Theoretical Limits of Damping
Attainable by Smart Beams
with Rate Feedback

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ABSTRACT
Using a generally accepted model we present a comprehensive analysis (within the page limitation) of an Euler-Bernoulli beam with PZT sensor-actuator and pure rate feedback. The emphasis is on the root locus — the dependence of the attainable damping on the feedback gain. There is a critical value of the gain beyond which the damping decreases to zero. We construct the time-domain response using semigroup theory, and show that the eigenfunctions form a Riesz basis, leading to a "modal" expansion.

1. INTRODUCTION
In this paper we present a comprehensive analysis of an Euler-Bernoulli beam with PZT sensor-actuator along its entire length. The sensor output is a charge in a condenser and the actuator input is the current, a differentiator circuit being then an essential component, yielding "rate feedback." We use a generally accepted model. Tzou et al. present purely computational results and seem to be unaware of a purely theoretical analysis given earlier by Chen et al. The most important design parameter is the control gain and the damping attainable — we construct a full root-locus analysis (omitting details to keep within the page limitation). We also unearth a curious phenomenon — the existence of a deadbeat mode (real eigenvalue) not noticed hitherto. We show that the eigenvalues are the roots of an entire function of order one-half, proving in particular the existence of a countably infinite number of eigenvalues. We also show that the eigenfunctions form a Riesz basis. We also construct the Green's function for the nonhomogeneous eigenvalue problem. As in Chen et al. we use the theory of semigroups of operators to obtain the time-domain solution. Our proof of the exponential stability is different from that in Chen et al., as is our choice of the function space. We note that a similar analysis for a Timoshenko model (a "smart string") is given in Balakrishnan, where there is a critical value of the gain at which there are no eigenvalues and the semigroup is actually nilpotent ("disappearing" solution).

2. MAIN RESULTS
The Euler-Bernoulli model formulates as
\[
\begin{align*}
    cf'''(t,s) + mf(t,s) &= 0, \quad 0 < s < L, \quad 0 < t \\
    f(t,0) &= 0 = f'(t,0); \quad f'''(t,L) = 0 \\
    cf''(t,L) + \alpha f'(t,L) &= 0
\end{align*}
\]
where \( f(t,s) \) is the displacement and the superdots indicate derivative with respect to \( t \) and the primes indicate derivative with respect to \( s \). It is convenient to set
\[ \nu^2 = \frac{m}{c}. \]

