Generic Equations for Constructing Smooth Paths Along Circles and Tangent Lines With Application to Airport Ground Paths

L. Keith Barker
Langley Research Center, Hampton, Virginia

December 1998
Since its founding, NASA has been dedicated to the advancement of aeronautics and space science. The NASA Scientific and Technical Information (STI) Program Office plays a key part in helping NASA maintain this important role.

The NASA STI Program Office is operated by Langley Research Center, the lead center for NASA’s scientific and technical information. The NASA STI Program Office provides access to the NASA STI Database, the largest collection of aeronautical and space science STI in the world. The Program Office is also NASA’s institutional mechanism for disseminating the results of its research and development activities. These results are published by NASA in the NASA STI Report Series, which includes the following report types:

- **TECHNICAL PUBLICATION.** Reports of completed research or a major significant phase of research that present the results of NASA programs and include extensive data or theoretical analysis. Includes compilations of significant scientific and technical data and information deemed to be of continuing reference value. NASA counter-part or peer-reviewed formal professional papers, but having less stringent limitations on manuscript length and extent of graphic presentations.

- **TECHNICAL MEMORANDUM.** Scientific and technical findings that are preliminary or of specialized interest, e.g., quick release reports, working papers, and bibliographies that contain minimal annotation. Does not contain extensive analysis.

- **CONTRACTOR REPORT.** Scientific and technical findings by NASA-sponsored contractors and grantees.

- **CONFERENCE PUBLICATION.** Collected papers from scientific and technical conferences, symposia, seminars, or other meetings sponsored or co-sponsored by NASA.

- **SPECIAL PUBLICATION.** Scientific, technical, or historical information from NASA programs, projects, and missions, often concerned with subjects having substantial public interest.

- **TECHNICAL TRANSLATION.** English-language translations of foreign scientific and technical material pertinent to NASA’s mission.

Specialized services that help round out the STI Program Office’s diverse offerings include creating custom thesauri, building customized databases, organizing and publishing research results . . . even providing videos.

For more information about the NASA STI Program Office, see the following:


- Email your question via the Internet to help@sti.nasa.gov

- Fax your question to the NASA Access Help Desk at (301) 621-0134

- Phone the NASA Access Help Desk at (301) 621-0390

- Write to:
  NASA Access Help Desk
  NASA Center for AeroSpace Information
  7121 Standard Drive
  Hanover, MD 21076-1320
NASA/TP-1998-208960

Generic Equations for Constructing Smooth Paths Along Circles and Tangent Lines With Application to Airport Ground Paths

L. Keith Barker
Langley Research Center, Hampton, Virginia

National Aeronautics and Space Administration
Langley Research Center
Hampton, Virginia 23681-2199

December 1998
Abstract

The primary purpose of this publication is to develop a mathematical model to describe smooth paths along any combination of circles and tangent lines. Two consecutive circles in a path are either tangent (externally or internally) or they appear on the same (lateral) or opposite (transverse) sides of a connecting tangent line. A path may start or end on either a segment or circle. The approach is to use mathematics common to robotics to design the path as a multilink manipulator. This approach allows a hierarchical view of the problem and keeps the notation manageable. A user simply specifies a few parameters to configure a path. Necessary and sufficient conditions automatically ensure the consistency of the inputs for a smooth path. Two example runway exit paths are given, and an angle to go assists in knowing when to switch from one path element to the next.

1. Introduction

A NASA goal is to improve the aviation system throughput in all weather conditions. One aspect of this problem is guidance along the runway and exit path. In this respect, coordinates along the centerline of a runway and the exit path must be known. Although others (refs. 1, 2, and 3) have modeled specific exit paths (taxiways) with linear segments and/or circles, a general formulation does not appear to exist for such paths.

This publication completely characterizes smooth paths along an arbitrary combination of circles and tangent lines. The approach herein is to configure the path as a multilink manipulator and use the mathematics common to robotics. This approach provides a ready-made high-level perspective of a complex system. A user simply specifies a few parameters in a computer program to configure a new path. Two examples of designing runway exit paths are given. Also, as a convenience, this publication introduces a parameter to tell when to switch from one path element to the next.

2. Symbols

- $A_{ij}^i$: homogeneous transformation matrix from axis system $i$ to axis system $j$ or transform from circle $C_i$ to circle $C_j$
- $a_i$: component of vector from $C_{i-1}$ to $C_i$ along $X_i$
- $C$: circle
- $ccw$: counterclockwise
- $cir$: circle
- $cw$: clockwise
- $e_i$: circle classification, either exterior ($i = 1$) or inward ($i = -1$)
- $(h_i,k_i,0)$: center of circle $C_i$ in general reference frame $(X_G,Y_G,Z_G)$
- $i,j$: index
- $L_{ij}^{i+1}$: transformation from circle $C_{i+1}$ to general reference frame with $l_{i+1} = 0$
- $l_i$: length of segment $P_i$, positive or negative
- $N$: number of circles in path
- $P_i$: tangent segment before circle $C_i$
- $p_i^{i-1}$: vector (in axis system $i-1$) from axis system $i-1$ to axis system $i$ or vector from center of circle $C_{i-1}$ to center of circle $C_i$
- $R_i$: radius of circle $C_i$
- $R_j^i$: Euler rotation matrix about $Z_j$ from axis system $i$ to axis system $j$
- $seg$: segment
(X_G, Y_G, Z_G) general reference frame

(X_i, Y_i, Z_i) axis system rigidly attached to center of circle C_i

(X_i', Y_i', Z_i') axis system parallel to (X_G, Y_G, Z_G) at center of circle C_i


x_i, y_i, z_i coordinates in circle axis system (X_i, Y_i, Z_i)

(x_i', y_i', 0) coordinates in circle axis system (X_i', Y_i', Z_i') parallel to (X_G, Y_G, Z_G)

(x_i^*, y_i^*, 0) starting point of motion on circle C_i

(x_i^#, y_i^#, 0) ending point of motion on circle C_i

ψ_{G,i} azimuth on circle C_i relative to axis X_G

ψ_i azimuth on circle C_i from north

ψ_{P,i} azimuth on segment P_i from north

ψ_X_G azimuth of axis X_G

ω_i index of rotation, clockwise (i = -1) or counterclockwise (i = 1) on circle C_{i-1}

Subscripts:

max maximum

target target value on path

3. Circle Classification

A known path can be approximated by measuring a series of points along the path and then linearly connecting the points (line segments). However, this requires many points to approximate closely a curvilinear path, and even then, the slope of the path where the segments connect is discontinuous.

The method discussed in this publication constructs a smooth path by using a combination of circles and lines. Any two successive circles in a path are either tangent or connected by a common tangent line. A path may begin or end on a circle or line. A path is the composite of pieces of travel along circles and tangent lines. A user specifies information about the circles and segments to generate the path. If two immediately consecutive circles in a path are tangent, except for the tangent point, either (1) the second circle is external to the first (disjoint interiors), (2) the second circle is inside the first circle, or (3) the first circle is inside the second circle.

Any two successive circles in a path (say, first and second circles) are either tangent or connected by a single common tangent line. The second circle is either an exterior circle or inward circle with respect to the first circle.
3.1. Exterior Circle

If two circles lie on opposite sides of a given tangent line, the second circle is an external (or transverse) circle with respect to the first, as shown in figure 1.

