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Nonreflecting Far-Field Boundary Conditions for Unsteady Transonic Flow Computation

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The approximate nonreflecting far-field boundary condition, as proposed by Engquist and Majda, is implemented in the computer code LTRAN2. This code solves the implicit finite-difference representation of the small-disturbance equations for unsteady transonic flows about airfoils. The nonreflecting boundary condition and the description of the algorithm for implementing these conditions in LTRAN2 are discussed. Various cases are computed and compared with results from the older, more conventional procedures. One concludes that the nonreflecting far-field boundary approximation allows the far-field boundary to be located closer to the airfoil, this permits a decrease in the computer time required to obtain the solution through the use of fewer mesh points.

I. Introduction

To compute time-dependent flows over an infinite region using finite-difference procedures requires either that the problem be reduced to the one in a finite domain via a coordinate transformation, or that boundaries be placed at a finite distance from the body and the boundary conditions modified along them. When the solution at infinity is simple, a coordinate transformation technique can be used. However, for many practical problems, especially when the flow is singular at infinity, the coordinate transformation method is not applicable. In such cases, computational simulations of unsteady flows in an unbounded region are performed on grids with finite dimensions. The artificial-wall effect created by these grid far-field boundaries must therefore be minimized so as not to degrade the resulting numerical solution.

Various approaches have been developed to reduce this spurious influence of far-field boundaries on interior solutions, some of which are listed below:

1) It is common to place these boundaries far enough (typically hundreds of chord lengths) from the airfoil such that reflected waves have been greatly reduced in amplitudes by numerical dissipation through the travel from these boundaries. Such a procedure naturally requires a large number of grid points.

2) In a series of papers, Engquist and Majda developed a nonreflecting boundary condition procedure. By absorbing waves incident on the boundary in certain directions, they designed approximate local conditions for simulating the far-field effects along those boundaries. Their lowest order approximation can be interpreted as a simplified form of the characteristic equation.

3) Bayliss and Turkel derived a far-field boundary condition procedure based on an asymptotic expansion to any order in the reciprocal of the distance from the origin. The dependence of the far-field distance is shown in the work of Fung, where the linearized unsteady small-disturbance transonic flow equation is solved to get a far-field condition. This condition is equivalent to a vortex solution with a time lag for waves originating at the airfoil and propagating to the far-field boundary.

4) A viscous damping method which absorbs incident waves on the far-field boundaries was tried by Bushby and Timpson. This method, however, requires a considerable number of grid points near the far-field boundary.

5) Guderley has shown that in a limited number of cases, far-field conditions can be derived analytically to investigate basic mathematical concepts. However, an extension of this to more general nonlinear problems is not clear at this point.

Unsteady transonic flow regions are encountered in the flight of many existing flight vehicles. Typical examples include a fluttering airfoil and spinning helicopter rotors. For unsteady small-disturbance transonic flows, Ballhaus and Goorjian have developed a time-accurate implicit finite-difference computer code, LTRAN2. Their boundary conditions at the top, bottom, and upstream boundaries (see Fig. 1) are perfectly reflecting conditions and are the proper ones for boundaries infinitely far away. In the present work, approximate nonreflecting boundary conditions are employed in this code to bring the far-field boundaries closer to the airfoil. As illustrated in Fig. 1, waves originating from the airfoil propagate to the far-field boundaries. The old boundary which is placed at a large distance from the airfoil leaves the near-field solution unaffected by reflected waves from the outer boundary. With nonreflecting conditions, the new boundary can be located closer to the airfoil, thus reducing the area of computation and thereby increasing the degree of resolution within the computational domain.

II. Governing Equation and Far-Field Boundary Conditions

A. Governing Equation

The unsteady, transonic small-disturbance equation for low reduced frequencies can be written as

\[ A \phi_{xx} = B \phi_{xx} + \phi_{yy} \]  \hspace{1cm} (1a)
approximations are made to get the following local conditions

\begin{align}
\text{upstream} & \quad \left\{ \begin{array}{l}
\text{1st approx:} \quad \phi_x - \frac{B}{A} \phi_y = 0 \\
\text{2nd approx:} \quad \phi_{xx} - \frac{A}{B} \phi_{yy} + \frac{1}{A} \phi_y = 0
\end{array} \right. \\
\text{downstream} & \quad \left\{ \begin{array}{l}
\text{or} \quad \phi_x = 0 \\
\phi_y = 0
\end{array} \right.
\end{align}

The first approximate condition absorbs waves normally incident on the boundary, while higher approximations absorb portions of obliquely incident waves.

