Ocean turbulence III: two-point closure model
Momentum, heat and salt vertical diffusivities
in the presence of shear

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Abstract

In papers I and II we have presented the results of the most updated 1-point closure model for the turbulent vertical diffusivities of momentum, heat and salt, $K_{m,h,s}$. In this paper, we derive the analytic expressions for $K_{m,h,s}$ using a new 2-point closure model that has recently been developed and successfully tested against some ~80 turbulence statistics for different flows. The new model has no free parameters. The expressions for $K_{m,h,s}$ are analytical functions of two stability parameters: the Turner number $R_P$ (salinity gradient/temperature gradient) and the Richardson number $R_i$ (temperature gradient/shear). The turbulent kinetic energy $K$ and its rate of dissipation may be taken local or non-local ($K-\epsilon$ model).

Contrary to all previous models that to describe turbulent mixing below the mixed layer (ML) have adopted three adjustable "background diffusivities" for momentum, heat and salt, we propose a model that avoids such adjustable diffusivities. We assume that below the ML, $K_{m,h,s}$ have the same functional dependence on $R_i$ and $R_P$ derived from the turbulence model. However, in order to compute $R_i$ below the ML, we use data of vertical shear due to wave-breaking measured by Gargett et al. (1981). The procedure frees the model from adjustable background diffusivities and indeed we use the same model throughout the entire vertical extent of the ocean.

Using the new $K_{m,h,s}$, we run an O-GCM and present a variety of results that we compare with Levitus and the KPP model.

Since the traditional 1-point (used in papers I and II) and the new 2-point closure models used here represent different modeling philosophies and procedures, testing them in an O-GCM is indispensable. The basic motivation is to show that the new 2-point closure model gives results that are overall superior to the 1-point closure in spite of the fact that the latter rely on several adjustable parameters while the new 2-point closure has none.

After the extensive comparisons presented in papers I and II, we conclude that the new model presented here is overall superior for it not only is parameter free but also
because is part of a more general turbulence model that has been previously successfully tested on a wide variety of other types of turbulent flows.
Introduction

a) O–GCM dynamic equations. Turbulent Diffusivities

The dynamic equations for the large scale mean velocity, temperature and salinity, $U_i$, $T$ and $S$, are given by (Semtner; 1995; Semtner and Chervin, 1992; Anderson and Willebrand, 1992)

\[
\frac{D}{Dt} U_i + \frac{\partial}{\partial x_j} \tau_{ij} = -\frac{\partial P}{\partial x_i} + \ldots \tag{1a}
\]

\[
\frac{D}{Dt} T + \frac{\partial}{\partial x_i} h_i = \frac{\partial}{\partial x_i} (\chi \frac{\partial T}{\partial x_i}) + \ldots \tag{1b}
\]

\[
\frac{D}{Dt} S + \frac{\partial}{\partial x_i} s_i = \frac{\partial}{\partial x_i} (\kappa \frac{\partial S}{\partial x_i}) + \ldots \tag{1c}
\]

where $D/Dt=\partial/\partial t + U_j \partial/\partial x_j$. $\chi$ and $\kappa$ are the temperature and salt kinematic diffusivities. Eqs.(1a–c) depend on turbulence via the the Reynolds stresses $\tau_{ij}=\bar{u}_i \bar{u}_j$ and the heat and salt fluxes $h_i=\bar{u}_i \bar{\theta}$, $s_i=\bar{u}_i \bar{\sigma}$, where $u_i$, $\theta$ and $\sigma$ represent the turbulent (fluctuating) components of the velocity, temperature and salinity fields. The dots represent external forces, rotation, etc. The vertical turbulent diffusivities $K_{m,h,s}$ (cm$^2$s$^{-1}$) enter via the relations:

\[
\bar{u} \bar{w} = -K_m \frac{\partial U}{\partial z}, \quad \bar{w} \bar{\theta} = -K_h \frac{\partial T}{\partial z}, \quad \bar{w} \bar{\sigma} = -K_s \frac{\partial S}{\partial z} \tag{1d}
\]

Historically, a value of $K_h=1$ was suggested by Munk (1966). Bryan (1991) and Colin de Verdiere (1988) later showed that the polar heat transport scales like $K_h^{2/3}$ and that in order to reproduce its observed value (~1 Petawatt), O–GCM require $K_h \sim 2$ (Holland, 1989; Mellor, 1989; Cummings et al., 1990). Chen et al. (1994) concluded that an improved mixed layer ameliorates the large scale equatorial circulation as well as the annual cycles of the SST (sea surface temperatures) while on a global scale McWilliams (1996) has recently stressed that O–GCMs are sensitive, among other things, to the vertical diffusivities. From high resolution profiles and tracer experiments (Moura 1989; Gargett, 1989; Ledwell et al. 1993; McPhee and Martison 1994; Toole et al. 1994; Polzin et al. 1997), it was concluded that in the ocean mixed layer $K_h$ can be as large as $(1-2)10^2$, while in the thermocline...
Kh~(0.11–0.6) with perhaps the tendency to vary as N⁻¹, where N is the Brunt–Vaisala frequency (Gargett and Holloway 1984; Gargett et al., 1989; Gargett 1984, 1993; Matear and Wong, 1997). Thus, diffusivities are not constant with depth and different processes contribute to the K's at different depths resulting in different values.

b) general constraints on the K's

Since the dimensions of the diffusivities are (length)²(time)⁻¹, to construct the diffusivities K's one needs two independent variables. Over the years, three expressions have been adopted:

1) Richardson law:
   \[ \text{Diffusivity} \sim \ell^{4/3} \epsilon^{-1/3} \]  

2) Prandtl law:
   \[ \text{Diffusivity} \sim K^{1/2} \ell \]  

3) Turbulence modeling:
   \[ \text{Diffusivity} \sim \frac{K^2}{\epsilon} \]  

where \( \ell \) is a typical length scale, K is the turbulent kinetic energy and \( \epsilon \) is the rate of dissipation of K. Richardson law (a precursor of Kolmogorov law) emphasizes the diffusive nature of large eddies while Prandtl law is reminiscent of kinetic theory of gases and mean free path arguments. Both expressions entail a length scale \( \ell \) which is usually difficult to quantify, especially in the case of stable stratification (Cheng and Canuto, 1994), whereas (1g) depends on K and \( \epsilon \) for which we shall derive two differential equations thus avoiding the need of an \( \ell \). On that basis alone, (1g) is preferable to both (1e–f). Even so, (1g) is still not satisfactory. In fact, consider the case of stable stratification when shear is the source of turbulence. In the absence of salt, the turbulent Prandtl number,

\[ \sigma_T = \frac{K_m}{K_h} \]  

is known to increase with stratification represented by the Richardson number Ri,

\[ \sigma_T = \sigma_T(Ri), \quad \frac{d}{dRi} \sigma_T(Ri) > 0 \]  

(Webster, 1964; Istweire and Helland, 1989; Schumann, 1991; Wang et al., 1996). If we limit the functional dependence of the K's to the forms (1e–g), we obtain \( \sigma_T=\text{constant} \)
which violates the second of (1i). For example, the model of Pacanowski and Philander does not satisfy (1i) since it yields a constant $\sigma_T$. The recent model for $K_{m,h}$ of D'Alessio et al. (1998) does not satisfy (1i) either. In addition, the function $S_m$ must satisfy the well known asymptotic limits of $Ri$=0 (shear driven flows)

$$Ri=0: \quad 2S_m \approx 0.1$$

as studies of shear flow indicate (Launder et al., 1975).

Finally, the model must provide a correct value of the critical Richardson number $Ri_{cr}$ above which turbulence can no longer be sustained by shear. While linear stability analysis implies that turbulence exists only for

$$Ri < Ri_{cr}, \quad Ri_{cr} = 1/4$$

even early laboratory data by Taylor (cited in Monin and Yaglom, 1971, Vol.I, pages 502–503) show that considerable turbulent exchange exists up to $Ri$$\sim$10. More recently, Martin (1985), Smart (1988) and Kundu et al. (1991) have found considerable mixing in the ocean up to $Ri$$\sim$1, a result also validated by the LES results of Wang et al. (1996). This means that a model for stably stratified turbulence must allow turbulence past $Ri_{cr}$=1/4. The original MY model (Mellor and Yamada, 1982) does not satisfy this criterion since it yields $Ri_{cr}$=0.19 nor do any of the subsequent models that have not significantly improved the physical content of the MY model. This brief discussion about $\sigma_T(Ri)$ and $Ri_{cr}$ suffices to show that there are some basic data that any model must satisfy before being considered a candidate for ocean turbulence.

