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AERODYNAMIC INDICIAL FUNCTIONS AND THEIR USE IN AEROELASTIC FORMULATION OF LIFTING SURFACES

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ABSTRACT
An investigation related to the use of linear indicial functions in the time and frequency domains, enabling one to derive the proper aerodynamic loads as to study the subcritical response and flutter of swept lifting surfaces, respectively, of the open/closed loop aeroelastic system is presented. The expressions of the lift and aerodynamic moment in the frequency domain are given in terms of the Theodorsen's function, while, in the time domain, these are obtained directly with the help of the Wagner's function. Closed form solutions of aerodynamic derivatives are obtained, graphical representations are supplied and conclusions and prospects for further developments are outlined.

NOMENCLATURE

\( a, a_o \) Dimensionless elastic axis position measured from the midchord, positive aft
\( c \) Chord length of wing, normal to the elastic axis, \( 2b \)
\( C_{t,a} \) Lift-curve slope
\( f_s, f_a \) Plunging and pitching deflection functions
\( h, h_o \) Plunging displacement and its amplitude, respectively
\( I_o \) Mass moment of inertia per unit length of wing
\( l \) Wing semi-span measured along the midchord line
\( l_h \) Dimensionless aerodynamic lift, \( l_h b/m U^2 \)

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\[ \rho \quad \text{Air density} \]
\[ \sigma \quad \text{Spanwise rate of change of bending} \]
\[ \tau \quad \text{Dimensionless time, } U/\dot{b} \]
\[ \tau_p \quad \text{Dimensionless positive phase duration of the pulse, measured from the time of the arrival} \]
\[ \omega, k \quad \text{Circular and reduced frequencies, } \omega b / U \]
\[ \omega_a, \omega_o \quad \text{Uncoupled frequency in plunging, } (K_a / m)^{1/2} \]
\[ \text{and pitching, } (K_a / I_a)^{1/2} \]
\[ \bar{\omega} \quad \text{Plunging-pitching frequency ratio, } \omega_a / \omega_o \]

**Subscript**

\(-1\), Circulatory terms of lift and aerodynamic moment
\(-2\), Non-circulatory terms of lift and aerodynamic moment
\(-n\) Quantity normal to the elastic axis
\(-\) Quantity associated with the swept wing

**Superscript**

\(-1\), Variables in Laplace transformed space
\(-1\), \(-2\), Derivatives with respect to the time \( t \), and the dimensionless time \( \tau \), respectively

**INTRODUCTION**

In this paper, the linear indicial functions in the time and frequency domains are used to determine the corresponding unsteady aerodynamic derivatives for swept lifting surfaces. Such a treatment of the problem enables one to approach either the open/closed loop aeroelastic response in the subcritical flight speed regime to arbitrary time-dependent external excitations (such as e.g. gusts, airblasts due to explosions or sonic-booms), or the flutter instability of actively controlled/uncontrolled swept wings. Moreover, this study is intended to extend its scope as to include, by an analogous procedure, the case of the various flight speed regimes, i.e. in addition to the incompressible one, also the compressible, transonic (within the linearized concept of indicial functions), and supersonic.

The representations of the aeroelastic governing equations in the time-domain is useful toward the approach of both the subcritical aeroelastic response of the actively controlled/uncontrolled swept aircraft wings. In other worlds, in this framework, the open/closed loops dynamic response of aeroelastic systems can be analyzed.

The unsteady aerodynamic lift and moment in incompressible flight speed regime are expressed for the swept aircraft wing in the time and the frequency domains by using the Wagner and Theodorsen functions, respectively. For the response of dynamic systems it is only necessary to express the lift and moment via the indicial Wagner's function. For the approach of the flutter problem, the Theodorsen's function helps the conversion of the expressions of both the aerodynamic loads and the unsteady aerodynamic derivatives in the frequency domain. Herein the case of a 2-D lifting surface, including the plunging and pitching degrees of freedoms is considered.

**PRELIMINARIES**

In the next developments an extensive use of variables in both the time and frequency domains will occur. As shown in Edwards, Ashley & Breakwell, for zero initial conditions, the aeroelastic equations can be converted from the time to the frequency domain via a Laplace transform. This results in the possibility of using the correspondence \( s \to ik \), where \( s \) and \( k \) are the Laplace variable and the reduced frequency, respectively. The Laplace transform operator \( \mathcal{L} \) is defined as:

\[ \mathcal{L}(\cdot) = \int_0^\infty (\cdot) e^{-st} d\tau. \quad (1) \]

In this sense, the Wagner's function \( \phi(\tau) \) is connected with Theodorsen's function \( C(k) \) via Laplace transform as:

\[ \frac{C(k)}{ik} = \frac{F(k)}{ik} + iG(k) = \int_0^\infty \phi(\tau) e^{-ik\tau} d\tau = \Phi(ik), \quad (2) \]

and vice-versa:

\[ \phi(\tau) = \mathcal{L}^{-1}\left\{ \frac{C(k)}{ik} \right\} \quad \text{Re}(ik) > 0. \quad (3) \]

and having in view the correspondence \( s \leftrightarrow ik \), we can also write:

\[ \Phi(ik) = \int_0^\infty \phi(\tau) e^{-ik\tau} d\tau = \Phi(s). \quad (4) \]

Using this relationship, it is possible to obtain the full expression of unsteady aerodynamic coefficients in
terms of the Theodorsen's function $C(k_\alpha)$ and its circulatory components $F(k_\alpha)$ and $G(k_\alpha)$.