For a precise formulation of the time-domain response we need to specify first the choice of function spaces. We pick \( L^2[0,L] \) for \( f(t, \cdot) \). Let \( A_o \) denote the operator defined by
\[ A_o f = cf''', \]
where
\[ D(A_o) = \{ f \mid f', f'', f''', f'''' \in L^2[0,L]; f(0) = 0 = f'(0) = f'''(L) \}. \]
Let
\[ \mathcal{H} = L^2[0, L] \times E^1. \]
Define the operator \( A \) with domain and range in \( \mathcal{H} \) by:
\[
x = \begin{bmatrix} f \\ b \end{bmatrix}, \quad Ax = \begin{bmatrix} Af \\ cf''(L) \end{bmatrix};
\]
with domain
\[
\mathcal{D}(A) = \left\{ \begin{bmatrix} f \\ b \end{bmatrix} : f \in \mathcal{D}(A_0) \text{ and } b = f'(L) \right\}.
\]
It is convenient to adopt the notation
\[ A_b x = cf''(L), \quad x \in \mathcal{D}(A). \]
Then for \( x \) in \( \mathcal{D}(A) \):
\[
[Ax, x] = \int_0^L cf''''(s) f(s) \, ds + cf''(L)f'(L)
\]
\[= c \int_0^L |f''(s)|^2 \, ds. \]
It is readily seen that \( A \) has dense domain and is self-adjoint and nonnegative definite, and has compact resolvent. Also zero is not an eigenvalue. Let \( \sqrt{A} \) denote the positive square root. On the product space
\[ \mathcal{D}(\sqrt{A}) \times L^2[0, L] \]
is introduced the "energy" inner product
\[
[Y, Z]_E = [\sqrt{A} y_1, \sqrt{A} z_1] + m[y_2, z_2]
\]
\[Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.\]
\[\mathcal{D}(\sqrt{A}) = \left\{ \begin{bmatrix} f \\ b \end{bmatrix} : f'' \in L^2[0, L] \text{ and } b = f'(L), \ f(0) = f'(0) = 0 \right\}.\]
For \( y_1 \) in \( \mathcal{D}(A) \), we see that
\[|Y, Y|_E = |A y_1, y_1| + m|y_2, y_2| \sim \text{"energy" (potential + kinetic).}\]
We denote the product space under this inner product by \( \mathcal{H}_E \) and note that it is a Hilbert space. Let \( A \) denote the operator defined by:
\[ AY = \begin{bmatrix} f_2 \\ -cf''(L) \\ -Af \end{bmatrix}, \quad Y = \begin{bmatrix} x \\ f_2 \end{bmatrix} = \begin{bmatrix} f_1(\cdot) \\ f'_1(L) \\ f_2(\cdot) \end{bmatrix}, \]
and
\[
\mathcal{D}(A) = \left\{ \begin{bmatrix} f_1(\cdot) \\ f'_1(L) \end{bmatrix} \in \mathcal{D}(A) \right\}, \quad \left\{ \begin{bmatrix} f_2(\cdot) \\ f_2''(L) \end{bmatrix} \in \mathcal{D}(\sqrt{A}) \right\}.
\]
Thus defined we can verify that
\[ \mathcal{D}(A) = \mathcal{D}(A^*) \]
and that \( A \) is dissipative:
\[
\frac{1}{2}[(A + A^*)Y, Y] = \text{Re}[AY, Y]_E = \frac{1}{\alpha |A_b|} ||A_b x||^2 = \frac{1}{\alpha} c^2 |f''(L)|^2.
\]
It is readily verified that $\mathcal{A}$ has a compact resolvent and that $\mathcal{A}$ generates a $C_0$ contraction semigroup. With these definitions, the system (1) goes over into the abstract formulation:

$$\dot{Y}(t) = \mathcal{A}Y(t). \quad (2)$$

This choice of the function space is technically different from that in Chen et al.$^5$

**Eigenvalues and eigenfunctions of $\mathcal{A}$**

Our primary interest is in the modal decomposition — the eigenvalues of $\mathcal{A}$ and the corresponding eigenfunctions. Or, equivalently, in the resolvent of $\mathcal{A}$. Let $\mathcal{R}(\lambda, \mathcal{A})$ denote the resolvent of $\mathcal{A}$. Let

$$\mathcal{R}(\lambda, \mathcal{A})Y = Z$$

where

$$Y = \begin{bmatrix} h_1 \\ b \\ h_2 \end{bmatrix}.$$  

Since $Z \in \mathcal{D}(\mathcal{A})$, we can write

$$Z = \begin{bmatrix} f_1(\cdot) \\ f'_1(L) \\ f_2(\cdot) \end{bmatrix}$$

and

$$(\lambda I - \mathcal{A})Z = Y$$

yields

$$\lambda f_1 - f_2 = h_1$$

$$\lambda f_2 + \frac{A_0 f_1}{m} = h_2$$

$$\lambda f'_1(L) + \frac{c f''_1(L)}{\alpha} = b.$$  

Hence

$$\left\{ \begin{array}{l}
\lambda \nu^2 f_1(s) + f_1'''(s) = \nu^2(h_2(s) + \lambda h_1(s)), \quad 0 < s < L \\
\lambda \alpha f'_1(L) + cf''_1(L) = \alpha b \\
f_1(0) = 0 = f'_1(0) = f''_1(L). \end{array} \right. \quad (3)$$

**Eigenvalues**  

First we consider the eigenvalue problem, setting

$$h_1 = 0 = h_2; \quad b = 0.$$  

Let

$$\gamma = \sqrt{-\lambda \nu} e^{i\theta/2} e^{i\pi/4}, \quad \gamma^4 = -\lambda^2 \nu^2$$

where

$$\lambda = |\lambda| e^{i\theta}.$$  

Then the solution satisfying the conditions at zero yields:

$$f_1(s) = a(\cosh \gamma s - \cos \gamma s) + b(\sinh \gamma s - \sin \gamma s), \quad 0 < s < L.$$  