3.2. Inward Circle

If the first and second circles lie on the same side of a given common tangent line, the second circle is an inward (or lateral) circle with respect to the first circle, as shown in figure 1. Notice that the second circle can be inside the first or vice versa.

3.3. Parameter for Circle Type

Symbolically, the circle classification is

\[ e_i = \begin{cases} 1 & (C_i \text{ exterior}) \\ -1 & (C_i \text{ inward}) \end{cases} \]  

(1)

4. Modeling Path as Multilink Manipulator

A robot manipulator (ref. 4) is a series of joints connected by links; for example, the sketch in figure 2 shows a manipulator arm with three rotational joints (shoulder, elbow, and wrist). Each joint only moves the part of the manipulator beyond that joint; that is, the elbow joint rotates the forearm but the wrist joint does not. To configure a manipulator, one changes the joint angles.

Designing a path is somewhat analogous to configuring a planar multilink manipulator if one thinks of the manipulator body as the path. Imagine, for example, that the manipulator in figure 2 is lying flat on a horizontal plane and think of the body of the manipulator as a road or path on this plane. A path of linear segments then follows. However, for a smooth path the segments should make a smooth transition into one another. Hence, imagine the joints as centers of circles and the links as tangent to these circles. Now, imagine moving along a path from the base of the manipulator toward the shoulder along link 0, which ends as a tangent to the shoulder circle. Proceed around the circumference of the shoulder circle for so many degrees (travel angle) and then travel toward the elbow along link 1, which is tangent to both the shoulder circle and the elbow circle, and so forth until the end of the path. Changing the joint angles changes the path. Really there are no joints on the path, but this publication relates the travel angle along the circular arcs to the joint angles to use mathematics common to robotics, which is the basic idea in this publication.

A known path can be approximated with a path along circles and tangent lines. To describe a path using circles and lines, one must know the radii of the circles, where the circles are located, the angular travel along each circle, and the length of any linear segment between circles.

4.1. Axis Systems

Figure 3 shows the initial layout of circles and linear segments to form a path. All axis systems are right-handed. As shown, all circles are exterior and all
segments are positive, but this initial configuration changes, depending on the type of circle and whether a segment is positive or negative. To change a circle from exterior to inward, simply move the external circle to the opposite side of its upper tangent line; to change a positive segment to a negative segment, simply shift the segment by its length in the opposite direction. As shown, there is a general reference frame \((X_G, Y_G, Z_G)\), plus each circle has its own rigidly attached axis system (at the center of the circle). When \(\Gamma = 0\), the axes \(X_G\) and \(Y_0\) are aligned. Note that the circle indices begin with zero; that is, the axis system of the first circle is \((X_0, Y_0, Z_0)\). All the axis systems for the circles are parallel when the circles angles are 0. The configuration of the circles changes to accommodate different path designs.

### 4.2. Circle Rotation Angle

Think of the center of each circle as a rotational joint and the distance between consecutive circle centers as a link (like in a manipulator robot). The first circle is stationary. A rotation at the center of the first circle (\(\theta_1\) about \(Z_0\)) translates and rotates the second circle and all remaining circles. If two circles happen to be tangent, the second circle moves along the circumference of the first circle. By convention, the circle angle \(\theta_1\) is always about \(Z_{i-1}\). See appendix A.
for an example of configuring a path with circle angles. Later, these circle angles will configure a path but will not be user inputs. (With respect to manipulators, circle angles are analogous to the joint angles.)

4.3. Segment Notation

If there is a segment prior to the first circle \( C_0 \), call it \( P_0 \). Afterwards, segment \( P_{i+1} \) follows circle \( C_i \). Therefore, the last segment \( P_N \) follows the last circle \( C_{N-1} \). Hence, there are \( N \) circles in a path and up to \( N + 1 \) linear segments.

5. Smooth Path Design

Figure 4 shows the six possible locations of the second of two immediately consecutive circles in a path. Various paths can be traced (arrows) along the configuration; however, not all are smooth. (A cusp can occur at the transition point between two circles or between a circle and a segment; otherwise, an abrupt change in direction occurs.) Smooth paths (no cusps) are the primary interest in this publication, but the basic analysis is not limited in this respect.

5.1. Direction of Motion on Circle

A counterclockwise (ccw) motion on a circle is positive, and a clockwise (cw) motion is negative. Symbolically, denote the direction of motion on a circle as

\[
\omega_i = \begin{cases} 
1 & (C_{i-1} \text{ ccw}) \\
-1 & (C_{i-1} \text{ cw}) 
\end{cases} 
\]  

(2)

5.2. Necessary and Sufficient Conditions for Smooth Path

A computer program can automatically assign information or issue an alert for invalid inputs in the design of a path. For a smooth transition from a circle to a segment, the segment length parameter with algebraic sign is

\[
\begin{align*}
    l_0 &= \text{Absolute length of segment } P_0 \\
    l_i &= \omega_i \text{ (Absolute length of segment } P_i) \\
    (i &= 1, 2, \ldots, N) 
\end{align*} 
\]  

(3)

For a smooth path, another relationship that must hold between immediately successive circles in a path is (fig. 4)

\[
\omega_{i+1} = -e_{i+1} \omega_i 
\]

(4)

where \( \omega_1 \) determines the direction of motion on the first circle and depends on the initial path direction.

Figure 4. Possible paths (arrows) along next circle.
(right or left). Equations (3) and (4) constitute necessary and sufficient conditions for a smooth path.

As an example in using equation (4), if the motion on the first of two consecutive circles is clockwise \((\omega_i = -1)\) and a counterclockwise motion \((\omega_{i+1} = 1)\) is desired on the second circle, then the second circle must be external, as shown by

\[
e_{i+1} = -\left(\frac{\omega_{i+1}}{\omega_i}\right) = -\left(\frac{1}{-1}\right) = 1
\]

For the same first circle \((\omega_i = -1)\), if the second circle were external \((e_{i+1} = 1)\), then the direction of motion on the second circle must be counterclockwise, as shown by

\[
\omega_{i+1} = -e_{i+1}\omega_i = -(1)(-1) = 1
\]

Consequently, beyond the first circle, a user only needs to specify either the circle classification or the direction of motion on the remaining circles.

6. Homogeneous Transformation Matrices Along Path

A homogeneous transformation matrix (ref. 4) completely characterizes the geometric relationship between two axis systems, which may be rotated and/or displaced from each other.

6.1. Transformation Between Consecutive Circles

The homogeneous transformation matrix from circle \(C_i\) to the previous circle \(C_{i-1}\) (for \(i = 1, 2, ..., N\)) is

\[
A_{i-1}^i = \begin{bmatrix}
\cos \theta_i & -\sin \theta_i & 0 & a_i \cos \theta_i - l_i \sin \theta_i \\
\sin \theta_i & \cos \theta_i & 0 & a_i \sin \theta_i + l_i \cos \theta_i \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(5)

where \(\theta_i\) is the circle angle (positive about \(Z_{i-1}\)) from \(X_{i-1}\) to \(X_i\). \(l_i\) is the distance (either plus or minus) along the tangent segment between the two circles, and

\[
a_i = R_{i-1} + e_i R_i
\]

is the component of the vector from \(C_{i-1}\) to \(C_i\) along \(X_i\). For the special case of tangent circles \((l_i = 0)\), the magnitude of \(a_i\) is the distance between circle centers. The transformation matrix between any two arbitrary circles in a path is a concatenation of the individual transformation matrices between pairs of circles in a path. See appendix B for a brief discussion of homogeneous matrices.