It is interesting to note that the reduced frequency parameter $k_\alpha$ for swept and for straight wings coincide:

$$k_\alpha = \frac{\omega h}{U_n} = \frac{\omega b \cos \Lambda}{U \cos \Lambda} = \frac{\omega b}{U} = k,$$

(5)

$$i \omega = i k_\alpha \tau .$$

(6)

This implies that the indicial Wagner's function $\phi(\tau)$ remains invariant to any change of the sweep angle.

**ANALYTICAL DEVELOPMENTS**

For swept wings, the total lift per unit span, can be expressed in the form:

$$\lambda L_h(\bar{y}, t) = \lambda L_c(\bar{y}, t) + \lambda L_{w_1}(\bar{y}, t) + \lambda L_{nc}(\bar{y}, t).$$

(7)

where the indices $c$ and $nc$ identify the various contributions associated with the circulatory and noncirculatory terms respectively. Using similar notations, the total moment per unit span about the elastic axis is:

$$\lambda M_a(\bar{y}, t) = \lambda M_c(\bar{y}, t) + \lambda M_{nc1}(\bar{y}, t) + \lambda M_{nc2}(\bar{y}, t) + \lambda M_{nc3}(\bar{y}, t).$$

(8)

$\lambda M_a(\bar{y}, t)$ being associated with the apparent moment of inertia. Herein the lift is positive in the negative $z$ direction (considered positive downward), while the moment is positive nose up. For the sake of convenience, herein the plunging coordinate is positive when is downward (see Fig. 2).

Using the expression of the lift (Eq. (7)), the equation for the moment, Eq. (8), can be cast as:

$$\lambda M_a(\bar{y}, t) = -(l/2 + a_n) b_n \lambda L_c(\bar{y}, t) - a_n b_n \lambda L_{w_1}(\bar{y}, t) + (l/2 - a_n) b_n \lambda L_{nc1}(\bar{y}, t) + \lambda M_{nc2}(\bar{y}, t) + \lambda M_{nc3}(\bar{y}, t).$$

(9)

**Basic Considerations**

Expressing the vertical displacement $Z$ of a point on the center line of the wing as, (Fig.2):

$$Z(\bar{x}, \bar{y}, t) = h + \bar{x} \alpha ,$$

(10)

where $h \equiv h(\bar{y}, t)$, $\alpha \equiv \alpha(\bar{y}, t)$ are the displacements in plunging and pitching, respectively, and the origin of the $\bar{x}$ axis coincides with the elastic center, the downwash velocity $w$ normal to the lifting surface becomes:

$$w(\bar{x}, \bar{y}, t) = w(\bar{x}, \bar{y}, t) = \frac{\partial Z}{\partial t} + U \frac{\partial Z}{\partial \bar{x}} .$$

(11)

In conjunction with the definition of $Z$, (Eq. 10), Eq. (11) becomes:

$$w(\bar{x}, \bar{y}, t) = h + \bar{x} \alpha + U \alpha \cos \Lambda$$

$$+ U \left( \frac{\partial h}{\partial \bar{y}} + \bar{x} \frac{\partial \alpha}{\partial \bar{y}} \right) \sin \Lambda .$$

(12)

where the superposed dots denote the derivatives with respect to time $t$. The quantity in Eq. (12) underscored by a solid line is usually discarded in the specialized literature (see Bisplinghoff, Ashley & Halfman), as being related with the wing camber effect (see also Flax). However, herein this quantity will be included. The in-plane coordinate $\bar{x}$ normal to the elastic axis (see Fig. 1) can be expressed as:

$$\bar{x} = b_n \left( l/2 - a_n \right) .$$

(13)

Consequently, using the dimensionless time $\tau \equiv U_n t/b_n$ Eq. (12) becomes:

$$w(\bar{x}, \bar{y}, \tau) = U_n \left( \frac{h'}{b_n} + \alpha + \frac{\partial h}{\partial \bar{y}} \tan \Lambda$$

$$+ (l/2 - a_n) \left( \alpha' + b_n \frac{\partial \alpha}{\partial \bar{y}} \tan \Lambda \right) \right) .$$

(14)

where $(\cdot)' \equiv \partial (\cdot) / \partial \tau .

In the following sections, the unsteady aerodynamic loads in incompressible flow can be obtained in time and, with the use of the Laplace transform space, in the frequency domain.

