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Solving ODE Initial Value Problems With Implicit Taylor Series Methods

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Introduction

By expanding the solution to the initial value problem

\[ y' = f(t, y) \quad y(t_0) = y_0 \]  \hspace{1cm} (1.1a,b)

in a Taylor series about \( t_0 \), one obtains a local solution which is valid within its radius of convergence \( R_0 \). If the series is evaluated at \( t_1 \), where \( t_1 < R_0 \), one obtains an approximation to \( y(t_1) \), and the solution may then be expanded in a new series about \( t_1 \). The solution may of course then be extended to the point \( t_2 \), and so forth, so that by a process of “analytic continuation” one obtains a piecewise polynomial solution to (1.1).

When the derivatives of \( y \) can be obtained cheaply, the method of Taylor series, or analytic continuation, offers several advantages over other methods. Foremost among these is that a Taylor series solution provides information that is not provided by other methods. This includes derivative information, local radius of convergence, and the location of poles in the complex plane. Another advantage is that both the stepsize and order can be easily changed, so that an optimal stepsize can always be chosen. Finally, Taylor series integration provides a piecewise continuous solution to the ODE, so that it is never necessary to interpolate at intermediate points.

Since analytic continuation is efficient only when high order derivatives can be obtained inexpensively, much effort has been made to develop recursive techniques that can generate the derivatives numerically. The first known attempt to incorporate recursion into the Taylor series method was by J.R. Airey in 1932 [1]. Miller was the first to employ a full recurrence scheme in which all series coefficients were calculated numerically, and the first to develop a systematic approach which was applicable to a class of differential equations [2,3]. Other early work includes that of Gibbons [4] and Moore [5], who developed programs that automatically generate the series coefficients for differential systems in canonical form. Other work concerning automated differentiation is that of Leavitt [6], Nikolaev [7], Barton et. al. [8,9], Norman [10], Kedem [11], Rall [12], Chang [13], Corliss and Chang [14], Irvine and Savageau [15], and Savageau and Voit [16].

Of the above, the most complete and systematic treatments of the Taylor series method are those of Barton, Willers, and Zahar [8,9], Chang [13,17,18] and Corliss and Chang [14].
and Irvine and Savageau [15]. Barton et al. were the first to develop and implement a
general purpose algorithm that automatically reduces a given differential system to canoni-
cal form and generates the series coefficients without any previous manipulations by the
user. Chang and Corliss developed a similar approach along with computer programs that
are fully portable. Their programs also provide information on the location and order of
primary singularities. Irvine and Savageau developed highly efficient series methods for
differential equations that can be expressed in the form of an “S-system”. Their approach
obviates the need to generate problem-dependent source code for each new problem.

Corliss and Chang and Irvine and Savageau compared their codes with several stan-
dard ODE solvers on the problem set of Hull et al. [19]. Compared to the IMSL routines
DVERK (Runge-Kutta method) and DGEAR (variable order Adams method), Corliss
and Chang report that their code was significantly faster for error tolerances in the range
10^{-9} to 10^{-12}. They conclude that the Taylor series method is superior to the variable
order Adams method and the Runge-Kutta method when stringent accuracies are required
and when solving small systems. Irvine and Savageau compared their method with two
standard Runge-Kutta schemes, a variable order Adams method, and Gear's method. On
an S-system from cellular biology, they report solution times that were significantly faster
than all four methods for error tolerances ranging from 10^{-1} to 10^{-15}. In some cases,
their code was faster by one or two orders of magnitude. On Hull’s problem set, their
results compared favorably both in terms of solution times and number of function eval-
uations. For small error tolerances, their code was up to 20 times faster. One concludes
that the Taylor series method is indeed competitive with other methods, especially when
high accuracies are required.

In this paper we propose an extension of the Taylor series method which improves its
accuracy and stability while also increasing its range of applicability. The basic idea is
to formulate the method as a generalized implicit method. Specifically, given an interval
[t_n, t_{n+1}], we consider a series solution to (1.1) which is expanded about the intermediate
point t_n + \mu h, where h is the stepsize and \mu is an arbitrary parameter called an expansion
coefficient.

The main results of the paper are the following. (i) We derive the discretization error
for a series solution to (1.1) which is expanded about the point t_n + \mu h. It is shown that
a series of degree k has exactly k expansion coefficients which raise its order of accuracy.
The increase is one order if k is odd, and two orders if k is even. For k odd, it will be seen
that a class of Turan type, one-stage implicit Runge-Kutta methods are recovered [20]. (ii)
We consider stability for the problem y' = \lambda y, \Re (\lambda) < 0, and identify several \lambda-stable
schemes. (iii) We show that, for all series of degree three or higher, local extrapolation
can be used to increase accuracy by two additional orders. (iv) We discuss variable step
schemes. (v) We present numerical results for a range of problems, including ODE's with
a singular point and stiff systems. Our results indicate that implicit Taylor series methods
are an effective integration tool for most problems.
Taylor Series of Degree One

We assume that \( f(t, y) \) is a real-valued function which is analytic at the point \((t_n, y_n)\), and that the problem

\[
y' = f(t, y), \quad y(t_n) = y_n, \tag{2.1a,b}
\]

has a unique, continuous solution which can be expressed as a power series that converges for all \( t \) in the interval \(|t - t_n| < R_n\). We denote the stepsize \( t_{n+1} - t_n \) by \( h \), where \( h < R_n \).

A Taylor series of first degree expanded about \( t_n \) gives the equation

\[
y(t_{n+1}) = y(t_n) + h f(t_n, y_n) + \frac{h^2}{2} f'(\xi_n, y(\xi_n)), \quad t_n < \xi_n < t_{n+1}, \tag{2.2}
\]

from which one obtains Euler's method

\[
y_{n+1} = y_n + h f(t_n, y_n) \tag{2.3}
\]

and its associated discretization error \( \frac{h^2}{2} f'(\xi_n, y(\xi_n)) := y''(\xi_n) \). The error may also be expressed by the expansion

\[
c_{n+1} := y(t_{n+1}) - y_{n+1} = \frac{h^2}{2} f_n' + \frac{h^3}{6} f''_n + \frac{h^4}{24} f'''_n + \cdots \tag{2.4}
\]

where \( f'_n := f'(t_n, y_n) \), and similarly for \( f''_n \) and \( f'''_n \).

If the series is expanded about \( t_{n+1} \), one obtains the backward Euler method

\[
y_{n+1} = y_n + h f(t_{n+1}, y_{n+1}), \tag{2.5}
\]

where \( y_{n+1} \) must be obtained implicitly. By expanding \( f(t_{n+1}, y_{n+1}) \) in a Taylor series about \((t_n, y_n)\), one derives from (2.5) the discretization error

\[
c_{n+1} := y(t_{n+1}) - y_{n+1} = -\left[ \frac{h^2}{2} f_n' + \frac{h^3}{6} (2 f''_n + 3 \frac{\partial f_n}{\partial y} f'_n) \\
+ \frac{h^4}{24} (3 f'''_n + 8 \frac{\partial f_n}{\partial y} f''_n + 12 \frac{\partial^2 f_n}{\partial y^2} f'_n) \right] + O(h^5). \tag{2.6}
\]

Both the forward and backward Euler methods provide a linear approximation to \( y \) on \( I_n := [t_n, t_{n+1}] \). This approximation can be written

\[
y_n = y_{n+\mu} + (t - t_{n+\mu}) f(t_{n+\mu}, y_{n+\mu}), \quad t_n \leq t \leq t_{n+1}, \tag{2.7}
\]

where \( \mu \) is zero or one, respectively.
If we define \( t_{n+\mu} := t_n + \mu h \), where \( 0 \leq \mu \leq 1 \), Eq. (2.7) becomes a generalized approximation to \( y \) on \( I_n \). In this case the approximation will pass through the intermediate value \( y_{n+\mu} \), which is unknown. By observing that \( y_n \) must equal \( y_n \) when \( t = t_n \), one can see that \( y_{n+\mu} \) must be determined from

\[
y_{n+\mu} = y_n + \mu h f(t_{n+\mu}, y_{n+\mu}).
\]

(2.8a)

Note that this is just the backward Euler method for the partial step from \( t_n \) to \( t_{n+\mu} \). Once \( y_{n+\mu} \) is determined from (2.8a), one sets \( t = t_{n+1} \), \( y_n = y_{n+1} \) in (2.7) to obtain

\[
y_{n+1} = y_{n+\mu} + (1 - \mu) h f(t_{n+\mu}, y_{n+\mu}).
\]

(2.8b)

One observes that this is just the forward Euler method for the partial step from \( t_{n+\mu} \) to \( t_{n+1} \). Thus, two steps are required to advance the solution to \( t_{n+1} \) - a backward Euler intermediate step followed by a forward Euler final step.

To determine an optimal expansion point \( t_{n+\mu} \), one obtains the truncation error for (2.8). Since (2.8a) is the backward Euler method with stepsize \( \mu h \), one uses (2.6) to obtain an expansion for \( y_{n+\mu} \), which may then be substituted into (2.8b). One gets from (2.6),

\[
y_{n+\mu} = y_n + \mu h f_n + (\mu h)^2 f'_n + \left( \frac{\mu h^3}{2} \right) \left( f''_n + \frac{\partial f_n}{\partial y} f'_n \right)
\]

\[
+ \frac{\mu h^4}{6} \left( f'''_n + 2 \frac{\partial f_n}{\partial y} f''_n + 3 \frac{\partial f'_n}{\partial y} f'_n \right) + O(h^5).
\]

(2.9)

Upon expanding \( f(t_{n+\mu}, y_{n+\mu}) \) about \((t_n, y_n)\), and evaluating powers of \((y_{n+\mu} - y_n)\) by way of (2.9), one obtains from (2.8b)

\[
y_{n+1} = y_n + h f_n + \mu h^2 f'_n + \frac{\mu^2 h^3}{2} \left( f''_n + \frac{\partial f_n}{\partial y} f'_n \right)
\]

\[
+ \mu^3 h^4 \left( f'''_n + 2 \frac{\partial f_n}{\partial y} f''_n + 3 \frac{\partial f'_n}{\partial y} f'_n \right) + O(h^5).
\]

(2.10)

The error expansion for (2.8) is then

\[
e_{n+1} := y(t_{n+1}) - y_{n+1} = (1 - 2\mu) \frac{h^2}{2} f'_n + \frac{h^3}{6} \left[ (1 - 3\mu^2) f''_n - 3\mu^2 \frac{\partial f_n}{\partial y} f'_n \right]
\]

\[
+ \frac{h^4}{24} \left[ (1 - 4\mu^3) f'''_n - \mu^3 \left( 8 \frac{\partial f_n}{\partial y} f''_n + 12 \frac{\partial f'_n}{\partial y} f'_n \right) \right] + O(h^5).
\]

(2.11)
Taking $\mu = \frac{1}{2}$, the marching scheme becomes

\[ y_{n+\frac{1}{2}} = y_n + \frac{h}{2} f(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) \]  
(2.12a)

\[ y_{n+1} = y_{n+\frac{1}{2}} + \frac{h}{2} f(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) \]  
(2.12b)

with discretization error

\[ e_{n+1} = \frac{h^3}{24} \left( f''_n - 3 \frac{\partial f_n}{\partial y} f'_n \right) \]

\[ + \frac{h^4}{48} \left( f'''_n - 2 \frac{\partial f_n}{\partial y} f''_n - 3 \frac{\partial^2 f_n}{\partial y^2} f'_n \right) + O(h^5). \]  
(2.13)

The choice $\mu = \frac{1}{2}$ thus leads to a second-order scheme. In view of the central expansion point $t_{n+\frac{1}{2}}$, we will refer to (2.12) as the “centered Euler method”.

It should be noted that, corresponding to (2.12), the linear expansion (2.7) approximates $y(t)$ with an error at most equal to $\frac{h^2}{8} |f'_n| + O(h^3)$. This is one-fourth the error bound on (2.7) corresponding to the forward and backward Euler methods.

We should also mention that the explicit halfstep (2.12b) can be equivalently expressed as

\[ y_{n+1} = y_n + hf(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}). \]  
(2.14)

When written in this form, it is clear that the centered Euler method is similar to the trapezoidal rule

\[ y_{n+1} = y_n + \frac{h}{2} [f(t_n, y_n) + f(t_{n+1}, y_{n+1})]. \]  
(2.15)

It is also similar to the one-stage Gauss implicit scheme [21], [22],

\[ y_{n+1} = y_n + k_1 \]  
(2.16a)

\[ k_1 = hf(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}), \]  
(2.16b)

which is equivalent to the implicit midpoint rule [22],

\[ y_{n+1} = y_n + hf(t_{n+\frac{1}{2}}, \frac{1}{2}(y_n + y_{n+1})). \]  
(2.17)

It can be shown in fact that the centered Euler method and the implicit midpoint rule are equivalent schemes. One can also show that the trapezoidal linear approximation to $y$ on $I_n$,

\[ y_T(t) = y_n + (t - t_n) \frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2}, \]  
(2.18)

approximates $y$ with a maximum error equal to that of the centered Euler method, $\frac{h^2}{8} |f'_n| + O(h^3)$. One concludes that these three schemes offer nearly identical benefits over the forward and backward Euler methods.
Taylor Series of Degree Two

Analogous to Eq. (2.7), a Taylor series of second degree expanded about the point \( t_{n+\mu} \) approximates \( y \) on \( I_n \) by

\[
y_n = y_{n+\mu} + (t - t_{n+\mu}) f(t_{n+\mu}, y_{n+\mu}) + \frac{(t - t_{n+\mu})^2}{2} f'(t_{n+\mu}, y_{n+\mu}).
\] (3.1)

As in the previous section, the expansion coefficient \( \mu \) defines a family of marching schemes. Choosing \( \mu = 0 \) and setting \( t = t_{n+1}, \ y_n = y_{n+1} \), one obtains the Taylor series method of order two

\[
y_{n+1} = y_n + h f(t_n, y_n) + \frac{h^2}{2} f'(t_n, y_n)
\] (3.2)

with local error

\[
e_{n+1} = \frac{h^3}{6} f''' + \frac{h^4}{24} f^{(iv)} + \frac{h^5}{5!} f^{(v)} + \cdots (3.3)
\]

Similarly, choosing \( \mu = 1 \) and setting \( t = t_n, \ y_n = y_{n+1} \), one obtains the implicit method

\[
y_{n+1} = y_n + h f(t_{n+1}, y_{n+1}) - \frac{h^2}{2} f'(t_{n+1}, y_{n+1})
\] (3.4)

By expanding both \( f(t_{n+1}, y_{n+1}) \) and \( f'(t_{n+1}, y_{n+1}) \) in a Taylor series about \((t_n, y_n)\), one derives from (3.4) the error expansion

\[
e_{n+1} = \frac{h^3}{6} f''' + \frac{h^4}{24} f^{(iv)} + \frac{h^5}{5!} f^{(v)} + \cdots (3.5)
\]

When \( 0 < \mu < 1 \), (3.1) leads to a two-halfstep, implicit/explicit marching scheme, analogous to (2.8). Setting \( t = t_n, \ y_n = y_{n+1} \), one obtains the implicit halfstep

\[
y_{n+\mu} = y_n + \mu h f(t_{n+\mu}, y_{n+\mu}) - \frac{(\mu h)^2}{2} f'(t_{n+\mu}, y_{n+\mu}).
\] (3.6a)

Setting \( t = t_{n+1}, \ y_n = y_{n+1} \), one obtains the explicit halfstep

\[
y_{n+1} = y_{n+\mu} + (1 - \mu) h f(t_{n+\mu}, y_{n+\mu}) + \frac{(1 - \mu) h^2}{2} f'(t_{n+\mu}, y_{n+\mu}).
\] (3.6b)

An error expansion for (3.6) can be derived by proceeding as in the previous section. Utilizing the fact that (3.6a) is just (3.4) for the partial step \( \mu h \), one obtains an expansion for \( y_{n+\mu} \) by way of Eq. (3.5). One gets

\[
y_{n+\mu} = y_n + \mu h f_n + \frac{(\mu h)^2}{2} f'_n - \frac{(\mu h)^4}{12} (f''' + 2 \frac{\partial f_n}{\partial y} f''_n)
\]
An expansion for $y_{n+1}$ is then obtained from (3.6b) by expanding $f(t_{n+\mu}, y_{n+\mu})$ and $f'(t_{n+\mu}, y_{n+\mu})$ about $(t_n, y_n)$, and evaluating powers of $(y_{n+\mu} - y_n)$ by way of (3.7). Retaining terms through $O(h^5)$, one finds that