6.2. Transformation From Circle to General Reference Frame

The homogeneous transformation matrix from any circle \(C_i\) in the path to the reference frame is

\[
A_G = A_G^0 A_G^1 A_G^2 ... A_G^{i-1}
\]

(7)

where, for the particular orientation of the general reference frame shown in figure 3,

\[
A_G^0 = \begin{bmatrix}
\cos \Gamma & -\sin \Gamma & 0 & h_0 \\
\sin \Gamma & \cos \Gamma & 0 & k_0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(8)

is the transformation from the first circle \(C_0\) and

\[
h_0 = x_{G,\text{exit}} - (\text{Exit direction}) R_0 \sin \Gamma
\]

(9)

\[
k_0 = y_{G,\text{exit}} + (\text{Exit direction}) R_0 \cos \Gamma
\]

(10)

are the center coordinates of \(C_0\) in the reference frame. Here, \(x_{G,\text{exit}}\) and \(y_{G,\text{exit}}\) are the coordinates of the exit point (beginning point on first circle in the path). The exit point is where the path first veers away from a reference direction (along runway), and the parameter that categorizes the exit direction is

\[
\text{Exit direction} = \begin{Bmatrix} 1 & (\text{Exiting to right}) \\ -1 & (\text{Exiting to left}) \end{Bmatrix}
\]

(11)
6.3. Recursive Relationship for Circle Center

The center of a circle is important in creating a path and later in following a path. Let \((h_i, k_i, 0)\) be the center point of circle \(C_i\). An equation for the circle center (zero coordinates in circle axis system) is

\[
\begin{bmatrix}
h_i \\
k_i \\
0 \\
1
\end{bmatrix} = \begin{bmatrix}
A_{i-1} & A_{i-1} \\
0 & 0 \\
0 & 0 \\
1 & 1
\end{bmatrix} \begin{bmatrix}
a_i \cos \theta_i - l_i \sin \theta_i \\
a_i \sin \theta_i + l_i \cos \theta_i \\
0 \\
1
\end{bmatrix}
+ P_{i-1}
\]

Or, more explicitly, the following vector recursive relationship is given:

\[
\begin{align*}
\begin{bmatrix}
h_i \\
k_i \\
0 \\
1
\end{bmatrix} &= R_G^{i-1} \begin{bmatrix}
a_i \cos \theta_i - l_i \sin \theta_i \\
a_i \sin \theta_i + l_i \cos \theta_i \\
0 \\
1
\end{bmatrix} + P_{i-1} \\
&= R_G^{i-1} \begin{bmatrix}
a_i \cos \theta_i - l_i \sin \theta_i \\
a_i \sin \theta_i + l_i \cos \theta_i \\
0 \\
1
\end{bmatrix} + P_{i-1}
\end{align*}
\]

(12)

To start the recursion,

\[
\begin{bmatrix}
h_0 \\
k_0 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
A_G(0,3) \\
A_G(0,3) \\
0 \\
0
\end{bmatrix}
\]

(14)

or equivalently,

\[
\begin{align*}
\begin{bmatrix}
h_0 \\
k_0 \\
0 \\
0
\end{bmatrix} &= \begin{bmatrix}
x_{G,exit} \\
y_{G,exit} \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
R_0 \sin \Gamma \\
R_0 \cos \Gamma \\
0 \\
0
\end{bmatrix} \\
&= \begin{bmatrix}
x_{G,exit} \\
y_{G,exit} \\
0 \\
0
\end{bmatrix} + \begin{bmatrix}
R_0 \sin \Gamma \\
R_0 \cos \Gamma \\
0 \\
0
\end{bmatrix}
\end{align*}
\]

(15)

where on the first circle \(C_0\), \(R_0\) is the radius, \((x_{G,exit}, y_{G,exit}, 0)\) is the initial point (exit), and \(\theta_1\) is the direction of travel.

7. Drawing Path

Several input formats could be used to define a path (user-preference issue). In any event, directly or indirectly from a user’s input, the following information about a path is known:

1. Exit point
2. Whether the path is to the right or left
3. The number of circles
4. Whether each circle is exterior or inward
5. Radius of each circle
6. Distance along each linear segment (with direction)
7. Angular travel (travel angle with direction) along each circle

A path can be drawn once a set of design parameters is known. Several reasons for drawing a path are

1. To show users what their design looks like, thereby adding confidence (validity) to the design
2. To provide visual feedback in the design process to changes in parameters
3. To overlay and adjust the path to known geometry (terrain)
4. To provide a graphical reference for motion (visual deviations from a path)

7.1. Path Along Circle

Figure 3 shows the configuration of the circles and segments in a path (for exterior circles and positive segments) when the circle angles are zero. Changing these circle angles from their zero values moves the circles (and segments) to other locations to
accommodate a particular path design. However, a more natural way to design a path is to specify the angular travel (travel angle) along each circle in the path instead of specifying the circle angles \( \theta_i \). (Analogously, in robotics, it is more natural for an operator to command the movements of an end-effector or gripper directly than to specify the joint angles individually in a manipulator.)

7.1.1. Relating Circle Angle to Travel Angle

Define the travel angle \( \lambda_i \) as the angular travel along the circumference of circle \( C_{i-1} \) (that is, arc travel along the circumference of a circle subtends the travel angle). A counterclockwise travel angle is positive.

Notice in figure 5 that motion begins from the rear (point on circle along \(-X_{i-1}\)) of an exterior circle and from the front (point on circle along \(X_{i-1}\)) of an inward circle. The circle angle \( \theta_i \), which moves circle \( C_i \) with respect to the circle \( C_{i-1} \), is a function of the travel angle and the circle classification. From figure 5, this relationship is seen to be

\[
\theta_i = 90^\circ (1 + e_{i-1}) + \lambda_i
\]

Hence, instead of specifying the circle angles to configure a path, a user specifies the angular travel along each circle and then calculates the circle angles, which is more convenient.

7.1.2. Path Coordinates Along Circle

Let the circle \( C_i \) be either an exterior or inward circle that rotates about circle \( C_{i-1} \) by the angle \( \lambda_{i-1} \). The path along circle \( C_{i-1} \) is the same whether there is a tangent line from \( C_{i-1} \) to \( C_i \). Therefore, for this calculation, assume that the two circles are tangent so that circle \( C_i \) touches and rotates along the circumference of circle \( C_{i-1} \). As \( \lambda_i \) varies, a plot of the touch points is the path along circle \( C_{i-1} \) (fig. 6). In the reference frame, the path coordinates along circle \( C_{i-1} \) are given by

\[
\begin{bmatrix}
 x_{G, i-1} \\
 y_{G, i-1}
\end{bmatrix} = A_{i-1}^i \begin{bmatrix} -e_i R_{i-1} \\ 0 \\ 1 \end{bmatrix} = A_{i-1}^i \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

(Path along \( C_{i-1} \))

where the vector on the right-hand side of equation (17) is the touch point in the axis system of circle \( C_i \) and the term in brackets varies with \( \theta_i \) which varies with \( \lambda_i \). The segment length \( l_i \) in \( A_{i-1}^i \) is set equal to 0 in this calculation, which changes at most two elements. Thus, equation (17) gives points along the circular track as an implicit function of the travel angle. To draw the circular track on each circle \( C_{i-1} \) (for \( i = 1, 2, \ldots, N \)), vary the travel angle \( \lambda_i \) from 0 to its final value. If desired, equation (17)
always touch point with respect to tangent circle, exterior or inward.