Since waves travel infinitely fast in the downstream direction, the following consistency conditions seem appropriate (see Krupp and Cole)\textsuperscript{4}:

\begin{align}
\text{top and bottom} & \quad \left\{ \begin{array}{l}
\text{1st approx:} \quad \phi_x \pm \sqrt{1 - B^2} \phi_y = 0 \\
\text{2nd approx:} \quad \phi_x \pm r_x \phi_y \pm r_y \phi_y = 0
\end{array} \right.
\end{align}

Here, ± corresponds to top and bottom boundaries, and \( r_x \) \( r_y \) govern the absorption of waves obliquely incident to the boundaries. The values \( r_x, r_y \) are discussed in more detail by Engquist and Majda.\textsuperscript{4}

Alternatively, a nonreflecting far-field condition can be designed by considering characteristic equations. Assuming \( B = \text{const} \) in Eq. (1b) at a large distance from the airfoil, the characteristic equation can be written as

\[ B \xi_x + \xi_x + \xi_y - A \xi_x + \xi_y = 0 \]

This equation is satisfied by the characteristic plane

\[ \xi(x,y,t) = r - \frac{x}{\sqrt{B^2 + A^2}} = \text{const} \]

where

\[ r = (x/\sqrt{B^2 + A^2}, y) \]

is a solution to Eq. (1a) which represents a traveling plane wave. From these, the following equation is derived by forcing linear combinations of the derivatives of \( \phi \) to be zero.

\[ \frac{x}{r} \phi_x + \frac{y}{r} \phi_y + \frac{1}{2} \left( \frac{x}{B} \frac{A}{r} + \frac{A}{\sqrt{B}} \right) \phi_y = 0 \]

This equation can be used as a nonreflecting far-field boundary condition. Since this equation requires the evaluation of \( x/r \) and \( y/r \) along the boundary at each time step, a further simplification is made. In the upstream region, \( x = -\infty \) with \( x \) finite, so that Eq. (12a) becomes

\[ \phi_x - (B/A) \phi_y = 0 \]

For the top and bottom boundaries, \( y = \pm \infty \) with \( x \) finite, so that Eq. (12a) becomes

\[ \phi_x \pm \frac{1}{2} (A/\sqrt{B}) \phi_y = 0 \]

Replacing \( \phi \) by \( \phi_y \), using Eq. (11b), this equation can be written equivalently as

\[ \phi_y \pm B \phi_y = 0 \]
These are the first approximate relations obtained by Engquist and Majda. Therefore, we may interpret Eqs. (4) and (8) as special forms of a far-field characteristic relation designed for simple, yet very efficient, applications.

III. Finite-Difference Scheme

In LTRAN2, Eq. (1a) is solved by a two-step procedure to advance from time step \( n + 1 \)

**x sweep:** \( A(\Delta t)^{-1} \delta_x (\phi^{n+1}_{x,i} - \phi^*_x) = D_x f_x + \delta_x \phi^n_x \)  

(13)

**y sweep:** \( A(\Delta t)^{-1} \delta_y (\phi^{n+1}_{y,j} - \phi^*_y) = \frac{1}{2} \delta_y (\phi^{n+1}_{x,i} - \phi^*_x) \)  

(14)

The operators in Eqs. (13) and (14) are defined by the following equations:

\[
\delta_x \phi^n_x = 2 (\phi_{x,i+1} - \phi_{x,i-1}) (x_{x,i} - x_{x,i-1})^{-1}  
\]

(first order)

\[
= (3 \phi_{x,i+1} - 4 \phi_{x,i} + \phi_{x,i-1}) (x_{x,i+1} - x_{x,i-1})^{-1}  
\]