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2} f'_n + 3 \mu (1 - \mu) \frac{h^3}{6} f''_n \quad + \quad \mu^2 \frac{h^4}{24} \left[ (6 - 8\mu) f'''_n - 4 \mu \frac{\partial f_n}{\partial y} f''_n \right] \quad + \quad \mu^3 \frac{h^5}{5!} \left[ (10 - 15\mu) f''''_n - 15 \mu \frac{\partial f_n}{\partial y} f'''_n - 10 \frac{\partial f'_n}{\partial y} f''_n \right] + O(h^6).$$

Subtracting the left hand side of (3.8) from $y(t_{n+1})$ and the right hand side from the Taylor series expansion of $y(t_{n+1})$, one obtains

$$e_{n+1} = \left( 1 - 3\mu + 3\mu^2 \right) \frac{h^3}{6} f''_n + \frac{h^4}{24} \left[ (1 - 6\mu^2 + 8\mu^3) f'''_n + 4 \mu^3 \frac{\partial f_n}{\partial y} f''_n \right] \quad + \quad \frac{h^5}{5!} \left[ (1 - 10\mu^3 + 15\mu^4) f''''_n + 15 \mu^4 \frac{\partial f_n}{\partial y} f'''_n + 10 \mu^3 \frac{\partial f'_n}{\partial y} f''_n \right] + O(h^6).$$

Setting the leading coefficient to zero,

$$1 - 3\mu + 3\mu^2 = 0,$$

one finds

$$\mu = \frac{1}{2} \pm i \frac{\sqrt{3}}{6},$$

so that the expansion point $t_{n+\mu} := t_n + \mu h$ must be moved into the complex plane. Upon substituting $\mu = \frac{1}{2} \pm i \frac{\sqrt{3}}{6}$ into (3.9), one obtains the complex truncation error

$$e_{n+1} = \frac{h^5}{720} \left( f''''_n - 5 \frac{\partial f_n}{\partial y} f'''_n \right) \quad \pm \quad i \left[ \frac{\sqrt{3}}{216} h^4 \left( f''''_n - 4 \frac{\partial f'_n}{\partial y} f'''_n \right) + \frac{\sqrt{3}}{432} h^5 \left( f''''_n - 3 \frac{\partial f_n}{\partial y} f'''_n - 4 \frac{\partial f'_n}{\partial y} f''_n \right) \right].$$

The above expression is valid so long as $f$ is analytic in a sufficiently large neighborhood of $(t_n, y_n)$ in the complex plane.

Equation (3.12) shows that for a real-valued function $f$, the real truncation error is $O(h^5)$. This is two orders higher than for any other value of $\mu$. We should note further that with $\mu$ defined as above, (3.6) becomes
\[y_{n+\mu} = \text{Re}(y_n) + \mu h f(t_{n+\mu}, y_{n+\mu}) - \frac{(\mu h)^2}{2} f'(t_{n+\mu}, y_{n+\mu})\]  
(3.13a)

\[y_{n+1} = y_{n+\mu} + \tilde{\mu} h f(t_{n+\mu}, y_{n+\mu}) + \frac{(\tilde{\mu} h)^2}{2} f'(t_{n+\mu}, y_{n+\mu}),\]  
(3.13b)

where \(\tilde{\mu}\) is the complex conjugate of \(\mu\). All operations in (3.13) must be performed in complex arithmetic, while the solution at each time level is the real part of \(y_n\).

If \(\mu\) is real, one can verify that \(\mu = \frac{1}{2}\) minimizes the truncation error \(e_{n+1}\) at a value of \(\frac{h^3}{24} f'' + O(h^4)\), in which case (3.6) becomes

\[y_{n+\frac{1}{2}} = y_n + \frac{h}{2} f(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) - \frac{h^2}{8} f'(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}})\]  
(3.14a)

\[y_{n+1} = y_{n+\frac{1}{2}} + \frac{h}{2} f(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) + \frac{h^2}{8} f'(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}).\]  
(3.14b)

It can also be shown that, corresponding to (3.14), the quadratic expansion (3.1) approximates \(y\) on \(I_n\) with a maximum error of \(\frac{h^3}{24} |f''| + O(h^4)\). The corresponding error bound for both (3.2) and (3.4) is \(\frac{h^3}{6} |f''| + O(h^4)\). As in the linear case, a centered expansion has a maximum error which is one-fourth that of an expansion around either endpoint. It should also be observed that, while (3.14) is only a second-order scheme, it provides a polynomial approximation to \(y\) on \(I_n\) whose accuracy is one order higher than that of (2.12).

**Taylor Series of Arbitrary Degree**

Let \(f^{(k)}(t, y)\) denote the \(k\)th total derivative of \(f\) with respect to \(t\), where \(f^{(0)}(t, y) := f(t, y)\). A Taylor series approximation of degree \(k\) to the exact solution \(y\) on \(I_n := [t_n, t_{n+1}]\) is

\[y_n = y_{n+\mu} + (t - t_{n+\mu}) f(t_{n+\mu}, y_{n+\mu}) + \frac{(t - t_{n+\mu})^2}{2!} f'(t_{n+\mu}, y_{n+\mu}) + \frac{(t - t_{n+\mu})^3}{3!} f''(t_{n+\mu}, y_{n+\mu}) + \cdots + \frac{(t - t_{n+\mu})^k}{k!} f^{(k-1)}(t_{n+\mu}, y_{n+\mu}),\]  
(4.1)

where \(t_{n+\mu} := t_n + \mu h\) and \(0 \leq \text{Re}(\mu) \leq 1\). As shown in the two previous sections, the expansion coefficient \(\mu\) defines a family of marching schemes.

Corresponding to \(\mu = 0\), one obtains the explicit Taylor series method of order \(k\),

\[y_{n+1} = y_n + h f(t_n, y_n) + \frac{h^2}{2!} f'(t_n, y_n) + \cdots + \frac{h^k}{k!} f^{(k-1)}(t_n, y_n).\]  
(4.2)
Corresponding to \( \mu = 1 \), one obtains the implicit series method

\[
y_{n+1} = y_n + hf(t_{n+1}, y_{n+1}) - \frac{h^2}{2!} f'(t_{n+1}, y_{n+1}) \\
+ \frac{h^3}{3!} f''(t_{n+1}, y_{n+1}) + \cdots + (-1)^{k-1} \frac{h^k}{k!} f^{(k-1)}(t_{n+1}, y_{n+1}).
\]

(4.3)

When \( 0 < \text{Re}(\mu) < 1 \), one obtains a two-halfstep implicit/explicit marching scheme analogous to (2.8) and (3.6), where the implicit and explicit halfsteps are,

\[
y_{n+\mu} = y_n + \mu hf(t_{n+\mu}, y_{n+\mu}) - \frac{(\mu h)^2}{2!} f'(t_{n+\mu}, y_{n+\mu}) \\
+ \frac{(\mu h)^3}{3!} f''(t_{n+\mu}, y_{n+\mu}) + \cdots + (-1)^{k-1} \frac{(\mu h)^k}{k!} f^{(k-1)}(t_{n+\mu}, y_{n+\mu})
\]

(4.4a)

\[
y_{n+1} = y_{n+\mu} + (1 - \mu) hf(t_{n+\mu}, y_{n+\mu}) + \frac{[(1 - \mu) h]^2}{2!} f'(t_{n+\mu}, y_{n+\mu}) \\
+ \frac{[(1 - \mu) h]^3}{3!} f''(t_{n+\mu}, y_{n+\mu}) + \cdots + \frac{[(1 - \mu) h]^k}{k!} f^{(k-1)}(t_{n+\mu}, y_{n+\mu}),
\]

(4.4b)

respectively.

The local error of (4.2) is

\[
e^{(k)}_{n+1} = \frac{h^k}{(k+1)!} f_n^{(k)} + \frac{h^{k+2}}{(k+2)!} f_n^{(k+1)} + \frac{h^{k+3}}{(k+3)!} f_n^{(k+2)} + \cdots
\]

(4.5)

The truncation error for (4.3) and (4.4) can also be derived. Corresponding to (4.3), one obtains

\[
e^{(k)}_{n+1} = (-1)^k \left\{ \frac{h^{k+1}}{(k+1)!} f_n^{(k)} + \frac{h^{k+2}}{(k+2)!} f_n^{(k+1)} + \frac{h^{k+3}}{(k+3)!} f_n^{(k+2)} + \cdots \right\} + O(h^{k+4}),
\]

(4.6)

for all \( k \geq 2 \). A proof is given in the Appendix. (The error for \( k = 1 \) is given by (2.6), and only differs from (4.6) in the last term.)
The discretization error for (4.4) can be derived in a two-step process. Recognizing that (4.4a) is equivalent to (4.3) with a stepsize of $\mu h$, one uses (4.6), with $h$ replaced by $\mu h$, to obtain an expansion for $y_{n+\mu}$. For $k$ odd, one gets

$$y_{n+\mu}^{(k)} = y_n + \mu h f_n + \frac{(\mu h)^2}{2!} f'_n + \cdots + \frac{(\mu h)^k}{k!} f^{(k-1)}$$

(4.7)

$$+ 2 \frac{(\mu h)^{k+1}}{(k+1)!} f^{(k)} + \frac{(\mu h)^{k+2}}{(k+2)!} \left[ (k+2) f^{(k+1)} + (k+2) \frac{\partial f_n}{\partial y} f^{(k)} \right]$$

$$+ \frac{(\mu h)^{k+3}}{(k+3)!} \left[ \frac{k^2 + 3 k + 4}{2} f^{(k+2)} + (k+1) (k+3) \frac{\partial f_n}{\partial y} f^{(k+1)} + \frac{(k+2) (k+3)}{2} \frac{\partial f'_n}{\partial y} f^{(k)} \right]$$

$$+ O(h^{k+4}) ,$$

and for $k$ even, one obtains

$$y_{n+\mu}^{(k)} = y_n + \mu h f_n + \frac{(\mu h)^2}{2!} f'_n + \cdots + \frac{(\mu h)^k}{k!} f^{(k-1)}$$

(4.8)

$$- \frac{(\mu h)^{k+2}}{(k+2)!} \left[ k f^{(k+1)} + (k+2) \frac{\partial f_n}{\partial y} f^{(k)} \right]$$

$$- \frac{(\mu h)^{k+3}}{(k+3)!} \left[ \frac{k (k+3)}{2} f^{(k+2)} + (k+1) (k+3) \frac{\partial f_n}{\partial y} f^{(k+1)} + \frac{(k+2) (k+3)}{2} \frac{\partial f'_n}{\partial y} f^{(k)} \right]$$

$$+ O(h^{k+4}) .$$

One then expands $f(t_{n+\mu}, y_{n+\mu})$ and its total derivatives in (4.4b) in a series about $(t_n, y_n)$. Powers of $(y_{n+\mu} - y_n)$ that appear are evaluated by way of (4.7) for $k$ odd and by (4.8) for $k$ even. It can be shown that $y_{n+1}^{(k)}$ agrees with the Taylor series expansion for $y(t_{n+1})$ through terms of order $k$. The $O(h^{k+1})$ term is a function of $\mu$, and for $k$ is odd, is given by

$$\frac{h^{k+1}}{(k+1)!} f^{(k)}_n \left\{ 2 \mu^{k+1} + \left[ (k+1) \mu^k (1 - \mu) \right. \right. \right.$$

(4.9)

$$+ \frac{(k+1) k}{2!} \mu^{k-1} (1 - \mu)^2 + \frac{(k+1) k (k-1)}{3!} \mu^{k-2} (1 - \mu)^3 + \cdots + (k+1) \mu (1 - \mu)^k \right\}$$
and for \( k \) even, is given by

\[
\frac{h^{k+1}}{(k+1)!} f^{(k)}_n \left\{ \left[ (k+1) \mu^k (1 - \mu) \right] \right.
+ \frac{(k+1) k}{2!} \mu^{k-1} (1 - \mu)^2 + \frac{(k+1) k (k-1)}{3!} \mu^{k-2} (1 - \mu)^3 + \cdots + (k+1) \mu (1 - \mu)^k \left\} \right.
\]

The sum in brackets in (4.9) and (4.10) can be simplified by making use of the following identity, which follows from the binomial theorem.

\[
\mu^k + k \mu^{k-1} (1 - \mu) + \frac{k (k-1)}{2!} \mu^{k-2} (1 - \mu)^2
+ \frac{k (k-1) (k-2)}{3!} \mu^{k-3} (1 - \mu)^3 + \cdots + k \mu (1 - \mu)^{k-1} + (1 - \mu)^k \equiv 1.
\]

Equation (4.11) holds for all \( k \geq 0 \).

Now (4.9) can be written

\[
\frac{h^{k+1}}{(k+1)!} f^{(k)}_n \left\{ \mu^{k+1} + \left[ \mu^{k+1} + (k+1) \mu^k (1 - \mu) \right] \right.
+ \frac{(k+1) k}{2!} \mu^{k-1} (1 - \mu)^2 + \frac{(k+1) k (k-1)}{3!} \mu^{k-2} (1 - \mu)^3 + \cdots + (k+1) \mu (1 - \mu)^k
+ (1 - \mu)^{k+1} \left\} - (1 - \mu)^{k+1} \right\}
\]

It follows from (4.11) that (4.12) equals

\[
\frac{h^{k+1}}{(k+1)!} f^{(k)}_n \left[ \mu^{k+1} - (1 - \mu)^{k+1} + 1 \right]
\]

Similarly, one rewrites (4.10) as

\[
\frac{h^{k+1}}{(k+1)!} f^{(k)}_n \left\{ - \mu^{k+1} + \left[ \mu^{k+1} + (k+1) \mu^k (1 - \mu) \right] \right.
+ \frac{(k+1) k}{2!} \mu^{k-1} (1 - \mu)^2 + \frac{(k+1) k (k-1)}{3!} \mu^{k-2} (1 - \mu)^3 + \cdots + (k+1) \mu (1 - \mu)^k
+ (1 - \mu)^{k+1} \left\} - (1 - \mu)^{k+1} \right\}
\]
which equals
\[
\frac{h^{k+1}}{(k+1)!} f_n^{(k)} \left[ 1 - \mu^{k+1} - (1 - \mu)^{k+1} \right].
\] (4.15)

The expansion for \( y_{n+1}^{(k)} \) is then, for \( k \) odd or even,
\[
y_{n+1}^{(k)} = y_n + h f_n + \frac{h^2}{2!} f_n' + \cdots + \frac{h^k}{k!} f_n^{(k-1)}
\]
\[+ \frac{h^{k+1}}{(k+1)!} f_n^{(k)} \left[ 1 + (-1)^{k+1} \mu^{k+1} - (1 - \mu)^{k+1} \right] + O(h^{k+2}). \] (4.16)

It follows that the discretization error of the marching scheme (4.4) is
\[
e_{n+1}^{(k)} = \left[ (1 - \mu)^{k+1} + (-1)^k \mu^{k+1} \right] \frac{h^{k+1}}{(k+1)!} f_n^{(k)} + O(h^{k+2}). \] (4.17)

Equation (4.17) is valid for \( 0 \leq \text{Re} (\mu) \leq 1 \), and for all \( k \geq 1 \).