Reference

Always touch point with respect to tangent circle, exterior or inward.

In essence, equation (17) uses the next circle to draw a path on the current circle. Consequently, a fictitious circle is required to draw the path along the last circle $C_{i-1}$. For $i = N$, equation (17) should give the path along the last circle $C_{N-1}$, but up to now, there is no matrix $A_{N-1}$ because there is no circle $C_N$. To use equation (17) to draw the path along the last circle $C_{N-1}$, think of a fictitious circle $C_N$ with arbitrary radius $R_N$ at the end of the final segment $P_N$. The homogeneous transformation for this fictitious circle is

$$A_N = \begin{bmatrix}
\cos \theta_N & -\sin \theta_N & 0 & a_N \cos \theta_N - l_N \sin \theta_N \\
\sin \theta_N & \cos \theta_N & 0 & a_N \sin \theta_N + l_N \cos \theta_N \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

(19)

where

$$a_N = R_{N-1} + e_N R_N$$

$$\theta_N = 90^\circ (1 + e_{N-1}) + \lambda_N$$

(20)

Here, $\lambda_N$ is the angular travel on circle $C_{N-1}$, and $\theta_N$ is the angular rotation angle of circle $C_N$ about circle $C_{N-1}$. Computation of this transformation is transparent and automatic with respect to a user.

7.2. Path Along Segment

A straight-line segment can occur

1. Prior to the first circle
2. Between two circles
3. After the last circle

In the first case, the initial point is known. Connecting this initial point and the exit point (which is the starting point on the first circle $C_0$) plots the first segment.

If there is a segment between two consecutive circles in a path, this segment connects the ending point of the first circle with the starting point of the second circle. Therefore, automatically, the segments will be drawn when the circle path points are connected.
Lastly, connecting the final path point on the last circle $C_{N-1}$ to the last point in a path plots the final segment $P_N$. If there is a final segment $P_N$ beyond the last circle, then the coordinates of the last point on this segment are the coordinates of the center of the fictitious circle $C_N$ when $R_N = 0$. Equation (13) gives the center of the fictitious circle $C_N$ as

$$
\begin{align*}
\begin{bmatrix}
h_N \\
k_N \\
0
\end{bmatrix} &= \begin{bmatrix}
h_{N-1} \\
k_{N-1} \\
0
\end{bmatrix} + R_N^{N-1} \begin{bmatrix}
a_N \cos \theta_N - l_N \sin \theta_N \\
a_N \sin \theta_N + l_N \cos \theta_N \\
0
\end{bmatrix} \\
&= \begin{bmatrix}
h_{N-1} \\
k_{N-1} \\
0
\end{bmatrix} + R_N^{N-1} \begin{bmatrix}
a_N \cos \theta_N - l_N \sin \theta_N \\
a_N \sin \theta_N + l_N \cos \theta_N \\
0
\end{bmatrix}
\end{align*}
$$

Temporarily, set $R_N = 0$. This means that the tangent point of the segment $P_N$ with the fictitious circle $C_N$ is the same as the center of the fictitious circle. Also, this makes $a_N = R_{N-1}$; additionally, let $\theta_N = \lambda_N$. Then, the last point on $P_N$ is given by

$$
(P_N)_{\text{last point}} = \begin{bmatrix}
h_{N-1} \\
k_{N-1} \\
0
\end{bmatrix} + R_N^{N-1} \begin{bmatrix}
a_N \cos \lambda_N - l_N \sin \lambda_N \\
a_N \sin \lambda_N + l_N \cos \lambda_N \\
0
\end{bmatrix}
$$

Geometrically, equation (22) says that a vector from the reference frame to the last point in a path is the vector sum of a vector from the origin of the reference frame to the center of the last circle plus a vector (transformed to the reference frame) from the center of the fictitious last circle to the last point on the path. (See appendix C for specific equations to compute the beginning or ending point on any segment in a path.)

7.3. Other Patterns Not Necessarily Smooth

Smooth paths are the primary interest in this publication, but the analysis applies to the design of a variety of patterns not just those that are smooth; see examples in appendix D.

8. Applications in Path Design

A computer program was written (in the C programming language) to compute some example paths. Two example runway exit paths are given for illustration: (1) the first path is along two exterior tangent circles and (2) the second approximates a spiral-type path.

For simplicity, assume that the runway (think of $P_0$ in fig. 3) is parallel to the reference axis $X_G$ and points north (that is, $\Gamma = \psi_{X_G} = 0$). Let the exit path be to the right (relative to motion just prior to the first circle). Although only examples of runway exit paths are given, the analysis applies equally well to a runway entrance and takeoff path.

8.1. Example 1: Runway Exit Path Using Two Exterior Tangent Circles

In reference 2, an airplane runway exit geometry has an initial constant radius of 2900 ft which makes a transition tangentially, after traveling 30°, into a final constant radius of 1500 ft. The travel angle on the second circle is also 30°. The final point on the exit path is approximately 2200 ft longitudinally and 600 ft laterally from the start of the exit turn.

Example 1 is a path along two tangent circles (both exterior). Because the exit path is to the right, the direction of travel on the first circle is clockwise about $z_0$ and the direction of travel on the second circle is counterclockwise about $z_1$. The appropriate parameters to generate the path are as follows:

Two circles, $N$ .................................................. 2
Radius of first circle, $R_0$ .................................. 2900
Radius of second circle, $R_1$ ............................ 1500
First circle is exterior, $e_0$ .................................. 1
Second circle is also exterior, $e_1$ ........................ 1
Angular travel on first circle, $\lambda_1$, deg ............. -30
Angular travel on second circle, $\lambda_2$, deg .......... 30

Assume that the exit turn starts at 3000 ft on the runway. The resulting path, which agrees with that in reference 2, is shown in figure 7. (For a left exit, set $e_0 = -1$ and interchange the travel angles.)
8.2. Example 2: Approximating Spiral-Type Path

Example 2 approximates the spiral-type path in reference 3. This approximation uses two circles at the beginning and at the end of the path with a linear segment in between. The first circle in each pair is exterior and the second is inward. The attraction of a spiral path is that it provides a smooth transition from the runway to the path. An airplane using a runway exit path that is initially spiral (decreasing radius) has a gradual buildup in the lateral acceleration for passenger comfort. Reference 3 uses 26 points (connected with linear segments) to approximate such a path to within a 2-ft error. Here, however, path parameters were varied until there was a reasonable match between the approximate and the ideal paths. No attempt was made to design the path to within any particular error criterion. The basic type of information is as follows (a user need not be aware of the particular variable symbol):