The general functional form of the turbulent diffusivities for momentum, heat and salt $K_{m,h,s}$ are of the form:

$$K_{m,h,s} = \frac{2K^2}{\epsilon} S_{m,h,s}$$

where $K$ and $\epsilon$ are the turbulent kinetic energy and its rate of dissipation which in principle are given by two dynamic equations (the $K-\epsilon$ model). The dimensionless structure functions $S_{m,h,s}$ must differ from one another so that:
In general we can write:

\[ \frac{K_m}{K_h} \neq \frac{K_h}{K_s} \] (2b)

where \( \alpha_T \) and \( \alpha_s \) are the volume expansion coefficients \( \alpha_T = -\rho^{-1} \partial \rho / \partial T \) and \( \alpha_s = \rho^{-1} \partial \rho / \partial S \) and where the shear \( \nabla U \) can be generated either by external sources like in the mixed layer ML or by internal wave-breaking processes below the ML. If one introduces the Turner number \( R_T \) and the Richardson number \( R_i \):

\[ R_T = g \frac{\partial \alpha}{\partial \sigma} (g \alpha_T \frac{\partial T}{\partial z})^{-1}, \quad R_i = N_h^2 / N_u^2 \] (2c)

where

\[ N_h^2 = g \alpha_T \frac{\partial T}{\partial z}, \quad N_u^2 = 2 \left( \sum_{ij} \Sigma_i \Sigma_j \right) \]

\[ N^2 = -g \frac{\partial \rho}{\partial z} = g \alpha_T \frac{\partial T}{\partial z} - g \alpha_s \frac{\partial S}{\partial z} = N_h^2 (1-R_T) \] (2d)

we can rewrite (2c) more concisely as

\[ S_{m,h,s} = S_{m,h,s}(R_T, R_i) \] (2e)

(Clearly, we could have also defined \( R_i \) in terms of \( N^2 \) rather than just the thermal gradient. We have chosen the latter for reasons of presentation of the results). One must distinguish the following four cases:

**SF (salt fingers, salty-warm over fresh-cold):**

\[ \frac{\partial S}{\partial z} > 0, \quad \frac{\partial T}{\partial z} > 0, \]

\[ R_T > 0, \quad R_i > 0 \]

\[ R_T < 1 \text{ Stable, } N^2 > 0, \quad R_T > 1 \text{ Unstable, } N^2 < 0 \] (3a)

**DC (diffusive convection, fresh-cold over salty-warm):**

\[ \frac{\partial S}{\partial z} < 0, \quad \frac{\partial T}{\partial z} < 0, \]

\[ R_T > 0, \quad R_i < 0 \]

\[ R_T < 1 \text{ Unstable, } N^2 < 0, \quad R_T > 1 \text{ Stable, } N^2 > 0 \] (3b)

**DS (doubly stable, fresh-warm over salty-cold):**

\[ \frac{\partial S}{\partial z} < 0, \quad \frac{\partial T}{\partial z} > 0, \]

\[ R_T < 0, \quad R_i > 0, \quad N^2 > 0, \quad \text{Stable} \]

(3c)
DU (doubly unstable, salty–cold over fresh–warm):

\[ \frac{\partial S}{\partial z} > 0, \quad \frac{\partial T}{\partial z} < 0, \]

\[ R_\rho < 0, \quad R_i < 0, \quad N^2 < 0, \quad \text{Unstable} \]  \hspace{1cm} (3d)

The stability/instability is predicated on the Brunt–Vaisala frequency \( N \) with \( N^2 > 0 \) (stable) and \( N^2 < 0 \) (unstable).

The general problem is to construct (2f) so as to encompass all four cases (3a–d). First, there is ample evidence from laboratory and oceanic field data that show that \( K_h \) is different from \( K_s \). In the SF case, the ratio \( K_s / K_h > 1 \) (Hamilton et al., 1989) while in the DC case, \( K_h / K_s > 1 \) (Kelley, 1984). Schmitt (1981) has also shown that the observed T–S relationship is not consistent with \( K_h = K_s \). For a discussion and review of the importance of these processes and their extent in different parts of the ocean, see Turner (1967, 1973, 1985), Schmitt (1994) and Zhang et al. (1998). In spite of this evidence, almost all O–GCM still assume

\[ K_s = K_h \]  \hspace{1cm} (3e)

Recently, attempts have been made to overcome (3e) but the task is not easy. The main difficulty is that in the absence of a model capable of encompassing all four cases, SF, DC, DS and DU, the only alternative is to employ laboratory and ocean data to build the functional form of the diffusivities to be used in an O–GCM. This is the approach employed by Large et al. (1994), Zhang et al. (1998, ZSH) and Merryfield et al. (1999, MHG) who used relations by Schmitt (1981) and Kelley (1984, 1990), among others.

There is, however, an internal limitation to such a procedure since the available data refer to SF and DC but not to DS and DU which are also important (Duffy and Caldera, 1999). Thus, away from the regions where SF and DC are active, the above authors take

\[ K_{m,h,s}(\text{DS, DU}) = 0 \]  \hspace{1cm} (3f)

or, more precisely, they use a background diffusivity which is chosen primarily on grounds of numerical stability but whose physical origin must be an internal–wave breaking phenomenon. This is clearly not a satisfactory situation especially in view of the fact that
Since the studies by ZHS and MHG have shown the importance of double diffusion, the above procedure is certainly better than (3e) but still not fully satisfactory. The goal of this paper is to consider the same problem but with a different methodology.

We develop a turbulence model to compute the three diffusivities for momentum, heat and salt to encompass the four processes SF, DC, DS and DU in the presence of an arbitrary shear. The inclusion of shear is quite relevant since is known to hamper the SF mechanism (Linden, 1971, 1974a, b; Kunze, 1990) and yet, the above procedures do not account for shear since they expressed $K_h$ and $K_s$ in terms of only one stability parameter $R_p$, rather than $R_p$ and the Richardson number $R_i$.

The model comes in three forms: 1) $K$ and $\epsilon$ are solutions of two dynamical equations, the so called $K-\epsilon$ model whose main but not unique advantage is to avoid the introduction of any mixing length, 2) only one of them satisfies a differential equation while the other is taken to be the local limit of its dynamic equation and 3) both $K$ and $\epsilon$ are taken as the local limit of their respective dynamic equations. As we shall show, in model 3) all the relations are algebraic and one must solve a cubic equation. All the numerical results correspond to case 3).

c) How to compute the diffusivities $K$'s

Since from the first O–GCM it clearly emerged that several key ocean features, e.g., polar heat transport, meridian circulation, T–S relationship (temperature–salinity), etc., depend sensitively on the vertical turbulent diffusivities of momentum $K_m$, heat $K_h$ and salt $K_s$ (F.O. Bryan, 1987; K.Bryan, 1991) even though until recently all codes employed the approximation $K_s=K_h$. The need to reliably compute such diffusivities has been dealt in the past with three quite distinct approaches.

d) descriptive–diagnostic method

One can adopt a descriptive–diagnostic approach whereby the O–GCM are used to
show that in order to obtain overall agreement with the data, the diffusivities $K$'s must lie in a restricted range of values. Such a diagnostic, a posteriori approach is however inadequate in climate studies where the predictive power of the model is the key feature on which the reliability of the model results is predicated.

e) turbulence models. These models are based on the Navier–Stokes equations and thus, at least in principle, have a built-in degree of generality, resiliency and predictive power. In particular, one-point closure turbulence models have the longest tradition (Chou, 1940, 1945) and, naturally, they were the first tools used to construct the turbulence variables needed in O–GCM. The pioneering work in this field is that of Mellor and Yamada (1982, MY; Mellor, 1989) who applied the 1980 state-of-the-art turbulence modeling to geophysical problems. However, almost 20 years later, most of the work in ocean turbulence still employs essentially the same 1980 state of the art turbulence modeling (Rosati and Miyakoda 1988; Galperin et al. 1989; Gaspar et al. 1990; Blanke and Delecluse 1992; Baum and Caponi, 1992; Ma et al. 1994; Kantha and Clayson 1994; Burchard and Baumert 1995; D'Alessio et al., 1998). This means that the basic difficulties that plagued the original MY model are still present. They can be identified with the closure of the pressure correlations terms, and the need to express the non-local third order moments (TOM) in terms of lower order moments. The pressure–correlations bring about adjustable constants which the model is unable to predict and the TOM are treated with the down gradient approximation which Moeng and Wyngaard (1989) have convincingly shown to be seriously inadequate (see paper I for details).

f) Non–turbulence models. On the grounds that the above uncertainties have not been satisfactorily resolved, Large et al. (1994) have suggested an alternative model (called KPP), not based on any turbulence model.
g) New turbulence models. It is generally agreed that, potentially, turbulence models have far greater generality, resiliency and predictability power that any ad-hoc-model. In papers I and II, we offered the most updated 1-point closure models that one can construct today. Here, we offer a new 2-point closure model that has been assessed over some ~80m turbulent statistics pertaining to large variety of flows. We derive the expressions for $K_m$, $K_h$ and $K_s$ and run the O-GCM with the new K's. The new model has no adjustable parameters.

II. New turbulence model

The turbulence model to be employed here (Canuto et al., 1996-1999, cited as CD) was tested on about 100 turbulent statistics belonging to a wide variety of flows: pure shear flows, Rayleigh–Benard convection, rotating turbulence, 2D turbulence, rotating convection, freely decaying turbulence, stably stratified turbulence, etc. In all cases, the model reproduces the data quite accurately and thus the model can be considered validated. In addition, the model has no adjustable parameters. Since this is the first application of this new turbulence model to ocean turbulence, it is necessary to begin by presenting the physical foundations of the model.