The constants $a$ and $b$ are then determined by the conditions at $L$:

$$a(\alpha \gamma (\sinh \gamma L + \sin \gamma L) + \gamma^2(\cosh \gamma L + \cos \gamma L)) + b(\alpha \gamma (\cosh \gamma L - \cos \gamma L) + \gamma^2(\sinh \gamma L + \sin \gamma L)) = 0$$

$$a \gamma^3(\sinh \gamma L - \sin \gamma L) + b \gamma^3(\cosh \gamma L + \cos \gamma L) = 0.$$
Let
\[ H(\lambda) = \begin{vmatrix} \lambda \alpha \gamma (\sinh \gamma L + \sin \gamma L) & \lambda \alpha \gamma (\cosh \gamma L - \cos \gamma L) \\ c_\gamma^2 (\cosh \gamma L + \cos \gamma L) & c_\gamma^2 (\cosh \gamma L + \sin \gamma L) \\ \gamma^3 (\sinh \gamma L - \sin \gamma L) & \gamma^3 (\cosh \gamma L + \cos \gamma L) \end{vmatrix} \] (4)
and
\[ D(\lambda) = \text{Det } H(\lambda). \]

Then
\[ D(\lambda) = (\gamma^4) [(\cosh \gamma L + \cos \gamma L)(\lambda \alpha (\sinh \gamma L + \sin \gamma L) + c_\gamma (\cosh \gamma L + \cos \gamma L)) - (\sinh \gamma L - \sin \gamma L)(\lambda \alpha (\cosh \gamma L - \cos \gamma L) + c_\gamma (\sinh \gamma L + \sin \gamma L))] = 2\gamma^4 [c_\gamma (1 + \cosh \gamma L \cos \gamma L) + \lambda \alpha (\sinh \gamma L \cos \gamma L + \cosh \gamma L \sin \gamma L)]. \] (5)

We note that zero is not an eigenvalue. The eigenvalues \( \{\lambda_k\} \) are thus determined by the nonzero roots of
\[ c_\gamma (1 + \cosh \gamma L \cos \gamma L) + \lambda \alpha (\sinh \gamma L \cos \gamma L + \cosh \gamma L \sin \gamma L) = 0 \]

Or, using
\[ \lambda = \frac{-i\gamma^2}{\nu} \]
we have
\[ (1 + \cosh \gamma L \cos \gamma L) - \frac{i\gamma \alpha}{\nu c} (\sinh \gamma L \cos \gamma L + \cosh \gamma L \sin \gamma L) = 0. \] (6)

**Theorem 2.1**
\( \mathcal{A} \) has exactly one real-valued eigenvalue.

**Proof**
Setting \( L = 1 \), and using \( \alpha \) to denote \( \frac{\alpha}{\nu} \), and expressing the trigonometric products in (6) as sums, we have
\[ f = 1 + \cosh \gamma \cos \gamma - i\gamma (\sinh \gamma \cos \gamma + \cosh \gamma \sin \gamma) \]
\[ f = 1 + \frac{1}{2} (\cos \gamma (1 + i) + \cos \gamma (1 - i)) - i\gamma \frac{1}{2} [\sinh \gamma (1 + i) + \sin \gamma (1 - i) + \sin \gamma (1 + i) + \sin \gamma (1 - i)]. \]
Hence making the 1:1 transformation
\[ \gamma = x(i - 1) \]
we obtain
\[ f(\gamma) = g(x) = 1 + \frac{1}{2} (\cos 2x + \cosh 2x) - \alpha x (\sin 2x + \sinh 2x) \] (7)
yielding an equivalent expression for determining the eigenvalues. Note that \( g(\cdot) \) is real-valued for real values of \( x \). Further
\[ g(0) = 2 \]
while, as \( x \to \infty \), \( (x \text{ real}) \), we note that
\[ g(x) \to -\infty. \]
Hence there is a positive real root. Denote it \( x_0 \). Then
\[ \lambda = -\gamma^2 i = -x^2 (i - 1)^2 i = -2x^2. \]
Hence
\[ \lambda_0 = -2x_0^2 \]

\[ \text{Due to J. Lin; private communication.} \]
is an eigenvalue. We note that \( x_1 \) is the only real-valued root of \( g(\cdot) \). Indeed, if there is a real-valued eigenvalue of \( A \), we must have, denoting it by \( \lambda_1 \),

\[
\lambda_1 = -2x_1^2
\]

and \( x_0 \) must be a root of \( g(\cdot) \). Hence

\[
x_0 = x_1.
\]

Or, \( \lambda_0 \) is the only real-valued eigenvalue of \( A \).

