Software for Statistical Analysis of Weibull Distributions With Application to Gear Fatigue Data: User Manual With Verification

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Summary

The Weibull distribution has been widely adopted for the statistical description and inference of fatigue data. This document provides user instructions, examples, and verification for software to analyze gear fatigue test data. The software was developed presuming the data are adequately modeled using a two-parameter Weibull distribution. The calculations are based on likelihood methods, and the approach taken is valid for data that include type I censoring. The software was verified by reproducing results published by others.

Introduction

This document provides user instructions, examples, and verification data for software to analyze gear fatigue test data. The software was developed on the presumption that the data are adequately modeled using a two-parameter Weibull distribution. It is based on the likelihood methods described by Meeker and Escobar (1998). The software can be used to determine

1. Maximum likelihood estimates of the Weibull distribution
2. Data for contour plots of relative likelihood for two parameters
3. Data for contour plots of joint confidence regions
4. Data for the profile likelihood of the Weibull distribution parameters
5. Data for the profile likelihood of any percentile of the distribution
6. Likelihood-based confidence intervals for parameters and/or percentiles of the distribution

The software can account for tests that are suspended without failure. The statistical terminology for suspended tests is “censoring.” The analytical approach for the software is valid for type I censoring, which is the removal of unfailed units at a prespecified time. Confidence regions and intervals are calculated using the likelihood ratio method. Guidance for the philosophy and interpretation of statistical confidence intervals can be found in the text of Hahn and Meeker (1991).

The Weibull distribution has been widely adopted for the statistical description and inference of fatigue data. The Weibull cumulative distribution function \( F \) is defined as

\[
F(t) = 1 - \exp \left( -\left( \frac{t}{\eta} \right)^\beta \right)
\]

where \( t \) is the time to failure, \( \beta \) is the shape parameter, and \( \eta \) is the scale parameter. For the Weibull distribution, the probability density function can have a variety of shapes, and the hazard function may be an increasing or decreasing function of time. The shapes of these curves depend only on the shape
TABLE I.—DATA SET 1
[Data from Krantz et al., 2000.]

<table>
<thead>
<tr>
<th>Test number</th>
<th>Test time, millions cycles</th>
<th>Test result</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>40.26</td>
<td>Failure</td>
</tr>
<tr>
<td>2</td>
<td>53.94</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>59.88</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>67.68</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>95.76</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>134.22</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>198.48</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>256.20</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>299.52</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>301.56</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>303.66</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>304.08</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>305.80</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>306.90</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>335.40</td>
<td></td>
</tr>
</tbody>
</table>

TABLE II.—DATA SET 2
[Data from Townsend and Shimski, 1994.]

<table>
<thead>
<tr>
<th>Test number</th>
<th>Test time, millions cycles</th>
<th>Test result</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.08</td>
<td>Failure</td>
</tr>
<tr>
<td>2</td>
<td>8.15</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>15.52</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>21.34</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>33.37</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>35.70</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>38.31</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>40.26</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>40.74</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>49.47</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>51.70</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>55.29</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>59.17</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>70.18</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>127.07</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>130.95</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>133.38</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>300.00</td>
<td>Suspended</td>
</tr>
<tr>
<td>19</td>
<td>300.00</td>
<td>Suspended</td>
</tr>
<tr>
<td>20</td>
<td>300.00</td>
<td>Suspended</td>
</tr>
</tbody>
</table>

parameter. Typically, surface fatigue data for gears have distributions with shape parameters less than three. Therefore, the probability distribution functions typically are skewed right. For a given Weibull distribution, one can determine the mode, mean, median, variance, and higher moments analytically by employing the gamma function (Cohen, 1973).

The software developed herein was written in the FORTRAN 90 programming language. Users of the software are required to write short main programs that provide the data to be analyzed and also call the appropriate subroutines. The examples provided in this manual were compiled using the Microsoft PowerStation Fortran Compiler, version 4.0 (Professional Edition).

The data analyzed as examples for this document (tables I and II) are those from gear fatigue experiments. These data have previously been analyzed using regression-based methods, and the results reported (Krantz et al., 2000). The data are again analyzed and reported in this document using software based on likelihood methods.

Verification information for the software is provided in appendix A. The software was verified by reproducing results published by Meeker and Escobar (1998). Appendix B provides descriptions and instructions for using subroutines. Some calculations require the use of routines from a standard math subroutine library (IMSL, 1997; IMSL is a registered trademark of Visual Numerics, Inc.) or an equivalent. Subroutine wmaxll (appendix B) is based on the code of Keats, Lawrence, and Wang (1997).

Estimates of Distribution Parameters

Data Set 1

The main program that analyzed the data of table I to provide maximum likelihood estimates of the Weibull distribution shape and scale parameters is presented below as source code listing 1. Allocatable dimensioning is used (lines 8 and 9) to make certain that the array sizes match the value of "n," the variable describing the number of tests (line 7). For the array "censor," a value of "zero" is given to indicate a test run to failure whereas a value of "one" is given to indicate a test suspended without failure (lines 10 to 15). A call to subroutine wmaxll provides maximum likelihood estimates of shape and scale parameters.
With the maximum likelihood values for the shape and scale available, subroutine \texttt{wblinverse} provides maximum likelihood estimates for any desired percentile of interest (lines 48 to 52).