Recent advances in phenomena related to turbulence (e.g., critical phenomena) have provided us with new tools and new perspectives on how to tackle the problem of turbulence. Like in many other fields, one begins with the case of homogeneous–isotropic turbulence for which an exact solution of the NSE found by Wyld (1961). The solution for the energy spectrum $E(k)$ can be shown to be equivalent to the solution of a stochastic Langevin–type dynamic equation for the turbulent velocity fields $u(k)$ of the form

$$\frac{\partial}{\partial t}u_i(k) = f_{ij}^{ext}(k) + f_{ij}^t(k) - \nu_d(k)k^2u_i(k)$$  \hspace{1cm} (4a)

where $E(k) \sim k^2u(k)u(k)$. Eq.(4a) has an appealing physical interpretation: external forces are represented by the first term whose specific form is obtained from the NSE for arbitrary flows (shear, convection, rotation, etc). The second term is a random turbulent force
representing the action of the large eddies on a given eddy $k$ even when the external forces are no longer operative (for example in the Kolmogorov region) and finally, the last term represents the dynamical viscosity felt by an eddy $k$ due to all smaller eddies. Though derived for homogeneous and isotropic turbulence, Eq.(4a) seems to capture the main ingredients that one needs to build the canonical picture of a supply, transfer and enhanced dissipation of energy that is at the base of Kolmogorov original picture of turbulent dynamics. The challenge is to give an explicit expression to all the terms and prove that in spite of its "idealized" origin, Eq.(4a) is valid quite in general and capable of reproducing real flows. It must be stressed that we are not the first to suggest a Langevin-type equation as a starting point to describe turbulence. Several previous models were in fact recast in terms of a Langevin-type equations but, to the best of our knowledge, this is the first case in which Eq.(4a) has been shown to be a viable, successful, operational tool to treat real turbulent flows. To construct the explicit form of the dynamical viscosity, CD first generalized the well developed and well tested renormalization group method (RNG) to derive the following expression for $\nu_d(k)$

$$\nu_d(k) = \nu_t(k) + \nu = (\nu^2 + \frac{2}{5} \int_{k}^{\infty} E(p)p^{-2}dp)^{\frac{1}{2}} \quad (4b)$$

where $\nu$ is the kinematic viscosity and $\nu_t(k)$ is the so-called turbulent viscosity. This expression tells two things: first, the integration from $k$ to $\infty$ means that $\nu_d(k)$ is contributed by all eddies smaller than $k$, as indeed expected on physical grounds and that while the Langevin equation is linear in the velocity fields, its "coefficients" like $\nu_d(k)$ are non linear since they depend on $E(k)$. Thus, while in the NSE the coefficients are numbers and the velocity fields enter non-linearly (giving rise to closure problems), in the Langevin equation the situation is reversed. This a key point that allows us to derive a closed system of equations for the second-order moments like Reynolds stresses, etc. Next, we consider the first term in (4a). It represents the external forces driving turbulence, is a linear function of the turbulent fields and its form can be derived directly from the NSE. In the
present paper we consider the turbulent fields of velocity, temperature and salinity \( u_i, \theta \) and \( \sigma \) in which case \( f^{\text{ext}} \) is the sum of three terms

\[
f^{\text{ext}}_i = f^u_i + f^\theta_i + f^\sigma_i
\]

where

\[
f^u_i = - U_{i,j}^+ u_j(k) + k_j U_{j,m} \left[ \frac{\partial}{\partial k_m} u_i(k) \right] k^j
\]

\[
f^\theta_i = - \alpha_T P_{ij}(k) g_j \theta(k)
\]

\[
f^\sigma_i = \alpha_S P_{ij}(k) g_j \sigma(k)
\]

Here, \( U_{i,j} \) is the mean velocity gradient, and the \( \alpha \)'s are the expansion coefficients (\( T \) and \( S \) are mean temperature and salinity)

\[
\alpha_T = - \rho^{-1} \frac{\partial \rho}{\partial T}, \quad \alpha_S = \rho^{-1} \frac{\partial \rho}{\partial S}
\]

which must be computed using the equation of state for seawater. The superscript \( \perp \) in (4d) means that only transverse components are allowed, that is, for any tensor \( A \),

\[
A^\perp_{ij} = \Pi_{ij}^\perp(k) A_{lm}^\perp P_{mj}^\perp(k)
\]

where \( P_{ij}^\perp = \delta_{ij} - k_i k_j k^2 \) is the projection operator. Analogously, for the \( \theta \) and \( \sigma \) fields, we have the Langevin equations:

\[
\frac{\partial}{\partial t} \theta(k,t) = f^\theta_i(k,t) - \chi^\perp_d(k,t) k^2 \theta(k,t) + f^\sigma_i(k,t)
\]

\[
\frac{\partial}{\partial t} \sigma(k,t) = f^\sigma_i(k,t) - \kappa^\perp_d(k,t) k^2 \sigma(k,t) + f^\sigma_i(k,t)
\]

with

\[
\chi^\perp_d(k) = \chi + \chi^\perp_t(k), \quad \kappa^\perp_d(k) = \kappa + \kappa^\perp_t(k)
\]

where \( \chi \) and \( \kappa \) are the heat and salt kinematic diffusivities. The stirring forces are:

\[
f^\theta_i(k,t) = - \beta^\theta_i u_i(k), \quad f^\sigma_i(k,t) = - \beta^\sigma_i u_i(k)
\]

where the \( \beta \)'s are the gradients of the mean temperature \( T \) and the mean salinity \( S \):

\[
\beta^\theta_i = \frac{\partial T}{\partial x_i}, \quad \beta^\sigma_i = \frac{\partial S}{\partial x_i}
\]

As Eqs.(1a–c) show, what one needs in practice the Reynolds stress \( \tau_{ij} \), which are the integral over all wavenumbers \( k \) of the two-point Reynolds stresses

\[
R_{ij}(k) \delta(k+k') = <u_i(k,t) u_j(k',t)>_s
\]
\[ \tau_{ij} = \] (8b)

where \( s \) means symmetrization with respect to \( k \) and \( k' \) and \( i,j \). The other required second-order moments are the temperature and salinity fluxes:

\[ j^\theta_i(k) \delta(k+k') = <u_i(k,t)\theta(k',t)>_s \] (9a)
\[ j^\sigma_i(k) \delta(k+k') = <u_i(k,t)\sigma(k',t)>_s \] (9b)

As it will be shown, these variables entail other turbulence variables, and precisely, the variances of both the temperature and salt fields,

\[ e_\theta(k) \delta(k+k') = \frac{1}{2} <\theta(k)\theta(k')> \] (10a)
\[ e_\sigma(k) \delta(k+k') = \frac{1}{2} <\sigma(k)\sigma(k')>= \] (10b)

as well as the temperature–salinity correlation

\[ e_{\theta\sigma}(k) \delta(k+k') = <\theta(k)\sigma(k')> \] (10c)

The spectra of energy, temperature and salt fields \( E(k), E_{\theta,\sigma}(k) \) are obtained by integrating over the directions of the vector \( k \) (times a factor \( k^2 \)) the density functions in the left hand sides of Eqs.(8)-(10), namely:

\[ E(k) = \frac{1}{2}k^2\int R_{ii}(k)dn_k \] (11a)
\[ E_{\theta,\sigma}(k) = k^2\int e_{\theta,\sigma}(k)dn_k \] (11b)

The dynamic equations for the second-order moments (8)-(10) are derived by multiplying the stochastic dynamic equations (4a) and (5a,b) by \( u_i(k'), \theta(k') \) and \( \sigma(k') \) and then averaging. The key difficulty is represented by the correlation of the first terms in (4a) and (5a,b) with the turbulent fields which physically represent the work performed by the turbulent forces \( f^i(k) \). To compute these terms, we adopted two assumptions which are in agreement with the general understanding of turbulent processes: 1) energy generated at the largest scales is transferred locally to high \( k \) regions in a manner analogous to the flow of gas in a pipe, a model that applies to temperature and salinity as well, 2) as turbulent energy is transferred to smaller scales, the correlation functions of different fields, as well as correlation functions between different components of the velocity fields, decreases with increasing \( k \) even faster than the energy spectrum itself. The first assumption leads to a
representation of the energy flux $\Pi(k)$ of the form:

$$\Pi(k) = r(k)E(k)$$  \hspace{1cm} (12)$$

where $r(k)$ is the rapidity with which energy is propagated among different eddies. Eq.(12) can be viewed as the analog of $j=\rho v$ with $j=\Pi(k)$, $\rho=E(k)$ and $v=r(k)$. The energy flux $\Pi(k)$ is related to the energy transfer function $T(k)$ by:

$$T(k) = -\frac{\partial}{\partial k} \Pi(k), \quad \int T(k) \, dk = 0$$  \hspace{1cm} (13a)$$

$$T(k) = -r(k) \frac{\partial}{\partial k} E(k) - \left[ \frac{\partial}{\partial k} r(k) \right] E(k)$$  \hspace{1cm} (13b)$$

On the other hand, the function $T(k)$ can be obtained by multiplying (4a) by $u_i(k')$ and separating the work of the non-linear interactions entering the first two terms in (4a).

$$T(k) = A^t(k) - 2\nu_t(k) k^2 E(k)$$  \hspace{1cm} (14)$$

where the work of the force $f^t$ is defined by

$$A_t(k) = k^2 \int d\kappa' \langle u_i(k',t) f^t_i(k,t) \rangle$$  \hspace{1cm} (15)$$

Comparing (13b) and (14), we obtain:

$$A_t(k) = -r(k) \frac{\partial}{\partial k} E(k), \quad \int r(k) = \int_0^k \nu_t(p) p^2 dp$$  \hspace{1cm} (16)$$