(second order)

\[
\delta_y \phi^n_y = 2 (\phi_{y,j+1} - \phi_{y,j-1}) (y_{y,j} - y_{y,j-1})^{-1}  
\]

\[- (\phi_{y,j+1} - \phi_{y,j}) (y_{y,j+1} - y_{y,j})^{-1} ] (y_{y,j} - y_{y,j-1})^{-1} \]

f_x = \frac{1}{2} (B_{x,j} \phi^*_{x,j} + (1 - M^2_x) \phi^*_{x,j}/\delta^{2/3})

B_{x,j} = (1 - M^2_x) \delta^{2/3} - (y_{j} + 1) M^2_x \phi^*_{x,j}

\phi_{y,j+1} = (\phi_{y,j+1} - \phi_{y,j}) (x_{y,j+1} - x_{y,j})^{-1}

D_x f_x = 2 (x_{x,i} - x_{x,i-1})^{-1} (1 - \epsilon_x) (f_{x,i} - f_{x,i-1})

+ \epsilon_x (f_{x,i} - f_{x,i-1})

\epsilon_x = [0] for \((B_{x,j}^* + \frac{1}{2} B_{x,j}^*)[\geq 0]\)

Here, \( \Delta t \) is the time step and \( j, \ell \) are the grid point indices in the \( x \) and \( y \) directions.

The first approximate nonreflecting far-field boundary conditions are readily incorporated into the above two-dimensional sweep procedure. For the \( x \) sweep, the upstream boundary condition [Eq. (12b)] becomes

\[
(\Delta t)^{-1} (\delta_x (\phi^{n+1}_{x,i} - \phi^*_x)) - \frac{1}{2} B_{x,i}^* \phi^*_{x,i} = \phi^{n+1} - \phi^*_{x,i} \]

(15)

Here

\[
B_{x,i}^* = [(1 - M^2_x)/\delta^{2/3}] - [(y_{i} + 1) M^2_x (\phi^*_{x,i} - \phi^*_{x,i})]
\]

\[
(x_{x,i} - x_{x,i-1})^{-1}, \quad j = I
\]

The downstream boundary condition is

\[
\phi^{n+1}_{x,i} + \phi^{n+1}_{x,i+1} - \phi^*_x = 0
\]

(16a)

or equivalently, since \( \phi^{n+1}_{x,i} - \phi^{n+1}_{x,i+1} = 0 \) from the previous iteration

\[
\phi^{n+1}_{x,i} + \phi^{n+1}_{x,i+1} - \phi^*_x = 0
\]

(16b)

Here, \( j_{\text{max}} \) and \( i_{\text{max}} \) are maximum indices in \( x \) and \( y \) directions and represent downstream and top boundaries, respectively.

Similarly, for the \( y \) sweep, the top and bottom boundary conditions become

\[
(\phi^{n+1}_{y,i} - \phi^*_y - \phi^*_{y,i-1}) / (y_{y,i} - y_{y,i-1})  
\]

\[
\times (\phi^{n+1}_{y,i} - \phi^*_y - \phi^*_{y,i-1}) / (y_{y,i} - y_{y,i-1}) = 0
\]

(17)

where

\[
\phi_{y,1+1} = \frac{1}{2} (\phi_y + \phi_{y,1})
\]

\[
\phi_{y,n+1} = \frac{1}{2} (\phi_y + \phi_{y,n})
\]

When \( j = 2 \), Eq. (17) requires \( \phi \) at \( j = 1 \). Therefore, for \( j = 2 \) we use Eq. (12b) also. Here, the upstream condition Eq. (15) is changed to

\[
(\Delta t)^{-1} (\phi^{n+1}_{y,i} - \phi^*_y) - \frac{1}{2} B_{y,i}^* \phi^*_{y,i} = \phi^{n+1} - \phi^*_{y,i} \]

\[
(\Delta t)^{-1} (\phi^{n+1}_{y,i} - \phi^*_y) + \phi^{n+1}_{y,i} - \phi^*_y = 0
\]

(18)

Then Eqs. (17) and (18) are solved simultaneously.