The main program produced the output presented below following source code listing 1. The maximum likelihood estimates for shape and scale parameters are 1.037 and 376.7, respectively.

\begin{verbatim}
 Line   Source Line  Microsoft Fortran PowerStation Compiler. Version 4.0

1  ! main program to analyze superfinish gear fatigue test results
2  implicit none
3  integer n,i
4  real, allocatable:: time(:)
5  integer, allocatable:: censor(:)
6  real shape, scale, frac, life
7  n = 15
8  allocate(time(n))
9  allocate(censor(n))
10  do i=1,8
11     censor(i) = 0
12  end do
13  do i=9,n
14     censor(i) = i
15  end do
16  open(20, file="superdat2.txt")
17  ! HERE WE NEED TO ENTER THE ARRAY OF TEST TIMES
18  time(1)= 40.26
19  time(2)= 53.94
20  time(3)= 59.88
21  time(4)= 67.68
22  time(5)= 95.76
23  time(6)= 134.22
24  time(7)= 198.48
25  time(8)= 256.20
26  time(9)= 299.52
27  time(10)= 301.56
28  time(11)= 303.66
29  time(12)= 304.08
30  time(13)= 305.80
31  time(14)= 306.90
32  time(15)= 335.40
33  do i=1,n
34     if (censor(i) .eq. 0 ) then
35        write(20,100) time(i)
36     else if (censor(i) .eq. 1 ) then
37        write(20,101) time(i)
38     end if
39  end do
40  call wmaxll(time,censor,n,shape,scale)
41  write(20,105)
42  write(20,106) shape, scale
43  100 format(f10.2, ' test failed ')
44  101 format(f10.2, ' test suspended ')
45  105 format( ' maximum likelihood estimates ')
46  106 format( ' shape = ',e12.4, ' scale = ',e12.4)
47  frac = 0.0
48  do i=1,9
49     frac = frac + 0.1
50  end do
51  call wblinverse(shape,scale,frac,life)
52  write(20,107) frac, life
53  107 format(f6.2, ' proportion life = ',f10.4)
54  close(20)
55  stop
56 end
\end{verbatim}

NASA/TM—2002-211109
Output produced by this main program was as follows:

40.26 test failed
53.94 test failed
59.88 test failed
67.68 test failed
95.76 test failed
134.22 test failed
198.48 test failed
256.20 test failed
299.52 test suspended
301.56 test suspended
303.66 test suspended
304.08 test suspended
305.80 test suspended
306.90 test suspended
335.40 test suspended

maximum likelihood estimates
shape = .10378E+01 scale = .37678E+03
.10 proportion life = 43.0405
.20 proportion life = 88.7278
.30 proportion life = 139.4505
.40 proportion life = 197.1552
.50 proportion life = 264.6016
.60 proportion life = 346.2899
.70 proportion life = 450.5648
.80 proportion life = 596.0463
.90 proportion life = 841.8331

Estimates of Distribution Parameters

Data Set 2

The main program that analyzed the data of table II to provide maximum likelihood estimates of the Weibull distribution shape and scale parameters is presented below as source code listing 2. The source code listing 2 mirrors source code listing 1, differing only in the lines defining the number and results of experiments (lines 7 to 32). The main program produced the output presented below following source code listing 2. The maximum likelihood estimates for shape and scale parameters are 0.8616 and 103.0, respectively.

Source Code Listing 2

Line Source Line Microsoft Fortran PowerStation Compiler, Version 4.0

1 ! main program to analyze superfinish gear fatigue test results
2 implicit none
3 integer n,i
4 real, allocatable:: time(:)
5 integer, allocatable:: censor(:)
6 real shape, scale, frac, life
7 n = 20
8 allocate(time(n))
9 allocate(censor(n))
10 do i=1,17
11 censor(i) = 0
12 end do
13 do i=18,n
14 censor(i) = 1
15 end do
16 open(20, file="superdat1.txt")
17 ! HERE WE NEED TO ENTER THE ARRAY OF TEST TIMES
18 time(1)= 7.08
19 time(2)= 8.15
20 time(3) = 15.52
21 time(4) = 21.34
22 time(5) = 33.37
23 time(6) = 35.70
24 time(7) = 38.32
25 time(8) = 40.26
26 time(9) = 40.74
27 time(10) = 49.47
28 time(11) = 51.70
29 time(12) = 55.29
30 time(13) = 59.17
31 time(14) = 70.81
32 time(15) = 127.07
33 time(16) = 133.38
34 time(17) = 300.00
36 time(19) = 300.00
37 time(20) = 300.00
38 do i = 1, n
39 if (censor(i) .eq. 0) then
40 write(20,100) time(i)
41 else if (censor(i) .eq. 1) then
42 write(20,101) time(i)
43 end if
44 end do
45 call wmaxll(time,censor,n,shape,scale)
46 write(20,105) shape, scale
47 100 format(f10.2,' test failed ')
48 101 format(f10.2,' test suspended ')
49 105 format(' maximum likelihood estimates ')
50 106 format(' shape = ',e12.4,' scale = ',e12.4)
51 frac = 0.0
52 do i = 1, 9
53 frac = frac + 0.1
54 call wbliinverse(shape,scale,frac,life)
55 write(20,107) frac,life
56 end do
57 107 format(f6.2,' proportion life = ',f10.4)
58 close(20)
59 stop
60 end