Number of circles in path, \( N \) ......................... 4
Exit direction, Exit direction = 1 ................ Right
Circle radius, ft, for —
First circle, \( R_0 \) ............................... 9100
Second circle, \( R_1 \) ............................... 1700
Third circle, \( R_2 \) ............................... 1500
Fourth circle, \( R_3 \) ............................... 800
Direction of motion on —
First circle, \( \omega_1 = -1 \) ......................... Clockwise
Second circle, \( \omega_2 = -1 \) ......................... Clockwise
Third circle, \( \omega_3 = 1 \) .................. Counterclockwise
Fourth circle, \( \omega_4 = 1 \) .................. Counterclockwise
Angular travel, deg, on —
First circle, \( \lambda_1 = -4^\circ \) ......................... 4
Second circle, \( \lambda_2 = -26^\circ \) .................. 26
Third circle, \( \lambda_3 = 8^\circ \) ......................... 8
Fourth circle, \( \lambda_4 = 21^\circ \) ......................... 21
Circle classification:
First circle, \( e_0 = 1 \) ......................... Exterior
Second circle, \( e_1 = -1 \) ........................ Inward
Third circle, \( e_2 = 1 \) ........................ Exterior
Fourth circle, \( e_3 = -1 \) ........................ Inward
Segment length, ft, for —
Initial segment (in front of first circle), \( l_0 \) ........ 0
Between first and second circle, \( l_1 \) ............ 0
Between second and third circle, \( l_2 \) ......... 416
Between third and fourth circle, \( l_3 \) ............ 0
After last (fourth) circle, \( l_4 \) ................. 0

The approximated path is shown in figure 8 and agrees well with the exact path.

9. Following a Path

A path is nothing more than a geometric curve (road). In trying to follow a path, one will deviate from the path; therefore, a correction is needed. A method to follow a path (refs. 1 and 2) is to orthogonally project the current position (moving point in reference frame) onto the path as a target point and then make a correction to move toward this target point to reduce the lateral deviation from the path. (Think of tracking your shadow on the path which is always directly below or above you.) This approach appears successful in controlling lateral deviation and is the motivation for this section. However, actually controlling motion to follow a path (refs. 1, 2, and 3) is not part of this publication.

9.1. Target Point on Circle

Imagine a new axis system \((X'_i, Y'_i, Z'_i)\) at the center of circle \(C_i\) that is always parallel to the general reference frame \((X_G, Y_G, Z_G)\) as shown in figure 9. Let \((x_G, y_G, 0)\) be a known, but arbitrary, point in the
general reference frame and project this point along the radius onto the circle. Refer to this point on the circle as the target point.

To compute the target point, first define a projection angle with respect to the \((X_i', Y_i', Z_i')\) axis system as

\[
\phi_i' = \tan^{-1}\left(\frac{x_{G} - h_i}{y_{G} - k_i}\right) \quad (-180^\circ < \phi_i' \leq 180^\circ) \quad (23)
\]

The target point lies along the same radial line, from the center of the circle, as the known point, which means that the two points have the same projection angle. Therefore, in this axis system, the target-point coordinates are

\[
(x_{i'})_{\text{target}} = R_i \sin \phi_i' \quad (24)
\]

\[
(y_{i'})_{\text{target}} = R_i \cos \phi_i' \quad (25)
\]

and, in general reference frame coordinates, the target point coordinates are

\[
(x_{G,i})_{\text{target}} = h_i + (x_{i'})_{\text{target}} \quad (26)
\]

\[
(y_{G,i})_{\text{target}} = k_i + (y_{i'})_{\text{target}} \quad (27)
\]
9.2. Target Point on Segment

The starting and ending points on a segment define a line. This section gives a set of equations to compute a target point on this line. In general, if two straight lines intersect at right angles, the intersection point is the orthogonal projection of any point on either line onto the other line. Let \((X_{G,1}, Y_{G,1}, 0)\) and \((X_{G,2}, Y_{G,2}, 0)\) be two points that define a line \(L\) in the general reference frame (for example, the two points may be the beginning and ending points of a segment). Let \(L_\perp\) be another line that (1) is perpendicular to \(L\) and (2) passes through an arbitrary but known point \((X_{G,a}, Y_{G,a}, 0)\). The intersection of the two lines defines the orthogonal projection of the known point. The orthogonal projection is the target point. Obviously, if \(L\) is parallel to \(X_G\), the target point is

\[
\begin{align*}
Y_G &= Y_{G,a} \\
0 &= 0
\end{align*}
\]

(28)

Otherwise, the target point is the intersection of the two lines:

\[
\begin{align*}
X_G &= mY_G + (Y_{G,1} - mY_{G,1}) \\
Y_G &= -\frac{1}{m}X_G + \left(Y_{G,a} + \frac{1}{m}X_{G,a}\right)
\end{align*}
\]

(29) (30)

Call the intersection point the target point and make the notational change \((x_G, y_G) = (x_{G,a}, y_{G,a})\) to get

\[
\begin{align*}
\begin{bmatrix}
x_G \\
y_G \\
0
\end{bmatrix}_{\text{target}} &= \frac{1}{1 + m^2} \\
\begin{bmatrix}
x_G \\
y_G \\
0
\end{bmatrix}_{\text{target}} &= \begin{bmatrix}
m(y_G - y_{G,1}) + x_G + m^2 x_{G,1} \\
m(x_G + x_{G,1}) + y_{G,1} + m^2 y_G \\
0
\end{bmatrix}_*(x_{G,1} \neq x_{G,2})
\end{align*}
\]

(31)

where

\[
m = \frac{Y_{G,2} - Y_{G,1}}{X_{G,2} - X_{G,1}}
\]

(32)

is the slope of the known line or segment and \((x_G, y_G)\) now denotes the current position.

9.3. Target Azimuth on Circle

The azimuth of a moving target point on a circle is useful in correcting the heading of current motion. The azimuth at an arbitrary point on circle \(C_i\) is

\[
\psi_i = \psi_{X_G} + \psi_{G,i} \quad (i = 0, 1, ..., N)
\]

(33)

where \(\psi_{X_G}\) is the known azimuth of the axis \(X_G\) and \(\psi_{G,i}\) is the azimuth of the moving point relative to \(X_G\). Next consider how \(\psi_{G,i}\) varies.

The azimuth on circle \(C_i\) with respect to \(X_G\) is shown in figure 10(a) as a function of the projection angle \(\phi_i\) and the direction of motion on the circle. For example, if \(\phi_i = 90^\circ\), figure 10(a) shows an azimuth of \(\psi_{G,i} = 90^\circ\) for a clockwise motion, whereas figure 10(b) shows an azimuth of \(\psi_{G,i} = 270^\circ\) for a counterclockwise motion.

