

# ATTITUDE REPRESENTATIONS FOR KALMAN FILTERING

F. Landis Markley\*

The four-component quaternion has the lowest dimensionality possible for a globally nonsingular attitude representation, it represents the attitude matrix as a homogeneous quadratic function, and its dynamic propagation equation is bilinear in the quaternion and the angular velocity. The quaternion is required to obey a unit norm constraint, though, so Kalman filters often employ a quaternion for the global attitude estimate and a three-component representation for small errors about the estimate. We consider these mixed attitude representations for both a first-order Extended Kalman filter and a second-order filter, as well for quaternion-norm-preserving attitude propagation.

## INTRODUCTION

The Kalman filter [1] is frequently employed in attitude estimation. A common application is to a spacecraft equipped with gyroscopes, with the filter solving for a quaternion parameterizing the attitude and a three-vector of gyro drifts [2-5]. Reference [5] discusses this application and provides extensive citations of the relevant literature. This paper is an extension of [5], so we begin with a brief review of the quaternion attitude representation

A quaternion is a four-component object with a three-vector part and a scalar part [6, 7]

$$q = \begin{bmatrix} q_0 \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}. \quad (1)$$

Quaternions representing spacecraft attitude are generally considered to have unit length:

$$|q|^2 \equiv |q|^2 + q_0^2 = 1. \quad (2)$$

The rotation matrix is a homogeneous quadratic function of the components of such a quaternion:

$$A(q) = (q_0^2 - |\mathbf{q}|^2)I + 2\mathbf{q}\mathbf{q}^T - 2q_0[\mathbf{q} \times], \quad (3)$$

where the 3x3 identity matrix is denoted by  $I$  and the cross product matrix is

$$[\mathbf{q} \times] \equiv \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix}. \quad (4)$$

The quaternion representation is 2-1 because Eq. (3) shows that  $q$  and  $-q$  represent the same rotation matrix.

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\*Guidance, Navigation and Control Systems Engineering Branch, NASA's Goddard Space Flight Center, Greenbelt, MD 20771



Since Eq. (3) and several other equations in spacecraft attitude estimation are nonlinear, an Extended Kalman Filter (EKF) is needed [8–10]. The linear quaternion measurement update provided by a straightforward EKF does not preserve the nonlinear normalization constraint of Eq. (2), a problem for which several solutions have been proposed [5, 11–13]. The most direct response is brute force normalization of the quaternion following an update. The norm constraint is only violated to second order in a correctly performed update, so normalization only changes the update to second order and is therefore outside the purview of the EKF. Other approaches have employed pseudo-measurements of the quaternion or of its norm [11, 12].

These modifications do not address an issue that has both conceptual and computational aspects. Unit quaternions reside on the three-dimensional sphere  $S^3$  embedded in four-dimensional Euclidean space  $E^4$ . If the quaternion errors are small, they lie approximately in the plane tangent to  $S^3$  at the true value  $q$  of the quaternion, which means that they are orthogonal  $q$ . This implies that  $q$  is an eigenvector of the covariance matrix with zero eigenvalue, so the covariance is not positive definite. This is acceptable in principle; but numerical errors can lead to loss of positive semidefiniteness and Kalman filter divergence. One approach is to relax the requirement of quaternion normalization and parameterize the attitude by

$$A = |q|^{-2} \left\{ (q_4^2 - |q|^2)I + 2\mathbf{q}\mathbf{q}^T - 2q_4[\mathbf{q} \times] \right\}, \quad (5)$$

which is an orthogonal matrix for any  $q$  [12]. This avoids both the normalization and zero covariance problems; the covariance is nonsingular because the norm of  $q$  is not known exactly. There is no firm conceptual foundation for a non-normalized quaternion or for pseudo-measurements, though, and performance of these methods has not been encouraging.

In fact there is a deeper conceptual problem with any conventional quaternion Kalman Filter. It would be natural to define the quaternion estimate  $\hat{q}(t)$  would be as an expectation value, defined as the integral

$$E\{q(t)\} \equiv \int_{E^4} q \rho(q,t) \delta(|q|^2 - 1) d^4q, \quad (6)$$

where  $\rho(q,t)$  is the probability distribution function (pdf) on  $S^3$ . The Dirac delta function enforces the unit norm constraint, giving  $\delta(|q|^2 - 1) d^4q$  as the volume element on  $S^3$ . Equation (6) is an unsatisfactory definition of  $\hat{q}(t)$ , however, since restricting the probability distribution in quaternion space to the surface of a unit sphere means that its expectation value must be *inside* the sphere. The integral cannot give a unit quaternion unless the pdf is concentrated at a point. In fact, since  $q$  and  $-q$  represent the same attitude, it would seem reasonable to take  $\rho(q,t) = \rho(-q,t)$ , which gives  $E\{q(t)\} = 0$  for any pdf. The same issues make it difficult to assign a meaning to the covariance of the quaternion error as an integral over  $S^3$ .

The widely adopted alternative, which has come to be known as the multiplicative quaternion Kalman filter, was developed in 1969 [2, 3], and has been used in NASA programs since 1978 [4, 5]. This uses the four-component quaternion representation for the attitude, but a three-component representation for the attitude errors. Several justifications have been provided for this non-standard approach to Kalman filtering. The approach in Section IX of [5] was to regard the filter as estimating a four-component quaternion, but to project the (rank 3)  $4 \times 4$  quaternion covariance and  $4 \times 3$  quaternion to gyro bias covariance down to  $3 \times 3$  matrices using a certain  $4 \times 3$  matrix. However, this formal procedure does not address the conceptual issues of defining the quaternion estimate and its covariance as integrals over  $S^3$ . The second approach, adopted in Section XI of [5] and in [13], regards the filter as estimating a three-component attitude error. It must be emphasized that these various approaches to the multiplicative Kalman filter lead to identical algorithms. The main purpose of this paper is to consider in some depth the relationship between the three-component attitude error vector and the quaternion. To set the stage, we begin with a brief discussion of attitude representations. We then generalize our results to a second-order filter. Finally, we propose similarly motivated techniques to preserve quaternion normalization during attitude propagation.

## ATTITUDE PARAMETERIZATIONS

Good reviews of attitude representations are available [6, 7]. We provide a brief discussion to establish conventions and notation for the representations of interest to this paper.

### Rotation Vector

Euler's Theorem [14] states that the most general motion of a rigid body with one point fixed is a rotation by an angle  $\phi$  about some axis, which we shall specify by a unit vector  $\mathbf{e}$ . We often combine the Euler axis and angle into a rotation vector  $\phi \equiv \phi \mathbf{e}$ . All rotations can be mapped to points inside and on the surface of a sphere of radius  $\pi$  in rotation vector space, where points at opposite ends of a diameter represent the same  $180^\circ$  rotation. It is sometimes convenient to consider a sphere of radius  $2\pi$ , which provides a 2-1 mapping of the rotation group, with each rotation corresponding to a point inside the sphere and a point outside. The kinematic equation for the rotation vector is transcendental, ill behaved for zero rotation angle, and singular for  $360^\circ$  rotations, so this is not a convenient global attitude representation. A globally nonsingular three-dimensional parameterization of the rotation group would be desirable, but this is known to be topologically impossible [15, 16].

### Rotation Matrix

The most fundamental representation of the attitude is the rotation matrix

$$R(\mathbf{e}, \phi) = \exp[(-\phi) \times] = (\cos \phi)I + (1 - \cos \phi)\mathbf{e}\mathbf{e}^T - \sin \phi[\mathbf{e} \times], \quad (7)$$

which obeys the kinematic relation

$$\dot{R} = -[\boldsymbol{\omega} \times]R(t), \quad (8)$$

where  $\boldsymbol{\omega}$  is the angular velocity vector. The bilinearity of Eq. (8) in  $R$  and  $\boldsymbol{\omega}$  is very convenient, and the skew-symmetry of the cross product matrix preserves the orthogonality of  $R$ . Numerical errors can cause the direction cosine matrix to lose its orthogonality, though, and orthogonality is not easily restored. This, along with its six redundant parameters, makes the rotation matrix inconvenient for numerical work.