**Unsteady Aerodynamic Loads in Incompressible Flow**

**Time Domain**

The circulatory components of the lift and moment expressed in term of Wagner's indicial function $\phi(\tau)$ (called also heredity function) obtained in the time domain are:
\( \lambda L_c(\bar{y}, \tau) = -C_{L_0} b_n \rho U_n^2 \int_0^\infty \phi(\tau - \tau_0) \frac{\partial \nu(x, \bar{y}, \tau_0)}{\partial \tau_0} d\tau_0 \) \tag{15}

\( \lambda M_c(\bar{y}, \tau) = \left( \frac{1}{2} + a_n \right) C_{L_0} b_n^2 \rho U_n^2 \int_0^\infty \phi(\tau - \tau_0) \frac{\partial \nu(x, \bar{y}, \tau_0)}{\partial \tau_0} d\tau_0 \) \tag{16}

As concern the aerodynamic non-circulatory components, using the dimensionless time \( \tau \) these are expressed as:

\( \lambda L_{nc1}(\bar{y}, \tau) = -\frac{1}{2} C_{L_0} \rho U_n^2 \left[ h^* - a_n b_n \alpha^* \right] \), \tag{17a}

\( \lambda L_{nc2}(\bar{y}, \tau) = -\frac{1}{2} C_{L_0} \rho U_n^2 b_n^2 \alpha^* \), \tag{17b}

\( \lambda L_{nc3}(\bar{y}, \tau) = -\frac{1}{2} C_{L_0} \rho U_n^2 b_n^2 \tan \Lambda \left[ (\delta_r + 1) \frac{\delta^*}{b_n} + \delta_r \right] \\
+ \delta_r \frac{\partial \sigma}{\partial \bar{y}} \tan \Lambda \right] + \frac{1}{2} C_{L_0} \rho U_n^2 b_n^2 \tan \Lambda \left[ (\delta_r + 1) \frac{\lambda^*}{b_n} \right] \\
+ \delta_r \frac{\partial \lambda}{\partial \bar{y}} \tan \Lambda \right) \). \tag{17c}

The moments induced by the non-circulatory components of lift are expressed using the dimensionless time as:

\( \lambda M_{nc1}(\bar{y}, \tau) = \frac{1}{2} C_{L_0} \rho U_n^2 \left[ h^* - a_n b_n \alpha^* \right] \right) \right) \left[ (\delta_r + 1) \frac{\sigma^*}{b_n} + \delta_r \lambda \right] + \delta_r \frac{\partial \sigma}{\partial \bar{y}} \tan \Lambda \right] \\
+ \delta_r \frac{\partial \lambda}{\partial \bar{y}} \tan \Lambda \right) \left[ (\delta_r + 1) \frac{\lambda^*}{b_n} \right] + \delta_r \frac{\partial \alpha}{\partial \bar{y}} \tan \Lambda \right) \right) \). \tag{18a}

\( \lambda M_{nc2}(\bar{y}, \tau) = -\frac{1}{2} C_{L_0} \rho U_n^2 b_n^2 \alpha^* \), \tag{18b}

\( \lambda M_{nc3}(\bar{y}, \tau) = \frac{1}{2} C_{L_0} \rho U_n^2 b_n^2 \tan \Lambda \left[ (\delta_r + 1) \frac{\lambda^*}{b_n} \right] \\
+ \delta_r \frac{\partial \alpha}{\partial \bar{y}} \tan \Lambda \right) \left[ (\delta_r + 1) \frac{\lambda^*}{b_n} \right] + \delta_r \frac{\partial \lambda}{\partial \bar{y}} \tan \Lambda \right) \). \tag{18c}

\( \lambda M_\alpha(\bar{y}, \tau) = -\frac{1}{16} \rho C_{L_0} b_n^2 U_n^2 \alpha^* \). \tag{18d}

where the spanwise rates of change of bending and twist, \( \sigma \) and \( \lambda \), respectively, are expressed as \( \sigma = \partial h/\partial \bar{y} \) and \( \lambda = \partial \alpha/\partial \bar{y} \). In these equations as well as in the following ones, the terms affected by the tracer \( \delta \), identify those generated by the last term in the expression of the downwash velocity (Eq. (12)), (term underscored by a solid line). In general these terms are discarded being considered negligibly small (see Bisplinghoff et al.\(^2\), and Flax\(^4\)), in what case \( \delta_r = 0 \), otherwise \( \delta_r = 1 \). Replacement of Eqs. (15) and (17) in Eq. (7) and of Eqs. (16) and (18) into Eq. (8), results in the unsteady lift and aerodynamic moment expressed in the time domain. Concerning the circulatory parts of the lift and aerodynamic moment, Eqs. (15) and (16), these can be explicitly determined by transforming these expressions in the Laplace domain using the relationship between the Laplace transform of the Wagner and Theodorsen functions, namely \( C(-is)/s = \mathcal{L}(\phi(t)) = \Phi(s) \), and afterwards inverting back the obtained expressions in the temporal space. Alternatively, in order to ease the computations, the available approximate expressions for \( \phi(t) \) (see Edwards et al.\(^3\)) can be used in the Laplace transform process.

The expressions of lift and aerodynamic moment in the time domain, \( L_\alpha(\bar{y}, \tau) \) and \( M_\alpha(\bar{y}, \tau) \) can be used to determine the subcritical aeroelastic response of swept wings. However, when the aeroelastic response of lifting surface to time-dependent external pulses, is needed, the unsteady aerodynamic loads in the time domain, \( L_\alpha \) and \( M_\alpha \), have to be supplemented by the ones corresponding to above mentioned pulses. This will be considered in the next developments and a simplified illustration of the capability of this method will be given in this work.