One may also derive the higher order terms for \( e_{n+1}^{(k)} \) in a manner similar to the above. The next two terms in the expansion are
\[
\left\{ \left[ (1 - \mu)^{k+1} + (k+1) \mu (1 - \mu)^{k+1} + (-1)^k \mu^{k+1} \right] f_n^{(k+1)}
\]
\[+ (-1)^k (k+2) \mu^{k+1} \frac{\partial f_n}{\partial y} f_n^{(k)} \right\} \frac{h^{k+2}}{(k+2)!}. \] (4.18)

and
\[
\left\{ \left[ (k+1) \mu + 1 \right] + \frac{(k+2)(k+1)}{2} \mu^2 \left[ (1 - \mu)^{k+1} + (-1)^k \mu^{k+1} \right] f_n^{(k+2)}
\]
\[+ (-1)^k (k+3)(k+1) \mu^{k+2} \frac{\partial f_n}{\partial y} f_n^{(k+1)}
\]
\[+ (-1)^k \frac{(k+3)(k+2)}{2} \mu^{k+2} \frac{\partial f_n'}{\partial y} f_n^{(k)} \right\} \frac{h^{k+3}}{(k+3)!}. \] (4.19)

From (4.17) it follows that the optimal expansion coefficients for a Taylor series method of degree \( k \) are the roots of the equation
\[
(1 - \mu)^{k+1} + (-1)^k \mu^{k+1} = 0,
\] (4.20)
or
\[
(1 - \mu)^{k+1} - \mu^{k+1} = 0 \text{ for } k \text{ odd}, (4.21a)
and
\[(1 - \mu)^{k+1} + \mu^{k+1} = 0 \quad \text{for } k \text{ even.} \tag{4.21b}\]

Since the left hand sides of (4.21a,b) are polynomials of degree \(k\), each equation has at most \(k\) distinct roots. One can show that the \(k\) roots of (4.21a) are
\[
\mu = \frac{1}{2} \pm i \frac{1}{2} \tan \left[ \frac{(m-1)\pi}{k+1} \right] \quad m = 1, 2, \ldots, \frac{k+1}{2} \tag{4.22a}
\]
and the \(k\) roots of (4.21b) are
\[
\mu = \frac{1}{2} \pm i \frac{1}{2} \tan \left[ \frac{(m-\frac{1}{2})\pi}{k+1} \right] \quad m = 1, 2, \ldots, \frac{k}{2}. \tag{4.22b}
\]

Equation (4.21a) thus has one real root, \(\mu = \frac{1}{2}\), while Eq. (4.21b) has no real roots.

Using (4.17) - (4.20), the discretization error corresponding to all \(\mu\) defined by (4.22) is
\[
e^{(k,m)}_{n+1} = \frac{h_{k+2}}{(k+2)!} \left[ (1 - \mu)^{k+1} f^{(k+1)}_n + (-1)^k (k+2) \mu^{k+1} \frac{\partial f_n}{\partial y} f^{(k)}_n \right]
\]
\[
+ \frac{h_{k+3}}{(k+3)!} \left[ (1 - \mu)^{k+1} \left[ (k+1) \mu + 1 \right] f^{(k+2)}_n + (-1)^k (k+3) (k+1) \mu^{k+2} \frac{\partial f_n}{\partial y} f^{(k+1)}_n \right]
\]
\[
+ \left( -1 \right)^k \frac{(k+3)(k+2)}{2} \mu^{k+1} \frac{\partial f_n'}{\partial y} f^{(k)}_n. \tag{4.23}\]

Equation (4.23) can be simplified by utilizing, from (4.20),
\[
(1 - \mu)^{k+1} = (-1)^{k+1} \mu^{k+1}. \tag{4.24}\]

One obtains
\[
e^{(k,m)}_{n+1} = (-1)^{k+1} \mu^{k+1} \left\{ \frac{h_{k+2}}{(k+2)!} \left[ f^{(k+1)}_n - (k+2) \frac{\partial f_n}{\partial y} f^{(k)}_n \right] \right. \\
\]
\[
+ \frac{h_{k+3}}{(k+3)!} \left[ \mu (k+1) \left[ f^{(k+2)}_n - (k+3) \frac{\partial f_n}{\partial y} f^{(k+1)}_n \right] \\
\]
\[
+ f^{(k+2)}_n - \frac{(k+3)(k+2)}{2} \frac{\partial f_n'}{\partial y} f^{(k)}_n \right\} + O(h^{k+4}). \tag{4.25}\]

Now if \(\mu\) is written in polar form, the expression for \(e^{(k,m)}_{n+1}\) can be simplified further. By defining \(\theta\) according to
\[
\theta = \pm \frac{(m-1)\pi}{k+1}, \quad \text{for } m = 1, 2, \ldots, \frac{k+1}{2} \quad \text{for } k \text{ odd} \tag{4.26a}
\]
and
\[ \theta = \pm \left( m - \frac{1}{2} \right) \pi, \quad \text{for } m = 1, 2, \ldots, \frac{k}{2} \] for \( k \) even \quad (4.26b)

one can write
\[ \mu = \frac{1}{2} \sec \theta e^{i\theta}. \quad (4.27) \]

One then obtains
\[ \mu^{k+1} = \left( \frac{1}{2} \sec \theta \right)^{k+1} e^{i(k+1)\theta}. \quad (4.28) \]

It follows that for \( k \) odd,
\[ \mu^{k+1} = (-1)^{m+1} \left( \frac{1}{2} \sec \theta \right)^{k+1}, \quad (4.29a) \]

and for \( k \) even,
\[ \mu^{k+1} = i (-1)^{m+1} \text{sgn}(\theta) \left( \frac{1}{2} \sec \theta \right)^{k+1}. \quad (4.29b) \]

It is significant that \( \mu^{k+1} \) is pure real when \( k \) is odd and pure imaginary when \( k \) is even. An inspection of (4.25) shows that if \( f \) is a real-valued function and \( k \) is odd, the \( O(h^{k+2}) \) error term is real. On the other hand, if \( f \) is real and \( k \) is even, the \( O(h^{k+2}) \) error term is pure imaginary. Consequently, the real truncation error is \( O(h^{k+2}) \) when \( k \) is odd, and \( O(h^{k+3}) \) when \( k \) is even. One thus obtains to leading order, for all odd \( k \geq 1 \) and for \( f \) real,
\[ e^{(k,m)}_{n+1} = (-1)^{k+m} \left( \frac{1}{2} \sec \theta \right)^{k+1} \frac{h^{k+2}}{(k+2)!} \left[ f^{(k+1)}_n - (k+2) \frac{\partial f_n}{\partial y} f^{(k)}_n \right] \quad (4.30) \]

where \( \theta \) and \( m \) are defined by (4.26a). For even \( k \), one obtains
\[ e^{(k,m)}_{n+1} = \quad (4.31) \]

\[ (-1)^{k+m+1} \frac{k+1}{2} |\tan \theta| \left( \frac{1}{2} \sec \theta \right)^{k+1} \frac{h^{k+3}}{(k+3)!} \left[ f^{(k+2)}_n - (k+3) \frac{\partial f_n}{\partial y} f^{(k+1)}_n \right] \]

where \( \theta \) and \( m \) are defined by (4.26b).

It is thus clear that, corresponding to the expansion coefficients derived above, a Taylor series method can have an accuracy which is one or two orders higher than the classical case in which \( \mu = 0 \). It also turns out that, due to the special form of the truncation error in (4.30) and (4.31), the accuracy of the solution can be raised two additional orders by local extrapolation. We will return to this point later in the paper.

One can now simplify the marching scheme (4.4) for all \( \mu \) defined by (4.22). Using the fact that
\[ 1 - \mu = \bar{\mu}, \quad (4.32) \]
(4.4) becomes
\[ y_{n+\mu} = \text{Re}(y_n) + \mu h f(t_{n+\mu}, y_{n+\mu}) - \frac{(\mu h)^2}{2!} f'(t_{n+\mu}, y_{n+\mu}) + \frac{(\mu h)^3}{3!} f''(t_{n+\mu}, y_{n+\mu}) - \cdots + \cdots + (-1)^{k-1} \frac{(\mu h)^k}{k!} f^{(k-1)}(t_{n+\mu}, y_{n+\mu}) \] (4.33a)

\[ y_{n+1} = y_{n+\mu} + \mu h f(t_{n+\mu}, y_{n+\mu}) + \frac{(\mu h)^2}{2!} f'(t_{n+\mu}, y_{n+\mu}) + \frac{(\mu h)^3}{3!} f''(t_{n+\mu}, y_{n+\mu}) - \cdots + \cdots + \frac{(\mu h)^k}{k!} f^{(k-1)}(t_{n+\mu}, y_{n+\mu}). \] (4.33b)

For the special case \( \mu = \frac{1}{2} \), one has \( \mu = \bar{\mu} \), so that (4.33) can be written

\[ y_{n+\frac{1}{2}} = y_n + \frac{1}{2} h f(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) - \frac{1}{2} \frac{h^2}{2!} f'(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) + \frac{1}{2^3} \frac{h^3}{3!} f''(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) - \cdots + \cdots + (-1)^{k-1} \frac{1}{2^k} \frac{h^k}{k!} f^{(k-1)}(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) \] (4.34a)

\[ y_{n+1} = y_n + h f(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) + \frac{1}{2} \frac{h^3}{3!} f''(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) + \frac{1}{2^3} \frac{h^5}{5!} f^{(4)}(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) - \cdots + \left[ \frac{1}{2^k} + (-1)^{k-1} \frac{1}{2^k} \right] \frac{h^k}{k!} f^{(k-1)}(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}). \] (4.34b)

Note that all odd-ordered derivatives have been eliminated in (4.34b).

Scheme (4.34) is equivalent to a class of Turan type, one-stage implicit Runge-Kutta methods [20]. However, unlike implicit Runge-Kutta methods, the implicit step (4.34a) only requires the solution of a single equation, instead of a system of \( k \) equations. This advantage becomes significant when \( k \) is much bigger than one.

From (4.30), the local error of the real marching scheme (4.34) is, for all odd \( k \),
\[
\begin{aligned}
\epsilon_{n+1}^{(k)} &= \frac{1}{2^{k+1}} \frac{h^{k+2}}{(k+2)!} \left[ f_n^{(k+1)} - (k+2) \frac{\partial f_n}{\partial y} f_n^{(k)} \right] + O(h^{k+3}). \quad (4.35a)
\end{aligned}
\]

For even \( k \), one obtains from (4.17)
\[
\begin{aligned}
\epsilon_{n+1}^{(k)} &= \frac{1}{2^k} \frac{h^{k+1}}{(k+1)!} f_n^{(k)} + O(h^{k+2}). \quad (4.35b)
\end{aligned}
\]
One also derives, corresponding to $\mu = \frac{1}{2}$, the following error bounds on the polynomial approximate $y_n(t)$ defined by (4.1).

$k$ odd:

$$|y(t) - y_n(t)| \leq \frac{1}{2^{k+1}} \frac{h^{k+1}}{(k+1)!} |f_n^{(k)}| + O(h^{k+2}), \quad t_n \leq t \leq t_{n+1}$$

(4.36a)

$k$ even:

$$|y(t) - y_n(t)| \leq \frac{1}{2^k} \frac{h^{k+1}}{(k+1)!} |f_n^{(k)}| + O(h^{k+2}), \quad t_n \leq t \leq t_{n+1}.$$  (4.36b)

Compared to the corresponding error bounds for $\mu = 0$ and $\mu = 1$, (4.36a) is more accurate by a factor of $2^{k+1}$, and (4.36b) is more accurate by a factor of $2^k$.

**Stability for $y' = \lambda y$, $\text{Re} \lambda < 0$**

Applying the marching scheme (4.4) to

$$y' = \lambda y, \quad y(t_o) = y_0$$

(5.1)

one gets

$$y_{n+1} = y_n \left[ \frac{1 + (1 - \mu) \lambda h + (1 - \mu)^2 \lambda^2 h^2 \frac{h^k}{k!} + \cdots + (1 - \mu)^k \lambda^k h^k h^k}{1 - \mu \lambda h + \mu^2 \lambda^2 h^2 \frac{h^k}{k!} - \cdots + \cdots + (-1)^k \mu^k \lambda^k h^k h^k} \right].$$

(5.2)

so that the discrete solution is

$$y_n = y_0 \left[ \frac{1 + (1 - \mu) \lambda h + (1 - \mu)^2 \lambda^2 h^2 \frac{h^k}{k!} + \cdots + (1 - \mu)^k \lambda^k h^k h^k}{1 - \mu \lambda h + \mu^2 \lambda^2 h^2 \frac{h^k}{k!} - \cdots + \cdots + (-1)^k \mu^k \lambda^k h^k h^k} \right]^n.$$

(5.3a)

Equation (5.3a) is valid for all $\mu$ satisfying $0 \leq \text{Re} \mu \leq 1$.

Since the exact solution to (5.1), $y(t) = y_0 e^{\lambda(t-t_o)}$, tends to zero as $t$ tends to infinity when $\text{Re} \lambda < 0$, it follows that the stability criterion is

$$\left| \frac{1 + (1 - \mu) \lambda h + (1 - \mu)^2 \lambda^2 h^2 \frac{h^k}{k!} + \cdots + (1 - \mu)^k \lambda^k h^k h^k}{1 - \mu \lambda h + \mu^2 \lambda^2 h^2 \frac{h^k}{k!} - \cdots + \cdots + (-1)^k \mu^k \lambda^k h^k h^k} \right| < 1.$$  (5.3b)

We consider the three cases $\mu = 0$, $\mu = 1$, and $\text{Re} \mu = \frac{1}{2}$, for which (5.3) simplifies as follows.
\(\mu = 0:\)

\[ y_n = y_0 \left( 1 + \lambda h + \frac{\lambda^2 h^2}{2!} + \cdots + \frac{\lambda^k h^k}{k!} \right)^n \] \hspace{1cm} (5.4a)

\[
\left| 1 + \lambda h + \frac{\lambda^2 h^2}{2!} + \cdots + \frac{\lambda^k h^k}{k!} \right| < 1 \hspace{1cm} (5.4b)
\]

\(\mu = 1:\)

\[ y_n = \frac{y_0}{\left[ 1 - \lambda h + \frac{\lambda^2 h^2}{2!} - \cdots + \frac{(-1)^k \lambda^k h^k}{k!} \right]^n}. \] \hspace{1cm} (5.5a)

\[
\left| 1 - \lambda h + \frac{\lambda^2 h^2}{2!} - \cdots + \frac{(-1)^k \lambda^k h^k}{k!} \right| > 1 \hspace{1cm} (5.5b)
\]

\(\text{Re} (\mu) = \frac{1}{2}:\)

\[ y_n = y_0 \left[ \frac{1 + \bar{\mu} \lambda h + \bar{\mu}^2 \lambda^2 h^2}{1 - \mu \lambda h + \mu^2 \lambda^2 h^2} + \cdots + \frac{\bar{\mu}^k \lambda^k h^k}{k!} \right]^n \] \hspace{1cm} (5.6a)

\[
\left| \frac{1 + \bar{\mu} \lambda h + \bar{\mu}^2 \lambda^2 h^2}{1 - \mu \lambda h + \mu^2 \lambda^2 h^2} + \cdots + \frac{\bar{\mu}^k \lambda^k h^k}{k!} \right| < 1 \hspace{1cm} (5.6b)
\]

**Degree-1 Taylor Series**

Letting \(\lambda h = w = u + iv,\) where \(u < 0,\) the stability criterion for Euler's method is

\[
|1 + w|^2 = (1 + u)^2 + v^2 < 1. \hspace{1cm} (5.7a)
\]

The above inequality describes the set of points inside a circle of radius one centered at the point (-1,0) in the complex \(\lambda h\) plane. For real \(\lambda,\) (5.7a) simplifies to

\[ h < \frac{2}{-\lambda}. \hspace{1cm} (5.7b)\]

From (5.5b), the stability requirement of the backward Euler method is

\[
|1 - w|^2 = (1 - u)^2 + v^2 > 1, \hspace{1cm} (5.8)
\]

which is clearly satisfied for all negative \(u.\) The backward Euler solution thus tends to zero and remains stable for any fixed \(h > 0.\) Schemes which possess this property for problem (5.1) are said to be \(A\)-stable [23],[24].

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From (5.6b), the stability requirement of the centered Euler method is

\[
\left| \frac{1 + \mu w}{1 - \mu w} \right|^2 < 1, \quad (5.9)
\]

where \( \mu = \bar{\mu} = \frac{1}{2} \). Since the centered Euler solution

\[
y_n = y_0 \left( \frac{1 + \frac{h}{\lambda} \frac{h}{2}}{1 - \frac{h}{\lambda} \frac{h}{2}} \right)^n
\]

is identical to that of the trapezoidal rule (which is known to be A-stable), it follows that the centered Euler method is also A-stable. (It should be noted that Dahlquist [25] has shown that the trapezoidal rule is the most accurate, A-stable scheme from the general class of linear multistep methods. The centered Euler method is, of course, a Taylor series method and therefore not in the same class. However, both methods give the same solution to \( y' = \lambda y \).)