Figure 10(c) shows a plot of \(\psi_{G,i}\) as a function of \(\phi_i\) and the direction of motion. Equations for computer computations are

\[
\begin{align*}
\psi_{G,i} &= 180^\circ - \phi_i \quad (C_i \text{ ccw}; -180^\circ \leq \phi_i \leq 180^\circ) \\
\psi_{G,i} &= -\phi_i \quad (C_i \text{ cw}; \phi_i \leq 0^\circ) \\
\psi_{G,i} &= 360^\circ - \phi_i \quad (C_i \text{ cw}; \phi_i > 0^\circ)
\end{align*}
\]

(34)

where \(\phi_i\) is given by equation (23). The scenario is that current motion near a circle changes the circle projection angle \(\phi_i\) in equation (23), and this changes the azimuth in equations (34), which changes the target azimuth in equation (33).
9.4. Target Azimuth on Segment

The constant target azimuth on a segment $P_i$ is

$$\psi_{P_i} = \psi_{X_G} + \tan^{-1}\left(\frac{Y_{G,2} - Y_{G,1}}{X_{G,2} - X_{G,1}}\right)$$

where $(x_{G,1}, y_{G,1})$ is the starting point on the segment and $(x_{G,2}, y_{G,2})$ is the ending point.

Alternatively, the azimuth on a segment prior to the first circle is $\psi_{P_1} = \psi_G + \Gamma$; the azimuth angle on any other segment following a circle is the same as the last azimuth angle on that circle (that is, the azimuth at the ending point of the circle).
10. Switching Motion to Next Path Element

10.1. Sample Flowchart for Referencing Motion to Path

Figure 11 shows a flowchart to reference motion to a path. The diagram sequences through the path along its various segments and circles. Assume there are \(N\) circles in a path, along with a final fictitious circle (for convenience). Then, for each path index \((i = 0, 1, ..., N)\), there is a segment followed by a circle. The fictitious circle follows the last segment. Initially, the first element is \(P_0\), but if the length of this segment is zero, the first circle \(C_0\) then becomes the reference for the current motion. After completing motion on \(C_0\), the loop index increases by one to the next segment in the path \(P_1\) and so forth. The loop continues until there are no more elements in the path.

10.2. Switching Motion to Next Path Element

A user supplies some criteria to decide when to switch from one path element to the next. Numerous possible switching criteria (a research issue) may exist; for example, each element in the path has its own angle to go before the next element starts. When this angle reaches 0, motion switches to the next element and so on.

This section defines an angle to go to the next successive element in a path. This switching angle is the central angle between the currently known position, a circle center, and the start (or end) of motion on the circle. Figure 12 indicates the switching angle for motion approaching an exterior circle. Bear in mind that the switching angle is only one indication of when to switch to the next path element; it does not indicate how close the motion is to the path at the instant of switching. Other parameters, such as the lateral position from the path, indicate the “closeness” of the motion to the path and also play an integral role in the switching decision.

10.2.1. Switching Motion Away From Segment

First, assume that motion is currently relative to the segment \(P_i\). Then, the angle to go tells when to switch motion to the upcoming circle \(C_i\). Let \((x_G, y_G, 0)\) be a known moving point in the general reference frame. A transformation of this point to the circle axis system is

\[
\begin{bmatrix}
  x_i \\
  y_i \\
  0 \\
  1
\end{bmatrix} = A_i^G
\begin{bmatrix}
  x_G \\
  y_G \\
  0 \\
  1
\end{bmatrix}
\]

The radial projection angle of the moving point in the circle axis system is

\[
\phi_i = \tan^{-1}\left(\frac{y_i}{x_i}\right) \quad (-180^\circ < \phi_i \leq 180^\circ)
\]
Current location of moving point $(x_G, y_G)$

Start of motion on exterior circle $(A_i = 0^\circ)$

Figure 12. Angular location of current position from starting point on exterior circle.

The central angle of travel before motion starts on circle $C_i$ is

$$\Lambda_i = - (\text{sign of } \omega_{i+1} \Phi_i)(1 + e_i)90^\circ + \omega_{i+1} \Phi_i \quad (38)$$

which is a function of the current projection angle, circle classification, and the direction of motion. Equation (38) results by considering the various possibilities of approaching the start of motion on a circle. In essence,

$$\begin{align*}
\Lambda_i &< 0 \quad (\text{Before motion starts on } C_i) \\
\Lambda_i & = 0 \quad (\text{Start motion on } C_i) \\
\Lambda_i &> 0 \quad (\text{Motion on } C_i)
\end{align*}$$

(If there is a segment $P_N$ following the last circle in a path, a user can conveniently choose to set $R_N = \left| \frac{1}{\lambda_i} \right|$ to make $\Lambda_N$ vary from $-45^\circ$ to $0^\circ$ as motion starts and stops on this last segment.) Define an angle to go that is positive before motion is to start on $C_i$ as

$$\text{Angle to go} = -\Lambda_i \quad (39)$$

10.2.2. Switching Motion Away From Circle

If motion is currently on circle $C_i$, then the angle to go should indicate when to switch motion away from $C_i$ onto the next upcoming path element (circle or segment). In this case the angle to go is to the end of motion on $C_i$. Although motion is on $C_i$, $\Lambda_i$ in equation (38) is positive and gives the central angle of travel on $C_i$. Because the desired angular travel on $C_i$ is known, the angle to go for switching motion away from circle $C_i$ is

$$\text{Angle to go} = \left| (\lambda_i)_{\text{max}} \right| - \Lambda_i \quad (40)$$

Normally, one knows the current position and then computes the angle to go to switch to the next element in a path.

11. Comments on Control

Knowing the angle to go and how this angle is currently changing with time, one can compute a time to go to switch to the next path element and, if desired, can start the switching maneuver a fraction of a second early for a better transition (less overshoot). In dealing with straight-line segments, reference 1 divides the distance to go by the current speed to get a time to go to the end of the segment. When the time to go is within 0.05 sec, tracking switches to the next segment in the path.

At a switching point in a path, there is a call for an instantaneous change in a velocity component; for example, in exiting from a runway onto a circular path element, there is a call for the velocity normal to the runway to change immediately from 0 to a value that is consistent with motion in a circle. Reference 2 starts to build the normal velocity component slightly before reaching the exit for a smoother transition off the runway, and reference 3 uses a spiral-type exit for smoothness. Error equations for controlling motion to follow a path are given in reference 1.

Reference 5 presents an algorithm, although not used, for lateral ride comfort limits. Current thinking (ref. 3) is that a lateral acceleration should not exceed 0.15g and that lateral jerk (derivative of acceleration) should not be in excess of 0.05 g/sec.

12. Concluding Remarks

This publication provides a general framework for the mathematical description of a smooth path along an arbitrary combination of circles and tangent lines.
The innovative approach is to model the path as a giant multilink manipulator and apply the mathematics common to robotics. Any two successive circles in a path are either tangent or connected by a common tangent line. There may be an initial segment prior to the first circle in the path, and a path may begin or end on a circle or line. A user specifies a few parameters to configure the path. Necessary and sufficient conditions ensure that user inputs are consistent with a smooth path. An immediate application is designing smooth paths for use in automating the exit of an aircraft from a runway. Two example runway exit paths are given, and this publication defines an angle to go for switching from one path element to the next.
Appendix A

Configuring Path With Circle Angles

Appendix A illustrates geometrically how an initial configuration of circles and tangent lines changes with circle angles in the design of a smooth path. However, in the main text circle angles are not explicit inputs in the design process.