**Laplace Transformed Space**

Several preliminaries related to Laplace transform applied to aeroelastic quantities will be given next.

\( \lambda \left\{ L_\alpha(\bar{y}, s), M_\alpha(\bar{y}, s) \right\} = \int_0^\infty \left\{ \lambda L_\alpha(\bar{y}, \tau), \lambda M_\alpha(\bar{y}, \tau) \right\} e^{-\tau} d\tau \) \tag{19}

We will express the Wagner's function and the plunging and pitching degrees of freedom in the Laplace transformed space as:

\( \phi(t) \rightarrow \Phi(s) \) \hspace{0.5cm} \( h(t) \rightarrow \dot{h}(s) \) \hspace{0.5cm} \( \alpha(t) \rightarrow \dot{\alpha}(s) \)

Considering zero initial condition, Laplace Transformed counterparts of Eqs. (15) and (16) are:
\[ L_c(y,s) = -C_{L_{ao}} b_n \rho U_n^2 \left( s^2 \frac{\dot{h}}{b_n} + s\dot{\alpha} + s\sigma \tan \Lambda + \right) \]
\[ \left( \frac{1}{2} - a_n \right) \left( s^2 \dot{\alpha} + b_n s\Lambda \tan \Lambda \right) \phi(s) \]  
\[ (20) \]
\[ M_c(y,s) = \left( \frac{1}{2} + a_n \right) C_{L_{ao}} b_n \rho U_n^2 \left( s^2 \frac{\dot{h}}{b_n} + s\dot{\alpha} + s\sigma \tan \Lambda \right) \]
\[ + \left( \frac{1}{2} - a_n \right) \left( s^2 \dot{\alpha} + b_n s\Lambda \tan \Lambda \right) \phi(s). \]  
\[ (21) \]
Herein, and in the next expressions, the plunging and pitching motions in the Laplace domain are expressed as \( h(y,s) \) and \( \dot{\alpha} \).

The Laplace transformed counterpart of Eqs. (17) can be expressed as:
\[ L_{nc1}(y,s) = -\frac{1}{2} C_{L_{ao}} \rho U_n^2 s^2 \left( \hat{h} - a_n b_n \dot{\alpha} \right), \]  
\[ (22a) \]
\[ L_{nc2}(y,s) = -\frac{1}{2} C_{L_{ao}} \rho U_n^2 b_n s\dot{\alpha}, \]  
\[ (22b) \]
\[ L_{nc3}(y,s) = -\frac{1}{2} C_{L_{ao}} \rho U_n^2 b_n^2 \tan \Lambda \left[ (1 + \delta) \frac{\sigma}{b_n} s + \delta, \lambda \right] \]
\[ + \delta, \frac{\partial \sigma}{\partial y} \tan \Lambda \right] + \frac{1}{2} C_{L_{ao}} \rho U_n^2 b_n^3 \tan \Lambda \left[ (1 + \delta) \frac{\lambda}{b_n} s \right] \]
\[ + \delta, \frac{\partial \lambda}{\partial y} \tan \Lambda \right]. \]  
\[ (22c) \]

The Laplace transformed counterpart of Eqs. (18) can be written as:
\[ M_{nc1}(y,s) = \frac{1}{2} C_{L_{ao}} \rho U_n^2 s^2 \left( \hat{h} - a_n b_n \dot{\alpha} \right), \]  
\[ (23a) \]
\[ M_{nc2}(y,s) = -\frac{1}{2} C_{L_{ao}} \rho U_n^2 b_n \left( \frac{1}{2} - a_n \right) s\dot{\alpha}, \]  
\[ (23b) \]
\[ M_{nc3}(y,s) = -\frac{1}{2} C_{L_{ao}} \rho U_n^2 b_n^2 \frac{1}{2} \lambda \tan \Lambda \]
\[ + \frac{1}{2} C_{L_{ao}} \rho U_n^2 b_n^3 \frac{1}{8} \lambda \tan \Lambda \]
\[ \left[ (1 + \delta) \frac{\sigma}{b_n} s + \delta, \lambda \right. \]
\[ \lambda + \frac{\partial \sigma}{\partial y} \tan \Lambda \right] \left. - \frac{1}{2} C_{L_{ao}} \rho U_n^2 b_n^4 \left( \frac{1}{8} + a_n^2 \right) \tan \Lambda \right] \]
\[ \left[ (1 + \delta) \frac{\lambda}{b_n} s + \delta, \frac{\partial \lambda}{\partial y} \tan \Lambda \right], \]  
\[ (23c) \]

The equations of lift and aerodynamic moment help us to perform the conversion in the frequency domain. This will be done in the next section.

**Frequency Domain**

Upon replacing \( s \rightarrow ik \) in Eqs. (22) and (23); using the relationship between Laplace transform of Wagner and Theodorsen's functions (Eq. (2)); representing the time dependence of displacement quantities as:
\[ \alpha(y, \tau) = f_a(y) e^{ik \tau}, \]  
\[ h(y, \tau) = f_h(y) e^{ik \tau}, \]  
\[ M_a(y, \tau) = C_{M_a} e^{ik \tau}, \]  
\[ (24) \]
\[ (25) \]

the equations for the unsteady lift and moment amplitudes can be expressed in the frequency domain. These expressions that coincide with the ones obtained differently by Barmby et al, can be used in the flutter analysis of swept aircraft wings.