The higher-order approximations can be applied in a similar way, however, at the expense of increased computing time.

IV. Computed Results for Various Test Cases

In the production version of LTRAN2, the default grid boundaries are located 857 chords from the airfoil in \( y \) direction and 200 chords from the leading edge, with smoothly stretched grid spacings in both directions (113, 97 mesh in \( x, y \) directions). The large grid solutions in Figs. 2-7 are obtained using this grid. However, depending on the particular problem, equivalent results can be obtained with smaller grids, and the minimum number of mesh points required is shown in the figures. Therefore, identical solutions are obtained when far-field boundaries are placed farther away than the minimum required.

In computing the large grid solutions, two different sets of far-field boundary conditions are imposed independently, namely: I) perfectly reflecting conditions as given by Eq. (3); and II) the first approximate conditions as given by Eqs. (4), (6), and (8). These boundary conditions yield essentially identical results, i.e., within plottable accuracy. Since the large grid results can be duplicated using different combinations of grids and boundary conditions, it is reasonable to assume that these are solutions of Eq. (1) with far-field boundaries at infinity.

A. Step Change in Angle of Attack

An impulsively started airfoil in plunging motion creates a pressure pulse. This is simulated numerically by a step change in the angle of attack. Due to the low-frequency nature of the governing equation, the lift and moment coefficients gradually increase after the initial change in the angle of attack. A problem such as this provides a good test case for investigating the influence of the computational far-field distances on the flow solution near the airfoil.

Starting from the standard large grid, the far-field boundary distances are reduced by removing outer grid lines in the \( x \) (both upstream and downstream equally) and \( y \) directions. Both the old reflecting boundary conditions [Eq. (3)] and the new nonreflecting ones are tested with these reduced grids. In Figs. 2 and 3, the lift and moment coefficients for an NACA 64A006 airfoil are plotted on a time scale in units of chord lengths of airfoil traveled. The indiscernible responses to a unit change in the angle of attack \( \alpha \) are \( C_L \) and \( C_m \), as described in Ref. 11, and can be regarded as...
C/\alpha \text{ and } C_m/\alpha \text{ in this case. By using the old conditions, the computed results start to deviate from the large grid solutions when the mesh gets smaller than } 103 \times 81 \text{ at } M_\infty = 0.80 \text{ and } 0.85. \text{ However, applying the first approximate nonreflecting conditions, the mesh can be reduced to } 93 \times 71 \text{ at } M_\infty = 0.80 \text{ without significant deviations from the large grid results. Similar results are obtained at } M_\infty = 0.85, \text{ as shown in Fig. 2.}

Since the y boundary can be brought to 3.8 chords from the leading edge using the first approximate condition, the second approximation was tried to bring the y boundary closer. However, there was practically no improvement to justify the additional computation required to implement the second approximation. In Table 1, the minimum mesh and the computational efficiency required to use the old and the first approximate boundary conditions are shown. Using the nonreflecting boundary conditions, the computational efficiencies gained are 19 and 10\% at } M_\infty = 0.80 \text{ and } 0.85, \text{ respectively.}

### B. Oscillating Flap Case

Another test of the nonreflecting boundary conditions was performed for the case of an oscillating trailing-edge flap. The configuration consisted of an NACA 64A006 airfoil with a sinusoidally oscillating flap. The three types of possible shock-wave motions, type A (sinusoidal shock-wave motion) and type B (interrupted shock-wave motion) are shown in Figs. 4 and 5. In both cases, large grid results can be obtained with a grid as small as } 93 \times 71. \text{ Again the minimum } x \text{ boundary distance is very small. As expected, the influence of the } y \text{ boundary is very pronounced. By the time the } y \text{ boundary distance is reduced to 7.6 chords from the airfoil, the old reflecting conditions cause significant deviation of the lift and the moment coefficient from the large grid solution.}