### Quaternions

The unit quaternion is a convenient parameterization of the attitude with only one redundant parameter. It is related to the axis and angle of rotation by

$$q = \begin{bmatrix} \mathbf{e} \sin(\phi/2) \\ \cos(\phi/2) \end{bmatrix}. \quad (9)$$

The half-angle formulas of trigonometry establish the consistency of Eqs. (3) and (7). The four components of  $q$  are the Euler symmetric parameters or the Euler-Rodrigues parameters. They first appeared in paper by Euler [17] and in unpublished notes by Gauss [18], but Rodrigues' classic paper of 1840 first demonstrated their general usefulness [19]. Rodrigues was also the first to emphasize the product rule for successive rotations, the non-commutativity of rotations, the commutativity of infinitesimal rotations, and the generation of finite rotations from infinitesimal ones. Hamilton introduced the quaternion as an abstract mathematical object in 1844 [20], but there is some question as to whether he correctly understood the relation between quaternions and rotations [21].

We follow Reference [7] in writing the quaternion product as

$$p \otimes q \equiv \begin{bmatrix} p_1 \mathbf{q} + q_1 \mathbf{p} - \mathbf{p} \times \mathbf{q} \\ p_1 q_1 - \mathbf{p} \cdot \mathbf{q} \end{bmatrix}. \quad (10)$$

This differs from the historical multiplication convention, denoted by  $p q$  without an infix operator [6, 20], by the sign of the cross product in the vector part. The two products are related by  $p \otimes q = q p$ . The convention adopted here has the useful property that

$$R(p)R(q) = R(p \otimes q). \quad (11)$$

With the historical convention, the quaternion ordering on the right side of the above equation would be the reverse of the order on the left side. With either convention, though, the product of two quaternions is bilinear in the elements of the component quaternions, a property shared with the direction cosine matrix but no other attitude representation. In fact, Eq. (11) means that the rotation group and the quaternion group are "almost isomorphic." The qualifier "almost" is due to the 2-1 nature of the mapping [22].

We use an overbar to denote the quaternion representation of a three-vector:

$$\bar{\mathbf{v}} \equiv \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix}. \quad (12)$$

With this convention, the kinematic equation for the quaternion can be written in the alternative forms

$$\dot{q} = \frac{1}{2} \bar{\boldsymbol{\omega}} \otimes q = \frac{1}{2} \boldsymbol{\Omega}(\boldsymbol{\omega}) q = \frac{1}{2} \Xi(q) \boldsymbol{\omega}, \quad (13)$$

where  $4 \times 4$  matrix  $\boldsymbol{\Omega}(\boldsymbol{\omega})$  and the  $4 \times 3$  matrix  $\Xi(q)$  are defined by [5]

$$\boldsymbol{\Omega}(\boldsymbol{\omega}) \equiv \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix} \quad (14)$$

and

$$\Xi(q) \equiv \begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix}. \quad (15)$$

The skew-symmetry of  $\boldsymbol{\Omega}(\boldsymbol{\omega})$  preserves the normalization of  $q$ , but this normalization may be lost due to computational errors. If so, it can be restored trivially by  $q = q/|q|$ , which is a much simpler operation than orthogonalizing an approximately orthogonal direction cosine matrix.

### Gibbs Vector or Rodrigues Parameters

The three Rodrigues parameters are defined by [6, 7, 19]

$$\mathbf{g} \equiv \frac{\mathbf{q}}{q_4} = \frac{\mathbf{e} \sin(\phi/2)}{\cos(\phi/2)} = \mathbf{e} \tan(\phi/2), \quad (16)$$

This has the inverse relation

$$q = \frac{1}{\sqrt{1 + \mathbf{g}^2}} \begin{bmatrix} \mathbf{g} \\ 1 \end{bmatrix}, \quad (17)$$

where we use an italic letter to represent the magnitude of a three-vector, except when the latter is the vector part of a quaternion. These parameters also give the Cayley parameterization [23], and Gibbs arrayed them in a "vector semitangent of version" [24]. It is little wonder that we now call it the Gibbs vector. The Gibbs vector is a gnomonic projection, a 2-1 mapping of the S3 quaternion space onto three-dimensional Euclidean  $\mathbf{g}$  space, with  $q$  and  $-q$  mapping to the same point, as shown in Fig. 1. Since  $q$  and  $-q$  represent the same rotation, the Gibbs vector parameterization is a 1-1 representation of the rotations. The Gibbs vector is infinite for 180° rotations (the  $q_4 = 0$  equator of S3), which is undesirable for a global representation of rotations.

### Modified Rodrigues Parameters

These parameters, defined by [7]

$$\mathbf{p} \equiv \frac{\mathbf{q}}{1 + q_4} = \frac{\mathbf{e} \sin(\phi/2)}{1 + \cos(\phi/2)} = \mathbf{e} \tan(\phi/4). \quad (18)$$

were introduced by Wiener [25] and rediscovered by Marandi and Modi [26], who established their interpretation as a stereographic projection. The inverse relation is

$$q = \frac{1}{1 + p^2} \begin{bmatrix} 2\mathbf{p} \\ 1 - p^2 \end{bmatrix}. \quad (19)$$

The relation between the quaternion and the modified Rodrigues parameters is like a stereographic projection of S3 quaternion space, as shown in Fig. 2. One hemisphere of S3 projects to interior of the unit sphere in three-dimensional  $\mathbf{p}$  space, and the other hemisphere of S3 projects to the exterior of the unit  $\mathbf{p}$ -sphere.

All rotations can be represented by modified Rodrigues parameters inside and on the surface of the unit ball. The two parameters  $\mathbf{p}$  and  $-\mathbf{p}/p^2$  represent the same rotation. Thus the origin and the point at infinity both represent a zero rotation. The properties of the modified Rodrigues parameters are very similar to those of the rotation vector, but no trigonometric functions are needed to express the rotation matrix in terms of the former parameters.

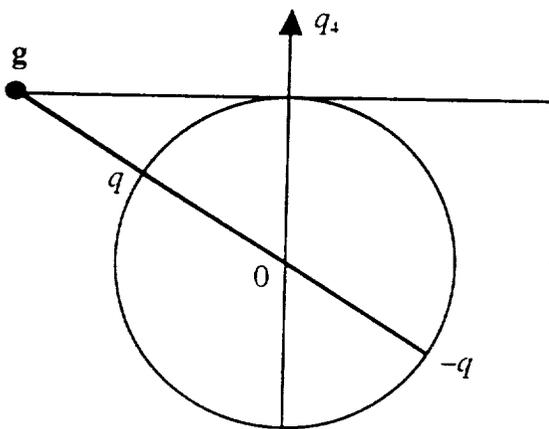


Figure 1. Gibbs Vector as a Gnomonic Projection

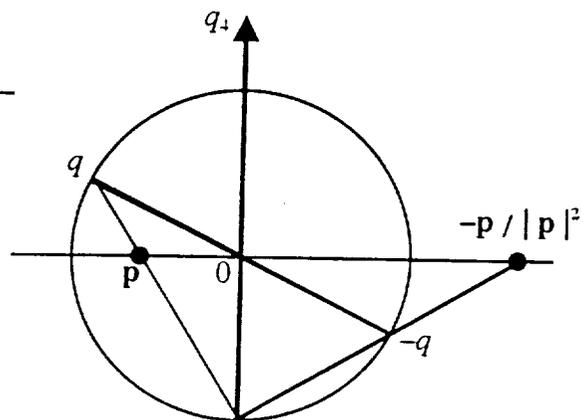


Figure 2. Modified Rodrigues Parameters as a Stereographic Projection

## EXTENDED KALMAN FILTER

The multiplicative EKF represents the attitude quaternion by

$$q(t) = \delta q(\mathbf{a}(t)) \otimes q_{ref}(t), \quad (20)$$

where  $q_{ref}(t)$  is some normalized reference quaternion and  $\delta q(\mathbf{a}(t))$  is a unit quaternion parameterizing the rotation from the reference attitude parameterized by  $q_{ref}(t)$  to the true attitude parameterized by  $q(t)$ . We choose the reference quaternion so that  $\delta q$  is close to the identity quaternion and can be represented by a three-dimensional parameterization, which we denote by  $\mathbf{a}$  for attitude. The two attitude representations  $\mathbf{a}(t)$  and  $q_{ref}(t)$  in Eq. (20) are clearly redundant. We choose this form so that the three-component  $\mathbf{a}(t)$  can keep the statistics straight while assuring exact quaternion normalization, and the four-component  $q_{ref}(t)$  can provide a globally nonsingular attitude representation.