In this analysis \( f_h(\bar{y}) \) and \( f_a(\bar{y}) \) are chosen to be the decoupled eigenmodes in plunging and twisting of the counterpart structure, and are determined as to fulfill identically the boundary conditions. Using the spanwise dimensionless coordinate \( \eta = \bar{y} / l \), these are expressed as:
\[ f_h(\bar{y}) = F_h(\eta) = C_h \left( \cos \beta_1 \eta - \cos \beta_1 \eta \right), \]  
\[ f_a(\bar{y}) = F_a(\eta) = C_2 \sin \beta_2 \eta, \]  
\[ \beta_1 = 0.5969 \pi \text{ and } \beta_2 = \pi / 2. \]  
\[ (26) \]
\[ (27) \]

where for the first bending and torsion we have \( \beta_1 = 0.5969 \pi \) and \( \beta_2 = \pi / 2 \).

The constants \( C_1 \) and \( C_2 \) are chosen as to normalize \( f_h(\bar{y}) \) and \( f_a(\bar{y}) \), and so to get the unitary maximum deflection at the wing tip. The uncoupled first bending and torsion mode shapes needed for the evaluations of the aerodynamic lift and moment are shown in Fig. 3.

The following expressions will be useful in the next developments:
\[ a' = \frac{\lambda'}{\sigma} = \frac{\partial f_a(\bar{y})}{\partial \eta} \alpha_n e^{ik \tau}, \]  
\[ (28) \]
\[ \sigma' = \frac{\partial^2 h}{\partial \tilde{y} \partial \tau} = \frac{\partial f}{\partial \tilde{y}} h_{ik} e^{a \tau}, \]  
\[ \frac{\partial \lambda}{\partial \tilde{y}} = \frac{\partial^2 f_a}{\partial \tilde{y}^2} \alpha e^{a \tau}, \]  
\[ \frac{\partial \sigma}{\partial \tilde{y}} = \frac{\partial^2 f_a}{\partial \tilde{y}^2} h_{ik} e^{a \tau}. \]  

**Unsteady Aerodynamic Derivatives in the Frequency Domain**

At this point, a careful inspection of Eqs. (7) - (8), suggests the following representations for the lift and aerodynamic moment:

\[ \lambda A_h(y, k, \tau) = \frac{1}{2} \rho U^2 b_2 \left( \frac{k}{b_2} k H + k H H + k H \right) + k H H \alpha, \]  
\[ \lambda A_m(y, k, \tau) = \frac{1}{2} \rho U^2 b_2 \left( \frac{k}{b_2} A \right) \alpha + \frac{k}{b_2} A \alpha + A \alpha \alpha \]  
Herein the dimensionless unsteady aerodynamic coefficients \( H, A \) have been introduced, and \( k \) has been included as to render the quantities in brackets nondimensional. Herein, \( b_2 \) is the half-chord of the airfoil and \( U_2 \) is the component of the flow speed, both normal to the elastic axis.

In a simplified context, such a mixed form of the lift and moment was used in Simiu & Scanlan. The unsteady aerodynamic derivatives for swept wings are obtainable from the previous equations of lift and aerodynamic moment, by assuming harmonic time dependence of displacements quantities. In such a way, the frequency domain counterpart of Eqs. (32)-(33), expressed in compact form, becomes:

\[ \lambda L_h(y, k, \tau) = C_{L_a} \left( \frac{1}{2} + a \right) \Phi(ik) \left[ k_n f_h - ib_n \frac{\partial f_h}{\partial y} \tan \Lambda \right], \]  
\[ \lambda M_a(y, k, \tau) = C_{L_a} \left( \frac{1}{2} + a \right) \Phi(ik) \left[ k_n f_a - ib_n \frac{\partial f_a}{\partial y} \tan \Lambda \right], \]  

where, for the sake of convenience, the unsteady aerodynamic derivative are written as:

\[ \tilde{H}_1 = H_1; \tilde{H}_2 = H_2; \tilde{H}_3 = (H_3 - H_2); \tilde{H}_4 = (H_4 - H_3), \]  
\[ \tilde{A}_1 = A_1; \tilde{A}_2 = A_2; \tilde{A}_3 = (A_3 - A_2); \tilde{A}_4 = (A_4 - A_3). \]  

The unsteady aerodynamic derivatives in the frequency domain for swept wing will be obtained from the following equations, expressed in terms of Wagner’s function \( \Phi(ik) \). Comparing Eqs. (36) with that expressing the lift in the frequency domain, yields:

\[ L_1 = \frac{1}{2} C_{L_a} f_h - \frac{1}{2} C_{L_a} b_n \tan \Lambda \left[ \left( \delta \right) + 1 \right] \frac{\partial f_h}{\partial y} \frac{\partial b_n}{\partial y} \tan \Lambda, \]  
\[ L_2 = -\frac{1}{2} C_{L_a} \alpha f_h - \frac{1}{2} C_{L_a} a f_a \tan \Lambda \left[ \left( \delta \right) + 1 \right] \frac{\partial f_h}{\partial y} \frac{\partial a b_n}{\partial y} \tan \Lambda. \]  