For the fields $\theta$ and $\sigma$, assumption 1) leads to relations analogous to Eqs.(15)-(16), namely:

$$\langle \theta(k') f^t_{\theta}(k) \rangle_s = (4\pi k^2)^{-1} A_{\theta} \delta(k+k')$$  \hspace{1cm} (17)$$

$$\langle \sigma(k') f^t_{\sigma}(k) \rangle_s = (4\pi k^2)^{-1} A_{\sigma} \delta(k+k')$$  \hspace{1cm} (18)$$

where the expressions for the functions $A_{\theta,\sigma}(k)$ are analogous to Eq.(16), namely,

$$A_{\theta,\sigma}(k) = -r_{\theta,\sigma}(k) \frac{\partial}{\partial k} E_{\theta,\sigma}(k)$$  \hspace{1cm} (19)$$

$$\int r_{\theta}(k) = \int_0^k \chi_t(p) p^2 dp$$  \hspace{1cm} (20)$$

$$\int r_{\sigma}(k) = \int_0^k \kappa_t(p) p^2 dp$$  \hspace{1cm} (21)$$

The functions $\chi_d, \kappa_d$ satisfy the differential equation ($F$ stands for $\chi_d$ and/or $\kappa_d$)

$$\frac{dF}{d\nu_d} = \frac{10}{3} \nu_d (\nu_d + F)^{-1}$$  \hspace{1cm} (22)$$

In the present paper, we shall be mostly interested in situations where

$$\nu_d > \nu, \quad \chi_d > \chi, \quad \kappa_d > \kappa$$  \hspace{1cm} (23)$$

and thus it follows from Eq.(22) that
\[
\chi_d \cdot \kappa_d = \sigma_t^{-1} \nu_d(k) \tag{24}
\]
where the turbulent Prandtl number \( \sigma_t = 0.72 \) and \( \nu_d(k) \) is given by Eq. (4b). As for assumption 2) above, it leads to the following relations:
\[
< \mathbf{f}_1(t) \mathbf{u}_j(k', t) > = -(8\pi k^2)^{-1} \mathbf{A}_t(k) \mathbf{P}_{ij}(k) \delta(k+k')
\]
\[
<u_j(k')\theta(k')f_{\theta,\sigma}(k)> = 0
\]
\[
<\theta(k')f_{\theta,\sigma}(k)> = 0
\]
\[
<\sigma(k')f_{\theta,\sigma}(k)> = 0
\]  
\[(25a)\]
Finally, the dynamic equations for \( E(k) \) is:
\[
\frac{\partial}{\partial t} E(k) = A_{ext}(k) + A_t(k) - 2k^2 \nu_d(k) E(k) \tag{25b}
\]
The information we have derived concerning the terms in the basic equations (4a) and (5a,b) is sufficient to allow us to derive a closed system of equations for the second-order moments (8)–(10).

III. Dynamic Equations for the Second-Order Moments

Multiplying the stochastic equations (4a) and (5a,b) by the fields \( u, \theta \) and \( \sigma \), it is straightforward to derive the following dynamic equations for the second-order correlation functions:
\[
\frac{1}{2} \frac{\partial}{\partial t} R_{ij}(k) = (8\pi k^2)^{-1} \mathbf{A}_t(k) \mathbf{P}_{ij}(k) - [\mathbf{S}_{im}^\perp \mathbf{R}_{mj}(k)]_s - \\
- \frac{1}{2} [\mathbf{V}^\perp, \mathbf{R}(k)]_{ij} + \frac{1}{2} [\mathbf{D} \mathbf{R}(k)]_{ij}^\perp - k^2 \nu_d(k) R_{ij}(k) \\
- \alpha_s^T [\mathbf{g}_{ij}^\perp \theta(k)]_s + \alpha_s^T [\mathbf{g}_{ij}^\perp \sigma(k)]_s \tag{26}
\]
\[
\frac{1}{2} \frac{\partial}{\partial t} j^\perp_i(k) = - \mathbf{S}_{im}^\perp j^\perp_m(k) - \mathbf{V}_{im}^\perp j^\perp_m(k) + [\mathbf{D} j^\theta(k)]_i^\perp \\
- k^2 [\nu_d(k) + \chi_d(k)] j^\theta_i(k) - \beta_m^\theta R_{im}(k) \\
- [2\alpha_s e_o(k) + \alpha_s e_o(k)] g_{ij} P_{ij}(k) \tag{27}
\]
\[
\frac{\partial}{\partial t} e_\theta(k) = (4\pi k^2)^{-1} \mathbf{A}_\theta(k) - \beta^\theta_{i1} j_1^\perp(k) - 2k^2 \chi_d(k) e_\theta(k) \tag{28}
\]
\[
\frac{\partial}{\partial t} e_{\theta\sigma}(k) = - \beta^\sigma_{i1} j_1^\perp(k) - \beta_{i1}^\sigma j_1^\perp(k) - 2k^2 [\chi_d(k) + \kappa_d(k)] e_{\theta\sigma}(k) \tag{29}
\]
In addition there are two equations which can be obtained from Eqs.(27) and (28) with the substitutions:

\[ j^\theta_i - j_\sigma_i, \quad \beta^\theta_i - \beta_\sigma_i, \quad \alpha_T \rightarrow \alpha_s, \quad e^\theta - e_\sigma, \quad A^\theta - A_\sigma, \quad \chi d^{-\kappa_d} \]  

(30)

In Eqs.(26)-(27) we have introduced the tensors

\[ U_{i,j} = S_{i,j} + V_{i,j}, \quad 2S_{i,j} = U_{i,j} + U_{j,i}, \quad 2V_{i,j} = U_{i,j} - U_{j,i} \]  

(31)

where \( S_{i,j} \) and \( V_{i,j} \) are the mean shear and vorticity and \( D \) is the operator

\[ \hat{D} = k_j(S_j\ell + V_j\ell)\frac{\partial}{\partial k_\ell} \]  

(32)

Eqs.(26)-(29) must be solved together with the incompressibility condition

\[ k_iR_{ij}(k) = k_jR_{ij}(k) = \kappa_{ij} = 0 \]  

(33)

This suggests that one may solve the above equations in spherical coordinates where one can write (for any \( j \))

\[ j(k) = j_\phi(k)e_\phi + j_\theta(k)e_\theta \]  

(34)

where \( e_\phi, e_\theta \) are the basic units vectors. Due to condition (33), the only non-zero components of \( R_{ij} \) lie in the plane perpendicular to \( k \) so that their number is three and the general decomposition has the form:

\[ R_{ij}(k) = e(k)P_{ij}(k) + a(k)\rho_{ij}(k) + b(k)\eta_{ij}(k) \]  

(35)

where \( e(k) \) is the energy density in \( k \)-space while the orthogonal tensors \( \rho \) and \( \eta \) should be chosen so as to avoid singularities in the functions \( a(k) \) and \( b(k) \) which invariably arise when the tensors \( \rho \) and \( \eta \) depend on \( \phi \) as \( \theta \approx 0 \). This point and the details of the decomposition are fully discussed in Sec.III of Canuto and Dubovikov (1996b). In the 3D case, the final equations for the scalar functions \( e(k), a(k), b(k) \) etc are quite cumbersome. Thus, we shall consider only the case of direct physical interest: the mean fields \( U_i, T, \) and salt depend only on the coordinate where there is a non-zero gravity. The final result is:

\[ \frac{\partial}{\partial t}e(k) = (4\pi k^2)^{-1}A_\ell(k) - 2k^2\nu_\ell^d(k)e(k) + M_0 e(k) + M_\ell a(k) + M_\eta b(k) - \lambda^\phi_\ell^\theta (k) - \lambda^\theta_\ell^\phi (k) + \lambda^\sigma_\ell^\phi (k) + \lambda^\sigma_\ell^\theta (k) \]  

(36)
\[ \frac{\partial}{\partial t} a(k) = -2k^2\nu_d(k) a(k) + M_0 a(k) + M_a e(k) + M_{ab} b(k) - \\
+ \delta_1 j^{\phi}(k) + \delta_2 j^{\theta}(k) + \delta_3 j^{\sigma}(k) + \delta_4 j^\phi(k) \]

with

\[ \delta_1 = \lambda_0^\phi \cos 2\phi + \lambda_0^\theta \sin 2\phi, \quad \delta_2 = -\lambda_0^\theta \cos 2\phi + \lambda_0^\phi \sin 2\phi \]
\[ \delta_3 = -\lambda_0^\sigma \cos 2\phi - \lambda_0^\sigma \sin 2\phi, \quad \delta_4 = \lambda_0^\sigma \cos 2\phi - \lambda_0^\sigma \sin 2\phi \]

\[ (37a) \]

\[ \frac{\partial}{\partial t} b(k) = -2k^2\nu_d(k) b(k) + M_0 b(k) + M_b e(k) - M_{ab} a(k) - \\
+ \gamma_1 j^{\phi}(k) + \gamma_2 j^{\theta}(k) + \gamma_3 j^{\sigma}(k) + \gamma_4 j^\phi(k) \]

with

\[ \gamma_1 = \lambda_0^\sigma \sin 2\phi - \lambda_0^\sigma \cos 2\phi, \quad \gamma_2 = -\lambda_0^\sigma \sin 2\phi - \lambda_0^\sigma \cos 2\phi \]
\[ \gamma_3 = -\lambda_0^\sigma \sin 2\phi + \lambda_0^\sigma \cos 2\phi, \quad \gamma_4 = \lambda_0^\sigma \sin 2\phi + \lambda_0^\sigma \cos 2\phi \]

\[ (38a) \]

\[ \frac{\partial}{\partial t} j^{\phi}(k) = -2k^2\nu_d(k) j^{\phi}(k) + M_0 j^{\phi} + M_0 j^{\theta} - Sz(1-\sin^2\phi) \]
\[ - 2\lambda_0^\sigma e^{\phi}(k) + \lambda_0^\sigma e^{\theta}(k) + \epsilon_1 a(k) + \epsilon_2 b(k) \]

\[ (39a) \]

\[ \epsilon_1 = \beta_0^\phi \sin 2\phi - \beta_0^\phi \cos 2\phi, \quad \epsilon_2 = -\beta_0^\phi \sin 2\phi - \beta_0^\phi \cos 2\phi \]
\[ (39b) \]

\[ \frac{\partial}{\partial t} j^{\theta}(k) = -2k^2\nu_d(k) j^{\theta}(k) + M_0 j^{\theta} + S \theta - \beta_0^\theta e(k) - \\
- 2\lambda_0^\sigma e^{\theta}(k) + \lambda_0^\sigma e^{\phi}(k) + \eta_1 a(k) + \eta_2 b(k) \]

\[ (40a) \]

\[ \eta_1 = \beta_0^\phi \cos 2\phi + \beta_0^\phi \sin 2\phi, \quad \eta_2 = -\beta_0^\phi \cos 2\phi + \beta_0^\phi \sin 2\phi \]
\[ (40b) \]

\[ \frac{\partial}{\partial t} e(k) = (4\pi k^2)^{-1} A_0(k) - 2k^2\sigma_0^{-1} \nu_d(k) e(k) - \\
- \beta_0^\phi j^{\phi}(k) - \beta_0^\phi j^{\theta}(k) \]

\[ (41) \]

\[ \frac{\partial}{\partial t} e^{\sigma}(k) = -2k^2\sigma A_0 e^{\theta}(k) - \beta_0^\phi e^{\phi}(k) - \\
- \beta_0^\phi e^{\phi}(k) - \beta_0^\phi e^{\phi}(k) - \beta_0^\phi e^{\phi}(k) \]

\[ (42) \]