Output produced by this main program was as follows:

7.08 test failed
8.15 test failed
15.52 test failed
21.34 test failed
33.37 test failed
35.70 test failed
38.32 test failed
40.26 test failed
40.74 test failed
49.47 test failed
51.70 test failed
55.29 test failed
59.17 test failed
70.81 test failed
127.07 test failed
133.38 test failed
300.00 test suspended
300.00 test suspended
300.00 test suspended
maximum likelihood estimates
shape = .8616E+00 scale = .1030E+03
.10 proportion life = 7.5571
.20 proportion life = 18.0554
.30 proportion life = 31.1160

NASA/TM—2002-211109
Joint Confidence Regions

Data Set 1

The main program that analyzed the data of table I to determine joint confidence regions for the simultaneous estimates of the shape parameter and 10-percent life is presented below as source code listing 3. Output files were written to provide data for likelihood ratios and confidence levels (lines 25 to 28). Allocatable dimensioning is used (lines 30 and 31) to make certain that the array sizes match the value of \( n \), the variable describing the number of tests (line 29). For the array "censor," a value of "zero" is given to indicate a test run to failure whereas a value of "one" is given to indicate a test suspended without failure (lines 32 to 37). The range of values for the shape and scale parameters must be provided (lines 59 to 62). A call to subroutine _tjlone provides the data required for contour plots. The subroutine writes results to FORTRAN units numbered 9 and 10. The output from the program is two data files, each containing 1681 lines. Each line consists of three numbers corresponding to a 10-percent life value, a shape parameter value, and either the likelihood ratio or the likelihood-ratio-based joint confidence value. The joint confidence values are displayed as a contour plot (fig. 1), providing a graphical display of plausible values for simultaneous estimates of the 10-percent life and the shape parameter.

![Figure 1.—Likelihood-ratio-based joint confidence regions for data of table I.](image)
Source Code Listing 3

Line  Source Line  Microsoft Fortran PowerStation Compiler, Version 4.0

1  ! a main program to calculate likelihood based probability contours  
2  ! and relative likelihood contours of Weibull distributions from censored data  
3  !  
4  ! this program presumes that the two parameters to  
5  ! be plotted are the shape parameter and the 10% life  
6  ! written by Tim Krantz on 2-11-2001  
7  ! version 1.0  
8  ! implicit none  ! declare all variables  
9  integer n  ! the number of tests  
10  real, allocatable:: time(:)  ! the test times  
11  integer, allocatable:: censor(:)  ! the censoring information  
12  real tenmax,tenmin  ! the max and min values for the 10 percent life  
13  real shapemin,shapemax  ! the max and min values for the shape factor  
14  integer i  ! a do loop counter  
15  ! change program as needed for a particular case  
16  ! change only lines between the two lines of stars  
17  ! the outputs for this program will be written to these two files  
18  ! column 1 is the shape parameter  
19  ! column 2 is the 10 percent life  
20  ! column 3 is the likelihood ratio or the confidence number as a percentage  
21  ! file for unit #10 is the confidence number data  
22  ! file for unit #9 is the likelihood ratio data  
23  OPEN (10, FILE = "super-prob.txt")  
24  OPEN (9, FILE = "super-ratio.txt")  
25  n = 15  
26  allocate (time(n))  
27  allocate (censor(n))  
28  do i=1,8  
29     censor(i) = 0  
30  end do  
31  do i=9,n  
32     censor(i) = 1  
33  end do  
34  ! HERE WE NEED TO ENTER THE ARRAY OF TEST TIMES  
35  time(1)= 40.2600  
36  time(2)= 53.9400  
37  time(3)= 59.8800  
38  time(4)= 67.6800  
39  time(5)= 95.7600  
40  time(6)= 134.220  
41  time(7)= 198.480  
42  time(8)= 256.200  
43  time(9)= 299.520  
44  time(10)= 301.560  
45  time(11)= 303.660  
46  time(12)= 304.080  
47  time(13)= 305.800  
48  time(14)= 306.900  
49  time(15)= 335.400  
50  ! here we set the bounds for 10 percent life and shape factors  
51  ! tenmax is the largest 10 percent life value  
52  ! tenmin is the smallest 10 percent life value  
53  ! shapemax is the largest shape value  
54  ! shapemin is the smallest shape value  
55  tenmax = 140.  
56  tenmin = 2.0  
57  shapemax = 2.  
58  shapemin = 0.4  
59  ! none of the lines below need to be changed to run a particular case  
60  ! the call to this subroutine checks for errors in the censoring array  
61  call ccheck(n,censor)  
62  ! now we ask for the solution  
63  call wjlone(time,censor,n,shapemax,shapemin,tenmax,tenmin)
The main program that analyzed the data of table II to determine joint confidence regions for the simultaneous estimates of the shape parameter and 10-percent life is presented below as source code listing 4. The source code listing 4 mirrors source code listing 3. The output from the program is two data files, each containing 1681 lines. Each line consists of three numbers corresponding to a 10-percent life value, a shape parameter value, and either the likelihood ratio or the likelihood-ratio-based joint confidence value. The joint confidence values are displayed as a contour plot (fig. 2), providing plausible values for simultaneous estimates of the 10-percent life and the shape parameter.