From a similar analysis, comparing the Eqs. (39) with that expressing the aerodynamic moment in frequency domain, yields:

\[ M_1 = -\frac{1}{2} C_{L_a} f_h - \frac{1}{2} C_{L_a} b_n \tan \Lambda \left[ \left( \delta \right) + 1 \right] \frac{\partial f_h}{\partial y} \frac{\partial b_n}{\partial y} \tan \Lambda, \]  
\[ M_2 = -\frac{1}{2} C_{L_a} \alpha f_h - \frac{1}{2} C_{L_a} a f_a \tan \Lambda \left[ \left( \delta \right) + 1 \right] \frac{\partial f_h}{\partial y} \frac{\partial a b_n}{\partial y} \tan \Lambda. \]
Using in Eqs (39)-(40) equations (2), separating the real and the imaginary parts of the above expressions, the unsteady aerodynamic derivatives result as:

\[ \dot{H}_1 = -C_{La} \left( \frac{F(\kappa_n)}{k_n} f_\alpha + b_n \tan \Lambda \frac{\partial \alpha}{\partial \Lambda} \left( \frac{G(\kappa_n)}{k_n} + \frac{1}{2} \right) \left( 1 + \delta_r \right) \right), \]

\[ \dot{H}_2 = -C_{La} \left( \left( \frac{1}{2} - a_n \right) F(\kappa_n) f_\alpha + \frac{G(\kappa_n)}{k_n} \right) \frac{1}{2} a_n + \frac{1}{2} a_n b_n \tan^2 \Lambda \left( \frac{2G(\kappa_n)}{k_n} \left[ \frac{1}{2} - a_n \right] - \frac{1}{2} a_n \left( 1 + \delta_r \right) \right), \]

\[ \dot{H}_3 = C_{La} \left( \left( \frac{1}{2} - a_n \right) F(\kappa_n) f_\alpha + \frac{G(\kappa_n)}{k_n} \right) \frac{1}{2} a_n + \frac{1}{2} a_n b_n \tan \Lambda \left( \frac{1}{2} - a_n \right) + \frac{1}{2} \delta_r, \]

\[ \dot{H}_4 = C_{La} \left( \frac{1}{2} f_\alpha + \frac{G(\kappa_n)}{k_n} \right) f_\alpha - b_n \tan \Lambda \left( F(\kappa_n) \frac{\partial \alpha}{\partial \Lambda} \left( \frac{1}{2} - a_n \right) + \frac{1}{2} a_n b_n \tan \Lambda \right), \]

\[ \dot{A}_1 = \frac{C_{La}}{k_n} \left( \left( \frac{1}{2} + a_n \right) \frac{G(\kappa_n)}{k_n} f_\alpha + b_n \tan \Lambda \left( \frac{1}{2} - a_n \right) \right), \]

\[ \dot{A}_2 = \frac{C_{La}}{2k_n} \left( \left( \frac{1}{2} + a_n \right) \frac{G(\kappa_n)}{k_n} f_\alpha + b_n \tan \Lambda \left( \frac{1}{2} - a_n \right) \right), \]

\[ \dot{A}_3 = \frac{C_{La}}{4k_n^2} \left( \left( \frac{1}{2} + a_n \right) \frac{G(\kappa_n)}{k_n} f_\alpha + b_n \tan \Lambda \left( \frac{1}{2} - a_n \right) \right), \]

\[ \dot{A}_4 = -C_{La} \left( \frac{1}{2} - a_n \right) f_\alpha + \frac{1}{2} a_n \left( \frac{G(\kappa_n)}{k_n} f_\alpha - \frac{b_n}{k_n} \right) \tan \Lambda \]

\[ \left( \frac{1}{2} + a_n \right) \frac{\partial \alpha}{\partial \Lambda} \left( \frac{1}{2} - a_n \right) + \frac{1}{2} a_n b_n \tan \Lambda, \right), \]

The coupling terms, due to the sweep effects, are separated in the overall expressions of aerodynamic derivatives. For \( \Lambda = 0 \) the expressions of aerodynamic coefficients corresponding to straight wings are obtained. The unsteady aerodynamic derivatives recorded above coincide with the ones obtained differently in Barmby et al.\(^2\), where the spanwise rates of change of bending and twist, \( \sigma \) and \( \lambda \), that are associated with the sweep effect, also intervene. When specialized for \( \delta_r = 0 \), there coincide with the ones by Bisplinghoff et al.\(^2\).