**Degree-2 Taylor Series**

The explicit stability requirement (5.4b) is

\[
\left| 1 + w + \frac{w^2}{2} \right|^2 < 1, \quad (5.11a)
\]

which reduces to

\[
2u + 2u^2 + u(u^2 + v^2) + \frac{1}{4}(u^2 + v^2)^2 < 0. \quad (5.11b)
\]

The above inequality is satisfied for all \((u, v)\) such that

\[
-2 < u < 0 \quad (5.12a)
\]

\[
|v| < \sqrt{2 \sqrt{-u(u+2)} - u(u+2)}. \quad (5.12b)
\]

See Figure 1.

The fully implicit stability criterion is

\[
\left| 1 - w + \frac{w^2}{2} \right|^2 > 1, \quad (5.13a)
\]

or

\[
2u - 2u^2 + u(u^2 + v^2) - \frac{1}{4}(u^2 + v^2)^2 < 0. \quad (5.13b)
\]

Since (5.13b) is satisfied for all \( u < 0 \), an implicit Taylor series of degree two is A-stable.
Stability requirement (5.6b) is
\[
\left| \frac{1 + \tilde{\mu} w + \tilde{\mu}^2 \frac{w^2}{4}}{1 - \mu w + \mu^2 \frac{w^2}{4}} \right| < 1.
\]
(5.14a)

Letting \( \mu_i := \text{Im}(\mu) \), it can be shown that (5.14a) holds if
\[
u \left[ 2 + 4 \mu_i v + \left( \frac{1}{4} + \mu_i^2 \right) \left( u^2 + v^2 \right) \right] < 0.
\]
(5.14b)

Inequality (5.14b) is clearly satisfied for all negative \( u \) when \( \mu_i = 0 \). For \( \mu_i = \pm \sqrt{3} \), corresponding to (3.12), (5.14b) can be written
\[
u \left[ 1 + \frac{u^2}{3} + \left( \frac{\sqrt{3}}{3} v \pm 1 \right)^2 \right] < 0,
\]
so that a degree-two series is A-stable for both \( \mu = \frac{1}{2} \) and \( \mu = \frac{1}{2} \pm i \frac{\sqrt{3}}{6} \).

**Degree-3 Taylor Series**

Proceeding as above, one derives the following stability constraints for Taylor series of degree three.

\( \mu = 0 \):
\[
2 u + 2 u^2 + \frac{4}{3} u^3 + \frac{1}{3} (u^4 - v^4) + \frac{1}{4} (u^2 + v^2)^2
+ \frac{1}{6} u (u^2 + v^2)^2 + \frac{1}{36} (u^2 + v^2)^3 < 0
\]
(5.16)

\( \mu = 1 \):
\[
2 u - 2 u^2 + \frac{4}{3} u^3 - \frac{1}{3} (u^4 - v^4) - \frac{1}{4} (u^2 + v^2)^2
+ \frac{1}{6} u (u^2 + v^2)^2 - \frac{1}{36} (u^2 + v^2)^3 < 0
\]
(5.17)

\( \mu = \frac{1}{2} + i \mu_i \):
\[
u \left[ 2 + 4 \mu_i v + \frac{u^2}{3} + 4 \mu_i^2 v^2 + \frac{1}{6} \mu_i^2 u^2 v^2 + \frac{1}{3} \left( \frac{u^2}{4} + \mu_i^2 v^2 \right) \left( \frac{v^2}{4} + \mu_i^2 u^2 \right)
+ \frac{4}{3} \mu_i v (u^2 + v^2) \left( \frac{1}{4} + \mu_i^2 \right) + \frac{1}{6} (u^4 + v^4) \left( \frac{1}{16} + \mu_i^4 \right) \right] < 0
\]
(5.18)

Because of the complexity of the above expressions, direct analysis is difficult. However, one can determine numerically that, corresponding to (5.16), an explicit series of degree three has the stability region shown in Figure 2. One can also show that near \( v = \pm \sqrt{3} \), (5.17) is not satisfied as \( u \to 0^- \), so that a degree-three, fully implicit series is not A-stable. See Figure 3.

Corresponding to (5.18), one considers stability for \( \mu_i = 0 \) and \( \mu_i = \pm \frac{1}{2} \). (See (4.22a).)

For \( \mu_i = 0 \), one can see directly that (5.18) is always satisfied when \( u \) is negative, so that a degree-three series is A-stable for \( \mu = \frac{1}{2} \). However, for \( \mu_i = \pm \frac{1}{2} \), one can show that (5.18) is not satisfied for all negative \( u \). When \( \mu_i = \pm \frac{1}{2} \), one obtains the instability region.
\[- \sqrt{4 \sqrt{-(v + 2)} - (v + 2)^2} \leq u < 0 \quad (5.19a)\]

\[-4.5198421 \leq v \leq -2, \quad (5.19b)\]

and when \( \mu_t = -\frac{1}{2} \), one obtains

\[- \sqrt{4 \sqrt{v - 2} - (v - 2)^2} \leq u < 0 \quad (5.20a)\]

\[2 \leq v \leq 4.5198421. \quad (5.20b)\]

See Figure 4.

**Degree-4 Taylor Series**

For Taylor series of degree four, one must rely primarily on numerical means to determine stability. We obtained the results shown in Figures 5 and 6 for explicit (\( \mu = 0 \)) and fully implicit (\( \mu = 1 \)) series, respectively. On the other hand, it is possible to show rigorously that, for \( \mu = \frac{1}{2} \), the degree-four series is A-stable. However, corresponding to the two complex pairs \( \mu = \frac{1}{2} \pm i \frac{1}{2} \tan \frac{\pi}{10} \) and \( \mu = \frac{1}{2} \pm i \frac{1}{2} \tan \frac{3\pi}{10} \) defined by (4.22b), the degree-four series is not A-stable. See Figure 7.

**Summary of A-Stable Taylor Series**

To the best of our knowledge, there are no A-stable Taylor series of the form (4.4) with degree higher than four. (There are, however, A-stable schemes which use derivatives of arbitrarily high order [26],[27].) Table I summarizes the A-stable Taylor series identified in this section.

<table>
<thead>
<tr>
<th>Taylor Series</th>
<th>Expansion Coefficient</th>
<th>Order of Accuracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degree-1</td>
<td>( \mu = 1 ) (Backward Euler)</td>
<td>1st Order</td>
</tr>
<tr>
<td></td>
<td>( \mu = \frac{1}{2} ) (Centered Euler)</td>
<td>2nd Order</td>
</tr>
<tr>
<td>Degree-2</td>
<td>( \mu = 1 )</td>
<td>2nd Order</td>
</tr>
<tr>
<td></td>
<td>( \mu = \frac{1}{2} )</td>
<td>2nd Order</td>
</tr>
<tr>
<td></td>
<td>( \mu = \frac{1}{2} \pm i \frac{\sqrt{3}}{6} )</td>
<td>4th Order</td>
</tr>
<tr>
<td>Degree-3</td>
<td>( \mu = \frac{1}{2} )</td>
<td>4th Order</td>
</tr>
<tr>
<td>Degree-4</td>
<td>( \mu = \frac{1}{2} )</td>
<td>4th Order</td>
</tr>
</tbody>
</table>

Table I A-Stable Taylor series with their orders of accuracy.
Numerical Results - Uniform Step size

Our main objectives in this section are to verify the order-of-accuracy results presented earlier in the paper and to introduce extrapolation schemes for Taylor series of degree three and higher. We also discuss some details concerning the numerical implementation of the marching scheme (4.33). For a set of model problems, we consider Problems A1 - A5 from Hull et. al. [19], which are shown below.

<table>
<thead>
<tr>
<th>Problem</th>
<th>ODE</th>
<th>Initial Condition</th>
<th>Exact Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( y' = -y )</td>
<td>( y(0) = 1 )</td>
<td>( y = e^{-t} )</td>
</tr>
<tr>
<td>2.</td>
<td>( y' = -\frac{y^3}{2} )</td>
<td>( y(0) = 1 )</td>
<td>( y = \frac{1}{\sqrt{1 + t}} )</td>
</tr>
<tr>
<td>3.</td>
<td>( y' = y \cos t )</td>
<td>( y(0) = 1 )</td>
<td>( y = e^{\sin t} )</td>
</tr>
<tr>
<td>4.</td>
<td>( y' = \frac{y}{4} (1 - \frac{y}{20}) )</td>
<td>( y(0) = 1 )</td>
<td>( y = \frac{20}{1 + 19e^{-t/4}} )</td>
</tr>
<tr>
<td>5.</td>
<td>( y' = \frac{y-t}{y+t} )</td>
<td>( y(0) = 4 )</td>
<td>( r = 4e^\pi/2 e^{-\frac{t}{4}} \ t = r \cos \theta \ y = r \sin \theta )</td>
</tr>
</tbody>
</table>

Degree-1 Taylor Series

Corresponding to \( k = 1 \) and \( \mu = \frac{1}{2} \), (4.33) becomes the centered Euler method (2.12a,b)

\[
y_{n+\frac{1}{2}} = y_n + \frac{h}{2} f(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}})
\]

\[
y_{n+1} = y_{n+\frac{1}{2}} + \frac{h}{2} f(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}).
\]

For nonlinear \( f \), the implicit halfstep (2.12a) must be solved by iteration. One can iterate on (2.12a) directly using corrector iteration,

\[
y_{n+\frac{1}{2}}^{(l)} = y_n + \frac{h}{2} f(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}^{(l-1)}),
\]

\[
(6.1)
\]
where \( y_{n+\frac{1}{2}}^{(0)} = y_n \). One can alternatively use Newton iteration,

\[
y_{n+\frac{1}{2}}^{(i)} = y_{n+\frac{1}{2}}^{(i-1)} - \frac{y_{n+\frac{1}{2}}^{(i-1)} - \frac{h}{2} f(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}^{(i-1)}) - y_n}{1 - \frac{h}{2} \frac{\partial f(t_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}^{(i-1)})}{\partial y}}.
\]  

(6.2)

For each method, the computational work per step depends on the number of iterations \( M \) required to solve (2.12a). See Tables II and III. Corresponding results for the implicit midpoint rule and trapezoidal rule are also shown for comparison.

We solved Problems 1 - 5 on the interval \([0,20]\), with stepsize \( h = 0.5, 0.25, 0.125, \ldots \). Figure 8 shows the reduction of numerical error with stepsize. The corresponding orders of accuracy are shown in Table IV. All calculations were performed in double precision FORTRAN 77 on a Dell Dimension XPS R400 computer running under the LINUX operating system.

<table>
<thead>
<tr>
<th>Operation</th>
<th>Centered Euler</th>
<th>Implicit Midpoint</th>
<th>Trapezoidal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Function Evaluations</td>
<td>( M + 1 )</td>
<td>( M )</td>
<td>( M + 1 )</td>
</tr>
<tr>
<td>Additions and Subtractions</td>
<td>( M + 1 )</td>
<td>( 2M )</td>
<td>( 2M )</td>
</tr>
<tr>
<td>Multiplications</td>
<td>( M + 1 )</td>
<td>( 2M )</td>
<td>( M )</td>
</tr>
<tr>
<td>Divisions</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table II  Computational Work Per Step – Corrector Method

<table>
<thead>
<tr>
<th>Operation</th>
<th>Centered Euler</th>
<th>Implicit Midpoint</th>
<th>Trapezoidal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Function Evaluations</td>
<td>( 2M + 1 )</td>
<td>( 2M )</td>
<td>( 2M + 1 )</td>
</tr>
<tr>
<td>Additions and Subtractions</td>
<td>( 4M + 1 )</td>
<td>( 5M )</td>
<td>( 5M )</td>
</tr>
<tr>
<td>Multiplications</td>
<td>( 2M + 1 )</td>
<td>( 3M )</td>
<td>( 2M )</td>
</tr>
<tr>
<td>Divisions</td>
<td>( M )</td>
<td>( M )</td>
<td>( M )</td>
</tr>
</tbody>
</table>

Table III  Computational Work Per Step – Newton’s Method

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Table IV  Numerical order of accuracy for Problems 1 - 5.

<table>
<thead>
<tr>
<th>Method</th>
<th>Problem:</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Centered Euler</td>
<td>2.00</td>
<td>1.98</td>
<td>2.02</td>
<td>1.99</td>
<td>2.06</td>
</tr>
<tr>
<td>Trapezoidal Rule</td>
<td>2.00</td>
<td>2.01</td>
<td>2.00</td>
<td>2.00</td>
<td>2.02</td>
</tr>
<tr>
<td>Implicit Midpoint</td>
<td>2.00</td>
<td>2.00</td>
<td>1.95</td>
<td>2.00</td>
<td>1.98</td>
</tr>
</tbody>
</table>

**Degree-2 Taylor Series**

Setting \( k = 2 \) and \( \mu = \frac{1}{2} \pm i \frac{1}{2} \tan \frac{\pi}{6} = \frac{1}{2} \pm i \frac{\sqrt{3}}{6} \) in (4.33), one recovers the marching scheme (3.13a,b)

\[
y_{n+\mu} = \text{Re}(y_n) + \mu h f(t_{n+\mu}, y_{n+\mu}) - \frac{(\mu h)^2}{2} f'(t_{n+\mu}, y_{n+\mu})
\]

\[
y_{n+1} = y_{n+\mu} + \bar{\mu} h f(t_{n+\mu}, y_{n+\mu}) + \frac{(\bar{\mu} h)^2}{2} f'(t_{n+\mu}, y_{n+\mu}),
\]

where the expansion point \( t_{n+\mu} := t_n + \mu h \) is in the complex plane. Taking the "plus" sign for \( \mu \), the integration path takes the form shown in Figure 9.

Analogous to (6.1), corrector iteration for (3.13a) is given by

\[
y_{n+\mu}^{(l)} = \text{Re}(y_n) + \mu h f(t_{n+\mu}, y_{n+\mu}^{(l-1)}) - \frac{(\mu h)^2}{2} f'(t_{n+\mu}, y_{n+\mu}^{(l-1)})
\]

(6.3)

where \( y_{n+\mu}^{(0)} = \text{Re}(y_n) \). Similarly, Newton iteration for (3.13a) is

\[
y_{n+\mu}^{(l)} = \frac{y_{n+\mu}^{(l-1)} - \mu h f(t_{n+\mu}, y_{n+\mu}^{(l-1)}) + \frac{(\mu h)^2}{2} f'(t_{n+\mu}, y_{n+\mu}^{(l-1)}) - \text{Re}(y_n)}{1 - \mu h \frac{\partial f}{\partial y}(t_{n+\mu}, y_{n+\mu}^{(l-1)}) + \frac{(\mu h)^2}{2} \frac{\partial f'}{\partial y}(t_{n+\mu}, y_{n+\mu}^{(l-1)})}.
\]

(6.4)

Letting \( M \) denote the number of iterations required to solve (3.13a), the work per step is shown in Table V, where all operations must be performed in complex arithmetic. Note that the evaluation of \( \frac{\partial f}{\partial y} \) in the denominator of (6.4) is not counted as a function evaluation, since it is part of the evaluation of \( f' \) in the numerator.

Application of the degree-2 Taylor series to Problems 1 - 5 on the interval \([0,20]\) produced the error reduction results shown in Figure 10. Table VI shows the corresponding orders of accuracy.
Operation | Corrector Iteration | Newton Iteration
---|---|---
Function Evaluations | 2 M + 2 | 3 M + 2
Additions and Subtractions | 2 M + 2 | 6 M + 2
Multiplications | 2 M + 2 | 4 M + 2
Divisions | 0 | M

Table V  Computational work per step for a degree-2 Taylor series with $\mu = \frac{1}{2} \pm i \frac{\sqrt{3}}{6}$.

<table>
<thead>
<tr>
<th>Problem:</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expansion Coefficient</td>
<td>$\mu = \frac{1}{2} + i \frac{\sqrt{3}}{6}$</td>
<td>4.04</td>
<td>3.96</td>
<td>4.12</td>
<td>3.97</td>
</tr>
</tbody>
</table>

Table VI  Numerical order of accuracy for Problems 1 - 5 using a degree-2 Taylor series.
Degree-3 Taylor Series

From (4.22a) and (4.30), the marching scheme (4.33) is fourth-order-accurate when
\( k = 3 \) and
\[
\mu = \frac{1}{2}
\]  
(6.5)
or
\[
\mu = \frac{1}{2} \pm i \frac{1}{2}.
\]  
(6.6)
Since \( \mu = \frac{1}{2} \) is real and has the smallest truncation error, it is the preferred expansion coefficient.

The degree-3 Taylor series can be implemented using corrector or Newton iteration, analogous to (6.3) and (6.4), respectively. The work per step is shown in Table VII.