We have defined the following variables \( z=\cos\theta)\):

\[ \hat{M}_0 = \frac{1}{4} S(1-z^2) \cos 2\phi + D \]

\[ (43) \]
$$-4S^{-1} \dot{M}_a = (1+z)^2 + (1-z)^2 \cos 4\phi$$
$$-4S^{-1} \dot{M}_b = (1-z)^2 \sin 4\phi$$
$$S^{-1} \dot{M}_{ab} = (1-z)(1-\sin 2\phi) -z$$
$$\dot{M}_\theta = -\frac{1}{4} Sz^2 \cos 2\phi + \ddot{D}, \quad \dot{M}_\phi = \frac{1}{4} S \cos 2\phi + \ddot{D} \quad (44a)$$
$$2S^{-1} \ddot{D} = \cos 2\phi \left(1-z^2\right) \left(k \partial_k -z \partial_z\right) + (1-\sin 2\phi) \partial_\phi$$
$$p = \frac{1}{2}(1+\sigma_l^{-1}) = 1.2 \quad (44b)$$
$$\beta_\phi^{\theta, \sigma} = -2\frac{1}{4}(\sin \phi - \cos \phi) \beta_\theta, \sigma$$
$$\beta_\theta^{\theta, \sigma} = 2\frac{1}{4} z(\cos \phi + \sin \phi) \beta_\theta, \sigma \quad (44c)$$
$$\lambda_\theta^{\theta, \sigma} = -2\frac{1}{4} z(\sin \phi + \cos \phi) g_\alpha_{T,s}$$
$$\lambda_\phi^{\theta, \sigma} = 2\frac{1}{4} (\sin \phi - \cos \phi) g_\alpha_{T,s} \quad (44d)$$

Finally, the equations for
$$j_\phi^\sigma(k), \quad j_\phi^\sigma(k), \quad e_\sigma(k) \quad (44e)$$
are obtained from Eqs.(40)–(41) via the substitution (30). In the above and subsequent equations,
$$S = (2S_{ij} S_{ij})^{\frac{1}{2}}, \quad N^2 = g_\alpha_{T} \partial_T ^z, \quad R_i = \frac{N^2}{S^2} \quad (44f)$$
S, N and Ri are the mean shear, the Brunt–Vaisala frequency and the Richardson number.

IV. Spectra of the Second–Order Moments

The above equations do not contain free parameters and can be solved numerically once the initial density functions are known. Concrete examples were discussed in Canuto and Dubovikov (1996c) for different types of shear driven flows for which we can compare the model results against DNS/LES and laboratory data. The model reproduces such data quite well. The case of buoyancy driven flows was also studied (Canuto and Dubovikov, 1997a) and in that case too, the model results were in good agreement with a variety of data. This performance of the model gives us confidence that the extension discussed here is based on solid foundations. Our final goal is to express the one–point second order
moments like the Reynolds stresses, the temperature and salinity fluxes in terms of the gradients of the large scale fields. Since a numerical solution of the above dynamic equations would not be of practical use, especially if the results are to be used in an O-GCM, an analytical model is needed which by necessity requires that we make some approximations. We shall employ the same philosophy that was employed earlier when we treated shear and buoyancy driven flows. The gist of the method is as follows. In the inertial ranges, Eqs.(36)–(42) can be solved perturbatively since we have the small parameters

$$S(k^2\nu_t)^{-1}, \quad g_\theta \beta^\theta(k^2\chi_t)^{-2}, \quad g_s \beta^\sigma(k^2\kappa_t)^{-2}$$

Once the inertial spectra are obtained, we extrapolate them all the way back to the infrared cut-off $k_0$ below which we assume that $E(k)=0$. Upon integrating the resulting spectra over all $k$'s, we obtain the desired one-point correlation functions. In the case of shear-driven flows, the results obtained with this method differ less than 15% from those obtained by numerically solving the full equations (Canuto et al., 1999c). We shall carry out the procedure without salinity which we shall incorporate in the next paper. To calculate the spectra, we first need to integrate the $k$-space densities, Eqs.(8)–(10), over all directions of $k$. In turn, the densities are expressed in terms of the variables in Eq.(36)–(42) as described in detail in Canuto and Dubovikov (1998b). Furthermore, we shall consider first the stationary case. To the zeroth-order in the smallness parameters (45), the only non-zero results are the isotropic components of the energy densities $\bar{e}(k)$, $\bar{e}_\theta(k)$ and $\bar{e}_\sigma(k)$ which are obtained by retaining only the first two terms in Eqs.(36) and (41) and in the equation for $e_\sigma$ which is obtained from (41) and the substitution (30). The results are:

$$4\pi k^2\bar{e}(k) = E(k) = Ko^2/3 k^{5/3}$$

$$4\pi k^2\bar{e}_\theta(k) = E_\theta(k) = Ba e \theta e^{-1/3} k^{5/3}$$

$$4\pi k^2\bar{e}_\sigma(k) = E_\sigma(k) = Ba e \sigma e^{-1/3} k^{5/3}$$

with

$$Ko = \frac{5}{3}, \quad Ba = e_t Ko$$

(48)
where \( Ba \) is the Batchelor constant and the \( \epsilon \)'s represent the rate of dissipation of the total kinetic and potential energies for the temperature and salinity fields. To compute the first order terms in the smallness parameters (45), we must consider the first and third terms in (37a), (38a), the first, fourth and fifth terms in (39a), (40a) and in the corresponding equation for \( j_\sigma \), whereas in (36) the second and third terms should be taken into account provided that only the component \( \tilde{e} = e - \tilde{e} \) is retained in the second term since the \( \tilde{e} \)-part has already been used in the zeroth-order. The results of the above procedure, as well as further intermediate steps, are presented for the simplified case in the absence of salinity, whereas the final results will include salinity.

**Reynolds stress spectrum**

\[
R(k) = \frac{2}{3}IE(k) + C_\sigma(k)\sigma + C_\Sigma\Sigma + C_t(k)t + C_\lambda(k)\lambda
\]  

(49a)

where the basis tensors are:

\[
\sigma = S^{-1}S, \quad \Sigma = S^{-2}S^2 - \frac{1}{6}I
\]

\[
t = S^{-2}[V,S], \quad \lambda = g^{-2}g_ig_j - \frac{1}{3}I
\]

(49b)

The coefficients \( C \)'s are given by (superscripts indicate the perturbative order at which they were obtained and \( \beta = \beta_0 \)):

\[
C^1_\sigma(k) = -\frac{20}{27}S\epsilon^{1/3}k^{-7/3}
\]

(50a)

\[
C^3_\sigma(k) = 0.92(1 - \sigma_t\epsilon\theta^{-1}\alpha_Tg\beta^{-1})N^2S\epsilon^{-1/3}k^{-11/3}
\]

(50b)

\[
C^2_t(k) = \frac{28}{27}S^2k^{-3}
\]

(50c)

\[
C^4_t(k) = -1.08(1 - \sigma_t\epsilon\theta^{-1}\alpha_Tg\beta^{-1})N^2S^2\epsilon^{-2/3}k^{-13/3}
\]

(50d)

\[
C^2_\Sigma(k) = -3.3S^2k^{-3}
\]

(50e)

\[
C^2_\lambda(k) = -1.3N^2(1 - 1.44\epsilon\theta^{-1}\alpha_Tg\beta^{-1})k^{-3}
\]

(50f)

where the Brunt–Vaisala frequency \( N \) is:

\[
N^2 = g\alpha_T\frac{\partial T}{\partial z}
\]

(50g)

**Heat flux spectrum**:

\[
J(k) = J_g(k)g_0 + J_uU_0
\]

(51a)

\[
g_0 = g^{-1}g, \quad U_0 = U^{-1}U
\]

(51b)
\[ J_g^1(k) = 0.93\beta \left( 1 - 1.44\epsilon \theta^{-1} \alpha_T g \beta^{-1} \right) \epsilon^{1/3} k^{-7/3} \]  
\[ J_u(k) = \frac{10}{27} \beta^{-1} \left[ 1 + \frac{17}{20} \right] \theta^{-1} \left( 1 - 1.44\epsilon \theta^{-1} \alpha_T g \beta^{-1} \right) \theta^{-1} \beta^{-1} \]  
V. One-point Reynolds stresses and temperature flux