Figure A1 shows an example of configuring a path with circles and tangent lines. In figure A1(a), the initial configuration, prior to designing a path, is when the circle angles are zero. The configuration then changes with circle angles to design a path. The first two circles are exterior (the first by definition), and the third is an inward circle. There is a tangent segment prior to the first circle and another tangent segment between the second and third circles. Figure A1(b) shows the configuration after a rotation of \( \theta_1 = 110^\circ \) about the center of the first circle, and figure A1(c) shows the final configuration after a rotation of \( \theta_2 = -110^\circ \) about the center of the second circle. The example smooth path is then drawn along the circles and segments. The amount of travel along the last circle is \( \theta_3 = 60^\circ \).
Figure A1. Constructing smooth path along circular arcs and tangent segments using circle angles.
Appendix B

Homogeneous Transformation Matrices

Homogeneous transformation matrices contain both orientation and displacement information between axis systems and are useful in following the path geometry. See reference 4 for application of homogeneous transformation matrices to robotics. Consider the two axis systems $a$ and $b$ shown in sketch B1:

The two axis systems are related by the following $4 \times 4$ homogeneous transformation matrix:

$$ A^b_a = \begin{bmatrix} R^b_a & p^a_b \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (B1) $$

where $R^b_a$ is a $3 \times 3$ rotational matrix that transforms a three-dimensional vector in axis system $b$ to axis system $a$:

$$ \begin{bmatrix} x \\ y \\ z \end{bmatrix}_b = R^b_a \begin{bmatrix} x \\ y \\ z \end{bmatrix}_a \quad (B2) $$

and

$$ p^a_b = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}_a \quad (B3) $$

is a $3 \times 1$ position vector directed from the origin of the axis system $a$ to the origin of axis system $b$. 

20
B1. Inverse Transformation

The homogeneous transformation matrix to transform quantities from axis system \( a \) to axis system \( b \) is the inverse of equation (B1):

\[
A_b^a = [A_a^b]^{-1} = \begin{bmatrix}
R_b^a & 0 & -R_b^a p_b^a \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  

(B4)

where the superscript \( T \) represents the transpose of the matrix.

B2. Transforming Points and Vectors

A homogeneous transformation matrix transforms the coordinates of a point and the components of a vector as follows:

\[
\begin{pmatrix}
  x \\
y \\
z \\
1
\end{pmatrix}_{i-1} = A_{i-1}^i
\begin{pmatrix}
  x \\
y \\
z \\
1
\end{pmatrix}_i
\]  

(Transforming coordinates of point)  

(B5)

\[
\begin{pmatrix}
v_x \\
v_y \\
v_z \\
0
\end{pmatrix}_{i-1} = A_{i-1}^i
\begin{pmatrix}
v_x \\
v_y \\
v_z \\
0
\end{pmatrix}_i
\]  

(Transforming vector, such as velocity)  

(B6)

B3. Specific Partitioning (Circle-to-Circle Matrix)

The homogeneous transformation matrix from circle \( C_i \) to the previous circle \( C_{i-1} \) is

\[
A_{i-1}^i = \begin{bmatrix}
R_{i-1}^i & 0 & p_{i-1}^i \\
0 & 0 & 1
\end{bmatrix}
\]  

(B7)

where

\[
R_{i-1}^i = \begin{bmatrix}
cos \theta_i & -sin \theta_i & 0 \\
sin \theta_i & cos \theta_i & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  

(B8)
is an inverse Euler rotation matrix about \( Z_{i-1} \), and

\[
p_{i-1} = \begin{pmatrix}
a_i \cos \theta_i - l_i \sin \theta_i \\
a_i \sin \theta_i + l_i \cos \theta_i \\
0
\end{pmatrix}
\]  \hspace{1cm} (B9)

is the position vector from the center of circle \( C_{i-1} \) to the center of circle \( C_i \) \( (p_{i-1}) \) has components in \( (X_{i-1}, Y_{i-1}, Z_{i-1}) \).
Appendix C

Path Elements at Starting and Ending Points

In switching from one path element to the next, one needs to know where the current element ends and the next begins. Appendix C gives equations for calculating the beginning and ending points on all circles and segments in a path.

For a given path configuration, the homogeneous transformation matrix $A_G^i$ from circle $C_i$ to the reference frame is constant. In addition, for later convenience, define the matrix

$$L_{G}^{i+1} \equiv A_G^i \left[ A_G^{i+1} \right]_{l_{i+1}=0}$$

which is the same as matrix $A_G^{i+1}$, except that $l_{i+1}$ is 0 in $A_G^{i+1}$; that is, except for the two elements:

$$L_{G}^{i+1} (0,3) = a_{i+1} \cos \theta_{i+1}$$
$$L_{G}^{i+1} (1,3) = a_{i+1} \sin \theta_{i+1}$$

C1. Circles

The starting point of motion on circle $C_i$ is

$$\begin{cases}
  x_{G,i}^* \\
  y_{G,i}^* \\
  0 \\
  1
\end{cases} = A_G^i \begin{bmatrix}
  -e_i R_i \\
  0 \\
  0 \\
  1
\end{bmatrix} \quad (i = 0, 1, \ldots, N)$$

where the superscript * identifies a starting point. If desired, equation (C3) expands to the more explicit vector expression:

$$\begin{cases}
  x_{G,j}^* \\
  y_{G,j}^* \\
  0
\end{cases} = \begin{bmatrix}
  h_{i-1} \\
  k_{i-1} \\
  0
\end{bmatrix} + R_G^{j-1} \begin{bmatrix}
  -e_j R_i \cos \theta_j + a_i \cos \theta_j - l_i \sin \theta_j \\
  -e_j R_i \sin \theta_j + a_i \sin \theta_j + l_i \cos \theta_j \\
  0
\end{bmatrix}$$

$$= \begin{bmatrix}
  h_{i-1} \\
  k_{i-1} \\
  0
\end{bmatrix} + R_G^{j-1} \begin{bmatrix}
  R_{j-1} \cos \theta_j - l_i \sin \theta_i \\
  R_{j-1} \sin \theta_j + l_i \cos \theta_i \\
  0
\end{bmatrix}$$

$$= \begin{bmatrix}
  h_{i-1} \\
  k_{i-1} \\
  0
\end{bmatrix} + R_G^{j-1} \begin{bmatrix}
  R_{j-1} \cos \theta_j - l_i \sin \theta_i \\
  R_{j-1} \sin \theta_j + l_i \cos \theta_i \\
  0
\end{bmatrix}$$

23
On the imaginary circle $C_N$, the starting point is the same as the ending point because there is no motion on this circle; that is

\[
\begin{align*}
\begin{bmatrix}
X_{G,N}^# \\
Y_{G,N}^#
\end{bmatrix}
&= \begin{bmatrix}
0 \\
0
\end{bmatrix} \\
\begin{bmatrix}
X_{G,N}^* \\
Y_{G,N}^*
\end{bmatrix}
&= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\end{align*}
\] (C5)

where the superscript # identifies the ending point. The ending points for all other circles are given by

\[
\begin{align*}
\begin{bmatrix}
X_{G,i}^# \\
Y_{G,i}^#
\end{bmatrix} = L_G^{i+1} \begin{bmatrix}
-e_{i+1}R_{i+1} \\
0 \\
0 \\
1
\end{bmatrix} & \quad (i = 0, 1, \ldots, N - 1) \quad (C6)
\end{align*}
\]