We now show that one can construct an extrapolation scheme using (6.5) - (6.6) which is sixth-order-accurate. Let \( y_{n+1}^{(3,1)} \) denote the numerical solution at \( t_{n+1} \) corresponding to \( \mu = \frac{1}{2} \), and let
\[
e_{n+1}^{(3,1)} := y(t_{n+1}) - y_{n+1}^{(3,1)}
\]  
(6.7)
denote its truncation error. Similarly, let \( y_{n+1}^{(3,2)} \) denote the numerical solution at \( t_{n+1} \) corresponding to \( \mu = \frac{1}{2} + i \frac{1}{2} \), and let
\[
e_{n+1}^{(3,2)} := y(t_{n+1}) - y_{n+1}^{(3,2)}.
\]  
(6.8)
Then one can see from (4.25) and (4.29a) that for a real-valued function \( f \), \( e_{n+1}^{(3,1)} \) and the real part of \( e_{n+1}^{(3,2)} \) are related according to
\[
\text{Re}(e_{n+1}^{(3,2)}) = -4e_{n+1}^{(3,1)} + O(h^7).
\]  
(6.9)
Taking the real part of (6.8) and using (6.9), one gets
\[
-4e_{n+1}^{(3,1)} + O(h^7) = y(t_{n+1}) - \text{Re}(y_{n+1}^{(3,2)}).
\]  
(6.10)
Multiplying (6.7) by \( \frac{4}{5} \) and (6.10) by \( \frac{1}{5} \) and then adding the resulting equations and rearranging, one obtains
\[
\frac{4}{5} y_{n+1}^{(3,1)} + \frac{1}{5} \text{Re}(y_{n+1}^{(3,2)}) = y(t_{n+1}) + O(h^7).
\]  
(6.11)
The extrapolated solution
\[
y_{n+1}^{(3,e)} := \frac{4}{5} y_{n+1}^{(3,1)} + \frac{1}{5} \text{Re}(y_{n+1}^{(3,2)})
\]  
(6.12)
is thus sixth-order-accurate.

In Figure 11 we show the stability region of the above extrapolation for problem (5.1). Figure 12 presents error reduction results for Problems 1 - 5 for \( \mu = \frac{1}{2} \), \( \mu = \frac{1}{2} + i \frac{1}{2} \), and extrapolation. The numerical orders of accuracy are shown in Table VIII.
<table>
<thead>
<tr>
<th>Operation</th>
<th>Corrector Iteration</th>
<th>Newton Iteration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Function Evaluations</td>
<td>3 M + 3</td>
<td>4 M + 3</td>
</tr>
<tr>
<td>Additions and Subtractions</td>
<td>3 M + 3</td>
<td>8 M + 3</td>
</tr>
<tr>
<td>Multiplications</td>
<td>3 M + 3</td>
<td>6 M + 3</td>
</tr>
<tr>
<td>Divisions</td>
<td>0</td>
<td>M</td>
</tr>
</tbody>
</table>

Table VII  Computational work per step for a degree-3 Taylor series with $\mu = \frac{1}{2}$ or $\mu = \frac{1}{2} \pm i \frac{1}{2}$.

<table>
<thead>
<tr>
<th>Expansion Coefficient</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu = \frac{1}{2}$</td>
<td>3.99</td>
<td>3.98</td>
<td>4.01</td>
<td>3.99</td>
<td>4.04</td>
</tr>
<tr>
<td>$\mu = \frac{1}{2} + i \frac{1}{2}$</td>
<td>4.01</td>
<td>3.91</td>
<td>4.02</td>
<td>3.97</td>
<td>3.94</td>
</tr>
<tr>
<td>extrapolation</td>
<td>6.11</td>
<td>5.89</td>
<td>6.03</td>
<td>6.02</td>
<td>6.20</td>
</tr>
</tbody>
</table>

Table VIII  Numerical order of accuracy for Problems 1 - 5 using a degree-3 Taylor series.
Degree-4 Taylor Series

From (4.22b), one obtains the expansion coefficients

\[ \mu = \frac{1}{2} \pm i \frac{1}{2} \tan \frac{\pi}{10} \]  
\hfill (6.13)

and

\[ \mu = \frac{1}{2} \pm i \frac{1}{2} \tan \frac{3\pi}{10}, \]  
\hfill (6.14)

for which (4.33) is sixth-order-accurate. One can verify from (4.31) that (6.13) has a truncation error which is about 47 times smaller than that of (6.14).

Using a corrector or Newton iteration implementation, the degree-4 Taylor series requires the work per step shown in Table IX.

As in the previous section, extrapolation can be used to raise the accuracy of the degree-4 Taylor series by two additional orders. Rather than go through the details, we simply state the results.

Let \( y_{n+1}^{(4,1)} \) denote the discrete solution at \( t_{n+1} \) when \( \mu = \frac{1}{2} + i \frac{1}{2} \tan \frac{\pi}{10} \), and let \( y_{n+1}^{(4,2)} \) denote the discrete solution when \( \mu = \frac{1}{2} + i \frac{1}{2} \tan \frac{3\pi}{10} \). Define \( p \) and \( q \) by

\[ p := \tan \frac{\pi}{10} \left( \sec \frac{\pi}{10} \right)^5 \]  
\hfill (6.15a)

and

\[ q := \tan \frac{3\pi}{10} \left( \sec \frac{3\pi}{10} \right)^5. \]  
\hfill (6.15b)

Then the extrapolated solution

\[ y_{n+1}^{(4,c)} := \frac{q y_{n+1}^{(4,1)} + p y_{n+1}^{(4,2)}}{p + q} \]  
\hfill (6.16)

satisfies

\[ y_{n+1}^{(4,c)} = y(t_{n+1}) + O(h^9) \]  
\hfill (6.17)

so that \( y_{n+1}^{(4,c)} \) is eighth-order-accurate.

The stability region of the above extrapolation for problem (5.1) is shown in Figure 13. Figure 14 shows reduction of error with stepsize for Problems 1 - 5 using \( \mu = \frac{1}{2} + i \frac{1}{2} \tan \frac{\pi}{10}, \mu = \frac{1}{2} + i \frac{1}{2} \tan \frac{3\pi}{10} \), and extrapolation. Table X shows the corresponding orders of accuracy.
<table>
<thead>
<tr>
<th>Operation</th>
<th>Corrector Iteration</th>
<th>Newton Iteration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Function</td>
<td>4 M + 4</td>
<td>5 M + 4</td>
</tr>
<tr>
<td>Evaluations</td>
<td>4 M + 4</td>
<td>10 M + 4</td>
</tr>
<tr>
<td>Additions and Subtractions</td>
<td>4 M + 4</td>
<td>10 M + 4</td>
</tr>
<tr>
<td>Multiplications</td>
<td>4 M + 4</td>
<td>8 M + 4</td>
</tr>
<tr>
<td>Divisions</td>
<td>0</td>
<td>M</td>
</tr>
</tbody>
</table>

Table IX  Computational work per step for a degree-4 Taylor series with \( \mu = \frac{1}{2} \pm i \frac{1}{2} \tan \frac{\pi}{10} \) or \( \mu = \frac{1}{2} \pm i \frac{1}{2} \tan \frac{3\pi}{10} \).

<table>
<thead>
<tr>
<th>Problem:</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expansion</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Coefficient</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mu = \frac{1}{2} + i \frac{1}{2} \tan \frac{\pi}{10} )</td>
<td>6.00</td>
<td>5.94</td>
<td>6.01</td>
<td>5.99</td>
<td>5.89</td>
</tr>
<tr>
<td>( \mu = \frac{1}{2} + i \frac{1}{2} \tan \frac{3\pi}{10} )</td>
<td>6.00</td>
<td>5.80</td>
<td>5.92</td>
<td>6.00</td>
<td>6.27</td>
</tr>
<tr>
<td>extrapolation</td>
<td>7.98</td>
<td>7.62</td>
<td>8.01</td>
<td>7.97</td>
<td>7.89</td>
</tr>
</tbody>
</table>

Table X  Numerical order of accuracy for Problems 1 - 5 using a degree-4 Taylor series.
Extrapolation for Taylor Series of Arbitrary Degree

According to our work in the two previous sections, a degree-3 Taylor series with extrapolation provides a sixth-order method, and a degree-4 Taylor series with extrapolation provides an eighth-order method. In general, if \( k \) is odd, a degree-\( k \) Taylor series with extrapolation provides a \((k + 3)\)rd-order method, and if \( k \) is even, a degree-\( k \) Taylor series with extrapolation provides a \((k + 4)\)th-order method. The purpose of this section is to generalize these results.

Let \( y_{n+1}^{(k,e)} \) denote the extrapolated solution of degree \( k \) at \( t_{n+1} \). Let \( y_{n+1}^{(k,1)} \) and \( y_{n+1}^{(k,2)} \) denote the regular solution of degree \( k \) at \( t_{n+1} \) corresponding to \( \mu = \mu_1 \) and \( \mu = \mu_2 \), respectively, where \( \mu_1 \) and \( \mu_2 \) are defined as follows. For \( k \) odd, let

\[
\mu_1 = \frac{1}{2}
\]

and

\[
\mu_2 = \frac{1}{2} + i \frac{1}{2} \tan \frac{\pi}{k+1}.
\]

For \( k \) even let

\[
\mu_1 = \frac{1}{2} + i \frac{1}{2} \tan \frac{\pi}{2(k+1)}
\]

and

\[
\mu_2 = \frac{1}{2} + i \frac{1}{2} \tan \frac{3\pi}{2(k+1)}.
\]

The extrapolated solution for all \( k \) is then

\[
y_{n+1}^{(k,e)} := \frac{q y_{n+1}^{(k,1)} + p y_{n+1}^{(k,2)}}{p + q}
\]

where \( p \) and \( q \) are defined below. For all odd \( k \geq 3 \),

\[
p = 1
\]

and

\[
q = \left( \sec \frac{\pi}{k+1} \right)^{k+1}.
\]

For all even \( k \geq 4 \),

\[
p = \tan \frac{\pi}{2(k+1)} \left[ \sec \frac{\pi}{2(k+1)} \right]^{k+1}
\]

and

\[
q = \tan \frac{3\pi}{2(k+1)} \left[ \sec \frac{3\pi}{2(k+1)} \right]^{k+1}.
\]

Using the above formulas, we implemented extrapolation schemes of degree five and six. Figure 15 shows the corresponding stability regions for problem (5.1). Application to Problem 1 produced the results shown in Figures 16.a,b, where the numerical orders of accuracy are 8.01 and 9.87, respectively.
Numerical Results - Variable Stepsize

The purpose of this section is to briefly introduce a variable step approach that can be implemented using an implicit Taylor series method. Here we present results for Problems 1 - 5, and in the next two sections we consider singular equations and stiff systems.

To develop a variable step scheme, one must have a method for estimating the local error and a procedure for controlling the stepsize. For an implicit series of degree three or higher, the local error can be obtained by extrapolation. For a degree-two series, the error can be obtained by integrating twice at each step – once with (3.13) and once with (3.14). The difference between the two solutions is an $O(h^3)$ error estimate. Similarly, one integrates with (2.12) and (2.5) when using a series of first degree.

The stepsize may be adjusted as follows. Let $e_n$ denote the error at step $n$, and let $\tau$ be the error tolerance. Then if $e_n$ is $O(h^p)$ and $h_n$ is a successful stepsize, an estimate for $h_{n+1}$ is

$$h_{n+1} = \xi \frac{\tau}{e_n} \left( \frac{1}{h_n} \right)^\frac{1}{p}$$  \hspace{1cm} (7.1)

where $\xi$ is an adjustable parameter less than one [28]. (See [29] for a discussion of stepsize control for long Taylor series methods.)

In what follows, we present numerical results from four variable step schemes, which we identify as follows. Scheme 1 is the degree-1 Taylor series (2.12) combined with (2.5). Scheme 2 is the degree-2 series (3.13) combined with (3.14). Schemes 3 and 4 are the degree-3 and -4 series with extrapolation, respectively. See Table XI.

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Taylor Series</th>
<th>Order of Accuracy</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Degree 1</td>
<td>2nd</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>Degree 2</td>
<td>4th</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>Degree 3</td>
<td>6th</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>Degree 4</td>
<td>8th</td>
<td>7</td>
</tr>
</tbody>
</table>

Table XI Variable step schemes with their orders of accuracy and $\rho$ values.

We integrated Problems 1 - 5 on the interval [0,20]. For Schemes 2 - 4, the initial stepsize was determined from $\frac{h^k}{h} f^{(k-1)} < \tau$, where $k$ is the degree of the corresponding Taylor series and the total derivative $f^{(k-1)}$ was evaluated near $t_0$ but not at $t_0$. For Scheme 1, we used $\frac{h^2}{2} \lambda^2 < \tau$, where $\lambda = \frac{df}{dy}$ [30]. Subsequent stepsizes were determined from (7.1), subject to $\frac{h_{n+1}}{h_n} \leq h_{\text{fact}}$, where $h_{\text{fact}} = 2.0$. We took $\xi$ to be 0.9 for Schemes 2 - 4 and 0.7 for Scheme 1. Unsuccessful stepsizes $h_{n+1}'$ were reduced using $h_{n+1} := \nu h_{n+1}'$. 

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where \( \nu = 0.7 \). At the initial step we took \( \nu = 0.1 \). All parameters identified above were maintained at their stated values for all calculations.

For each problem, we varied the tolerance from \( 10^{-2} \) to \( 10^{-12} \), and determined the maximum absolute error and number of integration steps per tolerance. The results are shown in Tables XII - XVI. The average number of repeat steps per tolerance is stated at the bottom of each table.

### Table XII.a

<table>
<thead>
<tr>
<th>Tolerance</th>
<th>Scheme 1</th>
<th>Scheme 2</th>
<th>Scheme 3</th>
<th>Scheme 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10E-01</td>
<td>0.0740</td>
<td>0.0437</td>
<td>0.0291</td>
<td>0.0150</td>
</tr>
<tr>
<td>0.10E-02</td>
<td>0.0789</td>
<td>0.0453</td>
<td>0.0328</td>
<td>0.0209</td>
</tr>
<tr>
<td>0.10E-03</td>
<td>0.0807</td>
<td>0.0447</td>
<td>0.0331</td>
<td>0.0212</td>
</tr>
<tr>
<td>0.10E-04</td>
<td>0.0814</td>
<td>0.0454</td>
<td>0.0327</td>
<td>0.0212</td>
</tr>
<tr>
<td>0.10E-05</td>
<td>0.0816</td>
<td>0.0454</td>
<td>0.0329</td>
<td>0.0210</td>
</tr>
<tr>
<td>0.10E-06</td>
<td>0.0816</td>
<td>0.0452</td>
<td>0.0327</td>
<td>0.0213</td>
</tr>
<tr>
<td>0.10E-07</td>
<td>0.0817</td>
<td>0.0450</td>
<td>0.0329</td>
<td>0.0208</td>
</tr>
<tr>
<td>0.10E-08</td>
<td>0.0817</td>
<td>0.0439</td>
<td>0.0331</td>
<td>0.0200</td>
</tr>
<tr>
<td>0.10E-09</td>
<td>0.0817</td>
<td>0.0355</td>
<td>0.0297</td>
<td>0.0196</td>
</tr>
<tr>
<td>0.10E-10</td>
<td>0.0817</td>
<td>0.0229</td>
<td>0.0231</td>
<td>0.0171</td>
</tr>
<tr>
<td>0.10E-11</td>
<td>0.1690</td>
<td>0.0128</td>
<td>0.0166</td>
<td>0.0147</td>
</tr>
</tbody>
</table>

Table XII.a Maximum error in units of the error tolerance for Problem 1.