One possible parameterization of  $\mathbf{a}$  is the rotation vector  $\phi \mathbf{e}$ , which we denote  $\mathbf{a}_\phi$ , so from Eq. (9),

$$\delta q = \begin{bmatrix} (\mathbf{a}_\phi / a_\phi) \sin(a_\phi / 2) \\ \cos(a_\phi / 2) \end{bmatrix}. \quad (21a)$$

We allow  $\mathbf{a}_\phi$  to range over a ball of radius  $\pi$  to cover the entire rotation group, but expect it to be close to the origin if  $q_{ref}$  is close to  $q$ . This parameterization has the advantage that the covariance will include the angular variances in radians<sup>2</sup>, but it is numerically inconvenient. A special form, such as a Taylor series, must be used near  $\mathbf{a}_\phi = \mathbf{0}$ . We can retain the interpretation of the covariance matrix in the small angle approximation by requiring  $\mathbf{a}$  to have the same first-order limit as a function of the rotation vector. This leads to the second parameterization of  $\mathbf{a}$  as twice the vector part of  $\delta q$ , which is the parameterization used in Section XI of [5], except for the factor of two:

$$\delta q = \frac{1}{2} \begin{bmatrix} \mathbf{a}_q \\ \sqrt{4 - a_q^2} \end{bmatrix}. \quad (21b)$$

where  $\mathbf{a}_q$  ranges over a ball of radius 2. A third parameterization is twice the Gibbs vector; from Eq. (17),

$$\delta q = \frac{1}{\sqrt{4 + a_g^2}} \begin{bmatrix} \mathbf{a}_g \\ 2 \end{bmatrix}, \quad (21c)$$

where  $\mathbf{a}_g$  ranges over all of E3. A fourth alternative is four times the vector of modified Rodrigues parameters, so from Eq. (19),

$$\delta q = \frac{1}{16 + a_p^2} \begin{bmatrix} 8\mathbf{a}_p \\ 16 - a_p^2 \end{bmatrix}, \quad (21d)$$

where  $\mathbf{a}_p$  ranges over a ball of radius 4. This parameterization has the computational advantage of not requiring any transcendental functions. These four definitions of  $\mathbf{a}$  have the same second-order approximation,

$$\delta q = \begin{bmatrix} \mathbf{a}/2 \\ 1 - a^2/8 \end{bmatrix}. \quad (22)$$

Thus they are equivalent for an EKF, which uses a linear approximation; but they differ in third and higher orders in  $\mathbf{a}$ . It is worthwhile to note that Eq. (22) only holds to *first* order if the components of  $\mathbf{a}$  are taken to be Euler angle rotations about three orthogonal axes, as in [2, 3]. Thus, an Euler angle parameterization will lead to the same EKF, but will give different results for a second order filter.

Equations (20) and (21) map a quaternion pdf from a hemisphere of  $S^3$  with  $q_{ref}(t)$  at its pole onto a pdf of  $\mathbf{a}$  over the appropriate subset of  $E^3$  as defined above for the particular parameterization chosen. There is no loss of information in mapping only one hemisphere of  $S^3$ , since the other hemisphere contains the same information due to the 2-1 nature of the quaternion representation and the assumed symmetry  $\rho(q,t) = \rho(-q,t)$ . We then compute  $\hat{\mathbf{a}}(t)$ , the expectation value of  $\mathbf{a}$ , in the conventional way as the integral of the product of  $\mathbf{a}$  and its pdf. This allows us to *define* the best estimate of the quaternion by

$$\hat{q}(t) \equiv \delta q(\hat{\mathbf{a}}(t)) \otimes q_{ref}(t), \quad (23)$$

which is a unit quaternion, and thus an acceptable estimate of the attitude. It appears that we are free to choose the reference quaternion  $q_{ref}(t)$  at our convenience, but this choice is not arbitrary, since the value of  $\hat{q}(t)$  depends on the choice of  $q_{ref}(t)$ . The examples in Appendix A show that the best estimate is obtained by choosing the reference quaternion  $q_{ref}(t)$  so that the expectation value  $\hat{\mathbf{a}}(t)$  is identically zero. With this choice, which we adopt in this paper, Eq. (23) shows that  $q_{ref}(t)$  is identically equal to  $\hat{q}(t)$ , so we will denote it as such. This means in turn that  $\delta q(\mathbf{a}(t))$  is a representation of the attitude error. This choice has the additional computational advantage that it obviates the need to propagate  $\hat{\mathbf{a}}(t)$ .

### State Propagation

The filter dynamics are found by differentiating

$$\delta q(t) = q(t) \otimes \hat{q}^{-1}(t), \quad (24)$$

inserting Eq. (13) and the identity

$$d\hat{q}^{-1}/dt = -\hat{q}^{-1} \otimes \dot{\hat{q}} \otimes \hat{q}^{-1}, \quad (25)$$

which gives

$$d(\delta q)/dt = \frac{1}{2} \bar{\omega} \otimes \delta q - \delta q \otimes \dot{\hat{q}} \otimes \hat{q}^{-1}. \quad (26)$$

Now let us consider the Gibbs vector parameterization for specificity. This parameterization has the advantage that it takes values over all of  $E^3$  rather than only over a finite subset. Solving Eqs. (20) and (21c) for  $\mathbf{a}_i$  gives

$$\mathbf{a}_i(t) = 2(\delta q)_v / (\delta q)_s. \quad (27)$$

Equations (21c), (26) and (27) give

$$\begin{aligned} \dot{\mathbf{a}}_i(t) = & \left\{ \frac{1}{2} \bar{\omega}(t) \otimes \begin{bmatrix} \mathbf{a}_i(t) \\ 2 \end{bmatrix} - \begin{bmatrix} \mathbf{a}_i(t) \\ 2 \end{bmatrix} \otimes \dot{\hat{q}}(t) \otimes \hat{q}^{-1}(t) \right\}_v \\ & - \frac{1}{2} \left\{ \frac{1}{2} \bar{\omega}(t) \otimes \begin{bmatrix} \mathbf{a}_i(t) \\ 2 \end{bmatrix} - \begin{bmatrix} \mathbf{a}_i(t) \\ 2 \end{bmatrix} \otimes \dot{\hat{q}}(t) \otimes \hat{q}^{-1}(t) \right\}_s \mathbf{a}_i(t) \equiv \mathbf{f}(\mathbf{x}(t), t). \end{aligned} \quad (28)$$

Now we specialize to the case of attitude and gyro drift estimation [5]. In this case, the angular rate vector is given in terms of the gyro output vector  $\mathbf{u}(t)$  and gyro drift vector  $\mathbf{b}(t)$  in spacecraft body coordinates by

$$\omega(t) = \mathbf{u}(t) - \mathbf{b}(t) - \boldsymbol{\eta}_1(t), \quad (29)$$

where  $\boldsymbol{\eta}_1(t)$  is a zero-mean white noise process. The gyro drift vector obeys

$$\dot{\mathbf{b}}(t) = \boldsymbol{\eta}_2(t), \quad (30)$$

where  $\boldsymbol{\eta}_2(t)$  is an independent zero-mean white noise process.

We want to construct an extended Kalman filter for the six-component state vector

$$\mathbf{x}(t) \equiv \begin{bmatrix} \mathbf{a}(t) \\ \mathbf{b}(t) \end{bmatrix}, \quad (31)$$

where we suppress the subscript on  $\mathbf{a}$  in equations that hold for any three-dimensional parameterization. This state vector obeys the propagation equation

$$\dot{\mathbf{x}}(t) \equiv \begin{bmatrix} \mathbf{f}(\mathbf{x}(t), t) \\ \boldsymbol{\eta}_2(t) \end{bmatrix}, \quad (32)$$

where the time dependence of the reference quaternion is implicitly included in the time argument of  $\mathbf{f}(\mathbf{x}(t), t)$ . The expectation value of this equation propagates the state estimate in the absence of measurements. In the usual linear EKF approximation, which is to ignore correlations in computing the expectation value,

$$\dot{\hat{\mathbf{x}}}(t) = \begin{bmatrix} \dot{\hat{\mathbf{a}}} \\ \dot{\hat{\mathbf{b}}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}(\hat{\mathbf{x}}(t), t) \\ \mathbf{0} \end{bmatrix}, \quad (33)$$

We want to choose  $\hat{q}(t)$  so that  $\hat{\mathbf{a}}(t)$  is identically zero, and the expectation value of Eqs. (28) and (29) with  $\hat{\mathbf{a}}(t)$  and  $\hat{\mathbf{b}}(t)$  equal to zero gives

$$\{\dot{\hat{q}}(t) \otimes \hat{q}^{-1}(t)\}_v = \frac{1}{2}[\mathbf{u}(t) - \hat{\mathbf{b}}(t)] \equiv \frac{1}{2}\hat{\boldsymbol{\omega}}(t). \quad (34)$$

The normalization of  $\hat{q}(t)$  requires

$$\{\dot{\hat{q}}(t) \otimes \hat{q}^{-1}(t)\}_4 = 0, \quad (35)$$

so

$$\dot{\hat{q}}(t) = \frac{1}{2}\overline{\hat{\boldsymbol{\omega}}}(t) \otimes \hat{q}(t) = \frac{1}{2}\Omega(\hat{\boldsymbol{\omega}}(t))\hat{q}(t) = \frac{1}{2}\Xi(\hat{q}(t))\hat{\boldsymbol{\omega}}(t), \quad (36)$$

Appendix B shows that Eq. (36) does not depend on our choice of three-dimensional parameterizations of the attitude. This quaternion propagation equation is the same as the equation derived by more conventional methods, but we have *derived* it from the requirement that  $\hat{\mathbf{a}}(t)$  be identically zero. The usual approach must either postulate Eq. (36) *a priori* or derive it from dubious arguments using Eq. (6).