![Figure 2.—Likelihood-ratio-based joint confidence regions for data of table II.](image)

**Source Code Listing 4**

<table>
<thead>
<tr>
<th>Line</th>
<th>Source Line</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>! a main program to calculate likelihood based probability contours</td>
</tr>
<tr>
<td>2</td>
<td>! and relative likelihood contours of Weibull distributions from censored data</td>
</tr>
<tr>
<td>3</td>
<td>! this program presumes that the two parameters to</td>
</tr>
<tr>
<td>4</td>
<td>! be plotted are the shape parameter and the 10% life</td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>
version 1.0
written by Tim Krantz on 2-11-2001
implicit none
integer n ! the number of tests
real, allocatable:: time(:) ! the test times
integer, allocatable:: censor(:) ! the censoring information
real tenmax, tenmin ! the max and min values for the 10 percent life
real shapemax, shapemin ! the max and min values for the shape factor
to be used in calculations
integer i ! a do loop counter
integer i ! change program as needed for a particular case
change only lines between the two lines of stars
***************************************************************
the outputs for this program will be written to these two files
the file formats are as follows (each has 3 columns)
column 1 is the shape parameter
column 2 is the 10 percent life
column 3 is the likelihood ratio or the confidence number as a percentage
file for unit #10 is the confidence number data
file for unit #9 is the likelihood ratio data
OPEN (10, FILE = "base-prob.txt")
OPEN (9, FILE = "base-ratio.txt")

n = 20
allocate(time(n))
allocate(censor(n))
do i=1,17
censor(i) = 0
end do
do i=18,n
censor(i) = 1
end do
HERE WE NEED TO ENTER THE ARRAY OF TEST TIMES
time(1) = 7.08
time(2) = 8.15
time(3) = 15.52
time(4) = 21.34
time(5) = 33.37
time(6) = 35.70
time(7) = 38.32
time(8) = 40.26
time(9) = 40.74
time(10) = 49.47
time(11) = 51.70
time(12) = 55.29
time(13) = 59.17
time(14) = 70.81
time(15) = 127.07
time(16) = 130.95
time(17) = 133.38
time(18) = 300.00
time(19) = 300.00
time(20) = 300.00
here we set the bounds for 10 percent life and shape factors
tenmax is the largest 10 percent life value
tenmin is the smallest 10 percent life value
shapemax is the largest shape value
shapemin is the smallest shape value
tenmax = 30.
tenmin = 1.5
shapemax = 1.4
shapemin = 0.4
none of the lines below need to be changed to run a particular case

the call to this subroutine checks for errors in the censoring array
call ccheck(n,censor)
now we ask for the solution
call wjalone(time,censor,n,shapemax,shapemin,tenmax,tenmin)
close(10)
close(9)
stop
end
Profile Likelihood and Profile-Likelihood-Based Confidence Intervals

Data Set 1

The main program that determined the profile likelihood and profile-likelihood-based confidence intervals for the 10-percent life for the data of table I is presented below as source code listing 5. The main program determines the profile likelihood over a range of values for a specified percentile of interest, in this case the 10th percentile. Any percentile can be investigated by changing one line in the code for variable "pfrac" (line 49). Allocatable dimensioning is used (lines 18 and 19) to make certain that the array sizes match the value of "n," the variable describing the number of tests (line 17). For the array "censor," a value of "zero" is given to indicate a test run to failure whereas a value of "one" is given to indicate a test suspended without failure (lines 24 to 29). Array "time" (lines 31 to 45) provides the test times. The subroutine ph,percent is called (line 56) to determine the relative likelihood ratio for a specified t0-percent life. By calling the subroutine repeatedly within a do-loop (lines 50 to 59), data for a profile-likelihood plot are calculated. The output from the main program was plotted using two vertical scales to show both the profile-likelihood ratio values and the confidence level (fig. 3). Table III lists some profile-likelihood ratios and the corresponding confidence level. A 90-percent confidence interval for the 10-percent life is shown graphically on figure 3. The lower and upper bounds can be determined more precisely by running the program again with an appropriate range of values for the 10-percent life (figs. 4 and 5).

<table>
<thead>
<tr>
<th>CONFIDENCE LEVEL AND CORRESPONDING PROFILE-LIKELIHOOD RATIOS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confidence level</td>
</tr>
<tr>
<td>0.50</td>
</tr>
<tr>
<td>0.60</td>
</tr>
<tr>
<td>0.70</td>
</tr>
<tr>
<td>0.80</td>
</tr>
<tr>
<td>0.85</td>
</tr>
<tr>
<td>0.90</td>
</tr>
<tr>
<td>0.95</td>
</tr>
<tr>
<td>0.97</td>
</tr>
<tr>
<td>0.99</td>
</tr>
</tbody>
</table>

Figure 3.—Profile-likelihood ratios and confidence levels for 10-percent life for data of table I.
The subroutine `plvpercent` called by the main program determines the shape and scale factors corresponding to the largest likelihood value by an iterative search method. This method requires that maximum and minimum values for the shape parameter be provided to begin the search. The minimum and maximum values used in this example were hardcoded in the subroutine as 0.05 and 8.0 respectively. Users who wish to analyze data sets that require inspection of shape values outside the just-stated range need to modify the subroutine (appendix B).

