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Accuracy Study of the Space-Time CE/SE Method for Computational Aeroacoustics Problems Involving Shock Waves
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ACCURACY STUDY OF THE SPACE-TIME CE/SE METHOD FOR COMPUTATIONAL AEROACOUSTICS PROBLEMS INVOLVING SHOCK WAVES

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Abstract

The space-time conservation element and solution element (CE/SE) method is used to study the sound-shock interaction problem. The order of accuracy of numerical schemes is investigated. The linear model problem governed by the 1-D scalar convection equation, sound-shock interaction problem governed by the 1-D Euler equations, and the 1-D shock-tube problem which involves moving shock waves and contact surfaces are solved to investigate the order of accuracy of numerical schemes. It is concluded that the accuracy of the CE/SE numerical scheme with designed 2nd-order accuracy becomes 1st order when a moving shock wave exists. However, the absolute error in the CE/SE solution downstream of the shock wave is on the same order as that obtained using a fourth-order accurate essentially non-oscillatory (ENO) scheme. No special techniques are used for either high-frequency low-amplitude waves or shock waves.

1. Introduction

In the computational aeroacoustic problem, the propagation of disturbances of small amplitude needs to be captured. Thus a stringent requirement is placed on a numerical algorithm. And further, when a shock wave and acoustic waves exist at the same time, it becomes more challenging for a numerical simulation. In the current field of CAA, the high-order accuracy finite-difference method is popularly used for the propagation of high-frequency low-amplitude waves because a traditional 2nd-order accurate scheme is not accurate enough for this kind of problem. When a shock wave exists, the numerical scheme also has to be able to accurately capture the shock wave without oscillations. Traditional high-order accurate shock-capturing methods are classified into linear and nonlinear methods. The ENO schemes belong to the latter class. The order of accuracy of an ENO scheme is investigated for a sound-shock interaction problem in [1]. A very interesting conclusion drawn in [1] is that the accuracy of the solution downstream of the shock wave drops to first order if the shock wave is not located at a mesh point. The designed accuracy downstream of the shock wave can be achieved by using subcell resolution in a numerical algorithm such that the shock wave is located exactly on a mesh point. However, such a strategy is not practical for general multidimensional problems, since the shock wave could be curved and moving in the entire domain. Therefore, the advantage of using high-order accurate methods in the study of unsteady flows with shocks is questionable.

The space-time CE/SE method is an innovative numerical method for solving conservation laws. It is different in both concept and methodology from the well-established traditional methods such as the finite difference, finite volume, finite element and spectral methods. It is designed from a physicist's perspective to overcome several key limitations of the traditional numerical methods.

Simplicity, generality and accuracy are pursued in the development of the CE/SE method. Its salient properties are summarized briefly as follows. First, the concepts of conservation element and solution element are introduced to enforce both local and global flux conservations in space and time instead of in space only. Second, all the dependent variables and their spatial derivatives are considered as individual unknowns to be solved for simultaneously at each grid point. Third, no approximation techniques other than Taylor's series expansion, no monotonicity constraints, and no characteristic-based techniques are used in the design of the scheme. A detailed description of this method and the accompanying analysis are set forth in [2–4].
A variety of numerical tests have been performed previously to illustrate the robustness of this method. For the CE/SE Euler solver, highly accurate numerical solutions have been obtained for various flow problems involving discontinuities, such as shock waves, contact surfaces and even their interactions [5,6]. Moreover, applications of the same Euler solver to computational aeroacoustics (CAA) problems reveal that the accuracy of the results is comparable to that of a 4th-order compact difference scheme even though the current solver is only 2nd-order accurate. Further, the non-reflecting boundary conditions can be implemented in a simple way without involving characteristic variables. The solver can be applied to subsonic, transonic, and supersonic flows in the same form without using the characteristic-based techniques. Results show that the present solver can handle both continuous and discontinuous flows very well [7-14].

In this paper, the test problems used in [1] are solved by using the CE/SE method. Problems governed by the linear convection equation and the Euler equations are studied. The order of accuracy of the CE/SE numerical schemes is investigated. Numerical solutions are compared with those obtained by using the ENO scheme.