### Table XII.b

<table>
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<tr>
<th>Tolerance</th>
<th>Scheme 1</th>
<th>Scheme 2</th>
<th>Scheme 3</th>
<th>Scheme 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10E-01</td>
<td>24</td>
<td>11</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>0.10E-02</td>
<td>68</td>
<td>18</td>
<td>11</td>
<td>9</td>
</tr>
<tr>
<td>0.10E-03</td>
<td>205</td>
<td>32</td>
<td>15</td>
<td>11</td>
</tr>
<tr>
<td>0.10E-04</td>
<td>642</td>
<td>61</td>
<td>21</td>
<td>14</td>
</tr>
<tr>
<td>0.10E-05</td>
<td>2023</td>
<td>124</td>
<td>31</td>
<td>17</td>
</tr>
<tr>
<td>0.10E-06</td>
<td>6391</td>
<td>258</td>
<td>46</td>
<td>23</td>
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<tr>
<td>0.10E-07</td>
<td>20204</td>
<td>545</td>
<td>70</td>
<td>30</td>
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<tr>
<td>0.10E-08</td>
<td>63886</td>
<td>1164</td>
<td>108</td>
<td>39</td>
</tr>
<tr>
<td>0.10E-09</td>
<td>202023</td>
<td>2497</td>
<td>167</td>
<td>53</td>
</tr>
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<td>0.10E-10</td>
<td>638849</td>
<td>5368</td>
<td>260</td>
<td>71</td>
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<td>0.10E-11</td>
<td>2020214</td>
<td>11553</td>
<td>407</td>
<td>96</td>
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</table>

Table XII.b Number of integration steps per tolerance for Problem 1. Average number of repeat steps is 0.00 for all four schemes.
### Table XIII.a Maximum error in units of the error tolerance for Problem 2.

<table>
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<tr>
<th>Tolerance</th>
<th>Scheme:</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10E-01</td>
<td></td>
<td>0.0447</td>
<td>0.0179</td>
<td>0.0129</td>
<td>0.0026</td>
</tr>
<tr>
<td>0.10E-02</td>
<td></td>
<td>0.0509</td>
<td>0.0195</td>
<td>0.0330</td>
<td>0.0127</td>
</tr>
<tr>
<td>0.10E-03</td>
<td></td>
<td>0.0530</td>
<td>0.0149</td>
<td>0.0447</td>
<td>0.0220</td>
</tr>
<tr>
<td>0.10E-04</td>
<td></td>
<td>0.0536</td>
<td>0.0090</td>
<td>0.0491</td>
<td>0.0312</td>
</tr>
<tr>
<td>0.10E-05</td>
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<td>0.0538</td>
<td>0.0047</td>
<td>0.0463</td>
<td>0.0389</td>
</tr>
<tr>
<td>0.10E-06</td>
<td></td>
<td>0.0539</td>
<td>0.0023</td>
<td>0.0391</td>
<td>0.0439</td>
</tr>
<tr>
<td>0.10E-07</td>
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<td>0.0539</td>
<td>0.0011</td>
<td>0.0300</td>
<td>0.0453</td>
</tr>
<tr>
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<td>0.0539</td>
<td>0.0005</td>
<td>0.0217</td>
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<td>0.0149</td>
<td>0.0384</td>
</tr>
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<td>0.0100</td>
<td>0.0325</td>
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<td>0.2639</td>
<td>0.0043</td>
<td>0.0067</td>
<td>0.0267</td>
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</table>

### Table XIII.b Number of integration steps per tolerance for Problem 2. Average number of repeat steps is 0.00 for all four schemes.

<table>
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<th>Scheme:</th>
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<th>3</th>
<th>4</th>
</tr>
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<tbody>
<tr>
<td>0.10E-01</td>
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<td>10</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
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<td>17</td>
<td>10</td>
<td>8</td>
</tr>
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<td>10</td>
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<td>121</td>
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<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
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<td>2.7849</td>
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</tr>
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<td>0.6472</td>
<td>0.0394</td>
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</tr>
<tr>
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<td>0.9516</td>
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</tr>
<tr>
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<td>0.0019</td>
<td>0.0953</td>
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<td>0.10E-10</td>
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<td>0.0205</td>
<td>0.2030</td>
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<td>6.7590</td>
<td>0.2318</td>
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<td>0.1470</td>
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</table>

Table XIV.a Maximum error in units of the error tolerance for Problem 3.

<table>
<thead>
<tr>
<th>Tolerance</th>
<th>Scheme:</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
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<td>189</td>
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<td>0.10E-02</td>
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<tr>
<td>0.10E-03</td>
<td>1833</td>
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<tr>
<td>0.10E-04</td>
<td>5785</td>
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<td>81</td>
<td>45</td>
</tr>
<tr>
<td>0.10E-05</td>
<td>18282</td>
<td>786</td>
<td>124</td>
<td>59</td>
</tr>
<tr>
<td>0.10E-06</td>
<td>57802</td>
<td>1676</td>
<td>189</td>
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<td>0.10E-07</td>
<td>182774</td>
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</tr>
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</tr>
<tr>
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<td>200</td>
</tr>
<tr>
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<td>5779651</td>
<td>35699</td>
<td>1129</td>
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<tr>
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<td>18276843</td>
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<td>1784</td>
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</tr>
</tbody>
</table>

Table XIV.b Number of integration steps per tolerance for Problem 3. Average number of repeat steps is 13.27, 39.55, 21.55, and 18.64, respectively.
#### Table XV.a

<table>
<thead>
<tr>
<th>Tolerance</th>
<th>Scheme 1</th>
<th>Scheme 2</th>
<th>Scheme 3</th>
<th>Scheme 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10E-01</td>
<td>1.0845</td>
<td>0.1886</td>
<td>0.1124</td>
<td></td>
</tr>
<tr>
<td>0.10E-02</td>
<td>0.0447</td>
<td>0.0563</td>
<td>0.1249</td>
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</tr>
<tr>
<td>0.10E-03</td>
<td>0.0358</td>
<td>0.0960</td>
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<td></td>
</tr>
<tr>
<td>0.10E-04</td>
<td>0.0253</td>
<td>0.1353</td>
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<td></td>
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<td>0.1105</td>
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</tr>
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</tr>
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<td>0.0655</td>
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</tr>
<tr>
<td>0.10E-08</td>
<td>0.0049</td>
<td>0.0670</td>
<td>0.0680</td>
<td></td>
</tr>
<tr>
<td>0.10E-09</td>
<td>0.0031</td>
<td>0.0706</td>
<td>0.0458</td>
<td></td>
</tr>
<tr>
<td>0.10E-10</td>
<td>0.0064</td>
<td>0.0375</td>
<td>0.0276</td>
<td></td>
</tr>
<tr>
<td>0.10E-11</td>
<td>0.1315</td>
<td>0.0426</td>
<td>0.0249</td>
<td></td>
</tr>
</tbody>
</table>

Maximum error in units of the error tolerance for Problem 4.

#### Table XV.b

<table>
<thead>
<tr>
<th>Tolerance</th>
<th>Scheme 1</th>
<th>Scheme 2</th>
<th>Scheme 3</th>
<th>Scheme 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10E-01</td>
<td>61</td>
<td>15</td>
<td>9</td>
<td>8</td>
</tr>
<tr>
<td>0.10E-02</td>
<td>188</td>
<td>26</td>
<td>12</td>
<td>9</td>
</tr>
<tr>
<td>0.10E-03</td>
<td>587</td>
<td>48</td>
<td>16</td>
<td>11</td>
</tr>
<tr>
<td>0.10E-04</td>
<td>1851</td>
<td>94</td>
<td>22</td>
<td>13</td>
</tr>
<tr>
<td>0.10E-05</td>
<td>5845</td>
<td>193</td>
<td>31</td>
<td>17</td>
</tr>
<tr>
<td>0.10E-06</td>
<td>18476</td>
<td>406</td>
<td>45</td>
<td>21</td>
</tr>
<tr>
<td>0.10E-07</td>
<td>58421</td>
<td>864</td>
<td>68</td>
<td>27</td>
</tr>
<tr>
<td>0.10E-08</td>
<td>184736</td>
<td>1849</td>
<td>102</td>
<td>35</td>
</tr>
<tr>
<td>0.10E-09</td>
<td>584179</td>
<td>3973</td>
<td>157</td>
<td>46</td>
</tr>
<tr>
<td>0.10E-10</td>
<td>1847330</td>
<td>8546</td>
<td>244</td>
<td>61</td>
</tr>
<tr>
<td>0.10E-11</td>
<td>5841766</td>
<td>18398</td>
<td>382</td>
<td>81</td>
</tr>
</tbody>
</table>

Number of integration steps per tolerance for Problem 4. Average number of repeat steps is 1.82, 5.64, 2.00, and 2.45, respectively.
### Table XVI.a  Maximum error in units of the error tolerance for Problem 5.

<table>
<thead>
<tr>
<th>Tolerance</th>
<th>Scheme 1</th>
<th>Scheme 2</th>
<th>Scheme 3</th>
<th>Scheme 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10E-01</td>
<td>0.8047</td>
<td>0.4356</td>
<td>0.2230</td>
<td>0.0720</td>
</tr>
<tr>
<td>0.10E-02</td>
<td>0.8404</td>
<td>0.3574</td>
<td>0.4007</td>
<td>0.1866</td>
</tr>
<tr>
<td>0.10E-03</td>
<td>0.8525</td>
<td>0.2637</td>
<td>0.5048</td>
<td>0.4553</td>
</tr>
<tr>
<td>0.10E-04</td>
<td>0.8564</td>
<td>0.1598</td>
<td>0.4884</td>
<td>0.3063</td>
</tr>
<tr>
<td>0.10E-05</td>
<td>0.8576</td>
<td>0.0965</td>
<td>0.4550</td>
<td>0.5675</td>
</tr>
<tr>
<td>0.10E-06</td>
<td>0.8580</td>
<td>0.0512</td>
<td>0.3541</td>
<td>0.5609</td>
</tr>
<tr>
<td>0.10E-07</td>
<td>0.8582</td>
<td>0.0293</td>
<td>0.2658</td>
<td>0.5781</td>
</tr>
<tr>
<td>0.10E-08</td>
<td>0.8584</td>
<td>0.0154</td>
<td>0.1848</td>
<td>0.5165</td>
</tr>
<tr>
<td>0.10E-09</td>
<td>0.9120</td>
<td>0.0109</td>
<td>0.1254</td>
<td>0.4395</td>
</tr>
<tr>
<td>0.10E-10</td>
<td>0.6662</td>
<td>0.0266</td>
<td>0.0828</td>
<td>0.3670</td>
</tr>
<tr>
<td>0.10E-11</td>
<td>12.1506</td>
<td>0.1926</td>
<td>0.0400</td>
<td>0.3304</td>
</tr>
</tbody>
</table>

### Table XVI.b  Number of integration steps per tolerance for Problem 5. Average number of repeat steps is 0.00, 3.36, 0.00, and 0.00, respectively.

<table>
<thead>
<tr>
<th>Tolerance</th>
<th>Scheme 1</th>
<th>Scheme 2</th>
<th>Scheme 3</th>
<th>Scheme 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10E-01</td>
<td>66</td>
<td>13</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>0.10E-02</td>
<td>203</td>
<td>23</td>
<td>11</td>
<td>8</td>
</tr>
<tr>
<td>0.10E-03</td>
<td>635</td>
<td>41</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>0.10E-04</td>
<td>2003</td>
<td>82</td>
<td>21</td>
<td>13</td>
</tr>
<tr>
<td>0.10E-05</td>
<td>6328</td>
<td>169</td>
<td>30</td>
<td>16</td>
</tr>
<tr>
<td>0.10E-06</td>
<td>20005</td>
<td>355</td>
<td>45</td>
<td>21</td>
</tr>
<tr>
<td>0.10E-07</td>
<td>63254</td>
<td>757</td>
<td>68</td>
<td>28</td>
</tr>
<tr>
<td>0.10E-08</td>
<td>200021</td>
<td>1620</td>
<td>104</td>
<td>36</td>
</tr>
<tr>
<td>0.10E-09</td>
<td>632515</td>
<td>3479</td>
<td>161</td>
<td>49</td>
</tr>
<tr>
<td>0.10E-10</td>
<td>2000183</td>
<td>7484</td>
<td>251</td>
<td>65</td>
</tr>
<tr>
<td>0.10E-11</td>
<td>6325129</td>
<td>16113</td>
<td>393</td>
<td>88</td>
</tr>
</tbody>
</table>
Numerical Results - ODE's with a Singular Point

Three ODE's with a regular singularity at the origin are Bessel's equation,

\[ x^2 y'' + x y' + (x^2 - k^2) y = 0 \]  (8.1)

Coulomb’s equation,

\[ x^2 y'' + \left[ -L(L+1) - 2 \eta x + x^2 \right] y = 0 \]  (8.2)

and the hypergeometric equation (Gauss’s equation)

\[ x (1-x) y'' + \left[ \gamma - (\alpha + \beta + 1) x \right] y' - \alpha \beta y = 0. \]  (8.3)

The above equations are typical of a large class of linear, second-order equations from mathematical physics with a regular singular point. A general approach for integrating such equations has been presented by Holubec and Stauffer [31]. Haftel et. al. [32] presented a similar approach for coupled systems. We show here that implicit Taylor series methods can directly integrate such problems as a special case. We consider the following set of model problems.

<table>
<thead>
<tr>
<th>Problem</th>
<th>ODE</th>
<th>Initial Condition</th>
<th>Exact Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.a</td>
<td>(8.1) with ( k = 0 )</td>
<td>( y(0) = 1 ) ( y'(0) = 0 )</td>
<td>( J_0 )</td>
</tr>
<tr>
<td>6.b</td>
<td>(8.1) with ( k = 1 )</td>
<td>( y(0) = 0 ) ( y'(0) = 1/2 )</td>
<td>( J_1 )</td>
</tr>
<tr>
<td>7.</td>
<td>(8.2) with ( \eta = 1 ) ( L = 0 )</td>
<td>( y(0) = 0 ) ( y'(0) = e^{-\frac{x}{2}} \sqrt{\pi} \sinh \pi )</td>
<td>Coulomb Wavefunction</td>
</tr>
<tr>
<td>8.a</td>
<td>(8.3) with ( \alpha = 0.4 ) ( \beta = 1.0 ) ( \gamma = 0.5 )</td>
<td>( y(0) = 1 ) ( y'(0) = 0.8 )</td>
<td>Hypergeometric Function</td>
</tr>
<tr>
<td>8.b</td>
<td>(8.3) with ( \alpha = 0.5 ) ( \beta = 0.5 ) ( \gamma = 1.5 )</td>
<td>( y(0) = 1 ) ( y'(0) = 1/6 )</td>
<td>Hypergeometric Function</td>
</tr>
</tbody>
</table>

We first looked at order of accuracy near the singular point. Each problem was reformulated as a system of two first order equations in the usual way. Problems 6 and 7
were integrated on [0,1], and Problem 8 on [0, ½]. Error reduction results are plotted in Figures 17 - 19, and the corresponding orders of accuracy are summarized in Table XVII. It is evident that a significant loss of accuracy occurred on Problems 6.b and 7. The degree-4 Taylor series lost from one to four orders of accuracy, while the degree-2 and -3 series lost from one to two orders. However, it should be noted that the orders of accuracy above are as high or higher than an explicit Taylor series of the same degree would be for a nonsingular problem. We also observe from Figures 17 - 19 that without exception the extrapolated solution for the degree-3 and -4 Taylor series is more accurate than the non-extrapolated solutions.

To look at variable step performance, we used Schemes 1 - 4 to solve Problems 6.a,b on the interval [0,20]. Each scheme was used as described in Section 7, with appropriate modifications for a system. The results shown in Tables XVIII - IXX are indeed comparable to those in Tables XII - XVI for the nonsingular Problems 1 - 5.