### Measurement Model and Covariance Propagation

Processing a set of measurements at a discrete time  $t_k$  in a standard Kalman filter for the state vector defined in Eq. (31) produces a post-update estimate  $\mathbf{x}(t_k+)$  in which the three-component attitude estimate  $\mathbf{a}(t_k+)$  will not be zero, in general. In order to avoid the need to propagate two different representations of the attitude, the attitude information is moved from  $\mathbf{a}(t_k+)$  to  $\hat{q}(t_k+)$ , after which  $\hat{\mathbf{a}}(t_k+)$  is set to zero. Since the value of the true quaternion is not changed by these operations, Eq. (20) requires

$$\delta q(\hat{\mathbf{a}}(t_k+)) \otimes \hat{q}(t_k-) = \delta q(\mathbf{0}) \otimes \hat{q}(t_k+). \quad (37)$$

Since  $\delta q(\mathbf{0})$  is the identity quaternion, this gives

$$\hat{q}(t_k+) = \delta q(\hat{\mathbf{a}}(t_k+)) \otimes \hat{q}(t_k-). \quad (38)$$

This procedure of moving the update information from  $\hat{\mathbf{a}}(t_k+)$  to  $\hat{q}(t_k+)$  is called a *reset*.

The reset operation is assumed not to modify the covariance. This seems logical, since the total information content of the estimate is neither increased nor decreased by the reset; it is merely moved from one part of the attitude representation to another. The discrete reset operation is a cause for some concern, though, so we want to find some means to avoid it.

We can eliminate the discrete reset operation by keeping  $\hat{\mathbf{a}}(t) \equiv \mathbf{0}$  at all times, even during the update. We accomplish this by considering each attitude measurement update to be spread out over an infinitesimal time interval, rather than being instantaneous. Thus the  $k$ th attitude measurement update is assumed to take place over the infinitesimal time interval from  $t_k$  to  $t_k + \varepsilon$ . Since the measurements are actually discrete and not truly continuous, we are really only interested in the limit that  $\varepsilon \rightarrow 0$ . In the reset terminology, the filter I am presenting here is a Continuous Reset Extended Kalman Filter. It is difficult to resist the temptation to call it the CoREK Filter, to contrast it with the other (inCoREK) filters.

For non-instantaneous measurements, Eq. (33) is replaced by [8-10]

$$\dot{\hat{\mathbf{x}}}(t) = \begin{bmatrix} \mathbf{f}(\hat{\mathbf{x}}(t), t) \\ \mathbf{0} \end{bmatrix} + P(t)H^T(t)R^{-1}(t)[\mathbf{z}(t) - \mathbf{h}(\hat{\mathbf{x}}(t), t)], \quad (39)$$

where  $\mathbf{z}(t)$  is an  $m$ -dimensional vector of attitude measurements and  $\mathbf{h}(\hat{\mathbf{x}}(t), t)$  is an  $m$ -dimensional vector of measurement models with the time dependence of the reference quaternion  $\hat{q}(t)$  implicitly included in its time argument. The covariance can be partitioned into  $3 \times 3$  submatrices as

$$P(t) \equiv E\{(\mathbf{x} - \hat{\mathbf{x}})(\mathbf{x} - \hat{\mathbf{x}})^T\} = \begin{bmatrix} P_a(t) & P_c(t) \\ P_c^T(t) & P_b(t) \end{bmatrix}, \quad (40)$$

and the  $m \times 6$  measurement sensitivity matrix is

$$H = \begin{bmatrix} \frac{\partial \mathbf{h}}{\partial \mathbf{a}} & \frac{\partial \mathbf{h}}{\partial \mathbf{b}} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{h}}{\partial \mathbf{a}} & \mathbf{0}_{m \times 3} \end{bmatrix}, \quad (41)$$

since the attitude measurements are assumed not to depend explicitly on the gyro drifts. This gives

$$P(t)H^T(t)R^{-1}(t)[\mathbf{z}(t) - \mathbf{h}(\hat{\mathbf{x}}(t), t)] = \begin{bmatrix} P_a(t)(\frac{\partial \mathbf{h}}{\partial \mathbf{a}})^T R^{-1}(t)[\mathbf{z}(t) - \mathbf{h}(\hat{\mathbf{x}}(t), t)] \\ P_c^T(t)(\frac{\partial \mathbf{h}}{\partial \mathbf{a}})^T R^{-1}(t)[\mathbf{z}(t) - \mathbf{h}(\hat{\mathbf{x}}(t), t)] \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{\alpha}(\hat{\mathbf{x}}(t), t) \\ \boldsymbol{\beta}(\hat{\mathbf{x}}(t), t) \end{bmatrix}. \quad (42)$$

The propagation of the Gibbs-vector-based representation including measurements is

$$\begin{aligned} \dot{\hat{\mathbf{a}}}_s(t) = & \left\{ \frac{1}{2} \bar{\boldsymbol{\omega}}(t) \otimes \begin{bmatrix} \hat{\mathbf{a}}_s(t) \\ 2 \end{bmatrix} - \begin{bmatrix} \hat{\mathbf{a}}_s(t) \\ 2 \end{bmatrix} \otimes \dot{\hat{q}}(t) \otimes \hat{q}^{-1}(t) \right\}_v \\ & - \frac{1}{2} \left\{ \frac{1}{2} \bar{\boldsymbol{\omega}}(t) \otimes \begin{bmatrix} \hat{\mathbf{a}}_s(t) \\ 2 \end{bmatrix} - \begin{bmatrix} \hat{\mathbf{a}}_s(t) \\ 2 \end{bmatrix} \otimes \dot{\hat{q}}(t) \otimes \hat{q}^{-1}(t) \right\}_s \hat{\mathbf{a}}_s(t) + \boldsymbol{\alpha}(\hat{\mathbf{x}}(t), t), \end{aligned} \quad (43)$$

Equation (43) with  $\hat{\mathbf{a}}_s(t)$  and  $\dot{\hat{\mathbf{a}}}_s(t)$  equal to zero gives

$$\{\dot{\hat{q}}(t) \otimes \hat{q}^{-1}(t)\}_v = \frac{1}{2} [\bar{\boldsymbol{\omega}}(t) + \boldsymbol{\alpha}(\hat{\mathbf{x}}(t), t)]. \quad (44)$$

so, with Eq. (35),

$$\dot{\hat{q}}(t) = \frac{1}{2} [\bar{\boldsymbol{\omega}}(t) + \boldsymbol{\alpha}(\hat{\mathbf{x}}(t), t)] \otimes \hat{q}(t) = \frac{1}{2} \boldsymbol{\Omega}(\bar{\boldsymbol{\omega}}(t) + \boldsymbol{\alpha}(\hat{\mathbf{x}}(t), t)) \hat{q}(t) = \frac{1}{2} \Xi(\hat{q}(t)) [\bar{\boldsymbol{\omega}}(t) + \boldsymbol{\alpha}(\hat{\mathbf{x}}(t), t)]. \quad (45)$$

As before, these results hold for a general three-dimensional attitude representation obeying Eq. (22). Equation (45) and the drift bias equation

$$\dot{\hat{\mathbf{b}}} = \boldsymbol{\beta}(\hat{\mathbf{x}}(t), t) = P_r^T (\partial \mathbf{h} / \partial \mathbf{a})^T R^{-1}(t) [\mathbf{z}(t) - \mathbf{h}(\hat{\mathbf{x}}(t), t)]. \quad (46)$$

are the estimate propagation equations including measurements, since we have chosen the reference quaternion so that  $\hat{\mathbf{a}}(t)$  is identically zero.

