_0 + n_0 (t - t_0) + n_0 \delta \mathfrak{z}$$

with respect to E gives

$$\frac{\bar{r}}{a_0} = n_0 \frac{dt}{dE} + \frac{dn_0 \delta \mathfrak{z}}{dE}, \tag{95}$$

and, eliminating dt from Equation 94,

$$\frac{dn_0 \delta \mathfrak{z}}{dE} = \frac{\bar{W} + \nu^2}{1 - \nu^2} \cdot \left(\frac{\bar{r}}{a_0} - \frac{dn_0 \delta \mathfrak{z}}{dE} \right) - \frac{\gamma}{\sqrt{1 - e_0^2}} \left(\frac{\bar{r}}{a_0} \right)^2 .$$

Solving with respect to $dn_0 \delta \mathfrak{z}/dE$ gives

$$\frac{dn_0 \delta \mathfrak{z}}{dE} = \frac{\bar{W} + \nu^2}{1 + \bar{W}} \cdot \frac{\bar{r}}{a_0} - \frac{1 - \nu^2}{1 + \bar{W}} \cdot \frac{\gamma}{\sqrt{1 - e_0^2}} \left(\frac{\bar{r}}{a_0} \right)^2 . \tag{96}$$

Eliminating $dn_0 \delta \mathfrak{z}/dE$ from Equation 95,

$$n_0 \frac{dt}{dE} = \frac{\bar{r}}{a_0} \cdot \frac{1 - \nu^2}{1 + \bar{W}} \left(1 + \frac{\gamma}{\sqrt{1 - e_0^2}} \cdot \frac{\bar{r}}{a_0} \right). \tag{97}$$

Taking Equations 74 and 97 into account, we deduce from Equation 93 that

$$\begin{aligned} \frac{dW}{dE} = {} & h_0 \left[\frac{\bar{r}}{a_0 \sqrt{1 - e_0^2}} \frac{\partial \Omega}{\partial E} - \frac{e_0 \sin E}{\sqrt{1 - e_0^2}} r \frac{\partial \Omega}{\partial r} \right] \cdot \left[2 \frac{\bar{\rho}}{r} \cos(\bar{f} - \bar{\varphi}) - 1 + 2 \frac{h^2}{h_0^2} \right. \\ & \left. \cdot \frac{\bar{\rho}}{a_0} \cdot \frac{\cos(\bar{f} - \bar{\varphi}) - 1}{1 - e_0^2} \right] \cdot \frac{\bar{r}}{a_0 n_0} \cdot \frac{1 - \nu^2}{1 + \bar{W}} \left(1 + \frac{\gamma}{\sqrt{1 - e_0^2}} \cdot \frac{\bar{r}}{a_0} \right) \\ & + 2h_0 \frac{\bar{\rho}}{r} \sin(\bar{f} - \bar{\varphi}) \cdot \frac{\bar{r}}{a_0 n_0} \cdot \frac{1 - \nu^2}{1 + \bar{W}} \left(1 + \frac{\gamma}{\sqrt{1 - e_0^2}} \cdot \frac{\bar{r}}{a_0} \right) \cdot r \frac{\partial \Omega}{\partial r} \\ & + \frac{\gamma}{\sqrt{1 - e_0^2}} \cdot \left[\frac{\bar{\rho}}{a_0} \cdot \frac{\partial W}{\partial F} - \left(W + 1 + \frac{h_0}{h} \right) e_0 \sin F \right]; \end{aligned} \tag{98}$$

or, after expressing the results in terms of E and F,

$$\frac{dW}{dE} = \left(M \frac{\partial a_0 \Omega}{\partial E} + N r \frac{\partial a_0 \Omega}{\partial r}\right) \Lambda + \frac{S \gamma}{\sqrt{1 - e_0^2}} , \tag{99}$$