2. Numerical Test Problems

2.1. A linear model problem.

Consider the scalar equation

\[ \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \]  

(1)

where the wave speed \( a \) is

\[ a = \begin{cases} 
2, & x \leq x_s, \\
1, & x > x_s 
\end{cases} \]  

(2)

where \( x_s = 0.5 \) given in [1]. The initial conditions are described as

\[ u(x, 0) = \begin{cases} 
1/2, & x \leq x_s, \\
1, & x > x_s 
\end{cases} \]  

(3)

The domain is \( 0 \leq x \leq 1 \). The inflow boundary condition is

\[ u(0, t) = \frac{1}{2}(1 - \varepsilon \sin \omega t) \]  

(4)

where \( \varepsilon = 0.001 \) and \( \omega = 8\pi \). The period of the acoustic wave \( T_A \) is defined as \( 2\pi/\omega \). The outflow boundary condition is imposed by using the analytical solution given in [1]. The 1-D CE/SE \( \alpha-c-\alpha \) scheme constructed for Eq. (1) is used here with \( \epsilon = 0.5, \alpha = 1 \) and Courant number 0.8. The details of the scheme are given in [2] and won’t be repeated here.

First, an even number cells with 64, 128, 256, 512, 1024 uniform mesh intervals is used. The computed perturbation \( u'(x, t) = u(x, t) - u(x, 0) \) at \( t = 10T_A \) and its \( \log_{10} \) error for different cell numbers are shown in Fig. 1. It can be seen that the scheme is 2nd-order accurate in the entire domain except at the shock location. In Fig. 2, the same results are shown for an odd number cells with 65, 129, 257, 513 uniform mesh intervals. It is still 2nd-order accurate in the entire domain since the space-time CE/SE method uses a staggered mesh. There is a grid point located at \( x_s = 0.5 \) no matter whether an even or an odd number cells is used. An ENO-4-3 scheme, which is fourth-order accurate in space and third-order accurate in time, is used in [1]. For this linear problem, the accuracy of the ENO-4-3 solutions remains 4th order for an even number cells because there is a mesh point at the shock location, while the accuracy drops to first-order downstream of the shock wave for an odd number cells because the shock wave is within a cell.

Second, \( x_s = 0.5 \) is replaced by \( x_s = 0.5 + ds \) with \( ds = 0.00035 \), which results in no mesh points coinciding with the shock location even on the finest mesh used. The solution is shown in Fig. 3. The accuracy downstream of the shock wave drops to first or zeroth order. It was found that the dominant error is from the phase error due to \( ds \) which can not be resolved. From this test result, it can be concluded that the designed 2nd-order accuracy can not be achieved if a shock is within a cell for the CE/SE scheme applied to linear problems. In order to obtain more relevant conclusion about the performance of the CE/SE method, the physical problem governed by the Euler equations is considered next.

2.2. Shock-Sound Interaction

The shock-sound interaction problem described in [1] is governed by the 1-D Euler equations, whose conservative form is expressed as

\[ \frac{\partial u_m}{\partial t} + \frac{\partial f_m}{\partial x} = 0, \quad m = 1, 2, 3 \]  

(5)

in which

\[ u_1 = \rho, \quad u_2 = \rho u, \quad u_3 = \rho(\gamma - 1) + \rho u^2/2 \]  

(6)

and

\[ f_1 = u_2 \]  

(7)

\[ f_2 = (\gamma - 1)u_3 + (3 - \gamma)(u_2)^2/(2u_1) \]  

(8)

\[ f_3 = \gamma u_2 u_3/u_1 - (1/2)(\gamma - 1)(u_2)^2/(u_1)^2 \]  

(9)
where $\rho, v, p$, and $\gamma$ are the mass density, velocity, static pressure, and constant specific heat ratio, respectively.

The spatial domain is $0 \leq x \leq 1$. The main flow is from left to right. A shock wave is initially located at $x = 0.5$. The initial conditions of main flow variables at the left side (upstream) and right side (downstream) of the shock wave are described as follows:

\[(\rho, v, p)_L = (1.0, 2.0, 1.0/1.4) \quad x < 0.5 \quad (10)\]

\[(\rho, v, p)_R = (2.6666, 0.75, 4.5/1.4) \quad x > 0.5 \quad (11)\]

The acoustic disturbance is introduced at $x = 0$. The flow variables at the inlet are defined as

\[p(0, t) = p_L(1 - \varepsilon \sin \omega t) \quad (12)\]

\[\rho(0, t) = \rho_L[p(0, t)/\rho_L]^{1/\gamma} \quad (13)\]

\[v(0, t) = v_L - \frac{2}{\gamma - 1} (c(0, t) - c_L) \quad (14)\]

where $\omega = 36\pi$, $\varepsilon = 0.001$, and $c = \sqrt{\gamma p/\rho}$ being the local sound speed, thus $c_L = \sqrt{\gamma p_L/\rho_L}$ is the sound speed upstream of the shock wave. At the outlet ($x = 1$), the non-reflecting boundary condition is imposed to let flow propagate out the computational domain.