Substituting Eq. (45) into Eq. (28) gives, after some algebra,

$$\begin{aligned} \mathbf{f}(\mathbf{x}(t), t) = & -\hat{\boldsymbol{\omega}}(t) \times \mathbf{a}_r(t) + \Delta \boldsymbol{\omega}(t) - \boldsymbol{\alpha}(\hat{\mathbf{x}}(t), t) - \frac{1}{2} [\Delta \boldsymbol{\omega}(t) + \boldsymbol{\alpha}(\hat{\mathbf{x}}(t), t)] \times \mathbf{a}_r(t) \\ & + \frac{1}{2} \{ [\Delta \boldsymbol{\omega}(t) - \boldsymbol{\alpha}(\hat{\mathbf{x}}(t), t)] \cdot \mathbf{a}_r(t) \} \mathbf{a}_r(t), \end{aligned} \quad (47)$$

where

$$\Delta \boldsymbol{\omega}(t) \equiv \boldsymbol{\omega}(t) - \hat{\boldsymbol{\omega}}(t) = -\mathbf{b}(t) + \hat{\mathbf{b}}(t) - \boldsymbol{\eta}_1(t). \quad (48)$$

It follows that the covariance matrix obeys

$$\dot{P}(t) = F(t)P(t) + P(t)F^T(t) + G(t)Q(t)G^T(t) - P(t)H^T(t)R^{-1}(t)H(t)P(t), \quad (49)$$

where

$$F(t) = \begin{bmatrix} E\{\partial \mathbf{f} / \partial \mathbf{a}\} & E\{\partial \mathbf{f} / \partial \mathbf{b}\} \\ 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} = \begin{bmatrix} -[\hat{\boldsymbol{\omega}}(t) \times] & -I_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix}, \quad (50)$$

$$G(t) = \begin{bmatrix} E\{\partial \mathbf{f} / \partial \boldsymbol{\eta}_1\} & E\{\partial \mathbf{f} / \partial \boldsymbol{\eta}_2\} \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix} = \begin{bmatrix} -I_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix}, \quad (51)$$

$$Q(t) = \begin{bmatrix} Q_1(t) & 0_{3 \times 3} \\ 0_{3 \times 3} & Q_2(t) \end{bmatrix}, \quad (52)$$

and

$$E\{\boldsymbol{\eta}_i(t)\boldsymbol{\eta}_j^T(t')\} = \delta_{ij}\delta(t-t')Q_i(t) \text{ for } i, j = 1, 2. \quad (53)$$

Recalling that the attitude measurements are discrete and not truly continuous, we see that the covariance propagation during the intervals between the attitude measurement updates, when the last term in Eq. (49) is absent, is identical to [2-5], except for factors of  $\frac{1}{2}$  in some of the formulations.

### Covariance Update

In considering the  $k$ th measurement update, we will treat the  $m \times 3$  measurement noise matrix  $R(t)$  and  $m \times m$  measurement sensitivity matrix  $H(t)$  as constants during the infinitesimal interval:

$$\frac{\partial \mathbf{h}}{\partial \mathbf{a}} \equiv H_k. \quad (54)$$

and

$$R(t) = \varepsilon R_k, \quad (55)$$

where  $R_k$  is the equivalent discrete measurement noise matrix as on p. 122 of [8]. The appearance of  $\varepsilon$  in the measurement noise matrix is the reason we get a finite measurement update to the state and covariance in the  $\varepsilon \rightarrow 0$  limit. We can ignore all the non-measurement terms in the equations for the state estimates

and the covariance during the update, since these will give corrections of that go to zero in this limit. The covariance propagation equation during the update is

$$\dot{P}(t) = -P(t) \begin{bmatrix} H_k^T (\varepsilon R_k)^{-1} H_k & 0_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} P(t). \quad (56)$$

The solution of this is simply

$$P^{-1}(t) = P^{-1}(t_k) + \begin{bmatrix} H_k^T R_k^{-1} H_k & 0_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} \frac{t - t_k}{\varepsilon}. \quad (57)$$

Using the matrix inversion lemma [9] gives

$$P(t) = P(t_k) - P(t_k) \begin{bmatrix} H_k^T \\ 0_{3 \times m} \end{bmatrix} \left[ H_k P_a(t_k) H_k^T + \frac{\varepsilon R_k}{t - t_k} \right]^{-1} \begin{bmatrix} H_k & 0_{m \times 3} \end{bmatrix} P(t_k), \quad (58)$$

which reduces to the usual form at  $t_k + \equiv t_k + \varepsilon$ , the end of the finite-time update,

$$P(t_k +) = P(t_k) - P(t_k) \begin{bmatrix} H_k^T \\ 0_{3 \times m} \end{bmatrix} \left[ H_k P_a(t_k) H_k^T + R_k \right]^{-1} \begin{bmatrix} H_k & 0_{m \times 3} \end{bmatrix} P(t_k). \quad (59)$$

This update is also identical to the standard update [2-5], except for occasional factors of  $\frac{1}{\varepsilon}$ .

### Measurement Update

The quaternion estimation equation during the update, again ignoring terms that go to zero in the  $\varepsilon \rightarrow 0$  limit, is

$$\begin{aligned} \dot{\hat{q}}(t) &= \frac{1}{\varepsilon} \Xi(\hat{q}(t)) \alpha(\hat{q}(t), t) = \frac{1}{\varepsilon} \Xi(\hat{q}(t)) P_a(t) H_k^T (\varepsilon R_k)^{-1} [\mathbf{z}(t) - \mathbf{h}(\hat{q}(t), t)] \\ &= \frac{1}{\varepsilon} \Xi(\hat{q}(t)) \left\{ P_a(t_k) - P_a(t_k) H_k^T \left[ H_k P_a(t_k) H_k^T + \frac{\varepsilon R_k}{t - t_k} \right]^{-1} H_k P_a(t_k) \right\} H_k^T (\varepsilon R_k)^{-1} [\mathbf{z}(t) - \mathbf{h}(\hat{q}(t), t)] \\ &= \frac{1}{\varepsilon} \Xi(\hat{q}(t)) P_a(t_k) H_k^T \left\{ I - \left[ H_k P_a(t_k) H_k^T + \frac{\varepsilon R_k}{t - t_k} \right]^{-1} H_k P_a(t_k) H_k^T \right\} (\varepsilon R_k)^{-1} [\mathbf{z}(t) - \mathbf{h}(\hat{q}(t), t)] \\ &= \frac{1}{\varepsilon} \Xi(\hat{q}(t)) P_a(t_k) H_k^T \left[ (t - t_k) H_k P_a(t_k) H_k^T + \varepsilon R_k \right]^{-1} [\mathbf{z}(t) - \mathbf{h}(\hat{q}(t), t)]. \end{aligned} \quad (60)$$

Several equations have been used to obtain this result, Eqs. (42), (45), and (58) in particular. Taking the measurement to be constant,  $\mathbf{z}(t) = \mathbf{z}_k$ , and assuming that the only time dependence of  $\mathbf{h}(\hat{q}(t), t) = \mathbf{h}_k(\hat{q}(t))$  is through the quaternion, the measurement residual time dependence can be computed by

$$\frac{d}{dt} [\mathbf{z}_k - \mathbf{h}_k(\hat{q}(t))] = - \left. \frac{\partial \mathbf{h}_k(q)}{\partial q} \right|_{q=\hat{q}(t)} \dot{\hat{q}}(t). \quad (61)$$

We evaluate this partial derivative for the Gibbs vector parameterization of  $\mathbf{a}$ . Equation (5) can be written

$$\mathbf{a}_r(t) = \frac{2[q(t) \otimes \hat{q}^{-1}(t)]_v}{[q(t) \otimes \hat{q}^{-1}(t)]_s} = \frac{2\Xi^T(\hat{q}(t))q}{\hat{q}^T(t)q}. \quad (62)$$

Then, using the chain rule for differentiation and the identity

$$\Xi^T(\hat{q})\hat{q} = 0, \quad (63)$$

we find that

$$\left. \frac{\partial \mathbf{h}_k(q)}{\partial q} \right|_{q=\hat{q}} = \left. \frac{\partial \mathbf{h}_k}{\partial \mathbf{a}_f} \right|_{\mathbf{a}=0} \left. \frac{\partial \mathbf{a}_f}{\partial q} \right|_{q=\hat{q}} = 2 H_k \Xi^T(\hat{q}(t)). \quad (64)$$