where

$$\begin{aligned} M(1 - e_0^2) = \frac{h^2}{h_0^2} \Big[&-2 + 2e_0 \cos E + 2 \cos (F - E) \\ &- e_0 \cos (F - 2E) - e_0 \cos F \Big] + \frac{1}{1 + \nu} \Big[2e_0^2 \\ &- 2e_0 \cos E + e_0^2 \cos (F + E) + (2 - e_0^2) \cos (F - E) \\ &- 2e_0 \cos F \Big] + \left(- \frac{1}{2} e_0^2 + 2e_0 \cos E - 1 - \frac{1}{2} e_0^2 \cos 2E \right) , \end{aligned} \tag{100}$$

$$\begin{aligned} N(1 - e_0^2) = \frac{h^2}{h_0^2} &\Big[2e_0 \sin E - e_0 \sin F + e_0 \sin (F - 2E) \Big] \\ &+ \frac{1}{1 + \nu} \Big[-2e_0 \sin E - (2 - e_0^2) \sin (F - E) \\ &+ e_0^2 \sin (F + E) \Big] + \left(e_0 \sin E - \frac{1}{2} e_0^2 \sin 2E \right) , \end{aligned} \tag{101}$$

$$\Lambda = \frac{1 - \nu^2}{1 + \overline{W}} \left(1 + \frac{\bar{r}}{a_0} \cdot \frac{\gamma}{\sqrt{1 - e_0^2}} \right),$$

and

$$S = \frac{\bar{\rho}}{a_0} \frac{\partial W}{\partial F} - \left(W + 1 + \frac{h_0}{h} \right) e_0 \sin F .$$

In the first approximation, there is

$$M(1-e_0^2) = -3\left(1-\frac{1}{2}e_0^2\right) + 2e_0 \cos E - \frac{1}{2}e_0^2 \cos 2E$$

$$+ e_0^2 \cos(E+F) + (4-e_0^2)\cos(E-F)$$

$$- e_0 \cos(2E-F) - 3e_0 \cos F ,$$

$$N(1-e_0^2) = +e_0 \sin E - \frac{1}{2}e_0^2 \sin 2E + e_0^2 \sin(E+F)$$

$$+ (2-e_0^2)\sin(E-F) - e_0 \sin(2E-F) - e_0 \sin F ,$$

and

$$S = -2e_0 \sin F .$$

Before integration of Equation 99 is started, ψ must be determined in such a way that no term of the form $A \sin F$ is present in this equation. Otherwise, the integration will produce a secular term $AE \sin F$ in W and a term $AE \sin E$ of the mixed type in $\overline{W}$. As long as the inclination is not near 63.4 degrees, only periodic terms are permitted and small divisors will not make the process inoperable.

Perturbations of the Orbit Plane

It already has been pointed out that the motion of the orbit plane can be decomposed into the secular motion of the mean orbit plane and the small oscillations about this mean position of the orbit plane. In order to determine the secular motion as well as the small oscillations of the orbit plane, the differential equations for λ parameters must be formed. The classic equations for the variation of constants will be used:

$$\sin i \frac{d\theta}{dt} = h r \frac{\partial\Omega}{\partial Z} \sin(v-\sigma) , \tag{102}$$

$$\frac{d\sigma}{dt} = \frac{d\theta}{dt} \cos i , \tag{103}$$

$$\frac{di}{dt} = h r \frac{\partial\Omega}{\partial Z} \cos(v-\sigma) . \tag{104}$$

Differentiating Equations 42 and 43 gives us

$$2 \frac{dN}{dt} = - \frac{d\sigma}{dt} - \frac{d\theta}{dt} - 2\alpha \frac{dE}{dt} , \tag{105}$$

$$2 \frac{dK}{dt} = - \frac{d\sigma}{dt} + \frac{d\theta}{dt} + 2\eta \frac{dE}{dt} . \tag{106}$$