The case of the Reynolds stresses in the presence of shear was discussed in detail in Canuto and Dubovikov (1999c) and we shall not repeat it here. The final result for the one-point Reynolds stress, arrived at by integrating (49a) over wavenumbers from the infrared cutoff to infinity, is:

\[ \tau_{ij} = \overline{u_i u_j} - \frac{2}{3} \delta_{ij} \]  
\[ \tau_{ij} = C_{\sigma} \sigma + C_t t + C_{\lambda} \lambda \]  
where the basis tensors were defined in Eqs.(49b). The coefficients C's are:

\[ C_{\sigma} = -\frac{5}{9} \epsilon^{1/3} k^{-4/3} \sigma_m \]  
\[ C_t = \frac{14}{27} S^{2/3} k^{-2} \sigma_m \]  
\[ C_{\lambda} = -0.65 N^2 k^{-2} (1 - 1.44\epsilon \theta^{-1} \alpha_T g \beta^{-1}) \]  
\[ \sigma_m = 1 - 0.62(1 - 0.72 \epsilon \theta^{-1} \alpha_T g \beta^{-1}) N^2 \epsilon^{2/3} k^{-4/3} \]  
Upon integration of the thermal spectra (51) and (52), we obtain the two components:

\[ J_g = 0.7 \beta \epsilon^{1/3} k^{-4/3} (1 - 1.44\epsilon \theta^{-1} \alpha_T g \beta^{-1}) \]  
\[ J_u = 0.156 \beta S k^{-2} (1 + 1.43(1 - 1.44\epsilon \theta^{-1} \alpha_T g \beta^{-1})) \]  
We recall that \( k_0 \) is obtained via the relations:

\[ K = \int_{k_0}^{\infty} E(k) \, dk = \frac{5}{2} \epsilon^{2/3} k_0^{-2/3} \]  
\[ k_0 = \left( \frac{5}{2} \right)^{3/2} \epsilon K^{3/2} \]  
Some modifications are however required. As stated earlier, the above results correspond to the stationary case and thus, they imply that production = dissipation, that is, \[ P = \epsilon, \quad P_{\theta} = \epsilon \theta \]  
where P and \( P_{\theta} \) represent the rate production of turbulent kinetic energy and of temperature variance (that is, potential energy). To introduce the \( P/\epsilon \) dependence into the
equations, we must somehow keep the left hand side of Eq.(36). In carrying out the perturbative approach, we obtain the following differential equation for the functions \( e, a, b \) to the first order in the smallness parameter \((45)\):

\[
\frac{\partial}{\partial t} f_1(k) = \epsilon^{1/3} k^{2/3} \left[ f_0^0(k) - f_1(k) \right]
\]

where \( f_0^0 \) is the first order term representing any of the functions \( e, a, b \) in the stationary limit. The same equation holds true for the first order of the spectrum \( C_\sigma^1(k) \). To integrate \((57)\) over \( k \) for the component \( C_\sigma^1(k) \), we assume that its form is the same as in the stationary limit, that is, \( k^{7/3} \). Then, from \((57)\) we obtain:

\[
\frac{4}{3} \epsilon^{1/3} k^{2/3} \frac{\partial}{\partial t} C_\sigma^1(k) = C_\sigma^0 - C_\sigma^1(k) \tag{58a}
\]

Following a practice widely used in treating the Reynolds stresses, we shall assume that

\[
\frac{\partial}{\partial t} C_\sigma^1(k) = C_\sigma^1 K^{-1} = C_\sigma^1(P - \epsilon) \tag{58b}
\]

Thus, Eq.(58a) gives finally

\[
C_\sigma^1(k) = F C_\sigma^0(k) \tag{58c}
\]

\[
F^{-1} = 1 + \frac{1}{3}(P/\epsilon - 1) \tag{58d}
\]

Proceeding in an analogous fashion, we obtain

\[
J = J_0^0 \frac{F}{F_1} \tag{59a}
\]

\[
F_1 = 1 + 0.084 \left[ P/\epsilon - 1 + \sigma^1 \right] \tag{59b}
\]

We can continue the analysis of the \( P/\epsilon \) dependence to the next order of perturbation, as it was described in detail in Canuto and Dubovikov (1999c). The results will be presented below but before we must discuss another correction to \( C_\sigma^1 \) which is due to the fact that an unmodified extension of the initial expression for the first–order function \( \tilde{e}_1(k) \) up to \( k_0 \) results in a violation of the positiveness of the energy density \( e(k) \) in the region \( k_0 < k < k_1 \) where

\[
k_1 = \frac{8}{15} x k_0^{2/3}, \quad x = \frac{K S}{\epsilon} \tag{60a}
\]

Such a violation can be prevented by multiplying the function \( \tilde{e}_2(k) \) by \((k/k_1)^{2/3}\) in the region \( k_0 < k < k_1 \). This leads to the following correction factor to \( C_\sigma^1(k) \)
\[ F_c = \frac{9}{5} - 3x^{-1}(1 - \frac{15}{16}x^{-1}) \quad \text{for } x > 15/8 \]
\[ F_c = 1, \quad \text{for } x < 15/8 \quad (60b) \]

Next, we substitute the variables \( k_0 \) from Eq.(55d) and adopt \( P_\theta = \epsilon_\theta \) to obtain

\[ \epsilon_\theta = \beta 1 \quad (60c) \]

Substituting (55a), we obtain a linear equation in \( \epsilon_\theta / \epsilon \) whose solution is

\[ \frac{\epsilon_\theta}{\epsilon} = \frac{11}{100} \beta N^{-2} x^2 R_i (1 + 0.16 x^2 R_i)^{-1} \quad (61) \]

The last modification we have to carry out accounts for corrections of higher orders in \((N^2)^n\) to the function \( \sigma_m \), Eq.(54d) where we took into account only the first correction and thus \( \sigma_m \) has the form

\[ \sigma_m = 1 - \xi \quad (62a) \]

Since \( \sigma_m > 0 \), a consistent way to extrapolate (62a) to the regions where \( \xi < 1 \) in the spirit of Pade' approximations, we write

\[ 1 - \xi \to (1 + \xi)^{-1} \quad (62b) \]

VI. Reynolds stresses, temperature and salinity fluxes

After performing the modifications indicated above, we obtain the following results:

Reynolds Stresses:

\[ \tau_{ij} = \bar{u}_i \bar{u}_j - \frac{2}{3} K \delta_{ij} \quad (63a) \]
\[ \tau_{ij} = C_\sigma \sigma + C_t t + C_\lambda \lambda \quad (63b) \]

where the basis tensors are defined in Eqs.(49b) and:

\[ C_\sigma = -4 \frac{K^2}{\epsilon} S S_m \quad (63c) \]
\[ C_t = 4 \frac{\gamma_3 K^2}{\epsilon} \tau S S_m \quad (63d) \]
\[ C_\lambda = -8 \frac{3 \frac{K^2}{\epsilon} \tau N^2 S_h}{\gamma_4} \quad (63e) \]
\[ \tau = 2K \epsilon^{-1} \quad (63f) \]

In particular:

\[ \bar{u}w = -K_m \frac{\partial U}{\partial z}, \quad \bar{v}w = -K_m \frac{\partial V}{\partial z} \quad (63g) \]
\[ K_m = 2 \frac{K^2}{\epsilon} S_m \]  

**Vertical Heat Flux \( J_\theta \):**

\[ J_\theta = \overline{w_\theta} = -K_h \frac{\partial T}{\partial z} \]  

\[ K_h = 2 \frac{K^2}{\epsilon} S_h \]  

**Vertical Salt Flux \( J_\sigma \):**

\[ J_\sigma = \overline{w_\sigma} = -K \frac{\partial S}{\partial z} \]  

\[ K_\sigma = 2 \frac{K^2}{\epsilon} S_\sigma \]  

**Dimensionless Structure Function for Momentum:**

\[ S_m = \frac{1}{25} \int c \sigma_m \]  

where

\[ F_c = \frac{5}{9} F_c = 1 - \frac{5}{3} x^1 + \frac{25}{16} x^2 \quad \text{for } x \geq 15/8 \]

\[ F_c = \frac{5}{9}, \quad \text{for } x \leq 15/8 \]  

where the dimensionless variable \( x \) is given by

\[ x = \frac{KS}{\epsilon} \]  

The other dimensionless functions are:

\[ \sigma_m^{-1} = 1 + (\eta_\theta + \eta_\sigma) F_c^{-1} \]  

\[ 10^2 \eta_\theta = 5 F_3 F_1 x^2 R_i \left[ 1 + \frac{1}{12} F_1 F_1^{-1}(5 + 7 F_2 F_2^{-1})(1 - 0.16 x^2 F_2 R_i) \psi_1 \right] \]  

\[ 10^2 \eta_\sigma = -5 F_3 F_1 x^2 R_i \left[ 1 + \frac{1}{12} F_1 F_1^{-1}(5 + 7 F_2 F_2^{-1})(1 + 0.16 x^2 F_2 R_i) \psi_1 \right] \]  

\[ \psi_1 = 1 + 0.16 F_1 x^2 (1 - R_i) R_i - 0.013 F_3 x^2 R_i R_i x^2 \phi \]  

\[ \phi^{-1} = 1 + 0.08 x^2 (1 - R_i) R_i \]  

\[ R_\rho = (g \alpha s \frac{\partial T}{\partial z}) (g \alpha x \frac{\partial}{\partial z})^{-1} \]  

Furthermore:

\[ S_m = \frac{1}{36} \tilde{S}_m, \quad \tilde{\sigma}_m^{-1} = 1 + \tilde{\eta}_\theta + \tilde{\eta}_\sigma \]  

\[ 10^2 \tilde{\eta}_\theta = 4 F_3 F_4 x^2 R_i \left[ 1 + \frac{1}{12} F_2 F_2^{-1}(5 + 7 F_3 F_3^{-1})(1 - 0.16 x^2 F_2 R_i) \psi_2 \right] \]  