**Source Code Listing 5**

Line Source Line Microsoft Fortran PowerStation Compiler. Version 4.0

1 ! a main program to calculate likelihood based profile
2 ! likelihood curves of Weibull distributions from censored data
This program presumes that a percentile parameter is the one of interest written by Tim Krantz on 2-13-2001.

Implicit none declare all variables.

The following variables represent inputs: they are hard-coded in this main program.

Integer n the number of tests.

Real, allocatable:: time(:) the test times.

Integer, allocatable:: censor(:) the censoring information.

Integer i do loop counter.

Real blv, spblv, scblv.

Real pfrac.

Real tmin, tmax, dt.

n = 15

Allocate (time(n))

Allocate (censor(n))

Here we need to supply censoring information. A value of '0' means that a failure occurred. A value of '1' means that test was suspended (censored) with no failure.

Enter the values as integers.

Do i=1,8

censor(i) = 0
endo do.

Do i=9,n

censor(i) = 1
endo do.

Here we need to enter the array of test times.

Time(1) = 40.2600

Time(2) = 53.9400

Time(3) = 59.8800

Time(4) = 67.6800

Time(5) = 95.7600

Time(6) = 134.220

Time(7) = 198.480

Time(8) = 256.200

Time(9) = 299.520

Time(10) = 301.560

Time(11) = 303.560

Time(12) = 304.080

Time(13) = 305.800

Time(14) = 306.900

Time(15) = 335.400

The call to this subroutine checks for errors in the censoring array.

Call ccheck2(n, censor).

Open (10, FILE= "super_L10_profile.txt")

Pfrac = 0.1

Tmin = 4.

Tmax = 135.

Dt = (tmax-tmin) / 100.

Tmin = Tmin - Dt

Do i=1,101

Tmin = Tmin + Dt

Call plvpercent (time, censor, n, pfrac, blv, spblv, scblv)

Write (10, 103) Tmin, spblv, scblv, blv

103 Format (4e16.7)

End do.

Stop.

End.

Data Set 2

The main program that determined the profile likelihood and profile-likelihood-based confidence intervals for the 10-percent life for the data of table II is presented below as source code listing 6. The main program mirrors that of source code 5. The output from the main program was plotted using two vertical scales to show both the profile-likelihood ratio values and the confidence level (fig. 6). A 90-percent confidence interval for the 10-percent life is shown graphically on figure 6.
The subroutine plvpercent called by the main program determines the shape and scale factors corresponding to the largest likelihood value by an iterative search method. This method requires that maximum and minimum values for the shape parameter be provided to begin the search. The minimum and maximum values used in this example were hardcoded in the subroutine as 0.05 and 8.0 respectively. Users who wish to analyze data sets that require inspection of shape values outside the just-stated range need to modify the subroutine (appendix B).

Source Code Listing 6

```fortran
1 ! a main program to calculate likelihood based profile
2 ! likelihood curves of Weibull distributions
3 ! from censored data
4 !
5 ! this program presumes that a percentile parameter is the one of interest
6 ! version 1.0
7 ! written by Tim Krantz On 2-13-2001
8 implicit none ! declare all variables
9 ! the following variables represent inputs
10 ! they are hard-coded in this main program
11 integer n ! the number of tests
12 real, allocatable:: time(:) ! the test times
13 integer, allocatable:: censor(:) ! the censoring information
14 integer i ! do loop counter
15 real blv, spblv, scblv
16 real pfrac
17 real tmin, tmax, dt
18 n = 20
19 allocate(time(n))
20 allocate(censor(n))
21 ! HERE WE NEED TO SUPPLY CENSORING INFORMATION
22 ! A VALUE OF "0" MEANS THAT A FAILURE OCCURRED
23 ! A VALUE OF "1" MEANS THAT TEST WAS SUSPENDED (CENSORED) WITH NO FAILURE
24 do i = 1, n
25 censor(i) = 0
26 end do
27 do i = 18, n
28 censor(i) = 1
29 end do
30 ! HERE WE NEED TO ENTER THE ARRAY OF TEST TIMES
```

Figure 6.—Profile-likelihood ratios and confidence levels for 10-percent life for data of table II.
Weibull Plot With Confidence Intervals

Profile-likelihood-based confidence intervals can be determined for any percentile of interest using main programs similar to source code listings 5 and 6. The variable "pfrac" determines the percentile to be investigated. For example, using "pfrac = 0.30" one can calculate data for the 30-percent life. Collections of such confidence intervals were calculated, and the results are provided in figures 7 and 8. These two figures are plotted using Weibull coordinates so that the Weibull cumulative distribution frequency will plot as a straight line. The test data points are plotted at the positions of exact median ranks (Jaquelin, 1993).
Figure 7.—Weibull plot for data of table I with profile-likelihood-based confidence intervals. Intervals shown are pointwise 90-percent confidence intervals of percentile estimates.