### Aeroelastic Response of an Airfoil Featuring Plunging and Pitching Coupled Motions to Sonic-Boom Pressure Pulses.

An application on the aeroelastic response of an airfoil in an incompressible flow featuring plunging and pitching coupled motions to sonic-boom and blast pressure pulses will be given in the next developments. The aeroelastic governing system of equations of an airfoil featuring plunging and twisting degrees of freedom to sonic-boom, blast pressure pulses, expressed in dimensionless form, can be cast as:

\[ \xi''(\tau) + \chi_{\alpha} \alpha''(\tau) + 2 \xi'(\tau) \frac{\partial V}{\partial V} \chi'(\tau) \]

\[ + \left( \frac{\partial V}{\partial \tau} \right)^2 \xi(\tau) - \ell_n(\tau) = \ell_b(\tau), \]

\[ \left( \chi_{\alpha} / r_a^2 \right) \chi''(\tau) + \alpha''(\tau) + \left( 2 \frac{\partial V}{V} \right) \alpha'(\tau) \]

\[ + \alpha / V^2 - m_a(\tau) = 0 \]

In the above expressions this following dimensionless parameters have been used:

\[ l_n = l_n b / m U_n^2 ; \quad \varphi_n = \phi_n b / m U_n^2 ; \quad V = U_m / \omega_n ; \]

\[ \mu = \mu / \nu \rho b^2 ; \quad \gamma_n = \gamma_n / \nu \rho b u ; \quad \rho = \varphi_n / \omega_n ; \]

\[ \omega_n = \left( K_n / m \right)^{1/2} ; \quad \omega_n = \left( K_n / I_n \right)^{1/2} ; \quad \zeta_n = \zeta_n / 2 \omega_n ; \]

\[ r_a = \left( l_n / mb^2 \right)^{1/2} ; \quad \chi_n = \chi_n / mb ; \quad m_n = M_a b^2 / I_n U_n^2 \]

The dimensionless sonic-boom overpressure signature of the N-wave shock pulse, can be described as follow:

\[ l_n(\tau) = H(\tau) \rho_n \left[ 1 - \frac{\tau}{\tau_p} \right] - \delta_n H(\tau - \tau_p) \rho_n \left[ 1 - \frac{\tau}{\tau_p} \right] \]

Herein, the Heaviside step function \( H(\tau) \) has been introduced in order to describe the typical pressure time-history for blast or sonic-boom loads; \( \delta_n \) is a tracer.
that should be taken as one when the sonic-boom is considered, and zero when the blast load is included; \( \vartheta_p \) denotes the dimensionless peak reflected pressure in excess of the ambient one; \( \tau_p \) denotes the positive phase duration of the pulse measured from the time of impact of the structure; \( r \) denotes the shock pulse length factor. For \( r = 1 \) the N-shaped pulse degenerates into a triangular pulse which corresponds to an explosive pulse (Fig. 4a), and for \( r = 2 \) a symmetric N-shaped pulse is obtained. A depiction of \( l_b / \vartheta_m \) vs. time is displayed in Fig. 4b.

The Eqs. (43) and (44) can be converted in the Laplace transformed space and solved for their unknowns, \( \xi (= \mathcal{L} (\xi)) \) and \( \alpha (= \mathcal{L} (\alpha)); \) inverted back in time domain one obtain the plunging and pitching time-histories and the load factor time-history due to the sonic-boom pressure pulse, \( \xi(t) = \mathcal{L}^{-1} \{\xi(s)\} \) and \( \alpha(t) = \mathcal{L}^{-1} \{\alpha(s)\}, \) respectively.

When the dynamic response of the actively controlled lifting surface is analyzed, also the feedback control forces and moments, that are time dependent, have to be included in the Eqs. (43) and (44). This will be considered in the future work.

**RESULTS AND DISCUSSION**

Herein, an unified way enabling one to obtain the unsteady lift and aerodynamic moment in the time and frequency domains for swept aircraft wing was developed. This was done via the use of the indicial function approach. The time domain representation is essential towards determination of the dynamic aeroelastic response to time dependent external loads, and in the case of the application of a feedback control methodology, of the dynamic aeroelastic response to both external time dependent loads and control forces. The frequency domain representation is essential towards determination of the flutter instability.

The unsteady aerodynamic derivatives for different values of \( a_s \), \( \Lambda \) and \( C_{La} \) have been plotted in Figs. 5 and 6, as a function of \( k_n \). For swept wings, the local lift-curve slope \( C_{La} \) for sections normal to the elastic axis are obtained from the aerodynamics of swept wings (see Yates\(^{10}\)).

The maximum influence of the corrective term (identified by the tracer \( \delta \)) is present the first plunging coefficients \( H_1 \) where the aerodynamic coefficient changes also its sign. Usually, for all coefficients, the effect of these terms becomes higher for high sweep angles.

For swept wings, the local lift-curve slope \( C_{La} \) for sections normal to the elastic axis are expressed as:

\[
C_{La} = C_{La} / \cos \Lambda. \tag{49}
\]