It is shown in Appendix B that this result is independent of our choice of three-dimensional parameterization. Inserting Eqs. (60) and (64) into Eq. (61) gives

$$\frac{d}{dt} [\mathbf{z}_k - \mathbf{h}_k(\hat{q}(t))] = -H_k P_d(t_k) H_k^T \left[ (t - t_k) H_k P_d(t_k) H_k^T + \varepsilon R_k \right]^{-1} [\mathbf{z}_k - \mathbf{h}_k(\hat{q}(t))], \quad (65)$$

where we have used the identity

$$\Xi^T(q)\Xi(q) = I_{3 \times 3}. \quad (66)$$

The solution of Eq. (64) is

$$[\mathbf{z}_k - \mathbf{h}_k(\hat{q}(t))] = \varepsilon R_k \left[ (t - t_k) H_k P_d(t_k) H_k^T + \varepsilon R_k \right]^{-1} [\mathbf{z}_k - \mathbf{h}_k(\hat{q}(t_k))]. \quad (67)$$

The validity of this solution can be verified by differentiating and using the identity

$$H_k P_d(t_k) H_k^T \left[ (t - t_k) H_k P_d(t_k) H_k^T + \varepsilon R_k \right]^{-1} \varepsilon R_k = \varepsilon R_k \left[ (t - t_k) H_k P_d(t_k) H_k^T + \varepsilon R_k \right]^{-1} H_k P_d(t_k) H_k^T. \quad (68)$$

This identity is clearly true if  $t = t_k$ . If  $t \neq t_k$ , its truth is demonstrated by

$$\begin{aligned} & (t - t_k) H_k P_d(t_k) H_k^T \left[ (t - t_k) H_k P_d(t_k) H_k^T + \varepsilon R_k \right]^{-1} \varepsilon R_k \\ &= \left\{ I - \varepsilon R_k \left[ (t - t_k) H_k P_d(t_k) H_k^T + \varepsilon R_k \right]^{-1} \right\} \varepsilon R_k = \varepsilon R_k \left\{ I - \left[ (t - t_k) H_k P_d(t_k) H_k^T + \varepsilon R_k \right]^{-1} \varepsilon R_k \right\} \\ &= \varepsilon R_k \left[ (t - t_k) H_k P_d(t_k) H_k^T + \varepsilon R_k \right]^{-1} (t - t_k) H_k P_d(t_k) H_k^T. \end{aligned} \quad (69)$$

Inserting Eq. (67) into Eq. (60) gives

$$\dot{\hat{q}}(t) = \frac{1}{2} \Xi(\hat{q}(t)) P_d(t_k) \chi(t) = \frac{1}{2} \Omega(P_d(t_k) \chi(t)) \hat{q}(t), \quad (70)$$

where

$$\begin{aligned} \chi(t) &\equiv H_k^T \left[ (t - t_k) H_k P_d(t_k) H_k^T + \varepsilon R_k \right]^{-1} \varepsilon R_k \left[ (t - t_k) H_k P_d(t_k) H_k^T + \varepsilon R_k \right]^{-1} [\mathbf{z}_k - \mathbf{h}_k(\hat{q}(t_k))] \\ &= \frac{d}{dt} \left\{ (t - t_k) H_k^T \left[ (t - t_k) H_k P_d(t_k) H_k^T + \varepsilon R_k \right]^{-1} [\mathbf{z}_k - \mathbf{h}_k(\hat{q}(t_k))] \right\}. \end{aligned} \quad (71)$$

The solution of this equation in the  $\varepsilon \rightarrow 0$  limit is [6]

$$\hat{q}(t_k +) = \exp\left(\frac{1}{2} \Omega(\Phi_k)\right) \hat{q}(t_k), \quad (72)$$

where

$$\Phi_k = P_d(t_k) \int_{t_k}^{t_k + \varepsilon} \chi(t) dt = P_d(t_k) H_k^T \left[ H_k P_d(t_k) H_k^T + R_k \right]^{-1} [\mathbf{z}_k - \mathbf{h}_k(\hat{q}(t_k))]. \quad (73)$$

Similar computations starting with Eq. (46) give the time dependence of the bias as

$$\hat{\mathbf{b}}(t) = P_d^T(t_k) \chi(t), \quad (74)$$

which has the solution

$$\hat{\mathbf{b}}(t_k+) = \hat{\mathbf{b}}(t_k) + P_c^T(t_k) H_k^T \left[ H_k P_c(t_k) H_k^T + R_k \right]^{-1} [\mathbf{z}_k - \mathbf{h}_k(\hat{\mathbf{q}}(t_k))]. \quad (75)$$

This drift bias update is the standard EKF result [2-5]; but the quaternion update differs in preserving the quaternion norm exactly. The update vector  $\hat{\boldsymbol{\phi}}_k$  would be  $\hat{\mathbf{a}}(t_k+)$  in a conventional EKF with a discrete reset as in Eq. (38). In the treatment presented here,  $\hat{\boldsymbol{\phi}}_k$  appears as a rotation vector independent of which three-dimensional parameterization we choose to represent the attitude error. The conventional procedure is more efficient than the norm-preserving update derived here, however, especially as implemented in [4]; and these methods are equivalent to second order in the measurement residuals.

## SECOND ORDER FILTER

According to Maybeck [10], a *first-order filter with bias correction terms* obtains the essential benefit of a second order filter without the computational penalty of additional second moment calculations. This filter adds second-order corrections to the state propagation and measurement residual equations, but uses the EKF expressions for the covariance and gains. In the continuous measurement case, Eq. (39) is replaced by

$$\dot{\hat{\mathbf{x}}}(t) = \begin{bmatrix} \mathbf{f}(\hat{\mathbf{x}}(t), t) + \hat{\mathbf{b}}_p(t) \\ \mathbf{0} \end{bmatrix} + P(t) H^T(t) R^{-1}(t) [\mathbf{z}(t) - \mathbf{h}(\hat{\mathbf{x}}(t), t) - \hat{\mathbf{b}}_m(t)], \quad (76)$$

where

$$b_{pk}(t) \equiv \frac{1}{2} \text{tr} \left\{ \frac{\partial^2 f_k(\mathbf{x}, t)}{\partial \mathbf{x}^2} P(t) \right\} \Bigg|_{\mathbf{x}=\hat{\mathbf{x}}(t)} \quad (77)$$

and

$$b_{mk}(t) \equiv \frac{1}{2} \text{tr} \left\{ \frac{\partial^2 h_k(\mathbf{x}, t)}{\partial \mathbf{x}^2} P(t) \right\} \Bigg|_{\mathbf{x}=\hat{\mathbf{x}}(t)} = \frac{1}{2} \text{tr} \left\{ \frac{\partial^2 h_k(\mathbf{a}, t)}{\partial \mathbf{a}^2} P_a(t) \right\} \Bigg|_{\mathbf{a}=\hat{\mathbf{a}}(t)} \quad (78)$$

The second form of Eq. (78) results from the lack of explicit gyro bias dependence of the measurement. In parallel with Eq. (42), we write the measurement part of Eq. (76) as

$$\begin{aligned} P(t) H^T(t) R^{-1}(t) [\mathbf{z}(t) - \mathbf{h}(\hat{\mathbf{x}}(t), t) - \hat{\mathbf{b}}_m(t)] &= \begin{bmatrix} P_a(t) (\partial \mathbf{h} / \partial \mathbf{a})^T R^{-1}(t) [\mathbf{z}(t) - \mathbf{h}(\hat{\mathbf{x}}(t), t) - \hat{\mathbf{b}}_m(t)] \\ P_c^T(t) (\partial \mathbf{h} / \partial \mathbf{a})^T R^{-1}(t) [\mathbf{z}(t) - \mathbf{h}(\hat{\mathbf{x}}(t), t) - \hat{\mathbf{b}}_m(t)] \end{bmatrix} \\ &\equiv \begin{bmatrix} \boldsymbol{\alpha}'(\hat{\mathbf{x}}(t), t) \\ \boldsymbol{\beta}'(\hat{\mathbf{x}}(t), t) \end{bmatrix}. \end{aligned} \quad (79)$$

For the Gibbs vector parameterization, Eqs. (28), (40) and (77) give

$$\hat{\mathbf{b}}_p(t) = \frac{1}{4} P_a \{ \hat{\boldsymbol{\omega}}(t) - 2[\hat{\mathbf{q}}(t) \otimes \hat{\mathbf{q}}^{-1}(t)]_v \} + \boldsymbol{\omega}_c(t), \quad (80)$$

where  $\boldsymbol{\omega}_c$  has components

$$\omega_{ci}(t) \equiv -\frac{1}{2} \sum_{j,k=1,3} \varepsilon_{ijk} [P_c(t)]_{jk} \quad \text{for } i = 1, 2, 3, \quad (81)$$

with  $\varepsilon_{ijk}$  being the totally antisymmetric Levi-Civita density:

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \quad (82a)$$

$$\varepsilon_{321} = \varepsilon_{213} = \varepsilon_{132} = -1, \quad (82b)$$

$$\varepsilon_{ijk} = 0 \quad \text{if any two indices are equal.} \quad (82c)$$

Including these second order terms into Eq. (43) gives

$$\begin{aligned} \dot{\hat{\mathbf{a}}}_q(t) = & \left\{ \frac{1}{2} \bar{\boldsymbol{\omega}}(t) \otimes \begin{bmatrix} \hat{\mathbf{a}}_q(t) \\ 2 \end{bmatrix} - \begin{bmatrix} \hat{\mathbf{a}}_q(t) \\ 2 \end{bmatrix} \otimes \hat{q}(t) \otimes \hat{q}^{-1}(t) \right\}_v \\ & - \frac{1}{2} \left\{ \frac{1}{2} \bar{\boldsymbol{\omega}}(t) \otimes \begin{bmatrix} \hat{\mathbf{a}}_q(t) \\ 2 \end{bmatrix} - \begin{bmatrix} \hat{\mathbf{a}}_q(t) \\ 2 \end{bmatrix} \otimes \hat{q}(t) \otimes \hat{q}^{-1}(t) \right\}_4 \hat{\mathbf{a}}_q(t) \\ & + \frac{1}{2} P_d \{ \hat{\boldsymbol{\omega}}(t) - 2[\hat{q}(t) \otimes \hat{q}^{-1}(t)]_v \} + \boldsymbol{\omega}_c(t) + \boldsymbol{\alpha}'(\hat{\mathbf{x}}(t), t). \end{aligned} \quad (83)$$

The condition that  $\hat{\mathbf{a}}_q(t)$  and  $\dot{\hat{\mathbf{a}}}_q(t)$  are equal to zero is

$$[\hat{q}(t) \otimes \hat{q}^{-1}(t)]_v = \frac{1}{2} \{ \hat{\boldsymbol{\omega}}(t) + [I + \frac{1}{2} P_d(t)]^{-1} [\boldsymbol{\omega}_c(t) + \boldsymbol{\alpha}'(\hat{\mathbf{x}}(t), t)] \}. \quad (84)$$

It is shown in Appendix B that the factor of  $[I + \frac{1}{2} P_d(t)]^{-1}$  depends on the specific choice of the three-dimensional parameterization of the rotation. Since  $P_d(t)$  is second order in the estimation errors and this factor multiplies terms of first order in errors, it is consistent with a second-order filter to ignore it, giving

$$\dot{\hat{q}}(t) = \frac{1}{2} \{ \bar{\boldsymbol{\omega}}(t) + \bar{\boldsymbol{\omega}}_c(t) + \bar{\boldsymbol{\alpha}}'(\hat{\mathbf{x}}(t), t) \} \otimes \hat{q}(t) = \frac{1}{2} \Xi(\hat{q}(t)) \{ \hat{\boldsymbol{\omega}}(t) + \boldsymbol{\omega}_c(t) + \boldsymbol{\alpha}'(\hat{\mathbf{x}}(t), t) \}. \quad (85)$$

The time dependence of the gyro drift bias is given by

$$\dot{\hat{\mathbf{b}}} = \boldsymbol{\beta}'(\hat{\mathbf{x}}(t), t). \quad (86)$$

For time propagation between measurements, the  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$  terms are zero, so the only change from the EKF is the addition of the term involving  $\boldsymbol{\omega}_c(t)$  in the quaternion propagation. This is a second-order correction to the angular rate vector arising from the skew part of the covariance between the attitude errors and gyro drift bias errors. The measurement update equations in the  $\varepsilon \rightarrow 0$  limit are given by

$$\hat{q}(t_k+) = \exp\left(\frac{1}{2} \Omega(\boldsymbol{\phi}')\right) \hat{q}(t_k), \quad (87)$$

where

$$\boldsymbol{\phi}' = P_d(t_k) H_k^T \left[ H_k P_d(t_k) H_k^T + R_k \right]^{-1} [\mathbf{z}_k - \mathbf{h}_k(\hat{q}(t_k)) - \hat{\mathbf{b}}_m(t_k)], \quad (88)$$

and

$$\hat{\mathbf{b}}(t_k+) = \hat{\mathbf{b}}(t_k) + P_c^T(t_k) H_k^T \left[ H_k P_d(t_k) H_k^T + R_k \right]^{-1} [\mathbf{z}_k - \mathbf{h}_k(\hat{q}(t_k)) - \hat{\mathbf{b}}_m(t_k)]. \quad (89)$$

Except for the maintenance of quaternion normalization by Eq. (87), these results are equivalent to those obtained by Vathsal [27]. It should be pointed out that Vathsal also considered second order effects on the covariance and gain matrices. The derivation of Eqs. (87)–(89) assumes that  $\hat{\mathbf{b}}_m(t)$  is constant over the infinitesimal duration  $\varepsilon$  of the update; this approximation needs to be verified for a specific measurement model.

## QUATERNION INTEGRATION

Exact propagation of the quaternion using Eq. (13) would preserve the quaternion norm exactly, due to the antisymmetry of the matrix  $\Omega(\boldsymbol{\omega})$ . The simplicity of this propagation, especially its linearity in  $q$ , is often given as one of the advantages of the quaternion representation of the attitude. However, the quaternion norm is not preserved exactly in many numerical integration procedures, for example in commonly employed Runge-Kutta integrators. We propose a new solution to this problem and contrast it with several alternative strategies for enforcing quaternion normalization. The criterion for comparing these methods should be the attitude error rather than the quaternion normalization error, since the latter contains no attitude information.

## New Method

This method is based on the ideas about attitude representations presented in this paper. It uses a three-component representation of the attitude variation over an integration step, and the quaternion representation for the accumulated attitude. This gives up the advantage of the bilinear quaternion differential equation to preserve the quaternion norm. One alternative is to integrate the Gibbs vector kinematic equation [7]

$$\dot{\mathbf{g}} = \frac{1}{2}[\boldsymbol{\omega} - \boldsymbol{\omega} \times \mathbf{g} + (\boldsymbol{\omega} \cdot \mathbf{g})\mathbf{g}], \quad (90)$$

from  $t_n$  to  $t_{n+1}$ , with the initial value  $\mathbf{g}_n = \mathbf{0}$ . At the end of the integration step, the quaternion is updated by

$$q_{n+1} = q(\mathbf{g}_{n+1}) \otimes q_n \quad (91)$$

where  $q(\mathbf{g}_{n+1})$  is given by Eq. (17). The square root in this equation can be avoided by using the Modified Rodrigues Parameters, whose kinematic equation is [7]

$$\dot{\mathbf{p}} = \frac{1}{2}[(1 - \rho^2)\boldsymbol{\omega} - 2\boldsymbol{\omega} \times \mathbf{p} + 2(\boldsymbol{\omega} \cdot \mathbf{p})\mathbf{p}]. \quad (92)$$

This is integrated from  $t_n$  to  $t_{n+1}$ , with the initial value  $\mathbf{p}_n = \mathbf{0}$ , and then the quaternion is updated by

$$q_{n+1} = q(\mathbf{p}_{n+1}) \otimes q_n \quad (93)$$

where  $q(\mathbf{p}_{n+1})$  is given by Eq. (19).

## Direct Integration

The conventional approach is straightforward integration of Eq. (13). Since quaternion normalization is not automatically preserved, we normalize the quaternion by setting  $q = q/|q|$ . We can normalize the quaternion either once after the completion of the entire range of integration or after each integration step.

## Closed-Form Solution

This method is based explicitly on the linearity of Eq. (13). It gives the solution as a matrix exponential [6]

$$q_{n+1} = \exp\left(\frac{1}{2}\Omega(\boldsymbol{\phi}_n)\right)q_n = I \cos(\phi_n/2) + \phi_n^{-1} \Omega(\boldsymbol{\phi}_n) \sin(\phi_n/2), \quad (94)$$

where

$$\boldsymbol{\phi}_k = \int_{t_n}^{t_{n+1}} \boldsymbol{\omega}(t) dt. \quad (95)$$

This solution is only approximate unless the direction of  $\boldsymbol{\omega}$  is constant, which is why Eq. (72) only holds in the  $\varepsilon \rightarrow 0$  limit. Equation (95) gives the lowest order of a Magnus expansion [28], which is related to the Baker-Campbell-Hausdorff formula. The next higher approximation is

$$\boldsymbol{\phi}_k = \int_{t_n}^{t_{n+1}} \boldsymbol{\omega}(t) dt + (h^2/12)\boldsymbol{\omega}(t_n) \times \boldsymbol{\omega}(t_{n+1}), \quad (96)$$

where  $h = t_{n+1} - t_n$  is the integration step size.