If desired, equation (C6) readily expands to

\[
\begin{align*}
\begin{bmatrix}
X_{G,i}^# \\
Y_{G,i}^#
\end{bmatrix} = A_G^i \begin{bmatrix}
R_i \cos \theta_{i+1} \\
R_i \sin \theta_{i+1} \\
0 \\
1
\end{bmatrix} & \quad (i = 0, 1, \ldots, N - 1) \quad (C7)
\end{align*}
\]

C2. Segments

The initial point on any initial segment prior to the first circle in a path is either known or computable from the exit coordinates, segment length, and azimuth angle. The final point on such a segment is the known exit point (or the starting point on the first circle). Hence,

\[
(P_0)_{\text{start}} = \text{Known or computable point} \quad (C8)
\]

\[
(P_0)_{\text{end}} = \text{Exit point (known)} \quad (C9)
\]

Thereafter, the starting point on segment $P_i$ is the ending point on the preceding circle:

\[
(P_i)_{\text{start}} = (C_{i-1})_{\text{end}} = \begin{bmatrix}
X_{G,i-1}^# \\
Y_{G,i-1}^#
\end{bmatrix} \quad (i = 1, 2, \ldots, N) \quad (C10)
\]
The ending point on $P_i$ is the starting point on the upcoming circle:

\[
(P_i)_{\text{end}} = (C_i)_{\text{start}} = \begin{cases}
(x_{G,i}, y_{G,i}, z_{G,i}) \\
(i = 1, 2, \ldots, N)
\end{cases}
\]  

As a matter of interest, after the first segment $P_0$, the relationship between the end point of segment $P_i$ (which is also the relationship between the ending point on $C_{i-1}$ and the starting point on $C_i$) is

\[
(P_i)_{\text{start}} = (P_i)_{\text{end}} - R_i^{i-1} \begin{bmatrix}
-l_i \sin \theta_i \\
l_i \cos \theta_i \\
0
\end{bmatrix} 
\]  

\[
(i = 1, 2, \ldots, N)
\]  

(C12)
Appendix D

Interesting Patterns, Not Necessarily Smooth

One can generate some interesting patterns using circles and lines. Not necessarily designing patterns for smooth paths (as done in the main text) extends the variety of the patterns. The concern in appendix D is not whether a pattern is smooth but to generate some interesting patterns by varying mathematical parameters.

For all patterns, except one, in this appendix, \( x_{exit} = 3000 \), \( e_0 = 1 \), and \( e_N = -1 \). Also, not shown, is the circle direction of motion, which is the sign on maximum track angle; any other parameter not shown is 0. Equation (4) does not necessarily hold, which means there can be cusps in the pattern, but equation (3) still holds to give the signed segment length parameter.

D1. Patterns Using Four 90° Circular Arcs

Each of the three patterns shown in figure D1 uses four 90° circular arcs. The first pattern (fig. D1(a)) is a four-cusp pattern with \( N = 4 \), \( R_0 = R_1 = R_2 = R_3 = 2900 \), \( (\lambda_1)_{\text{max}} = (\lambda_2)_{\text{max}} = -90^\circ \), \( (\lambda_3)_{\text{max}} = (\lambda_4)_{\text{max}} = 90^\circ \), and \( e_1 = e_2 = e_3 = 1 \). The second pattern (fig. D1(b)) shows a two-cusp pattern
resulting from shifting the lower portion of figure D1(a) to the right by letting $l_2 = 6000$. Figure D1(c) is a spiraling pattern, with $N = 4$, $e_1 = e_2 = e_3 = -1$, and $(\lambda_1)_{\text{max}} = (\lambda_2)_{\text{max}} = (\lambda_3)_{\text{max}} = (\lambda_4)_{\text{max}} = -90^\circ$.

**D2. Square-Wave Pattern**

Setting the radius of a circle in a path equal to zero directly connects its tangent segments. For example, with zero radii, the remaining parameters which generate the square wave are $N = 5$, $e_1 = e_2 = e_3 = e_4 = 1$, $(\lambda_1)_{\text{max}} = (\lambda_2)_{\text{max}} = (\lambda_3)_{\text{max}} = -90^\circ$, $(\lambda_4)_{\text{max}} = 90^\circ$, $l_1 = l_2 = 1000$, $l_3 = 2000$, and $l_4 = 1000$. This pattern, the square-wave pattern, is shown in figure D2.

**D3. Interconnecting Ring Pattern**

In the text, the circle classification parameter is either 1 or -1. However, there is no need to impose this restriction in the quest for interesting patterns. For example, to draw two interconnecting rings, let $e_2 = 0$. (This removes all terms that multiply this parameter.) Other parameters for this pattern are $N = 2$, $R_0 = R_1 = 1000$, $(\lambda_1)_{\text{max}} = (\lambda_2)_{\text{max}} = 360^\circ$, and $l_1 = 1000$. The interconnecting ring pattern is shown in figure D3.

![Figure D2. Square-wave pattern.](image1)

![Figure D3. Interconnecting ring pattern.](image2)
References


**1. AGENCY USE ONLY (Leave blank)**

**2. REPORT DATE**
December 1998

**3. REPORT TYPE AND DATES COVERED**
Technical Publication

**4. TITLE AND SUBTITLE**
Generic Equations for Constructing Smooth Paths Along Circles and Tangent Lines With Application to Airport Ground Paths

**5. FUNDING NUMBERS**
WU 538-04-13-02

**6. AUTHOR(S)**
L. Keith Barker

**7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)**
NASA Langley Research Center
Hampton, VA 23681-2199

**8. PERFORMING ORGANIZATION REPORT NUMBER**
L-17745

**9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)**
National Aeronautics and Space Administration
Washington, DC 20546-0001

**10. SPONSORING/MONITORING AGENCY REPORT NUMBER**
NASA/TP-1998-208960

**11. SUPPLEMENTARY NOTES**

**12a. DISTRIBUTION/AVAILABILITY STATEMENT**
Unclassified—Unlimited
Subject Category: 03
Availability: NASA CASI (301) 621-0390

**12b. DISTRIBUTION CODE**

**13. ABSTRACT (Maximum 200 words)**
The primary purpose of this publication is to develop a mathematical model to describe smooth paths along any combination of circles and tangent lines. Two consecutive circles in a path are either tangent (externally or internally) or they appear on the same (lateral) or opposite (transverse) sides of a connecting tangent line. A path may start or end on either a segment or circle. The approach is to use mathematics common to robotics to design the path as a multilink manipulator. This approach allows a hierarchical view of the problem and keeps the notation manageable. A user simply specifies a few parameters to configure a path. Necessary and sufficient conditions automatically ensure the consistency of the inputs for a smooth path. Two example runway exit paths are given, and an angle to go assists in knowing when to switch from one path element to the next.

**14. SUBJECT TERMS**
Smooth paths; Runway exits; Taxiways; Circles; Homogeneous matrix; Robotics; Manipulator

**15. NUMBER OF PAGES**
33

**16. PRICE CODE**
A03

**17. SECURITY CLASSIFICATION OF REPORT**
Unclassified

**18. SECURITY CLASSIFICATION OF THIS PAGE**
Unclassified

**19. SECURITY CLASSIFICATION OF ABSTRACT**
Unclassified

**20. LIMITATION OF ABSTRACT**

NSN 7540-01-260-5500