Figure 8.—Weibull plot for data of table II with profile-likelihood-based confidence intervals. Intervals shown are pointwise 90-percent confidence intervals of percentile estimates.
Summary of Results

Software has been developed to analyze gear fatigue test data. The source code listings discussed provided examples illustrating how to define the test data to be analyzed and how to call the appropriate subroutines to analyze the data. The software can be used to determine

1. Maximum likelihood estimates of the Weibull distribution
2. Data for contour plots of relative likelihood for two parameters
3. Data for contour plots of joint confidence regions
4. Data for the profile likelihood of the Weibull distribution parameters
5. Data for the profile likelihood of any percentile of the distribution
6. Profile-likelihood-based confidence intervals for parameters and/or percentiles of the distribution

Contour plots of relative likelihood (not illustrated in this document) can be obtained from the output files as provided by source code listings 3 and 4. Profile-likelihood calculations for Weibull distribution parameters were not illustrated directly. A point for the profile-likelihood plot of the shape parameter can be determined using subroutine *wlratio* (appendix B). A profile likelihood for the scale parameter can be obtained using a source code in the manner of listings 5 and 6 and analyzing for the 63.2 percentile (“*pfrac*” = 0.632).
Appendix A
Verification of Software

The software was verified by reproducing results published by Meeker and Escobar (1998). The main programs to produce the figures in this appendix (figs. 9 to 14) were similar to the source code listings found in the main body of this document.

Figure 9.—Contours of equal likelihood ratios. Weibull shape parameter $\beta = 1/\sigma$ and Weibull scale parameter $\eta = \exp(\mu)$. In the manner of figure 8.1 from Meeker and Escobar (1998).

Figure 10.—Contours of equal joint confidence levels. Weibull shape parameter $\beta = 1/\sigma$ and Weibull scale parameter $\eta = \exp(\mu)$. In the manner of figure 8.4 from Meeker and Escobar (1998).
Figure 11.—Profile-likelihood method to determine confidence interval for scale parameter. Weibull scale parameter $\eta = \exp(\mu)$. In the manner of figure 8.5 from Meeker and Escobar (1998).

Figure 12.—Profile-likelihood method to determine confidence interval for shape parameter. Weibull shape parameter $\beta = 1/\sigma$. In the manner of figure 8.6 from Meeker and Escobar (1998).
Figure 13.—Contours of equal joint confidence levels for Weibull shape parameter and 10-percent life, $t_{0.1}$. Weibull shape parameter $\beta = 1/\sigma$. In the manner of figure 8.7 from Meeker and Escobar (1998).

Figure 14.—Profile-likelihood method to determine confidence interval for 10-percent life, $t_{0.1}$. In the manner of figure 8.8 from Meeker and Escobar (1998).
Appendix B
Subroutine Descriptions and User Instructions

Subroutine CCHECK

A subroutine to do "error" checking on censoring array, all values should be either 1 or 0 integers.

Usage

subroutine ccheck (n, censor)

Arguments

n (integer)  size of array "censor" (input)
censor (integer array)  array of dimension [n], censoring information (input)

Comments

If all elements in array "censor" equal either 0 or 1, no action is taken.
If any element does not meet the criteria, a line of text is written to FORTRAN units numbered 6, 9, and 10, and then execution of the main program is halted.

Other subroutines called

None
Subroutine CCHECK2

A subroutine to do error checking on censoring array. All values should be either 1 or 0 integers.

Usage

subroutine ccheck2 (n, censor)

Arguments

n (integer) size of array "censor" (input)
censor (integer array) array of dimension [n], censoring information (input)

Comments

If all elements in array "censor" equal either 0 or 1, no action is taken.
If any element does not meet the criteria, a line of text is written to FORTRAN units numbered 6 and 10, and then execution of the main program is halted.

Other subroutines called

None
Subroutine FIXSHAPEMAXLL

A subroutine to calculate the maximum likelihood estimates of the scale parameter for a two-parameter Weibull distribution, for a presumed shape and censored data.

Usage

subroutine fixshapemaxll (times, censor, n, shape, scale)

Arguments

<table>
<thead>
<tr>
<th>Argment</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>times</td>
<td>array of dimension [n], test times (input)</td>
</tr>
<tr>
<td>n</td>
<td>size of array &quot;censor&quot; (input)</td>
</tr>
<tr>
<td>censor</td>
<td>array of dimension [n], censoring information (input)</td>
</tr>
<tr>
<td>shape</td>
<td>presumed Weibull shape parameter (input)</td>
</tr>
<tr>
<td>scale</td>
<td>calculated Weibull scale parameter (output)</td>
</tr>
</tbody>
</table>

Comments

This routine is based on the code published by Keats and Lawrence (1997). Although arguments are passed as single precision, calculations within the subroutine are done in double precision.

For array "censor," a value of 0 indicates a test to failure whereas a value of 1 indicates a test suspended without failure (type I censoring).

Other subroutines called

None
Subroutine PLVPERCENT

A subroutine to calculate a point on a profile-likelihood curve for a Weibull distribution at a particular cumulative distribution function (CDF) percentile of interest and a presumed value for the time to failure at that percentile. This routine can be called repeatedly using different values for the assumed time to failure (atime) to generate data for a profile-likelihood curve.

Usage

subroutine plvpercent (time, censor, n, atime, pfrac, ratio, spblv, scblv)

Arguments

time (real array) array of dimension [n], test times (input)
censor (integer array) array of dimension [n], censoring information (input)
n (integer) size of array “censor” (input)
atime (real) assumed time to failure (input)
pfrac (real) assumed percentile of CDF corresponding to value of atime (input)
ratio (real) profile ratio for best likelihood value [blv] (output)
spblv (real) shape factor corresponding to blv (output)
sclblv (real) scale factor corresponding to blv (output)

Comments

For array “censor,” a value of 0 indicates a test to failure whereas a value of 1 indicates a test suspended without failure (type I censoring).