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\( 10^{2 \gamma} = -4 \dot{\psi} \left[ 1 + \frac{1}{12} \dot{F}^2 \dot{F}^{-1} \left( 5 + 7 \dot{F} \dot{F}^{-1} (1 + 0.16 x^2) \right) \psi \right] \) (67c)

\( \psi^{-1} = 1 + 0.16 \dot{F}^2 x^2 (1 - R) R_i - 0.013 \dot{F}^2 x^3 R_i^2 \phi \) (67d)

The functions \( \gamma \)'s and \( F \) are given by:

\( \gamma_3 = 0.12 \dot{F} \dot{F}^{-1}, \quad \gamma_4 = 0.095 \dot{F} \) (68a)

\( F_n^{-1} = 1 + \frac{2}{5} n(1+n)^{-1} \left( \frac{P}{\xi} - 1 \right) \) (68b)

\( F_n^{-1} = 1 + \frac{1}{6} n(1+n)^{-1} \left( \frac{P}{\xi} - 1 \right) \) (68c)

**Dimensionless Structure Function for Temperature and Salinity:**

\( S_h = 0.056 \dot{F}^2 (\dot{F}^{-1} - 0.08 x^2 \phi R_i \rho)_1 \) (68d)

\( S_o = 0.056 \dot{F}^2 (\dot{F}^{-1} + 0.08 x^2 \phi R_i \rho)_1 \) (68e)

### VII. Local and Non-Local Models

The above expressions require the knowledge of both the turbulent kinetic energy \( K \) and of its rate of dissipation \( \epsilon \). In principle, \( K \) and \( \epsilon \) satisfy the non-local dynamic equations:

\[
\frac{\partial K}{\partial t} + \frac{\partial F(K)}{\partial x} = - \tau_{ij} S_{ij} + g \alpha J_{\theta} - g \alpha_s J_{\sigma} - \epsilon
\]

\[
\frac{\partial \epsilon}{\partial t} + \frac{\partial F(\epsilon)}{\partial x} = \left[ -c_3 \tau_{ij} S_{ij} + c_1 (g \alpha T J_{\theta} - g \alpha_s J_{\sigma}) \right] \epsilon K^{-1} - c_2 \epsilon^2 K^{-1}
\]

The values of the coefficients \( c_{1,2,3} \) have been discussed at length by Burchard and Baumert (1995). \( F(K) \) and \( F(\epsilon) \) denote the fluxes of \( K \) and \( \epsilon \) and they are third-order moments. The most complete expressions for these third-order moments (TOM) are those given by Canuto et al. (1994) which were successfully tested in the PBL (planetary boundary layer) against results from LES (large eddy simulation). Use of these TOM, together with the above model for the diffusivities, would represent a very complete treatment of ocean mixed layer turbulence. It is the \( K-\epsilon \) model widely used in engineering turbulent flows.

Here, we present numerical results for the local limits of (69a,b) which we can write as

\( - \tau_{ij} S_{ij} + g \alpha J_{\theta} = \epsilon \) (69c)

\( \epsilon = K^{3/2} A^{-1} \) (69d)
where $A$ is a mixing length. The first equation has the simple physical interpretation of production=dissipation, while (69d) is a statement of Kolmogorov law. Using the above expressions for $\tau_{ij}$ and $J_0, Eq.$(69c) becomes the equation for the dimensionless variable $x$ defined in Eq.(66c). We obtain:

$$x^2 = \frac{1}{4}(S_m - RiS_h)^{-1} \quad (69e)$$

where both $S_{m,h}$ depend on $x$. The Richardson number $Ri$ is defined in Eq.(44f). For a given $Ri$, $x$ is thus uniquely determined, $x=x(Ri)$ and so are the structure functions

$$S_{m,h} = S_{m,h}(Ri) \quad (69f)$$

If we use the other local relation (69d), we finally obtain

$$K_{m,h} = 2A^2Sx^{-1}S_{m,h} \quad (69g)$$

which yields the turbulent diffusivities in units of $A^2S$. Finally, we note in the production=dissipation local case, there is further simplification:

$$F_{1,2,3,4}^1 = F_{1,2,3} = 1 \quad (69h)$$

We must stress, however, that the use of a local model is not required by the turbulence model we have presented being made here only for reasons of simplicity. The length scale $A$ is determined using the Deardorff-Blackadar formula:

$$A = 2^{-3/2}B \ell, \quad B = 24.7, \quad \ell = \min(\frac{4\sqrt{K}}{N}, \ell) \quad (69i)$$

$$\ell = \kappa \ell_0 (\ell_0 + \kappa \ell)^{-1}, \quad \ell_0 = 0.17H \quad (69j)$$

where $\frac{4}{3}q^2 = K$ is the turbulent kinetic energy, $N$ is the Brunt-Vaisala frequency, $\kappa=0.4$ is the von Karman constant and $H$ is the mixed layer depth. When used within the NCAR CSM Ocean Model, $H$ is determined as the depth where the buoyancy difference

$$g[\rho(H) - \rho(surface)]\rho(H)^{-1} = 3 \times 10^{-4} \text{ms}^{-2} \quad (69k)$$

VIII. Ocean GCM

We tested the new vertical diffusivities in a global ocean general circulation model, O-GCM. We used the NCAR CSM Ocean Model produced by the University Corporation
for Research. National Center for Atmospheric Research, Climate and Global Dynamics Division. They developed their model by modifying the MOM 1.1 GFDL code as described in the NCAR CSM Ocean Model Technical Note, "The NCAR CSM Ocean Model", by the NCAR Oceanography section. We employed the stand-alone 3°x3° configuration of the model detailed in their technical note with the default parameter values. It has 3.6° spacing in longitude and a variable spacing in latitude increasing from 1.8° at the equator to 3.4° at 17° N, S and then decreasing back to 1.8° for 60° N, S and poleward. There are 25 levels of increasing thickness in the vertical, with the surface level 6 meters thick. The option for the GM mesoscale eddy parameterization was enabled. Bulk forcing with a seasonal cycle plus a 1/2 year timescale restoring condition on the salinity is used, except under sea-ice where there is strong restoring. This configuration corresponds to the case B–K described in Large et al. (1997). It should be noted, however, that for determination of the length scale in our turbulence model we used the program's definition for mixed layer depth (a buoyancy difference from the surface of 3 10^{-4} ms^{-2}), which is different from that graphed as a diagnostic in Fig.5 of Large et al.(1997). We initialized our runs with annually averaged Levitus data and ran for 126 momentum years. As in Large et al. (1997) a 3504sec timestep for momentum is used, while for the first 96 momentum years the tracers are accelerated by a factor increasing from 10 at the surface to 100 for the deep ocean. We then set all timesteps equal for the remaining 30 years as they did.

First, we ran the NCAR program as is, with the option for the KPP mixing enabled, producing the KPP data presented in the figures below. Then, in place of the KPP module, we inserted a module which uses our new model for the diffusivities for momentum and heat with the salt diffusivity set equal to that of heat. To save computing time, we constructed tables of the dimensionless functions $S_{m,h}$ and of the dimensionless variable $y$ (obtained from solving Eq.69e),

$$y = \frac{x^2 \ell^2}{K} \equiv \frac{x^2 (\ell / \lambda)^2}{(70a)}$$

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vs. Ri. Then, for each point in space and time these were interpolated to the local Ri. To construct the diffusivities $K_{m,h}^{\text{model}}$, we used the results (63h) and (64b) which can be rewritten in terms of (70a) as

$$K_{m,h}^{\text{model}}/\ell^2S = \frac{1}{4}B_1 y^{-\frac{1}{2}}S_{m,h}$$  \hspace{1cm} (70b)

**IX. Below the Mixed Layer**

Below the ocean mixed layer, the external wind-generated shear is too small to generate turbulent mixing and yet, even in regions where both the temperature and the salinity gradients are stably stratified, it is usual to assume "background diffusivities" for viscosity, heat and salt diffusivity which are believed to be caused by internal wave breaking (Large et al., 1997). It would be preferable not to do so but rather model the physical processes causing this background mixing. Our main assumption is that the turbulence model has given us the correct functional dependence of the $K_{m,h,s}^{\text{model}}$ on $R_i$ and $R_\rho$, and that such diffusivities can thus be used below the ML. Since all the arguments discussed below, are valid for any of the three $K$'s, we shall use only the generic symbol $K$ and write succinctly

$$K = K(R_i, R_\rho)$$  \hspace{1cm} (71a)

The key problem is how to define and thus compute $R_i$. Here, we shall make use of the measured data (Gargett et al., 1981) on the vertical shear generated by the wave breaking phenomenon. By integrating over all wavenumbers one can compute the shear due to internal waves, $S_{wb}$. One can then form a corresponding $R_{iwb}$ as follows:

$$R_{iwb} = N^2/S_{wb}^2$$  \hspace{1cm} (71b)

where

$$N^2 = -g\rho^{-1}\frac{\partial \rho}{\partial z}$$  \hspace{1cm} (71c)