Substituting Equations 102 and 103 into the two last equations, we have

$$\frac{dN}{dt} = -\alpha \frac{dE}{dt} - \frac{1}{2} h r \frac{\partial \Omega}{\partial Z} \cot \frac{i}{2} \sin (v - \sigma) , \tag{107}$$

$$\frac{dK}{dt} = +\eta \frac{dE}{dt} + \frac{1}{2} h r \frac{\partial \Omega}{\partial Z} \tan \frac{i}{2} \sin (v - \sigma) . \tag{108}$$

Differentiating Equations 56 for λ parameters and substituting Equations 104, 107, and 108 yields the following equations:

$$\frac{d\lambda_1}{dt} = + \alpha \lambda_2 \frac{dE}{dt} + \frac{1}{2} a_0 h (1 + \nu) \frac{\partial \Omega}{\partial Z} (+\lambda_4 \ell - \lambda_3 m) , \tag{109}$$

$$\frac{d\lambda_2}{dt} = - \alpha \lambda_1 \frac{dE}{dt} + \frac{1}{2} a_0 h (1 + \nu) \frac{\partial \Omega}{\partial Z} (-\lambda_3 \ell - \lambda_4 m) , \tag{109a}$$

$$\frac{d\lambda_3}{dt} = + \eta \lambda_4 \frac{dE}{dt} + \frac{1}{2} a_0 h (1 + \nu) \frac{\partial \Omega}{\partial Z} (+\lambda_2 \ell + \lambda_1 m) , \tag{109b}$$

$$\frac{d\lambda_4}{dt} = - \eta \lambda_3 \frac{dE}{dt} + \frac{1}{2} a_0 h (1 + \nu) \frac{\partial \Omega}{\partial Z} (-\lambda_1 \ell + \lambda_2 m) . \tag{109c}$$

But in this case

$$r \frac{\partial \Omega}{\partial Z} = \frac{\partial \Omega}{\partial \psi} \cos i , \tag{110}$$

and

$$\frac{dt}{dE} = \frac{\bar{r}}{a_0 n_0} \Lambda . \tag{111}$$

Taking Equations 110 and 111 into consideration, the following final equations can be deduced from Equations 109 through 109c:

$$\frac{d\lambda_1}{dE} = +\alpha\lambda_2 + \frac{1}{2}\frac{h}{h_0}\cdot\frac{a_0}{\sqrt{1-e_0^2}}\frac{\partial\Omega}{\partial\psi}\cos i\,(+\lambda_4\ell - \lambda_3 m)\Lambda\,, \tag{112}$$

$$\frac{d\lambda_2}{dE} = -\alpha\lambda_1 + \frac{1}{2}\frac{h}{h_0}\cdot\frac{a_0}{\sqrt{1-e_0^2}}\frac{\partial\Omega}{\partial\psi}\cos i\,(-\lambda_3\ell - \lambda_4 m)\Lambda\,, \tag{112a}$$

$$\frac{d\lambda_3}{dE} = +\eta\lambda_4 + \frac{1}{2}\frac{h}{h_0}\cdot\frac{a_0}{\sqrt{1-e_0^2}}\frac{\partial\Omega}{\partial\psi}\cos i\,(+\lambda_2\ell + \lambda_1 m)\Lambda\,, \tag{112b}$$

$$\frac{d\lambda_4}{dE} = -\eta\lambda_3 + \frac{1}{2}\frac{h}{h_0}\cdot\frac{a_0}{\sqrt{1-e_0^2}}\frac{\partial\Omega}{\partial\psi}\cos i\,(-\lambda_1\ell + \lambda_2 m)\Lambda\,. \tag{112c}$$

The values of α and η must be determined from the second and the fourth equations in such a way that no constant terms are present on the right-hand sides. The system (Equations 112 through 112c) is solved by the method of iterations starting with

$$\lambda_1 = \sin\frac{i_0}{2}\,, \qquad \lambda_3 = 0\,,$$

$$\lambda_2 = 0\,, \qquad \lambda_4 = \cos\frac{i_0}{2}\,.$$

Each iteration step leads to an improved value of λ and α, η, and the process must be continued until the final values of λ_1, λ_2, λ_3, λ_4, α, and η are reached.