Data are rescaled within the subroutine to avoid extreme values.

The Weibull shape parameter resulting in the profile-likelihood value is found within the subroutine by an iterative method. The shape value is found to within a tolerance of 0.01. A range for the shape must be provided to start the iterative process, and the shape parameter resulting in the profile-likelihood value must be contained within that range. The subroutine uses a range from 0.05 to 8.0. These values could be adjusted to fit the needs of a particular data set, but one might encounter numerical problems for extreme scale parameter values since it is used as an exponent in calculations. The routine writes a line of text to unit number 6 if the routine fails to converge within 1000 iterations.

Other subroutines called

wmaxll
wlratio

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Subroutine WBLINVERSE

A subroutine to calculate an inverse Weibull cumulative distribution function (CDF) value.

Usage

subroutine wblinverse (shape, scale, frac, life)

Arguments

<table>
<thead>
<tr>
<th>Argument</th>
<th>Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>shape</td>
<td>(real)</td>
<td>Weibull shape parameter (input)</td>
</tr>
<tr>
<td>scale</td>
<td>(real)</td>
<td>Weibull scale parameter (input)</td>
</tr>
<tr>
<td>frac</td>
<td>(real)</td>
<td>Weibull CDF percentile of interest (input)</td>
</tr>
<tr>
<td>life</td>
<td>(real)</td>
<td>time to failure for provided percentile of CDF</td>
</tr>
</tbody>
</table>

Comments

None

Other subroutines called

None
Subroutine WFLSHAPE

A subroutine to calculate Weibull distribution likelihood-based profile data (likelihood ratio).

Usage

subroutine wflshape (z, censor, n, shapemax, shapemin)

Arguments

z (real array) array of dimension [n], test times (input)
censor (integer array) array of dimension [n], censoring information (input)
n (integer) size of array “censor” (input)
shapemax (real) maximum shape value of range to be calculated (input)
shapemin (real) minimum shape value of range to be calculated (input)

Comments

For array “censor,” a value of 0 indicates a test to failure whereas a value of 1 indicates a test suspended without failure (type I censoring).

No outputs are returned to the calling program. However, output is written to FORTRAN unit number 9. If a file for unit number 9 was not yet opened, the user will be prompted for a filename.

The likelihood profile ratio is calculated for 41 values of the shape factor, the values equally spaced between the values passed as shapemax and shapemin. The output written to unit number 9 consists of two columns of numbers of fixed length, where the numbers are the shape factor and the calculated likelihood ratio.

Data are rescaled within the routine to avoid extreme values.

Other subroutines called

wmaxll
fixshapemaxll
wlratio
Subroutine WJLONE

A subroutine to calculate Weibull distribution likelihood-based joint confidence ratios and corresponding confidence levels to be plotted as contour plots. This routine presumes that the Weibull distribution is parameterized by the shape parameter and 10-percent life.

Usage

subroutine wjone (z, censor, n, shapemax, shapemin, tenmax, tenmin)

Arguments

<table>
<thead>
<tr>
<th>Argument</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>z</td>
<td>real array, array of dimension [n], test times (input)</td>
</tr>
<tr>
<td>censor</td>
<td>integer array, array of dimension [n], censoring information (input)</td>
</tr>
<tr>
<td>n</td>
<td>integer</td>
</tr>
<tr>
<td>shapemax</td>
<td>real, maximum shape value of range to be calculated (input)</td>
</tr>
<tr>
<td>shapemin</td>
<td>real, minimum shape value of range to be calculated (input)</td>
</tr>
<tr>
<td>tenmax</td>
<td>real, maximum 10-percent life to be calculated (input)</td>
</tr>
<tr>
<td>tenmin</td>
<td>real, minimum 10-percent life to be calculated (input)</td>
</tr>
</tbody>
</table>

Comments

For array “censor,” a value of 0 indicates a test to failure whereas a value of 1 indicates a test suspended without failure (type I censoring).

No outputs are returned to the calling program. However, output is written to FORTRAN units numbered 9 and 10. If files were not yet opened, the user will be prompted for filenames.

The likelihood profile ratios and confidence levels are calculated for a full array of 41×41 values (1681 lines) of the shape parameter and 10-percent life, the values equally spaced between the ranges defined by the values passed as shapemax to shapemin and tenmax to tenmin, respectively. The output written to unit number 9 consists of three columns of numbers of fixed length, the numbers being the shape parameter, the 10-percent life, and the calculated likelihood ratio. The output written to unit number 10 consists of three columns of numbers of fixed length, where the numbers are the shape parameter, the 10-percent life, and the calculated joint confidence level.

Data are rescaled within the routine to avoid extreme values.

NOTE: Confidence levels less than 10 percent are written to the output file as equal to 10 percent. Confidence levels greater than 99.99 percent are written to the output file as equal to 99.99 percent.

Other subroutines called

wmaxll
wlratio
chidf (IMSL routine—see IMSL (1997); IMSL is a registered trademark of Visual Numerics, Inc.)
Subroutine WJLTWO

A subroutine to calculate Weibull distribution likelihood-based joint confidence ratios and corresponding confidence levels to be plotted as contour plots. This routine presumes that the Weibull distribution is parameterized by the shape parameter and scale parameter (or 63.2-percent life).