Gargett et. al. (1981, sec. 5) confirmed earlier arguments by Munk (1966) that $R_{iwb} \approx 1$. To those argument, we would like to add the following consideration. As the value of $R_{icr}$, above which there is no longer turbulent mixing, computed from our model is $O(1)$, if $R_{iwb}$
were $\gg 1$, there would be no turbulence generated by the internal waves at all. On the other hand if $Ri_{wb}$ were $\ll 1$, there would be a very strong turbulence producing a viscosity sufficient to damp out the waves themselves. The wave-generated turbulence is thus self-limiting. Since the turbulence model gives a precise value for $Ri_{cr}$, while the above argument only tells us that $Ri_{wb} \sim O(1)$, we shall write:

$$Ri_{wb} = cRi_{cr}$$

(71d)

where $c$ is a constant reasonably close to unity. We have found that $c=0.88$ gives a diffusivity close that measured by Ledwell et al. (1993). Since in the local model, the $K$'s are also proportional to the length scale $\Lambda$ or $\ell$, see Eq.(69i,j). Below the mixed layer, we thus need an analogous $\ell_{wb}$. We shall use the same formal expressions (69i,j) but with different $\ell_0(wb)$ which we compute as follows. Assuming a Kolmogorov spectrum at wavenumbers upward of a breakpoint $k_0$, and integrating, we obtain:

$$\ell_0(wb) = (3K_0)^{3/2}(B_1 k_0)^{-1}$$

(71e)

where $K_0=1.6$ is the Kolmogorov constant. We identify $k_0$ with the best value of Gargett et al. (1981) for the break in slope of the observed spectrum of internal waves, namely

$$k_0 = \frac{1}{10} \frac{2\pi}{\text{radians/meter}}$$

(71f)

Thus, $\ell_0(wb)$ is known and so is $\ell_{wb}$. Similarly, $y_{wb}$ is obtained by solving the production=dissipation, Eq.(69c). Thus, the complete wave-breaking expressions for the three diffusivities are:

$$K_{m,h,s}(wb) = \frac{1}{2} B_1 \ell^2(wb) S_{wb} y_{wb}^{1/2} S_{m,h,s}(Ri_{wb}, R_\rho)$$

(72a)

We add together the diffusivities calculated using the shear resolved in the ocean model and the background diffusivities, ensuring continuity in the transition between regions where external excited shear dominates and those where the internal wave shear does. We thus take the total diffusivities to be:

$$K_{m,h,s} = K_{m,h,s}(Ri,R_\rho) + K_{m,h,s}(wb, R_\rho)$$

(72b)

In the statically unstable case ($Ri < 0$), we set $K_{m,h,s}(wb)=0$. The very large mixing due to convective instability makes the background irrelevant in this situation in any case.
X. **O–GCM results**

In Fig.1 we present the dimensionless structure functions $S_{m,h}$ vs. $R_i$. They are given by Eqs.(66a) and (68d). The graphs correspond to the case without salinity. In Fig.2 we present the ratio $K_m/K_h$ (again for the case without salinity) versus the LES data (dots) of Wang et al. (1996). In Figs.3–8 we present various diffusivities and their ratios for the full case with salinity and shear. The latter is represented by $R_i$, defined as the ratio of $N^2$, Eq.(50h) and the shear squared. The numbers on the curves represent the Turner number $R_p$, defined in Eq.(66i). The nomenclature SF, DC, DS and DU was introduced and discussed in the Introduction.

In Figs.9–20 we present the results of the O–GCM which comprise both global temperature and salinity profiles as well as for the case of different ocean basins. In Figs.21–29 we present the diffusivities in different ocean basins. Particularly interesting is the case of the Canary Islands since one can compare the model results with the value of $0.11\pm0.02\text{cm}^2\text{s}^{-1}$ measured by Ledwell et al. (1993). In each case, we compare our results with Levitus data as well as with the results we have obtained by running the same code with the KPP model which is available only for the case $K_s=K_h$.

Finally, in Fig.30 we present the polar heat transport.

XI. **Discussion**

It seems natural to require that the reliability of a model used to describe a given phenomenon should precede rather than follow the application of the model. Stated somewhat differently, a model should be judged on more than its performance in a given instance. Regrettably, the 1-point closure models used thus far to treat vertical mixing and following the Mellor–Yamada seminal work, have suffered and still do, from the presence of several adjustable parameters which have not allowed a clear appraisal of their performance before their use in the ocean context. In that respect, the new 2-point model presented here is quite different for it is being used only after it has exhibited a pedigree based on its
performance on a wide variety of turbulent flows. Its credibility, manageability and resiliency makes the best candidate for the computation of the vertical diffusivities.

The work is however not complete. In fact, it is known that double-diffusion phenomena change a smooth gradient into one that exhibit a step-like structure. As of today no theory that we know of is capable of encompassing such phenomenon. It is a challenge that we shall undertake in future work.
Figure caption

Fig.1 The momentum diffusivity $K_m$ vs. Ri for different values of the Turner number $R_{\rho}$. The label DC and SF are defined in the Introduction.

Fig.2 Same as in Fig.1 for the DU and DS cases.

Fig.3. Heat diffusivity vs. Ri for different $R_{\rho}$ for the DC and SF cases.

Fig.4 Same as in Fig.3 for the DU and DS cases.

Fig.5 Salt diffusivity $K_s$ vs. Ri for the SF and DC cases

Fig.6 Same as in Fig.5 for the DU and DS cases

Fig.7 The turbulent Prandtl number $K_m/K_h$ vs. Ri for different $R_{\rho}$. The heavy line corresponds to the case of laboratory (see paper I, Figs. 3–4). Cases DC and SF.

Fig.8. The ratio of $K_m/K_h$ vs. Ri for different $R_{\rho}$. Cases DU and DS.

Fig.9 The ratio of $K_m/K_s$ vs. Ri for different $R_{\rho}$. DC and SF cases.

Fig.10 The ratio of $K_m/K_s$ vs. Ri for different $R_{\rho}$, DU and DS cases.

Fig.11 The ratio of $K_h/K_s$ vs. Ri for different $R_{\rho}$, DC and SF cases.

Fig.12 The ratio of $K_h/K_s$ vs. Ri for different $R_{\rho}$, DU and DS cases.

Fig.13 The efficiency ratio $\Gamma$ (paper II, Eq.59f) vs. Ri for different $R_{\rho}$, DC and SF cases

Fig.14 The global ocean temperature using the O–GCM discussed in VIII–IX with the new diffusivities. The Levitus (1994) data are the solid line. We have also run the O–GCM code with the KPP model ($K_s=K_h$) and the results are indicated by diamonds. The results with our new 1-point closure model with $K_s=K_h$ (paper II) are shown by squares while the present model results with $K_s\neq K_h$ are indicated by asterisks.

Fig.15. Global salinity

Fig.16. Artic ocean temperature

Fig.17. Artic ocean salinity

Fig.18. Atlantic ocean temperature

Fig.19. Artic ocean salinity
Fig. 20. Pacific ocean temperature
Fig. 21. Pacific ocean salinity
Fig. 22. Indian ocean temperature
Fig. 23. Indian ocean salinity
Fig. 24. Southern ocean temperature
Fig. 25. Southern ocean salinity
Fig. 26. The four diffusivities $K_{m,h,s,p}$ (cm$^2$s$^{-1}$) for the Papa station
Fig. 27. The four diffusivities $K_{m,h,s,p}$ (cm$^2$s$^{-1}$) for the Arctic ocean
Fig. 28. The four diffusivities $K_{m,h,s,p}$ (cm$^2$s$^{-1}$) for the Canary Islands.
Fig. 29. Same as in Fig. 26 but for the first 1 km
Fig. 30. Same as in Fig. 27 for the first 1 km
Fig. 31. Same as in Fig. 28 for the first 1 km. Ledwell et al. (1993) value of 0.11$\pm$0.02 cm$^2$s$^{-1}$
(see, however, the discussion in the main text)
Fig. 32. Same as in Fig. 26 for the first 40 m
Fig. 28. Same as in Fig. 22 for the first 40 m
Fig. 29. Same as in Fig. 23 for the first 60 m.
Fig. 30. Polar heat transport vs. latitude for three different models.
References


of rotation", *Phys. Fluids*, 9, 2132–2140,


Fig. 1
Fig. 2

$K_m$

$R_p = -10$

DU

DS

$R_i$

$K_m$

$R_i$

$R_p$

-10

-5

-2

0.5

-1

0

-0.1
Fig. 3

Diagram with the following axes:
- \( K_h \) on the y-axis
- \( R_i \) on the x-axis

Key features:
- \( R_p = 10 \)
- Lines indicating different values
- Regions labeled DC and SF

Legend:
- Numbers indicating specific lines or regions:
  - 0
  - 0.1
  - 0.2
  - 0.5
  - 1
  - 2
  - 5
Fig. 6
Fig. 7
Fig. 8

\[ \frac{K_m}{K_h} \]

\[ R_p = -10 \]

DU

DS

-5

-2

-1

-0.5

-0.2

0
Fig. 9
Fig. 10
Fig. 11

\[ \frac{K_h}{K_s} \]

DC

SF

\( R_p = 10 \)

Fig. 11
Fig. 12
Fig. 13

The graph shows the variation of $\Gamma$ with $R_i$ for different values of $R_p$. The region is divided into DC and SF zones. The curves represent different values of $R_p$, with $R_p = 10$ being a notable value.
Fig. 14

Global Temperature

- Levitus
- KPP
- 1 pt closure $K_s = K_h$
- 1 pt closure $K_s \neq K_h$
- 2 pt closure $K_s \neq K_h$ (Present model)
Arctic Ocean Temperature

Fig. 16
Arctic Ocean Salinity

Fig. 17
Fig. 18

Atlantic Ocean Temperature

Temperature

Temperature
Atlantic Ocean Salinity

Fig. 19
Pacific Ocean Temperature

Fig. 20
Pacific Ocean Salinity

Fig. 21
Indian Ocean Temperature

Fig. 22
Fig. 24

Southern Ocean Temperature
Southern Ocean Salinity

Salinity (ppt)

Fig. 25
Fig. 27
Canary Islands

Fig. 28
Fig. 29
Fig. 30

Arctic
Canary Islands

Fig. 31
Average Northward Heat Transport

Fig. 35