ON THE CONSTANTS OF INTEGRATION, DETERMINATION OF THE BASIC HANSEN FUNCTION, AND PERTURBATIONS IN THE MEAN ANOMALY

The constants of integration are the six given elements a_0, e_0, g_0, θ_0, i_0, and $\omega_0 = \pi_0 - \sigma_0$. No moment of time exists for which these elements are osculating. They must be compatible with the observations, and the development of the coordinates obtained on the basis of these elements must contain the periodic terms only. The system of basic elements is determined by repetition of the orbit correction; after each orbit correction the development of perturbations must be done anew in order to obtain a better representation. If the two theories are compared, the systems of elements used in different theories are similar but not identical. Such a comparison requires, generally speaking, great care and ingenuity. But in any given theory there is a unique system of elements compatible with the observations.

In the integration connected with determining the perturbations, some new additional constants of integration are introduced. They must be determined in such a way that no secular terms appear in the development of the coordinates. Implicitly, these additional constants are determined as functions of the basic elements. The analytical form of the dependence of these constants on the elements is not evident in the numerical theory.

In this development there will be series of the following four forms:

$$\Sigma\, C \cos (i E + 2 j \omega) , \tag{113}$$

$$\Sigma\, C \cos [i E + (2 j + 1) \omega] , \tag{113a}$$

$$\Sigma\, S \sin (i E + 2 j \omega) , \tag{113b}$$

$$\Sigma\, S \sin [i E + (2 j + 1) \omega] . \tag{113c}$$

In order to keep the secular terms absent in the development of the coordinates, no constant of integration is added if the integrated series has the form of Equations 113a, 113b, or 113c; but one is added if the integrated series has the form 113. That is clear because series in Equations 113b and 113c, being the sine series, do not contain any constant terms. The series 113a cannot contain any constant term because only the combinations $\pm\omega$, $\pm 3\omega$, $\pm 5\omega$, $\cdots$ can be present in the argument.

At each step of the process of iteration, we obtain for Equation 99 an expression of the form

$$\frac{dW}{dE} = \Sigma\, S \sin (i E + 2 j \omega + k F) + \Sigma\, C \cos [i E + (2 j + 1) \omega + k F] \tag{114}$$

or of the form

$$\frac{dW}{dE} = \Sigma\, S \sin(iE + 2j\omega) + \Sigma\, C \cos[iE + (2j + 1)\omega]$$

$$+ \left\{\Sigma\, S \sin(iE + 2j\omega) + \Sigma\, C \cos[iE + (2j + 1)\omega]\right\} \cos F$$

$$+ \left\{\Sigma\, C \cos(iE + 2j\omega) + \Sigma\, S \sin[iE + (2j + 1)\omega]\right\} \sin F. \qquad (114a)$$

If the series in Equation 114 or 114a are integrated, the additive constant of integration must be of the form

$$C_0 + C_1 \cos F$$

and the series for W has the form

$$W = C_0 + C_1 \cos F + \Sigma\, C \cos(iE + 2j\omega + kF)$$

$$+ \Sigma\, S \sin[iE + (2j + 1) + kF] . \qquad (114b)$$

The series for $\overline{W}$ becomes

$$\overline{W} = C_0 + C_1 \cos E + \Sigma\, C \cos(iE + 2j\omega) + \Sigma\, S \sin[iE + (2j + 1)\omega] . \quad (114c)$$

Equation 96 for $n_0 \delta z$ can be put in the form

$$\frac{dn_0 \delta z}{dE} = \overline{W}\, \frac{\bar{r}}{a_0} + \frac{\nu^2 - \overline{W}^2}{1 + \overline{W}}\, \frac{\bar{r}}{a_0} - \frac{y}{\sqrt{1 - e_0^2}}\, \frac{1 - \nu^2}{1 + \overline{W}}\, \frac{\bar{r}^2}{a_0^2} . \qquad (115)$$