## Constraint-Preserving Integrator

A recently proposed a second-order method assuring exact quaternion norm preservation is [29, 30]

$$q_{n+1} = \left[1 + (h/4)^2 \omega_{n+1}^2\right]^{-1} \left[1 + (h/4)\Omega(\boldsymbol{\omega}_{n+1})\right]^2 q_n. \quad (97)$$

Higher order methods have been proposed [30], but these require higher derivatives of the angular rates, which are not generally available without additional computations.

## SUMMARY

The major result of this paper is to clarify the relationship between the four-component quaternion representation of attitude and the three-component representation of attitude errors in the widely used extended Kalman filter that has become known as the multiplicative Kalman filter. We view this filter as based on an apparently redundant representation of the attitude in terms of a reference quaternion and a three-vector specifying the deviation of the attitude from the reference. This apparent redundancy is removed by constraining the reference quaternion so that the expectation value of the three-vector of attitude deviations is identically zero. It is therefore not necessary to compute this identically zero expectation value. The basic structure of the multiplicative Kalman filter follows from constraining the reference quaternion in this fashion: the reference quaternion becomes the attitude estimate, the three-vector becomes the attitude error vector, and the covariance of the three-vector becomes the attitude covariance. All these results are well known in practice, but the justification for using this mixed attitude representation has been unclear.

An explicitly norm-preserving measurement update of the quaternion has been developed in this paper. It is less efficient computationally than the conventional update, however, and it is mathematically equivalent through second order terms in the measurement residuals. We have also investigated a second-order filter in the new framework, but have merely reproduced results obtained previously.

The idea of representing the attitude in terms of a reference quaternion and a three-vector specifying the deviation of the attitude from the reference can also be used to develop norm-preserving quaternion integrators. Some algorithms in this area are proposed.

## APPENDIX A

We'll consider two toy problems to illustrate the assertions about pdfs below Eq. (23). In each case, we'll start with a pdf on S3 that obeys  $\rho(q, t) = \rho(-q, t)$  and has two well-defined maxima. We would like our estimate  $\hat{q}$  to have one of these values. We denote the reference quaternion by  $\bar{q}$ , since we won't assume that  $\hat{\mathbf{a}} = \mathbf{0}$  in this Appendix.

### Case 1

Consider a pdf concentrated at the poles  $\pm \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  of S3:

$$\rho(q) = \delta^{(3)}(\mathbf{q}). \quad (\text{A1})$$

Mapping this to  $\rho(\mathbf{a}_s)$  using Eq. (21c) gives

$$\rho(\mathbf{a}_s) = \delta^{(3)} \left( \frac{q_{ref,s} \mathbf{a}_s + 2\mathbf{q}_{ref} - \mathbf{a}_s \times \mathbf{q}_{ref}}{\sqrt{4 + |\mathbf{a}_s|^2}} \right). \quad (\text{A2})$$

If  $q_{ref,s} \neq 0$ , it's clear that  $\hat{\mathbf{a}}_s$  must be such that the argument of the delta function in Eq. (A2) is zero. This gives

$$\hat{\mathbf{a}}_s = -\frac{2\mathbf{q}_{ref}}{q_{ref,s}} \quad (\text{A3})$$

Then Eq. (23) gives

$$\hat{q} = \text{sign}(q_{ref,s}) \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (\text{A4})$$

the desired result. If  $q_{ref,s} = 0$ , then  $\hat{\mathbf{a}}_s = \mathbf{0}$  by symmetry, and  $\hat{q} = q_{ref}$ .

## Case 2

Consider a pdf with a very broad distribution centered at the poles  $\pm \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  of S3:

$$\rho(q) = \frac{4q_{ref}^2}{\pi^2}. \quad (A5)$$

Mapping this to  $\rho(\mathbf{a}_s)$  using Eq. (21c) gives

$$\rho(\mathbf{a}_s) = \frac{4}{\pi^2} \frac{(2q_{ref} - \mathbf{a}_s \cdot \mathbf{q}_{ref})^2}{4 + |\mathbf{a}_s|^2}. \quad (A6)$$

The expectation value of  $\mathbf{a}_s$ , using the appropriate volume element, is

$$\hat{\mathbf{a}}_s = \int \mathbf{a}_s \rho(\mathbf{a}_s) \frac{2d^3\mathbf{a}_s}{(4 + |\mathbf{a}_s|^2)^2} = \frac{8}{\pi^2} \int \frac{\mathbf{a}_s (2q_{ref} - \mathbf{a}_s \cdot \mathbf{q}_{ref})^2}{(4 + |\mathbf{a}_s|^2)^3} d^3\mathbf{a}_s = -4 q_{ref} \mathbf{q}_{ref}, \quad (A7)$$

after a tedious but straightforward integration. Then Eq. (23) gives

$$\hat{q} = \frac{1}{\sqrt{1 + 4q_{ref}^2 |\mathbf{q}_{ref}|^2}} \begin{bmatrix} (1 - 2q_{ref}^2) \mathbf{q}_{ref} \\ (1 + 2|\mathbf{q}_{ref}|^2) q_{ref} \end{bmatrix}. \quad (A8)$$

The special case of  $q_{ref} = \pm 1$  gives the desired result of Eq. (A4), but this is not true in general. It is worth noting, though, that Eq. (A8) gives  $\hat{q}_i^2 > 0.95$  (where the pdf is greater than 95% of its maximum value) for  $|q_{ref}| > 0.562$ .

## APPENDIX B

Instead of using the Gibbs vector, we will employ the vector part of the quaternion as in Eq. (21b). With this parameterization, Eq. (43) is replaced by

$$\dot{\hat{\mathbf{a}}}_q(t) = \left\{ \frac{1}{2} \bar{\hat{\omega}}(t) \otimes \left[ \frac{\hat{\mathbf{a}}_q(t)}{\sqrt{4 - \hat{a}_q^2(t)}} \right] - \left[ \frac{\hat{\mathbf{a}}_q(t)}{\sqrt{4 - \hat{a}_q^2(t)}} \right] \otimes \dot{\hat{q}}(t) \otimes \hat{q}^{-1}(t) \right\}_v + \alpha(\hat{\mathbf{x}}(t), t). \quad (B2)$$

Solving for  $\hat{\mathbf{a}}_q(t)$  and  $\dot{\hat{\mathbf{a}}}_q(t)$  equal to zero gives Eq. (44). Omitting the measurement term gives Eq. (34).

Substituting Eq. (45) into the right side of Eq. (B2) gives, after some algebra,

$$\mathbf{f}(\mathbf{x}(t), t) = -\hat{\omega}(t) \times \mathbf{a}_q(t) + \frac{1}{2} \sqrt{4 - \hat{a}_q^2(t)} [\Delta\omega(t) - \alpha(\hat{\mathbf{x}}(t), t)] - \frac{1}{2} [\Delta\omega(t) + \alpha(\hat{\mathbf{x}}(t), t)] \times \mathbf{a}_q(t). \quad (B3)$$

It follows that Eqs. (49)–(52) for the EKF are unchanged. Differentiating Eq. (B2) gives, with Eq. (77),

$$\mathbf{b}_p(t) = -\frac{1}{2} \text{tr} P_u(t) \{ \hat{\omega}(t) - 2[\dot{\hat{q}}(t) \otimes \hat{q}^{-1}(t)]_v \} + \omega_c(t), \quad (B4)$$

in place of Eq. (80), which changes Eq. (84) to

$$[\dot{\hat{q}}(t) \otimes \hat{q}^{-1}(t)]_v = \frac{1}{2} \{ \hat{\omega}(t) + [1 - \frac{1}{2} \text{tr} P_u(t)]^{-1} [\omega_c(t) + \alpha'(\hat{\mathbf{x}}(t), t)] \}. \quad (B5)$$

Equations (84) and (B5) agree if and only if we ignore products of  $P_u(t)$  and terms of first order in the errors.

This parameterization of  $\mathbf{a}$  gives

$$\mathbf{a}_v(t) = 2[q(t) \otimes \hat{q}^{-1}(t)]_v = 2\Xi^T(\hat{q}(t))q, \quad (\text{B6})$$

so evaluating the measurement partial derivative using the chain rule leads directly to Eq. (64).

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