We note that the corresponding eigenfunction is given by

\[
\phi_1(s) = (\cosh \gamma_0 - \cos \gamma_0)(\cosh \gamma_0 s - \cos \gamma_0 s) - (\sinh \gamma_0 - \sin \gamma_0)(\sinh \gamma_0 s - \sin \gamma_0 s)
\]

where

\[
\gamma_0 = x_0(i - 1), \quad \lambda = -2x_0^2.
\]

**Theorem 2.2 (Chen, et al.\textsuperscript{5})**

Let \( \{\lambda_k\} \) denote the eigenvalues, and assume that

\[
|\lambda_k| \to \infty.
\]

Then

\[
\lim_{k} \text{Re}\lambda_k = \frac{-c}{L\alpha}.
\]

**Proof**

See Chen et al.\textsuperscript{5} for a proof.

The authors of Chen et al. however do not appear to offer a proof of the fact that the eigenvalues \( \{\lambda_k\} \) are nonfinite in number. The fact that the resolvent is compact is not adequate to establish this; the compactness only assures that if nonfinite in number then \( \{\lambda_k\} \) can be arranged so that

\[
|\lambda_{k+1}| \geq |\lambda_k|
\]

and

\[
|\lambda_k| \to \infty \quad \text{as} \quad k \to \infty.
\]

For proving the fact that eigenvalues are denumerably infinite we can indicate a general technique.

**Theorem 2.3**

The eigenvalues \( \{\lambda_k\} \) are denumerably infinite and such that

\[
\sum_{i}^{\infty} \left| \text{Im} \left( \frac{1}{\lambda_k} \right) \right| < \infty.
\]

**Proof**

From (6) we see that for each \( \alpha \), the eigenvalues are the zeros of the function

\[
d(\lambda) = (1 + \cosh \gamma L \cos \gamma L) - i \left( \frac{\alpha}{\lambda} \right) \gamma(\sinh \gamma L \cos \gamma L + \cosh \gamma L \sin \gamma L).
\]

As a power series expansion will show, this is an entire function of the complex variable \( \lambda \). Moreover it is of exponential type, of order \( \frac{1}{2} \), and of completely regular growth. Further we can calculate that

\[
h(\theta) = \lim_{r \to \infty} \log |d(re^{i\theta})| = \sqrt{2} \max \left( |\sin \frac{\theta}{2}|, |\cos \frac{\theta}{2}| \right).
\]
Let \( n(r) \) denote the number of zeros of \( d(\cdot) \) in the circle of radius \( r \) centered at zero. Then by the theorem of R.P. Boas (see Levin\(^7\)) we have:

\[
\lim_{r \to \infty} \frac{n(r)}{r^{1/2}} = \frac{1}{4\pi} \int_0^{2\pi} h(\theta) \, d\theta > 0.
\]

Hence

\[
\lim_{r \to \infty} n(r) = \infty,
\]
or, the number of zeros is not finite. Moreover the function is of class A (see Levin\(^7\) for the definition) since

\[
\sup_{R > 0} \int_0^R \log |d(s) d(-s)| \left| \frac{1}{1 + s^2} \right| \, ds < M_d < \infty.
\]

The result (9) is a consequence. Q.E.D.