<table>
<thead>
<tr>
<th>Taylor Series</th>
<th>Problem:</th>
<th>6.a</th>
<th>6.b</th>
<th>7</th>
<th>8.a</th>
<th>8.b</th>
</tr>
</thead>
<tbody>
<tr>
<td>Degree-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mu = \frac{1}{2} )</td>
<td></td>
<td>2.00</td>
<td>1.89</td>
<td>1.85</td>
<td>2.01</td>
<td>1.92</td>
</tr>
<tr>
<td>Degree-2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mu = \frac{1}{2} + i \frac{\sqrt{3}}{6} )</td>
<td></td>
<td>3.93</td>
<td>2.00</td>
<td>3.02</td>
<td>3.82</td>
<td>4.07</td>
</tr>
<tr>
<td>Degree-3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mu = \frac{1}{2} )</td>
<td></td>
<td>3.99</td>
<td>3.89</td>
<td>3.82</td>
<td>4.01</td>
<td>3.88</td>
</tr>
<tr>
<td>( \mu = \frac{1}{2} + i \frac{1}{2} )</td>
<td></td>
<td>3.99</td>
<td>3.89</td>
<td>3.81</td>
<td>3.84</td>
<td>3.77</td>
</tr>
<tr>
<td>extrapolation</td>
<td></td>
<td>5.83</td>
<td>4.47</td>
<td>4.21</td>
<td>5.57</td>
<td>5.77</td>
</tr>
<tr>
<td>Degree-4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mu = \frac{1}{2} + i \frac{1}{2} \tan \frac{\pi}{10} )</td>
<td></td>
<td>5.92</td>
<td>4.01</td>
<td>4.99</td>
<td>5.84</td>
<td>5.75</td>
</tr>
<tr>
<td>( \mu = \frac{1}{2} + i \frac{1}{2} \tan \frac{3\pi}{10} )</td>
<td></td>
<td>5.92</td>
<td>4.10</td>
<td>5.02</td>
<td>5.70</td>
<td>5.62</td>
</tr>
<tr>
<td>extrapolation</td>
<td></td>
<td>5.83</td>
<td>3.94</td>
<td>4.83</td>
<td>7.20</td>
<td>7.48</td>
</tr>
</tbody>
</table>

Table XVII  Order of accuracy near a regular singular point.
### Table XVIII.a  Maximum error in units of the error tolerance for Problem 6.a.

<table>
<thead>
<tr>
<th>Tolerance</th>
<th>Scheme:</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>0.10E-01</td>
<td>1.0965</td>
<td>1.7756</td>
<td>0.4928</td>
<td>1.1900</td>
</tr>
<tr>
<td>0.10E-02</td>
<td>1.0755</td>
<td>0.5731</td>
<td>0.4334</td>
<td>1.0079</td>
</tr>
<tr>
<td>0.10E-03</td>
<td>1.0736</td>
<td>0.2519</td>
<td>0.2957</td>
<td>0.4966</td>
</tr>
<tr>
<td>0.10E-04</td>
<td>1.0733</td>
<td>0.1157</td>
<td>0.2076</td>
<td>0.3070</td>
</tr>
<tr>
<td>0.10E-05</td>
<td>1.0732</td>
<td>0.0536</td>
<td>0.1335</td>
<td>0.2350</td>
</tr>
<tr>
<td>0.10E-06</td>
<td>1.0732</td>
<td>0.0249</td>
<td>0.0849</td>
<td>0.1643</td>
</tr>
<tr>
<td>0.10E-07</td>
<td>1.0732</td>
<td>0.0116</td>
<td>0.0543</td>
<td>0.1179</td>
</tr>
<tr>
<td>0.10E-08</td>
<td>1.0733</td>
<td>0.0054</td>
<td>0.0340</td>
<td>0.0878</td>
</tr>
<tr>
<td>0.10E-09</td>
<td>1.0744</td>
<td>0.0024</td>
<td>0.0216</td>
<td>0.0624</td>
</tr>
<tr>
<td>0.10E-10</td>
<td>1.1015</td>
<td>0.0012</td>
<td>0.0130</td>
<td>0.0622</td>
</tr>
<tr>
<td>0.10E-11</td>
<td>1.3933</td>
<td>0.0212</td>
<td>0.0106</td>
<td>0.0892</td>
</tr>
</tbody>
</table>

### Table XVIII.b  Number of integration steps per tolerance for Problem 6.a. Average number of repeat steps is 0.00, 0.09, 0.64, and 0.45, respectively.

<table>
<thead>
<tr>
<th>Tolerance</th>
<th>Scheme:</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>0.10E-01</td>
<td>114</td>
<td>24</td>
<td>15</td>
<td>13</td>
</tr>
<tr>
<td>0.10E-02</td>
<td>332</td>
<td>51</td>
<td>23</td>
<td>18</td>
</tr>
<tr>
<td>0.10E-03</td>
<td>1021</td>
<td>107</td>
<td>34</td>
<td>22</td>
</tr>
<tr>
<td>0.10E-04</td>
<td>3198</td>
<td>226</td>
<td>52</td>
<td>28</td>
</tr>
<tr>
<td>0.10E-05</td>
<td>10083</td>
<td>481</td>
<td>81</td>
<td>37</td>
</tr>
<tr>
<td>0.10E-06</td>
<td>31854</td>
<td>1032</td>
<td>125</td>
<td>49</td>
</tr>
<tr>
<td>0.10E-07</td>
<td>100701</td>
<td>2217</td>
<td>196</td>
<td>66</td>
</tr>
<tr>
<td>0.10E-08</td>
<td>318415</td>
<td>4769</td>
<td>308</td>
<td>89</td>
</tr>
<tr>
<td>0.10E-09</td>
<td>1006888</td>
<td>10269</td>
<td>484</td>
<td>120</td>
</tr>
<tr>
<td>0.10E-10</td>
<td>3184029</td>
<td>22115</td>
<td>764</td>
<td>164</td>
</tr>
<tr>
<td>0.10E-11</td>
<td>10068755</td>
<td>47639</td>
<td>1206</td>
<td>225</td>
</tr>
</tbody>
</table>
### Table IXX.a

Maximum error in units of the error tolerance for Problem 6.b.

<table>
<thead>
<tr>
<th>Tolerance</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10E-01</td>
<td>1.1825</td>
<td>2.0417</td>
<td>0.5799</td>
<td>3.3158</td>
</tr>
<tr>
<td>0.10E-02</td>
<td>1.1569</td>
<td>0.6318</td>
<td>0.4310</td>
<td>0.8488</td>
</tr>
<tr>
<td>0.10E-03</td>
<td>1.1535</td>
<td>0.2763</td>
<td>0.3510</td>
<td>0.5983</td>
</tr>
<tr>
<td>0.10E-04</td>
<td>1.1521</td>
<td>0.1262</td>
<td>0.2365</td>
<td>0.3736</td>
</tr>
<tr>
<td>0.10E-05</td>
<td>1.1511</td>
<td>0.1003</td>
<td>0.1515</td>
<td>0.2587</td>
</tr>
<tr>
<td>0.10E-06</td>
<td>1.1505</td>
<td>0.1007</td>
<td>0.0958</td>
<td>0.1845</td>
</tr>
<tr>
<td>0.10E-07</td>
<td>1.1502</td>
<td>0.0993</td>
<td>0.0604</td>
<td>0.1307</td>
</tr>
<tr>
<td>0.10E-08</td>
<td>1.1501</td>
<td>0.0941</td>
<td>0.0378</td>
<td>0.0940</td>
</tr>
<tr>
<td>0.10E-09</td>
<td>1.1492</td>
<td>0.0832</td>
<td>0.0237</td>
<td>0.0736</td>
</tr>
<tr>
<td>0.10E-10</td>
<td>1.1454</td>
<td>0.0720</td>
<td>0.0172</td>
<td>0.2112</td>
</tr>
<tr>
<td>0.10E-11</td>
<td>1.3087</td>
<td>0.0774</td>
<td>0.0201</td>
<td>0.5614</td>
</tr>
</tbody>
</table>

### Table IXX.b

Number of integration steps per tolerance for Problem 6.b. Average number of repeat steps is 0.00, 0.09, 0.09, and 0.00, respectively.

<table>
<thead>
<tr>
<th>Tolerance</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10E-01</td>
<td>113</td>
<td>24</td>
<td>18</td>
<td>19</td>
</tr>
<tr>
<td>0.10E-02</td>
<td>326</td>
<td>50</td>
<td>25</td>
<td>23</td>
</tr>
<tr>
<td>0.10E-03</td>
<td>1000</td>
<td>106</td>
<td>37</td>
<td>27</td>
</tr>
<tr>
<td>0.10E-04</td>
<td>3131</td>
<td>224</td>
<td>55</td>
<td>34</td>
</tr>
<tr>
<td>0.10E-05</td>
<td>9869</td>
<td>477</td>
<td>83</td>
<td>42</td>
</tr>
<tr>
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</table>
Numerical Results - Stiff Equations

In this section we briefly consider stiff equations. For a set of model problems we use Problems 9 - 13 below.

Problem 9: (Enright and Pryce [33], Problem A3)

\[
\begin{align*}
y_1' &= -10^4 y_1 + 100 y_2 - 10 y_3 + y_4 \\
y_2' &= -10^3 y_2 + 10 y_3 - 10 y_4 \\
y_3' &= -y_3 + 10 y_4 \\
y_4' &= -0.1 y_4
\end{align*}
\]

\[y_1(0) = 1, \quad y_2(0) = 1, \quad y_3(0) = 1, \quad y_4(0) = 1, \quad t_{\text{end}} = 20\]

Problem 10: ([33], Problem B1)

\[
\begin{align*}
y_1' &= -y_1 + y_2 \\
y_2' &= -100 y_1 - y_2 \\
y_3' &= -100 y_3 + y_4 \\
y_4' &= -10^4 y_3 - 100 y_4
\end{align*}
\]

\[y_1(0) = 1, \quad y_2(0) = 0, \quad y_3(0) = 1, \quad y_4(0) = 0, \quad t_{\text{end}} = 20\]

Problem 11: ([33], Problem D4)

\[
\begin{align*}
y_1' &= -0.013 y_1 - 1000 y_1 y_3 \\
y_2' &= -2500 y_2 y_3 \\
y_3' &= -0.013 y_1 - 1000 y_1 y_3 - 2500 y_2 y_3
\end{align*}
\]

\[y_1(0) = 1, \quad y_2(0) = 1, \quad y_3(0) = 0, \quad t_{\text{end}} = 50\]
Problem 12: (Robertson [34])

\[
\begin{align*}
\dot{y}_1 &= -0.04y_1 + 10^4y_2y_3 \\
\dot{y}_2 &= 0.04y_1 - 10^4y_2y_3 - 3 \cdot 10^7y_2^2 \\
\dot{y}_3 &= 3 \cdot 10^7y_2^2 \\
\end{align*}
\]

\[y_1(0) = 1, \quad y_2(0) = 0, \quad y_3(0) = 0, \quad t_{\text{end}} = 40\]

Problem 13: ([33], Problem F3)

\[
\begin{align*}
\dot{y}_1 &= -10^7y_2y_1 + 10y_3 \\
\dot{y}_2 &= -10^7y_2y_1 - 10^7y_2y_5 + 10y_3 + 10y_4 \\
\dot{y}_3 &= 10^7y_2y_1 - 1.001 \cdot 10^4y_3 + 10^{-3}y_4 \\
\dot{y}_4 &= 10^4y_3 - 10.001y_4 + 10^7y_2y_5 \\
\dot{y}_5 &= 10y_4 - 10^7y_2y_5 \\
\end{align*}
\]

\[y_1(0) = 4 \cdot 10^{-6}, \quad y_2(0) = 1 \cdot 10^{-6}, \quad y_3(0) = 0, \quad y_4(0) = 0, \quad y_5(0) = 0 \]

\[t_{\text{end}} = 100\]

We integrated each system using Schemes 1 - 4, along with a trapezoidal extrapolation scheme which we now introduce.

Let \(y_n^T\) denote the trapezoidal solution (2.15), and let \(y_n^{(2)}\) denote the degree-2 Taylor series solution (3.14) with \(\mu = \frac{1}{2}\). One can show that the extrapolated solution

\[
y_n^{e} := \frac{2y_n^{(2)} + y_{n+1}^T}{3} \quad (9.7)
\]
is A-stable and fourth-order-accurate with truncation error

\[ y(t_{n+1}) - y_{n+1}^e = - \frac{h^5}{2880} \left[ f''''_{n} + 5 \frac{\partial f_{n}}{\partial y} f'''_{n} + 20 \left( \frac{\partial^2 f_{n}}{\partial t \partial y} + \frac{\partial^2 f_{n}}{\partial y^2} f_{n} \right) f''_{n} \right]. \]  

(9.8)

A variable stepsize for (9.7) can be obtained using (7.1) together with the \( O(h^3) \) error estimate \( e_n = |v_{n+1}^e - y_{n+1}^{(2)}| \). The above extrapolation with variable step implementation will be called Scheme 2T.

In integrating Problems 9 - 13, the parameters \( \xi, h_{\text{fact}}, \) and \( \nu \), introduced in Section 7, were assigned default values of 0.9, 2.0, and 0.7, respectively, except for Scheme 1 where \( \xi \) was set to 0.7. Any deviation from these values is noted at the bottom of Tables XX - XXIV. The local error estimate \( e_n \) was taken to be the largest absolute error in any solution component. The initial stepsize was determined as in Section 7 (with appropriate modifications for a system).

Tables XX - XXI demonstrate some degradation in the ability of Schemes 1 - 4 to meet the error tolerance. Whether this holds in general for stiff problems requires further investigation.

One also observes that the eighth-order-accurate Scheme 4 frequently required more steps than some of the lower order schemes. This contrasts with the nonstiff results presented earlier, and illustrates the well known need for stiff solvers to be able to vary their order.

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Table XX.a Maximum error in units of the error tolerance for Problem 9.
### Table XX.b
Number of integration steps per error tolerance for Problem 9. Average number of repeat steps is 0.18, 0.91, 1.18, 5.82, and 13.00, respectively. Changes to default values: Scheme 3, $ν = 0.5$; Scheme 4, $h_{\text{fact}} = 1.5$, $ν = 0.5$.

<table>
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### Table XXI.a
Maximum error in units of the error tolerance for Problem 10.

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See NASA/TM—2000-209400 for more information.
### Table XXI.b
Number of integration steps per error tolerance for Problem 10. Average number of repeat steps is 19.45, 26.64, 26.73, 19.18, and 10.00, respectively. Changes to default values: Scheme 1, \( \nu = 0.5 \); Scheme 2, \( \xi = 0.8 \); Scheme 2T, \( \xi = 0.8 \); Scheme 3, \( \xi = 0.8 \); Scheme 4, \( \xi = 0.8, \nu = 0.5 \).

<table>
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### Table XXII
Number of integration steps per error tolerance for Problem 11. Average number of repeat steps is 0.00, 1.45, 1.55, 0.09, and 10.36, respectively. Changes to default values: Scheme 3, \( \nu = 0.5 \); Scheme 4, \( \xi = 0.8, h_{fact} = 1.5, \nu = 0.5 \).

<table>
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<tr>
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Table XXIII  Number of integration steps per error tolerance for Problem 12. Average number of repeat steps is 0.00, 3.91, 2.82, 5.55, and 28.18, respectively. Changes to default values: Scheme 2, \( \nu = 0.5 \); Scheme 2T, \( \nu = 0.5 \); Scheme 3, \( _{_{-}} = 0.8, h_{\text{fact}} = 1.5, \nu = 0.5 \); Scheme 4, \( _{_{-}} = 0.8, h_{\text{fact}} = 1.5, \nu = 0.5 \).

Table XXIV  Number of integration steps per error tolerance for Problem 13. Average number of repeat steps is 0.00, 0.00, 0.00, 0.00, and 4.09, respectively. Changes to default values: Scheme 4, \( _{_{-}} = 0.8, h_{\text{fact}} = 1.5, \nu = 0.5 \).
Discussion and Conclusion

In this paper we have presented a new class of implicit Taylor series methods for solving ODE initial value problems. Based on the results presented herein, we conclude that the main advantages of implicit series methods are (i) ability to achieve high accuracy while maintaining stability, (ii) robustness, (iii) simplicity, and (iv) versatility (ability to solve a wide range of problems).

However, implicit series methods suffer the disadvantage of being computationally intensive. Table XXV summarizes the median number of Newton iterations per step for the results presented in the last three sections. The Newton convergence criteria for these results was that successive solution iterates differ by less than $10^{-14}$.

It is seen that a minimum of four Newton iterations are required per step – two iterations for each independent solution calculated. This means that, for a system of equations, at least two L-U decompositions must be performed at each step. Consequently, an implicit series method must be able to solve a given problem in relatively fewer steps to be competitive with other methods. For most nonstiff problems, this implies the need for a series with high order accuracy. For stiff problems, this implies the need to vary the order of accuracy during the calculation.