The variation of the unsteady aerodynamic derivatives \( \hat{H}_1 \) and \( \hat{A}_1 \) as a function of \( k_n \) are depicted. In these developments, all the terms, including the aerodynamic ones associated with \( \hat{h} \) and \( \hat{\alpha} \), usually neglected, have been retained. As a result, the coefficients \( H_5, H_6 \) and \( A_5, A_6 \) are also included. Whereas the aerodynamic coefficients \( \hat{H}_1 \) and \( \hat{A}_1 \) are the principal uncoupled aerodynamic damping coefficients in plunging and torsion, respectively, \( \hat{H}_2 \) and \( \hat{A}_2 \) are the coupled damping coefficients. For straight wings, these terms remains negative for all values of \( 2\pi/k_n \) and for different values of \( a_s \), but for swept wings only \( \hat{A}_1 \) continues to remain negative. The elastic axis position is not involved in the expression of the unsteady aerodynamic coefficients \( \hat{H}_1 \) and \( \hat{H}_4 \), a fact, which clearly appears from the equations. The variation of the aerodynamic derivatives as a function of \( k_n \) is, in general, a smooth one. Among these coefficients, only \( \hat{H}_2 \) features, with the variation of the reduced frequency \( k_n \), a change of sign. As concerns, the depiction of \( \hat{H}_1 \) and \( \hat{A}_1 \) versus \( 2\pi/k_n = (U_s/nb_s) \), this representation enables one to get an idea of the variation of the respective quantity with that of the normal freestream speed \( U_s \). The differences in the unsteady aerodynamic coefficients induced by the discard or inclusion of the terms generated from the downwash velocity (i.e. that underscored by a solid line in Eq. (12)) are properly indicated in Figs. 7. As shown, the corrective term does not modify the trend of coefficients \( \hat{A}_1 \) and \( \hat{A}_4 \). It should be mentioned that the expressions of the lift and aerodynamic moment in the frequency domain obtained by Barmby, Cunningham and Garrick\(^{1} \) coincide with the ones obtained here via indicial function approach.

The graphs depicting the aeroelastic response time-history to blast pulses (i.e. explosive and sonic-boom blasts) are displayed (Figs. 8 - 10). The graphs supply the dimensionless plunging displacement \( \xi = h/b \), and the load factor \( N = h^2/g \), where \( g \) is the acceleration of gravity. The predictions of \( \xi \) and \( N \) based on pure plunging and coupled plunging-twist...
models are depicted on the same plots, and the closeness of the two predictions becomes apparent from the graphs. Herein, the dotted and solid curves correspond to pure plunging and coupled plunging-pitch models, respectively. As a result, the coupling helps to reduce the amplitude of the aeroelastic response. The pitching displacement has its maximum for \( r = 0 \). The same trend is valid also for the load factor N. It should be indicated that the response to sonic-boom pressure pulse involves two different regimes; one for which \( 0 < r < 30 \) that corresponds to the forced motion, and the other one to \( r > 30 \) belonging to the free motion. The jump in the time-history of \( N \) is due to the discontinuity in the load occurring at \( r = 30 \). This jump doesn't appear for explosive pressure pulses, where \( r = 1 \). The increase of the mass ratio results in the increase of the plunging displacement amplitude and the decrease of the pitching amplitude. At the same time, for higher mass ratios, the differences in the plunging predictions based upon 1DOF and 2 DOF disappear (Figs. 8 and 9). Moreover, for higher mass ratios, the motion damps out at larger times. Figs. 10 highlights the effect of the structural damping coefficient in plunging and pitching.

Using the idea developed by Yates\textsuperscript{10}, a modified strip theory can be accommodated as to address the problem of the aeroelastic response (for open/closed loop aeroelastic systems), by capturing also the 3-D effects. Alternatively, an exact solution methodology enabling one to determine both the flutter instability and the aeroelastic dynamic response based on a double Laplace transform, in time and space, can be used. Such a method was devoted to the solution of aeroelastic eigenvalue problems in Karpouzian & Librescu\textsuperscript{5,6}. Librescu & Thangjitham\textsuperscript{7}. On this basis, the obtained results can be extended as to approach the subcritical aeroelastic response and flutter, respectively, of 3-D advanced lifting surfaces, in various flight speed regimes.

**CONCLUSIONS**

A unified treatment of the aeroelasticity of 2-D lifting surfaces in time and frequency domains has been presented and the usefulness in this context of the aerodynamic indicial functions concept was emphasized. Applications assessing the versatility of this approach enabling one to treat both subcritical aeroelastic responses and flutter instability were presented, and prospects for extending this treatment to 3-D aeroelastic problems are contemplated in forthcoming developments.

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**REFERENCES**

Fig. 1 Nonuniform Swept Wing

Fig. 2 Airfoil Section

Fig. 3 Bending-Mode and Torsion-Mode

Fig. 4 (a) Sonic-Boom and (b) Triangular Blast Pressure Pulses

Fig. 5 Unsteady aerodynamic derivative for swept lifting surfaces ($\Lambda = 45^\circ$) and different elastic axis position ($a_n$), vs. $2\pi/k_n$

Fig. 6 Unsteady aerodynamic derivative for swept lifting surfaces ($\Lambda = 15^\circ$) and different elastic axis position ($a_n$), vs. $2\pi/k_n$
Fig. 7 Results based on the discard of the corrective effect ($\delta_r = 0$), and including that effect ($\delta_r = 1$) vs. $2\pi/k_n$.

Fig. 8 Influence of mass ratio on the time-history response of (a) the nondimensional plunging displacement and (b) of the load factor, to sonic-boom pressure pulse

Fig. 9 Influence of mass ratio on the time-history response of (a) the nondimensional plunging displacement and (b) of the load factor, to explosive pressure pulse

Fig. 10 Influence of structural damping coefficient on the time-history response of (a) the nondimensional plunging displacement, (b) of the load factor and (c) of the nondimensional pitching displacement to sonic-boom pressure pulse