**Remark**

Applying Jensen's Theorem we have

\[
\frac{1}{2\pi} \int_0^{2\pi} d'(r e^{i\theta}) \frac{r e^{i\theta}}{d(r e^{i\theta})} \, d\theta = n(r).
\]

We can actually compute this as a quick means of locating eigenvalues. There is a jump of 2 corresponding to each eigenvalue and its conjugate. This is shown in Figure 1 for

\[
\frac{\alpha}{L_{CV}} = .01.
\]

**Eigenfunctions**

The eigenfunction corresponding to the eigenvalue \( \lambda_k \) is given by

\[
\Phi_k = A_k \begin{vmatrix} \phi_k \\ \phi'_k(L) \\ \lambda_k \phi_k \end{vmatrix}
\]

where

\[
\phi_k(s) = c_k(\cosh \gamma_k s - \cos \gamma_k s) + d_k(\sinh \gamma_k s - \sin \gamma_k s)
\]

where

\[
H(\lambda_k) \begin{vmatrix} c_k \\ d_k \end{vmatrix} = 0
\]

or, we may take

\[
c_k = (\cosh \gamma_k L - \cos \gamma_k L); \quad d_k = -(\sinh \gamma_k L - \sin \gamma_k L)
\]
or,

\[
\phi_k(s) = A_k[(\cosh \gamma_k L - \cos \gamma_k L)(\cosh \gamma_k s - \cos \gamma_k s) - (\sinh \gamma_k L - \sin \gamma_k L)(\sinh \gamma_k s - \sin \gamma_k s)].
\]  \(\text{(11)}\)

Correspondingly:

\[
\phi'_k(L) = 2A_k \gamma_k(\cosh \gamma_k L - \cos \gamma_k L) \sin \gamma_k L.
\]

The coefficient \( A_k \) may be chosen for appropriate normalization. For example we may make

\[
||\Phi_k|| = 1.
\]

Note that \( \lambda_k \) is an eigenvalue of \( A^* \) and the corresponding eigenvector is:

\[
\Psi_k = B_k \begin{vmatrix} \overline{\phi}_k(\cdot) \\ \overline{\phi}'_k(L) \\ -\lambda_k \overline{\phi}_k(\cdot) \end{vmatrix}
\]

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where \( B_k \) is again a “normalization” scalar. Note that

\[
[\Phi_k, \Psi_k]_E = \left( c \int_0^L \phi''_k(s) \, ds - m \lambda_k^2 \int_0^L \phi_k(s)^2 \, ds \right) A_k \overline{B_k}
\]

\[
= 4c A_k \overline{B_k} \gamma_k c_k d_k \int_0^L \left( \cosh \gamma_n s \cos \gamma_n s + \sinh \gamma_n s \sin \gamma_n s \right) \, ds
\]

\[
= 4c c_k d_k A_k \overline{B_k} \gamma_k \cosh \gamma_k L \sin \gamma_k L
\]

\[
\neq 0.
\]

In particular we may choose \( A_k, B_k \) so that

\[
[l_k, \psi_k] = 1.
\]

(12)

Further using a result of Gohberg and Krein\(^{10}\) (we omit the details) we can establish that \( \{ \Phi_k, \Psi_k \} \) with the normalization (12) actually yield a Riesz basis for \( \mathcal{H}_E \). In terms of this basis we have the (“modal”) expansion for the solution of (2)

\[
Y(t) = \sum_{k=1}^{\infty} a_k e^{\lambda_k t} \Phi_k
\]

(13)

where

\[
a_k = [Y(0), \Psi_k]_E
\]

and as an easy byproduct, using (8), we see that the semigroup generated by \( \mathcal{A} \) is exponentially stable (established in Chen et al. by different arguments).

**Root Locus**

Let us consider how the eigenvalues behave as the gain \( \alpha \) is varied. For this purpose it is convenient to define

\[
d(\lambda; \alpha) = M(\lambda) + \frac{\alpha}{cv} N(\lambda)
\]

where

\[
M(\lambda) = 1 + \cosh \gamma L \cos \gamma L
\]

\[
N(\lambda) = -\frac{i\gamma}{cv} \left( \sinh \gamma L \cos \gamma L + \cosh \gamma L \sin \gamma L \right).
\]

Because of the analytic dependence of \( d(\lambda; \alpha) \) on \( \alpha \), we can invoke the theory of algebraic or algebroidal functions\(^{8,9}\) and note that

\[
d(\lambda(\alpha); \alpha) = 0
\]

will define \( \lambda(\alpha) \) as a multivalued analytic function of \( \alpha \) with isolated singularities, if any. In particular this allows us to define the sequence \( \{ \lambda_k(\alpha) \} \), \( k = 1, 2, \ldots \) such that

\[
\lambda_k(0) = \frac{i \mu_k}{L^2 v}, \quad \mu_k = (2k - 1) \frac{\pi}{2} + \epsilon_k
\]