This form is convenient for the use of the method of iteration because the values of ν^2, $\overline{W}^2$ and $1 + \overline{W}$ can be taken from the previous iteration. Hence, Equation 115 takes the form

$$\frac{dn_0 \delta z}{dE} = (C_0 + C_1 \cos E)\,(1 - e_0 \cos E)$$

$$+ \Sigma\, A_{ij} \cos(iE + 2j\omega) + \Sigma\, B_{ij} \sin[iE + (2j + 1)\omega] . \qquad (115a)$$

The coefficients C_0 and C_1 must be determined in such a way that no constant term and no term of the form $K \cos E$ are contained in the last equation because such terms, when integrated, will produce terms already contained in Kepler's equation. Then

$$C_0 - \frac{1}{2} e_0 C_1 + A_{00} = 0 ,$$

$$- e_0 C_0 + C_1 + A_{10} = 0 ,$$

or

$$C_0 = - \frac{2A_{00} + e A_{10}}{2 - e_0^2} ,$$

$$C_1 = - \frac{2A_{00} e_0 + 2A_{10}}{2 - e_0^2} ,$$

and the coefficient A_{20} is corrected by an amount

$$- \frac{1}{2} e_0 C_1 .$$

The final forms of W, $\overline{W}$ and $dn_0 \delta z/dt$, after the constants C_0 and C_1 are determined and substituted into Equations 114b, 114c, and 115, remain the same as before:

$$W = \Sigma C \cos (iE + 2j\omega + kF) + \Sigma S \sin [iE + (2j + 1)\omega + kF] ,$$

$$\overline{W} = \Sigma C \cos (iE + 2j\omega) + \Sigma S \sin [iE + (2j + 1)\omega] ,$$

$$\frac{dn_0 \delta z}{dt} = \Sigma C \cos (iE + 2j\omega) + \Sigma S \sin [iE + (2j + 1)\omega] ;$$

and at each step of the iteration process we obtain for $n_0 \delta z$ a result of the form

$$n_0 \delta z = \Sigma S \sin (iE + 2j\omega) + \Sigma C \cos [iE + (2j + 1)\omega] .$$

Determination of the Perturbations in h/h_0 and in the Radius Vector

We have obtained for W:

$$W = \Sigma C \cos (iE + 2j\omega + kF) + \Sigma S \sin [iE + (2j + 1)\omega + kF] ,$$

where

$$\ell = -1,\ 0,\ +1\ .$$

This result can be presented in the form

$$W = \Xi + \Upsilon \cos F + \Psi \sin F\ ; \tag{116}$$

where Ξ is obtained as the part in Equation 114b independent of F, and γ can be obtained by putting $F = 0$ in the remaining part because

$$\Upsilon = (W - \Xi)_{F=0}\ . \tag{117}$$

The perturbations in h/h_0 can be obtained from

$$\Xi + e_0 \Upsilon = -1 - \frac{h_0}{h} + 2\, \frac{h}{h_0}\ . \tag{87}$$

Putting

$$\frac{h_0}{h} = 1 + \Delta\ , \tag{118}$$

we deduce from the last equation

$$\Delta = -\frac{1}{3}\,(\Xi + e_0 \Upsilon) + \frac{2}{3}\,\frac{\Delta^2}{(1 + \Delta)}\ , \tag{119}$$

or

$$\Delta = -\frac{(\Xi + e_0 \Upsilon)}{3} + \frac{2}{3}\,(\Delta^2 - \Delta^3 + \cdots)\ . \tag{120}$$

Also,

$$\frac{h}{h_0} = 1 + \frac{1}{2}\,(\Delta + \Xi + e_0 \Upsilon)\ . \tag{121}$$

Equation 79 can be put in the form

$$\nu = \frac{1}{2}\,(\Delta - \overline{W}) - \frac{1}{2}\,\nu\,(\Delta + \overline{W})\ . \tag{122}$$

This form is more convenient for the use of the process of iterations because Δ, $\overline{W}$ and ν in the second term can be taken from the previous iteration. The determination of h/h_0 and $1 + \nu$, consequently, does not require any additional integration after the basic function W is determined.