We conclude with the following remarks. First, for nonstiff problems, corrector iteration is an attractive alternative to Newton iteration. It is easier to implement, requires less derivative information, and for small error tolerances is often more computationally efficient. It works especially well for small, uniform stepsizes. Second, for problems without a singular point, Scheme 2T is preferred over Scheme 2. Although both schemes employ a degree-two Taylor series, use the same local error estimate (to $O(h^5)$), and generally require the same number of marching steps, Scheme 2T can be programmed in real arithmetic and is significantly faster as a result. Third, due to the length of this paper, we have not made any direct comparisons with other methods. Comparing the efficiency and reliability of integration methods has practically become a specialty in itself, and requires considerable care and discussion. The intent here is to present enough results to give some idea of performance on a range of problems. The next step toward a full comparison with other methods is to extend implicit series methods to become a variable order approach. This is proposed as a future research topic. Fourth, and lastly, implicit Taylor series methods offer a unique combination of high accuracy, stability, robustness, and versatility. The author is not aware of another existing method that can, without modification, solve both regular ODE’s and stiff systems, together with linear and nonlinear ODE’s with a singular point – all with an arbitrarily high order of accuracy. With computer speed and memory steadily increasing, algorithm features such as robustness, versatility, and ease of use are becoming increasingly important. For this reason, implicit Taylor series methods merit consideration as a standard integration tool for solving ordinary differential equations.
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<td>10.56</td>
</tr>
</tbody>
</table>

Table XXV  Median number of Newton iterations per timestep.
Appendix

Let \( y_{n+1} := y_{n+1}^{(k)} \) be the solution of

\[
y_{n+1} = y_n + h f(t_n + 1, y_{n+1}) - \frac{h^2}{2!} f'(t_n + 1, y_{n+1}) + \frac{h^3}{3!} f''(t_n + 1, y_{n+1}) \\
- \ldots + \ldots + (-1)^{k-1} \frac{h^k}{k!} f^{(k-1)}(t_n + 1, y_{n+1}).
\] (A.1)

Then for all \( k \geq 2 \),

\[
e_{n+1}^{(k)} := y(t_{n+1}) - y_{n+1}^{(k)} = \left\{ (-1)^k \left[ \frac{h^{k+1}}{(k+1)!} f_n^{(k)} + \frac{h^{k+2}}{(k+2)!} [(k + 1) f_n^{(k+1)} + (k + 2) \frac{\partial f_n}{\partial y} f_n^{(k)}] \\
+ \frac{h^{k+3}}{(k+3)!} \left[ \frac{(k + 1)(k + 2)}{2} f_n^{(k+2)} + (k + 1)(k + 3) \frac{\partial f_n}{\partial y} f_n^{(k+1)} \\
+ \frac{(k + 2)(k + 3)}{2} \frac{\partial f_n}{\partial y} f_n^{(k+1)} \right] \right. \right\} + O(h^{k+4}).
\] (A.2)

Proof:

From equation (A.1), one can expand the solution \( y_{n+1}^{(k+1)} \) in a series about the solution \( y_{n+1}^{(k)} \) to obtain

\[
y_{n+1}^{(k+1)} = y_n + h \left[ f(t_n + 1, y_{n+1}^{(k)}) + (y_{n+1}^{(k)} - y_{n+1}^{(k)}) \frac{\partial f}{\partial y} (t_{n+1}, y_{n+1}^{(k)}) \\
+ \frac{1}{2} \left( y_{n+1}^{(k)} - y_{n+1}^{(k)} \right)^2 \frac{\partial^2 f}{\partial y^2} (t_{n+1}, y_{n+1}^{(k)}) + \ldots \right] - \frac{h^2}{2!} \left[ f'(t_{n+1}, y_{n+1}^{(k)}) \\
+ \left( y_{n+1}^{(k)} - y_{n+1}^{(k)} \right) \frac{\partial f'}{\partial y} (t_{n+1}, y_{n+1}^{(k)}) + \frac{1}{2} \left( y_{n+1}^{(k)} - y_{n+1}^{(k)} \right)^2 \frac{\partial^2 f'}{\partial y^2} (t_{n+1}, y_{n+1}^{(k)}) + \ldots \right] \\
+ \ldots - \ldots + (-1)^k \frac{h^k}{(k+1)!} \left[ f^{(k)}(t_{n+1}, y_{n+1}^{(k)}) + (y_{n+1}^{(k)} - y_{n+1}^{(k)}) \frac{\partial f^{(k)}}{\partial y} (t_{n+1}, y_{n+1}^{(k)}) \\
+ \frac{1}{2} \left( y_{n+1}^{(k)} - y_{n+1}^{(k)} \right)^2 \frac{\partial^2 f^{(k)}}{\partial y^2} (t_{n+1}, y_{n+1}^{(k)}) + \ldots \right] \right) \right) + O(h^{k+4}).
\] (A.3)
from which one gets

\[ y_{n+1}^{(k+1)} = y_n + h f(t_{n+1}, y_{n+1}^{(k)}) - \frac{h^2}{2!} f'(t_{n+1}, y_{n+1}^{(k)}) \]

\[ + \ldots - \ldots + (-1)^k \frac{h^{k+1}}{(k+1)!} f^{(k)}(t_{n+1}, y_{n+1}^{(k)}) \]

\[ + \left( y_{n+1}^{(k+1)} - y_n^{(k)} \right) \left[ h \frac{\partial f}{\partial y} (t_{n+1}, y_{n+1}^{(k)}) - \frac{h^2}{2!} \frac{\partial f'}{\partial y} (t_{n+1}, y_{n+1}^{(k)}) \right] \]

\[ + \ldots - \ldots + (-1)^k \frac{h^{k+1}}{(k+1)!} \frac{\partial f^{(k)}}{\partial y} (t_{n+1}, y_{n+1}^{(k)}) \]

\[ + \ldots \]

\[ (A.4) \]

Now

\[ y_n + h f(t_{n+1}, y_{n+1}^{(k)}) - \frac{h^2}{2!} f'(t_{n+1}, y_{n+1}^{(k)}) \]

\[ + \ldots - \ldots + (-1)^{k-1} \frac{h^k}{k!} f^{(k-1)}(t_{n+1}, y_{n+1}^{(k)}) = y_{n+1}^{(k)} \]

from equation (A.1). Therefore (A.4) becomes

\[ y_{n+1}^{(k+1)} - y_{n+1}^{(k)} = (-1)^k \frac{h^{k+1}}{(k+1)!} f^{(k)}(t_{n+1}, y_{n+1}^{(k)}) \]

\[ + \left( y_{n+1}^{(k+1)} - y_n^{(k)} \right) \left[ h \frac{\partial f}{\partial y} (t_{n+1}, y_{n+1}^{(k)}) - \frac{h^2}{2!} \frac{\partial f'}{\partial y} (t_{n+1}, y_{n+1}^{(k)}) \right] \]

\[ + \ldots - \ldots + (-1)^k \frac{h^{k+1}}{(k+1)!} \frac{\partial f^{(k)}}{\partial y} (t_{n+1}, y_{n+1}^{(k)}) \]

\[ (A.6) \]

It follows from (A.6) that \( y_{n+1}^{(k+1)} - y_{n+1}^{(k)} \) is \( O(h^{k+1}) \). Retaining terms through \( O(h^{k+3}) \), one gets
\[
y^{(k+1)}_{n+1} - y^{(k)}_{n+1} = (-1)^k \frac{h^{k+1}}{(k+1)!} f^{(k)}(t_{n+1}, y^{(k)}_{n+1}) \]
\[
+ \left( y^{(k+1)}_{n+1} - y^{(k)}_{n+1} \right) \left[ h \frac{\partial f}{\partial y}(t_{n+1}, y^{(k)}_{n+1}) - \frac{h^2}{2} \frac{\partial f'}{\partial y}(t_{n+1}, y^{(k)}_{n+1}) + O(h^3) \right]
\]  
(A.7)

so that

\[
y^{(k+1)}_{n+1} - y^{(k)}_{n+1} = (-1)^k \frac{h^{k+1}}{(k+1)!} f^{(k)}(t_{n+1}, y^{(k)}_{n+1}) \left[ 1 - h \frac{\partial f}{\partial y}(t_{n+1}, y^{(k)}_{n+1}) + \frac{h^2}{2} \frac{\partial f'}{\partial y}(t_{n+1}, y^{(k)}_{n+1}) + O(h^3) \right].
\]  
(A.8)

For sufficiently small \( h \), one then obtains

\[
y^{(k+1)}_{n+1} - y^{(k)}_{n+1} = (-1)^k \frac{h^{k+1}}{(k+1)!} f^{(k)}(t_{n+1}, y^{(k)}_{n+1}) \left[ 1 + h \frac{\partial f}{\partial y}(t_{n+1}, y^{(k)}_{n+1}) + h^2 \left[ \frac{\partial f}{\partial y}(t_{n+1}, y^{(k)}_{n+1}) \right]^2 - \frac{1}{2} \frac{\partial f'}{\partial y}(t_{n+1}, y^{(k)}_{n+1}) \right] + O(h^{k+4}).
\]  
(A.9)

Now each term on the right hand side of (A.9) can be expanded in a series about \((t_n, y_n)\), so that

\[
y^{(k+1)}_{n+1} - y^{(k)}_{n+1} = (-1)^k \frac{h^{k+1}}{(k+1)!} \left[ f^{(k)}(t_n, y^{(k)}_{n+1}) \left( 1 + h \frac{\partial f}{\partial y}(t_{n+1}, y^{(k)}_{n+1}) + h^2 \left[ \frac{\partial f}{\partial y}(t_{n+1}, y^{(k)}_{n+1}) \right]^2 - \frac{1}{2} \frac{\partial f'}{\partial y}(t_{n+1}, y^{(k)}_{n+1}) \right] \right] + O(h^k).
\]  
(A.10)

For all \( k \geq 2 \), equation (3.8) shows that

\[
y^{(k)}_{n+1} - y_{n} = h f_n + \frac{h^2}{2} f_n' + O(h^3).
\]  
(A.11)

Substituting (A.11) into (A.10) and simplifying, one finally obtains

\[
y^{(k+1)}_{n+1} - y^{(k)}_{n+1} = (-1)^k \frac{h^{k+1}}{(k+1)!} \left[ f_n^{(k)} + h \left( f_n^{(k+1)} + \frac{\partial f_n}{\partial y} f_n^{(k)} \right) + \frac{h^2}{2} \left( f_n^{(k+2)} + 2 \frac{\partial f_n}{\partial y} f_n^{(k+1)} + \frac{\partial f_n'}{\partial y} f_n^{(k)} \right) \right].
\]  
(A.12)
Since

\[ y^{(k+1)}_{n+1} - y^{(k)}_{n+1} = y(t_{n+1}) - y^{(k)}_{n+1} = (y(t_{n+1}) - y^{(k+1)}_{n+1}) = e^{(k)}_{n+1} - e^{(k+1)}_{n+1} \] (A.13)

one obtains from (A.12)

\[
e^{(k+1)}_{n+1} - e^{(k)}_{n+1} = (-1)^{k+1} \frac{h^{k+1}}{(k+1)!} \left[ f^{(k)}_{n} + h \left( f^{(k+1)}_{n} + \frac{\partial f_{n}}{\partial y} f^{(k)}_{n} \right) \right]^2
\]

\[
+ \frac{h^2}{2} \left( f^{(k+2)}_{n} + 2 \frac{\partial f_{n}}{\partial y} f^{(k+1)}_{n} + \frac{\partial^2 f_{n}}{\partial y^2} f^{(k)}_{n} \right). \] (A.14)

Since \( n \) and \( h \) are both fixed, (A.14) is a linear difference equation on \( k \). One can verify by direct substitution that the solution to the difference equation is given by (A.2), plus an arbitrary constant which may be taken to be zero. Therefore (A.2) is valid for all \( k \geq 2 \).

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Figure 1.a  Stability region for Euler's method.
Figure 1.b Stability region for a degree-2 explicit Taylor series.
Figure 2  Stability region for a degree-3 explicit Taylor series.
Figure 3  Stability region for a degree-3 fully implicit Taylor series.
Figure 4.a  Stability region for a degree-3 Taylor series with $\mu = \frac{1}{2} + i \frac{1}{2}$. 
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Figure 8.c  Reduction of error for Problem 3. Results are shown for the Centered Euler Method, Trapezoidal Rule, and Implicit Midpoint Rule.
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Reduction of error for Problem 4 using a degree-2 Taylor series with $\mu = \frac{1}{2} + i \frac{\sqrt{3}}{6}$. 

Figure 10.d
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Figure 12.b  Reduction of error for Problem 2 using a degree-3 Taylor series with \( \mu_1 = \frac{1}{2} \), \( \mu_2 = \frac{1}{2} + i \frac{1}{2} \), and with extrapolation.
Figure 12.c  Reduction of error for Problem 3 using a degree-3 Taylor series with $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{2} + i \frac{1}{2}$, and with extrapolation.
Figure 12.d  Reduction of error for Problem 4 using a degree-3 Taylor series with $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{2} + i \frac{1}{2}$, and with extrapolation.
Figure 12.e Reduction of error for Problem 5 using a degree-3 Taylor series with $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{3}{2} + i \frac{1}{2}$, and with extrapolation.
Figure 13  Stability region for a degree-4 Taylor series with extrapolation.
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Figure 14. Reduction of error for Problem 2 using a degree-4 Taylor series with $\mu_1 = \frac{1}{2} + i \frac{1}{2} \tan \frac{\pi}{10}$, $\mu_2 = \frac{1}{2} + i \frac{1}{2} \tan \frac{3\pi}{10}$, and with extrapolation.
Figure 14.c Reduction of error for Problem 3 using a degree-4 Taylor series with $\mu_1 = \frac{1}{2} + i \frac{1}{2} \tan \frac{\pi}{10}$, $\mu_2 = \frac{1}{2} + i \frac{1}{2} \tan \frac{3\pi}{10}$, and with extrapolation.
Figure 14.d  Reduction of error for Problem 4 using a degree-4 Taylor series with $\mu_1 = \frac{1}{2} + i \frac{1}{2} \tan \frac{\pi}{10}$, $\mu_2 = \frac{1}{2} + i \frac{1}{2} \tan \frac{3\pi}{10}$, and with extrapolation.
Figure 14.e  Reduction of error for Problem 5 using a degree-4 Taylor series with $\mu_1 = \frac{1}{2} + i \frac{1}{2} \tan \frac{\pi}{10}$, $\mu_2 = \frac{1}{2} + i \frac{1}{2} \tan \frac{3\pi}{10}$, and with extrapolation.
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Reduction of error for Problems 6.a,b using a degree-2 Taylor series with 
\[ \mu = \frac{1}{2} + i \frac{\sqrt{3}}{6}. \]
Figure 17.c Reduction of error for Problem 6.a using a degree-3 Taylor series with $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{2} + i \frac{1}{2}$, and with extrapolation.
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and with extrapolation.
Figure 17.f  Reduction of error for Problem 6.b using a degree-4 Taylor series with \( \mu_1 = \frac{1}{2} + i\frac{1}{2} \tan \frac{\pi}{10} \), \( \mu_2 = \frac{1}{2} + i\frac{1}{2} \tan \frac{3\pi}{10} \), and with extrapolation.
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[Graph showing the relationship between Max Error and Stepsize h]
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Figure 19.c  Reduction of error for Problem 8.a using a degree-3 Taylor series with $\mu_1 = \frac{1}{2}$, $\mu_2 = \frac{1}{2} + i \frac{1}{2}$, and with extrapolation.
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Figure 19.f  Reduction of error for Problem 8.b using a degree-4 Taylor series with $\mu_1 = \frac{1}{2} + i \frac{1}{2} \tan \frac{\pi}{14}$, $\mu_2 = \frac{1}{2} + i \frac{1}{2} \tan \frac{3\pi}{14}$, and with extrapolation.
In this paper we introduce a new class of numerical methods for integrating ODE initial value problems. Specifically, we propose an extension of the Taylor series method which significantly improves its accuracy and stability while also increasing its range of applicability. To advance the solution from \( t_n \) to \( t_{n+1} \), we expand a series about the intermediate point \( t_{n+\mu h} = t_n + \mu h \), where \( h \) is the stepsize and \( \mu \) is an arbitrary parameter called an expansion coefficient. We show that, in general, a Taylor series of degree \( k \) has exactly \( k \) expansion coefficients which raise its order of accuracy. The accuracy is raised by one order if \( k \) is odd, and by two orders if \( k \) is even. In addition, if \( k \) is three or greater, local extrapolation can be used to raise the accuracy two additional orders. We also examine stability for the problem \( y'' = \lambda y, Re(\lambda) < 0 \), and identify several A-stable schemes. Numerical results are presented for both fixed and variable stepsizes. It is shown that implicit Taylor series methods provide an effective integration tool for most problems, including stiff systems and ODE’s with a singular point.