(the “clamped-free” beam modes) and

\[
\lim_{\alpha \to -\infty} \lambda_k(\alpha) = \frac{i (k \pi - \epsilon')^2}{L^2 v}
\]

(“clamped-rolling” modes) and the real root

\[
\lambda_0(\alpha)
\]

is such that

\[
\lim_{\alpha \to -\infty} \lambda_0(\alpha) = 0, \quad \lim_{\alpha \to 0} \lambda_0(\alpha) = -\infty.
\]

A plot of the locus of the real root is shown in Figure 2. Moreover

\[
\lambda'_k(\alpha) = \frac{-1}{cv} \left. \frac{N(\lambda)}{M'(\lambda) + \frac{\alpha}{cv} N'(\lambda)} \right|_{\lambda=\lambda_k(\alpha)}.
\]
In particular
\[ \lambda_k'(0) = -\frac{1}{cv} \left. \frac{N(\lambda)}{M'(\lambda)} \right|_{\lambda=\lambda_k(0)}. \]

We can show that
\[ \frac{d}{d\alpha} (\text{Re} \lambda_k(\alpha)) = -\frac{\mu_k^2}{L^2v} \left( \frac{2}{Lc\nu} \right), \quad \alpha = 0 \]
\[ = \frac{c}{2\alpha^2L}, \quad \alpha = +\infty \]
\[ \frac{d}{d\alpha} (\text{Im} \lambda_k(\alpha)) \geq 0. \]

A root locus of the first mode is shown in Figure 3. The damping \((= |\text{Re} \lambda_k|)\) increases with the gain until a critical value of the gain is reached and thereafter decreases to zero. Note that by virtue of (14) we have actually "proportional damping" for small gain. A plot of the critical value of the gain versus the mode number is given in Figure 4.

**Resolvent**

Let us now return to the resolvent — or solving (3). We note that
\[
g(\lambda, s) = \frac{1}{2\gamma^3} \int_0^s (\text{Sinh} \gamma(s-\sigma) - \text{Sin} \gamma(s-\sigma)) \nu^2 (h_2(\sigma) + \lambda h_1(\sigma)) \, d\sigma
\]
is a "particular" solution of
\[ \lambda^2 \nu^2 f_1 + f_1''' = \nu^2 (h_2 + \lambda h_1) \]
such that
\[ f_1(0) = f_1'(0) = 0. \]

Hence we can express the solution \(f_1(\lambda, s)\), where we have included \(\lambda\) to indicate the dependence on \(\lambda\), as:
\[ f_1(\lambda, s) = g(\lambda, s) + a(\lambda)(\text{Cosh} \gamma s - \text{Cos} \gamma s) + b(\lambda)(\text{Sinh} \gamma s - \text{Sin} \gamma s), \quad 0 < s < L \]
where the coefficients \(a(\lambda), b(\lambda)\) are determined from
\[
\begin{vmatrix}
a(\lambda) \\
b(\lambda)
\end{vmatrix} = H(\lambda)^{-1} \begin{vmatrix} ab - \alpha \lambda g'(\lambda, L) - cg''(\lambda, L) \\ -g'''(\lambda, L) \end{vmatrix}
\]
where the primes again denote derivatives with respect to the variable \(s\). Hence letting
\[ H(\lambda) = \begin{vmatrix} h_{11}(\lambda) & h_{12}(\lambda) \\ h_{21}(\lambda) & h_{22}(\lambda) \end{vmatrix} \]
and defining
\[ \bar{H}(\lambda) = \begin{vmatrix} h_{22}(\lambda) & -h_{12}(\lambda) \\ -h_{21}(\lambda) & h_{11}(\lambda) \end{vmatrix} \]
so that
\[ H(\lambda)\bar{H}(\lambda) = D(\lambda)I = \bar{H}(\lambda)H(\lambda), \]
\[ a(\lambda) = \frac{1}{D(\lambda)} \left[ h_{22}(\lambda)(ab - \alpha \lambda g'(\lambda, L) - cg''(\lambda, L)) + h_{12}(\lambda)g'''(\lambda, L) \right] \]
\[ b(\lambda) = \frac{1}{D(\lambda)} \left[ -h_{21}(\lambda)(ab - \alpha \lambda g'(\lambda, L) - cg''(\lambda, L)) - h_{11}(\lambda)g'''(\lambda, L) \right]. \]
We can cast the Green’s function in the form:

\[
    f_1(\lambda, s) = \int_0^s \frac{K(\lambda; s, \sigma)}{D(\lambda)} h(\sigma) \, d\sigma + \int_s^L \frac{K(\lambda; \sigma, s)}{D(\lambda)} h(\sigma) \, d\sigma
\]

\[
    + \frac{\alpha b}{D(\lambda)} [h_{22}(\lambda)(\cosh \gamma s - \cos \gamma s) - h_{21}(\lambda)(\sinh \gamma s - \sin \gamma s)]
\]

\[
    K(\lambda; \sigma, s) = (\cosh \gamma s - \cos \gamma s) \left[ \left( 2h_{12} - \frac{2\alpha \lambda}{\gamma^2} h_{22} \right) \cosh \gamma (L - \sigma) + \left( 2h_{12} + \frac{2\alpha \lambda}{\gamma^2} h_{22} \right) \cos \gamma (L - \sigma) \right.
\]

\[
    - \frac{2ch_{22}}{\gamma} \left( \sinh \gamma (L - \sigma) + \sin \gamma (L - \sigma) \right) \left. \right]
\]

\[
    + (\sinh \gamma s - \sin \gamma s) \left[ \left( \frac{2\alpha \lambda h_{21}}{\gamma^2} - 2h_{11} \right) \cosh \gamma (L - \sigma) + \left( -\frac{2\alpha \lambda h_{21}}{\gamma^2} - 2h_{11} \right) \cos \gamma (L - \sigma) \right.
\]

\[
    + \frac{2ch_{21}}{\gamma} \left( \sinh \gamma (L - \sigma) + \sin \gamma (L - \sigma) \right) \right], \quad s < \sigma \quad (15)
\]

\[
    h = m(h_2 + \lambda h_1)
\]

\[
    D(\lambda) = -2\alpha^2 \nu^2 [c_1(1 + \cosh \gamma L \cos \gamma L) + \lambda \alpha (\sinh \gamma L \cos \gamma L + \cosh \gamma L \sin \gamma L)].
\]

Finally

\[
    \mathcal{R}(\lambda, \mathcal{A}) \begin{bmatrix} h_1 \\ b \\ h_2 \end{bmatrix} = \begin{bmatrix} f_1(\lambda, s) \\ f_1(\lambda, 0) \\ \lambda f_1(\lambda, s) - h_1(s) \end{bmatrix}.
\]

Note that setting \( \alpha = 0 \) in (15) we get the Green’s function for the clamped/free-free beam. In particular

\[
    \mathcal{R}(0, \mathcal{A}) \begin{bmatrix} h_1 \\ h_1(0) \\ h_2 \end{bmatrix} = \begin{bmatrix} K h_2 \\ (K h_2)(0) \\ -h_1 \end{bmatrix} + \alpha h_1(0) \begin{bmatrix} \frac{L}{c} \\ \frac{K}{c} \\ 0 \end{bmatrix}
\]

where \( K h_2 \) is the function given by

\[
    \frac{m}{c} \int_0^s (L - \sigma) h_2(\sigma) \, d\sigma + \frac{m}{c} \int_s^L (L - \sigma) h_2(\sigma) \, d\sigma, \quad 0 < s < L.
\]

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**REFERENCES**


\[ \frac{\alpha}{L c \nu} = 0.01 \]

\[ L \sqrt{\nu |\lambda| \nu} \]

Figure 1: \( n(|\lambda|) \) Zeros.
\[ g = \frac{\alpha}{Lcv} \]

Figure 2: Deadbeat Mode (Real Eigenvalue).

Figure 3: Root Locus: First Mode \( \lambda = \sigma + i\omega \).
Figure 4: Critical Gain vs. Mode.