Determination of the λ Parameters

After the constants α and η are determined, Equations 112 through 112c give

$$\lambda_1 = \sin \frac{i_0}{2} + \frac{1}{2}(A + B) + \delta\lambda_1 , \tag{123}$$

$$\lambda_2 = \delta\lambda_2 , \tag{124}$$

$$\lambda_3 = \delta\lambda_3 , \tag{125}$$

$$\lambda_4 = \cos \frac{i_0}{2} + \frac{1}{2}(A - B) + \delta\lambda_4 . \tag{126}$$

The terms

$$\delta\lambda_1, \quad \delta\lambda_2, \quad \delta\lambda_3, \quad \delta\lambda_4$$

are obtained by the formal integration of Equations 109 through 109c. They do not contain the additive constants of integration, and they have the form

$$\delta\lambda_1 = \Sigma\, C \cos(iE + 2j\omega) + \Sigma\, S \sin[iE + (2j + 1)\omega] , \tag{127}$$

$$\delta\lambda_2 = \Sigma\, S \sin(iE + 2j\omega) + \Sigma\, C \cos[iE + (2j + 1)\omega] , \tag{128}$$

$$\delta\lambda_3 = \Sigma\, S \sin(iE + 2j\omega) + \Sigma\, C \cos[iE + (2j + 1)\omega] , \tag{129}$$

$$\delta\lambda_4 = \Sigma\, C \cos(iE + 2j\omega) + \Sigma\, S \sin[iE + (2j + 1)\omega] . \tag{130}$$

Only the series for λ_1 and λ_4 contain the additive constants of integration. The determination of these constants can be done in a more symmetrical way if they are written in the form $(A + B)/2$ and $(A - B)/2$.

Two conditions must be satisfied: (1) The principal term in the latitude must have the form

$$\sin i_0 \sin [\bar{f} + (\omega)] ; \tag{131}$$

(2) In addition,

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 = 1 . \tag{132}$$

Now, Equation 58 is

$$\psi = 2(\lambda_1\lambda_4 - \lambda_2\lambda_3) \sin [\bar{f} + (\omega)] + 2(\lambda_2\lambda_4 + \lambda_1\lambda_3) \cos [\bar{f} + (\omega)] , \tag{58}$$

and, as a consequence of the first condition, the constant part in $2(\lambda_1 \lambda_4 - \lambda_2 \lambda_3)$ is equal to $\sin i_0$. Substituting Equations 123 through 126 into 131 and 132, gives us

$$\frac{1}{2} (A^2 - B^2) + \left(\cos \frac{i_0}{2} + \sin \frac{i_0}{2}\right) A + \left(\cos \frac{i_0}{2} - \sin \frac{i_0}{2}\right) B$$
$$+ \ const. \ in \ 2(\delta\lambda_1 \delta\lambda_4 - \delta\lambda_2 \delta\lambda_3) = 0 ,$$

and

$$\frac{1}{2} (A^2 + B^2) + \left(\cos \frac{i_0}{2} + \sin \frac{i_0}{2}\right) A - \left(\cos \frac{i_0}{2} - \sin \frac{i_0}{2}\right) B$$
$$+ \ const. \ in \ (\delta\lambda_1^2 + \delta\lambda_2^2 + \delta\lambda_3^2 + \delta\lambda_4^2) = 0 ,$$

or

$$A^2 + 2A \left(\cos \frac{i_0}{2} + \sin \frac{i_0}{2}\right) + (11) = 0 , \tag{133}$$

$$B^2 - 2B \left(\cos \frac{i_0}{2} - \sin \frac{i_0}{2}\right) + (12) = 0 , \tag{134}$$

where

$$(11) = const. \ in \left[(\delta\lambda_1 + \delta\lambda_4)^2 + (\delta\lambda_2 - \delta\lambda_3)^2\right] ,$$
$$(12) = const. \ in \left[(\delta\lambda_1 - \delta\lambda_4)^2 + (\delta\lambda_2 + \delta\lambda_3)^2\right] .$$