In this problem, the shock wave moves around the initial location due to the interaction with the acoustic wave. Thus, the shock wave is not located at a mesh point. Therefore, the first-order accuracy was achieved downstream of the shock wave using the ENO-4-3 scheme, which is shown in Fig. 4. However, in this problem, the shock wave does not move outside the cell in which it is initially located. Thus, inclusion of subcell resolution in the ENO-4-3 scheme can be done within this cell to exactly resolve the shock location. The 4th order accuracy downstream of the shock wave was achieved, which is shown in Fig. 5. Subcell resolution can be achieved more easily for the linear model problem described in 2.1, since the shock wave does not move and remains at $x = 0.5$. However, if shock waves are moving across the cells as the solution evolves, the use of subcell resolution in a scheme would be impractical due to its cost and complexity.

This problem is solved here again using the 1D CE/SE Euler solver. The same initial conditions and the boundary condition at the inlet are used. At the outlet ($x = 1$), the non-reflecting boundary condition which is not based on the characteristic theory is imposed as

\[(u_m)_j^n = (u_m)_{j-1/2}^{n-1/2} \quad (u_m)_j^n = 0 \quad (15)\]

The boundary condition at the outlet is not mentioned for ENO scheme in [1]. Two parameters $\varepsilon$ and $\alpha$ for controlling numerical dissipations in the CE/SE Euler solver are set as 0.5 and 1, and the Courant number is 0.8 in the calculations. The computed acoustic wave solutions $p'(x, t) = p(x, t) - p(x, 0)$ and $p'(x, t) = p(x, t) - p(x, 0)$ at $t = 30T_\lambda$ are shown in Fig. 6 along with the exact solutions. The log$_{10}$ errors of both $p'(x, t)$ and $p'(x, t)$ are shown in Fig. 7. It can be concluded that the accuracy of the CE/SE solution is first order in the entire domain except at the shock location. The absolute error of the CE/SE solution downstream of the shock wave is on the same order as that of the ENO-4-3 solution shown in Fig. 4 using the same 512 uniform mesh intervals.

2.3. 1-D Shock-Tube Problem

Finally, the 1-D shock-tube problem named Sod's problem [2] is solved here in an accuracy study using the same solver mentioned in 2.2. In this problem, the shock wave and contact surface propagate through the entire spatial domain during the time interval of computation. The numerically computed solutions of pressure and density at $t = 1.00$ with $CFL = 0.44$ are shown in Fig. 8 along with the exact solution. The log$_{10}$ errors of pressure and density are shown in Fig. 9, respectively. It can be seen that the solution is first-order accurate in the entire domain except at the shock wave and contact surface locations. No solution obtained by an ENO scheme is available in [1].

3. Conclusions

Three test problems have been solved to investigate the accuracy of the CE/SE method for unsteady compressible flows with shock waves. Generally speaking, first-order accuracy can be obtained for the CE/SE method without using any special techniques for either high-frequency low-amplitude waves or shock waves. The absolute error in the CE/SE solution downstream of the shock wave is on the same order as that obtained by using fourth-order accurate ENO scheme for sound-shock interaction problem. It can be concluded that CE/SE method can produce accurate solutions for aero-acoustic problems involving shock waves in a simple way.

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References


Figure 1: The CE/SE solution at $t = 10T_\lambda$ (upper) and the pointwise error distribution using even numbered cells for a discontinuity located at $x = 0.5$.

Figure 2: The CE/SE solution at $t = 10T_\lambda$ (upper) and the pointwise error distribution using odd numbered cells for a discontinuity located at $x = 0.5$. 
Figure 3: The CE/SE solution at $t = 10T_\lambda$ (upper) and the pointwise error distribution using even numbered cells for a discontinuity located at $x = 0.5 + 0.00035$.

Figure 4: The ENO-4-3 pointwise error of pressure for the sound-shock interaction problem at $t = 30T_\lambda$.

Figure 5: The ENO-4-3-SR pointwise error of pressure for the sound-shock interaction problem at $t = 30T_\lambda$. 
Figure 6: The CE/SE solutions of the sound-shock interaction problem at $t = 30T_{\lambda}$.

Figure 7: The CE/SE pointwise errors for the sound-shock interaction problem.
Figure 8: The CE/SE solutions of the shock-tube problem (Sod's problem) at \( t = 0.24 \) with \( CFL = 0.44 \).

Figure 9: The CE/SE pointwise errors for the shock-tube problem with \( CFL = 0.44 \).
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**Supplementary Notes:**

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