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Indicial response to a step change in angle of attack:</th>
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<tr>
<td>NACA 64A006, } \alpha = 1 \text{ deg (} X_{max} \text{ and } Y_{max} \text{ are given in units of chord) }</td>
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<tr>
<td>Boundary condition</td>
<td>Reflecting</td>
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<td>\text{CPU time is normalized by the nonreflecting case.}</td>
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<th>Table 2</th>
<th>Sinusoidally oscillating trailing-edge flap:</th>
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<tr>
<td>NACA 64A006, } \theta = \sin \omega</td>
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<tr>
<td>Boundary condition</td>
<td>Reflecting</td>
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<td>\text{Mesh} \text{ } &amp; \text{ } \text{ } \text{ } &amp; \text{ } \text{ } \text{ } &amp; \text{ } \text{ } \text{ }</td>
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<td>\text{CPU time is normalized by the nonreflecting case.}</td>
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\( ^a \)CPU time is normalized by the nonreflecting case.
Fig. 4 Effect of far-field distance on $C_\alpha$ and $C_t$ for an NACA 64A006 airfoil with oscillating trailing-edge flap: a) type A, $M_\infty = 0.875$, $k = 0.468$; b) type B, $M_\infty = 0.854$, $k = 0.358$.

Fig. 5 Effect of far-field distance on upper surface pressure coefficient for an NACA 64A006 airfoil with oscillating trailing-edge flap: a) type A, $M_\infty = 0.875$, $k = 0.468$; b) type B, $M_\infty = 0.854$, $k = 0.358$. 
C. Impulsively Started Airfoil

To visualize the behavior of waves originating from the airfoil and then propagating throughout the computational domain, an NACA 64A006 airfoil is impulsively started from rest at time zero with $M_\infty = 0.85$. This is equivalently done in LTRAN 2 by turning on the freestream from rest. To see the reflection more clearly, the far-field boundaries are placed very close to the airfoil ($x_{\max} = 1.22c$, $y_{\max} = 1.08c$). The propagation of disturbances is demonstrated by pressure contour of the upper half plane of the computational domain. In Fig. 6, the results are shown at three sequential times measured by chord lengths of airfoil traveled.

Figure 6a is obtained from the large grid solution by looking through the window of 1.22 times $c$. In this sequence taken from a movie produced from the calculation, waves from the airfoil propagate outwardly only during the time observed as though the domain were infinite. In Fig. 6b, the old reflecting conditions are used. Physically, this is equivalent to placing solid walls on the boundaries. As time increases, the influence of the boundaries, especially the $y$ boundary, becomes more apparent and, at $t = 7.1$, the flow is shown to be choked. This illustrates a possible influence that a solid wind-tunnel wall can have on experimental results. When the nonreflecting conditions are used (Fig. 6c), the pressure contour map more closely resembles the large grid solution. As shown by the upper surface pressure coefficient plot in Fig. 7, the reflecting conditions contaminate the near-field solution much faster. Since the nonreflecting conditions used are simple approximations to perfectly absorbing conditions, some reflections still exist.

V. Conclusion

The first approximate nonreflecting far-field boundary condition procedure of Engquist and Majda has been successfully applied in solving various practical problems using the unsteady small-disturbance transonic flow code, LTRAN 2. (As discussed in Sec. 11B, their first approximation is a form of the characteristic equation. Therefore, the boundary condition procedure based on the first approximation alone may be regarded as a characteristic condition procedure.) It has been demonstrated successfully that the computational domain can be reduced considerably using their boundary condition procedure. However, since the mesh system is stretched in LTRAN 2, the number of computational grid points is not linearly proportional to the distance of the boundary from the airfoil. The gain in computer time after
implementing the new conditions is problem dependent and, for the cases tested using the first approximate conditions, the time saved is 10-24%. Further reduction in the computational mesh may be achieved by higher-order approximations. However, the benefit obtained by the reduced number of mesh is overshadowed by the increased computing time required to implement these high-order boundary conditions. Since the approximate conditions applied here are very simple and do not permit any waves to propagate in from the outside, modifications can be made to accommodate the waves which should come back into the domain. However, especially in the production version of a code, the simplicity of the first approximate nonreflecting conditions and the computational efficiency gained are more than adequate for applications in the design process.

References