Equations 133 to 134 are solved by the method of successive approximations *if* $i_0 \neq 90°$. It is preferable to use the p, q, s, system if the satellite is a polar one. We simply put

$$p = \tan\frac{i_0}{2} + \delta p \ ,$$

$$q = \delta q \ ,$$

$$s = \delta s \ ;$$

and δp, δq, δs are deduced by formal integration.

Decomposition of the Matrix of Rotation

In his lunar theory, Hansen developed the radius vector, the longitude, and the latitude into trigonometric series. In this case, however, the inclination can be large, and it is preferable to use the matrix of rotation instead of the development of the coordinates into series. Two types of rotation matrices appear in the present exposition:

(1) The matrix representing a rotation about the x-axis for the angle α. This matrix has the form

$$A_1[\alpha] = \begin{bmatrix} +1 & 0 & 0 \\ 0 & +\cos\alpha & -\sin\alpha \\ 0 & +\sin\alpha & +\cos\alpha \end{bmatrix} .$$

(2) The matrix representing a rotation about the z-axis:

$$A_3[\alpha] = \begin{bmatrix} +\cos\alpha & -\sin\alpha & 0 \\ +\sin\alpha & +\cos\alpha & 0 \\ 0 & 0 & +1 \end{bmatrix} .$$

The coordinates of the real satellite with respect to the system rigidly connected to the moving ellipse (the x-axis is directed to the perigee of the moving ellipse and the z-axis is normal to the orbit plane) are:

$$(1+\nu)\ a_0\ (\cos E - e_0), \qquad (1+\nu)\ a_0 \sqrt{1 - e_0^2}\ \sin E, \qquad 0.$$

The transformation from the system rigidly connected to the moving ellipse to the inertial system requires a triple rotation, and the position vector in the inertial system can be presented in the form

$$\mathbf{r} = (1 + \nu)\, \Gamma \cdot \begin{bmatrix} a_0 \; (\cos E - e_0) \\ a_0 \sqrt{1 - e_0^2} \; \sin E \end{bmatrix}, \tag{135}$$

$$\Gamma = A_3[\theta] \cdot A_1[i] \cdot A_3[\bar{\pi}_0 + \gamma\, \Delta E - \sigma] \; . \tag{136}$$

We had

$$\theta = (\theta) - N + K \; ,$$

$$\bar{\pi}_0 + \gamma\, \Delta E - \sigma = (\omega) + N + K \; ,$$

and we deduce

$$\Gamma = A_3[(\theta)] \cdot A_3[K - N] \cdot A_1[i] \cdot A_3[K + N] \cdot A_3[(\omega)] \; .$$

Taking

$$\lambda_1 = \sin \frac{i}{2} \cos N \, , \qquad \lambda_3 = \cos \frac{i}{2} \sin K \, ,$$

$$\lambda_2 = \sin \frac{i}{2} \sin N \, , \qquad \lambda_4 = \cos \frac{i}{2} \cos K$$

into account, we have

$$\Gamma = A_3[(\theta)] \cdot [\lambda_{ij}] \cdot A_3[(\omega)] \; , \tag{137}$$

where

$$\lambda_{11} = +\lambda_1^2 - \lambda_2^2 - \lambda_3^2 + \lambda_4^2 \, , \qquad \lambda_{21} = +2\,(\lambda_3 \lambda_4 - \lambda_1 \lambda_2) \, ,$$

$$\lambda_{12} = -2\,(\lambda_3 \lambda_4 + \lambda_1 \lambda_2) \, , \qquad \lambda_{22} = -\lambda_1^2 + \lambda_2^2 - \lambda_3^2 + \lambda_4^2 \, ,$$

$$\lambda_{13} = +2\,(\lambda_1 \lambda_3 - \lambda_2 \lambda_4) \, , \qquad \lambda_{23} = -2\,(\lambda_1 \lambda_4 + \lambda_2 \lambda_3) \, ,$$

$$\lambda_{31} = +2\,(\lambda_3 \lambda_1 + \lambda_2 \lambda_4) \, ,$$

$$\lambda_{32} = +2\,(\lambda_4 \lambda_1 - \lambda_2 \lambda_3) \, ,$$

$$\lambda_{33} = -\lambda_1^2 - \lambda_2^2 + \lambda_3^2 + \lambda_4^2 \, .$$

The numerical computation of the perturbations for a given moment of time is done by the method of iterations. Using Kepler's equation (Equation 40) and the development of $n_0 \delta z$, the values of E and $n_0 \delta z$ are obtained by the method of iterations, starting with $n_0 \delta z = 0$.

Then the values of λ_{ij}, Γ and $1 + \nu$ are evaluated and, finally, the coordinates x, y, z in the inertial system are evaluated.

CONCLUDING REMARKS

The theory described here is a numerical one and permits the full use of the large capacity of modern machines. The computation can be carried out to any desired order compatible with the accuracy of the basic data. Terms are retained or rejected on the basis of their numerical values, and the decision about the importance of a certain term is made by the machine automatically. Hansen's theory permits the easy inclusion of any number of gravitational sources, and the present program can be used without modification for such cases.

However, the numerical treatment has disadvantages in certain cases. If the eccentricity is small, the determination of the motion of the perigee is difficult because the eccentricity appears as a divisor. In the case of large eccentricity, difficulties arise through the presence of the factor $1 - e_0^2$ in the denominator and by the slow convergence of the series for $a_0/\bar{r}$. For these two extreme cases an analytical development would be preferable. In this connection we refer to the results obtained by Brouwer (Reference 8), Kozai (Reference 9), and Garfinkel (Reference 10).

In addition, there is the special problem involved in the treatment of the critical inclination. An extension of the present methods, designed for the solution of the critical inclination as well as the cases of very small and very large eccentricity, is now in progress.

REFERENCES

1. Hansen, P. A. "Auseinanderzetzung einer Zweckmässigen Method zur Berechnung der Absoluten Störungen der Kleinen Planeten," K. Sächsische Gesellschaft der Wissenschaften, Mathematische-Physikalishe Klasse, Abh. 5, 1857

2. Hansen, P. A., "Fundamenta Nova Investigationis Orbitae Verae Quam Luna Perlustrat," Gotha, 1838, pp. 1-133

3. Brouwer, D., "Outlines of General Theories of the Hill-Brown and Delaunay Types of Orbits of Artificial Satellites," Astron. J. 63(10): 433-438, November 1958

4. Whittaker, E. T., "A Treatise on the Analytical Dynamics of Particles and Rigid Bodies," Cambridge: Cambridge Univ. Press, 1904

5. Gibbs, J. W., "Vector Analysis," New York: Scribner's, 1901, p. 343

6. Brown, E. W. and Shook, C. A., "Planetary Theory," Cambridge: Cambridge Univ. Press, 1933, p. 70

7. Hill, G. W., "Note on Hansen's Theory of Perturbations," Am. J. Math., 4:256-259, 1881

8. Brouwer, D., "Solution of the Problem of Artificial Satellite Theory Without Drag," Astron. J. 64(9):378-397, November 1959

9. Kozai, Y., "The Motion of a Close Earth Satellite," Astron. J. 64(9):367-377, November 1959

10. Garfinkel, B., "The Orbit of a Satellite of an Oblate Planet," Astron. J. 64(9):353-367, November 1959