Usage

subroutine wjltwo (z, censor, n, shapemax, shapemin, scalemax, scalemin)

Arguments

<table>
<thead>
<tr>
<th>z</th>
<th>(real array)</th>
<th>array of dimension [n], test times (input)</th>
</tr>
</thead>
<tbody>
<tr>
<td>censor</td>
<td>(integer array)</td>
<td>array of dimension [n], censoring information (input)</td>
</tr>
<tr>
<td>n</td>
<td>(integer)</td>
<td>size of array “censor” (input)</td>
</tr>
<tr>
<td>shapemax</td>
<td>(real)</td>
<td>maximum shape value of range to be calculated (input)</td>
</tr>
<tr>
<td>shapemin</td>
<td>(real)</td>
<td>minimum shape value of range to be calculated (input)</td>
</tr>
<tr>
<td>scalemax</td>
<td>(real)</td>
<td>maximum scale value of range to be calculated (input)</td>
</tr>
<tr>
<td>scalemin</td>
<td>(real)</td>
<td>minimum scale value of range to be calculated (input)</td>
</tr>
</tbody>
</table>

Comments

For array “censor,” a value of 0 indicates a test to failure whereas a value of 1 indicates a test suspended without failure (type I censoring).

No outputs are returned to the calling program. However, output is written to FORTRAN units numbered 9 and 10. If files were not yet opened, the user will be prompted for filenames.

The likelihood profile ratios and confidence levels are calculated for a full array of 41x41 values (1681 lines) of the shape parameter and scale parameter, the values equally spaced between the ranges defined by the values passed as shapemax to shapemin and scalemax to scalemin, respectively. The output written to unit number 9 consists of three columns of numbers of fixed length, the numbers being the shape parameter, the scale parameter, and the calculated likelihood ratio. The output written to unit number 10 consists of three columns of numbers of fixed length, where the numbers are the shape parameter, the scale parameter, and the calculated joint confidence level.

Data are rescaled within the routine to avoid extreme values.

NOTE: Confidence levels less than 10 percent are written to the output file as equal to 10 percent. Confidence levels greater than 99.99 percent are written to the output file as equal to 99.99 percent.

Other subroutines called

wmaxil
wlratio
chidf (IMSL routine—see IMSL (1997); IMSL is a registered trademark of Visual Numerics, Inc.)
Subroutine WLOGLV

A subroutine to calculate the log-likelihood value of a single life test. Can be called repeatedly, with the outputs summed, to calculate the joint log-likelihood value of a data set.

Usage

subroutine wloglv (sc, sh, t, fail, llv)

Arguments

<table>
<thead>
<tr>
<th>Argument</th>
<th>Type</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>sc</td>
<td>(real)</td>
<td>Weibull scale parameter (input)</td>
</tr>
<tr>
<td>sh</td>
<td>(real)</td>
<td>Weibull shape parameter (input)</td>
</tr>
<tr>
<td>t</td>
<td>(real)</td>
<td>test time (input)</td>
</tr>
<tr>
<td>fail</td>
<td>(integer)</td>
<td>flag indicating test to failure of test censored (input)</td>
</tr>
<tr>
<td>llv</td>
<td>(real)</td>
<td>log-likelihood value (output)</td>
</tr>
</tbody>
</table>

Comments

For variable fail, a value of 0 indicates a test to failure whereas a value of 1 indicates a test suspended without failure (type I censoring).

Other subroutines called

None
Subroutine WLRATIO

A subroutine to calculate the numerator of a likelihood ratio of a data set using presumed Weibull shape and scale parameters.

Usage

subroutine wlratio (time, censor, n, shape, scale, lv)

Arguments

time (real array) array of dimension [n], test times (input)
censor (integer array) array of dimension [n], censoring information (input)
n (integer) size of array "censor" (input)
shape (real) presumed Weibull shape parameter (input)
scale (real) presumed Weibull scale parameter (input)
lv (real) likelihood value, or value of numerator (output)

Comments

For array "censor," a value of 0 indicates a test to failure whereas a value of 1 indicates a test suspended without failure (type I censoring).

Other subroutines called

wloglv
subroutine WMAXLL

A subroutine to calculate the maximum likelihood estimates of the scale parameter for a two-parameter Weibull distribution, for a presumed shape and censored data.

Usage

subroutine wmaxll (times, censor, n, shape, scale)

Arguments

times (real array) array of dimension [n], test times (input)
n (integer) size of array "censor" (input)
censor (integer array) array of dimension [n], censoring information (input)
shape (real) calculated Weibull shape parameter (output)
scale (real) calculated Weibull scale parameter (output)

Comments

This routine is based on the code published by Keats, Lawrence, and Wang (1997). Although arguments are passed as single precision, calculations within the subroutine are done in double precision.

For array "censor," a value of 0 indicates a test to failure whereas a value of 1 indicates a test suspended without failure (type I censoring).

The routine finds the maximum likelihood value for the shape parameter iteratively, where the shape parameter is the root of an equation. The routine is considered as converged if the step size of the Newton-Raphson method is less than 0.0003. If the routine fails to converge after 100 iterations, a line is written to FORTRAN unit number 15.

Other subroutines called

None
References

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Keats, J. Bert; Lawrence, Frederick P.; and Wang, F.K., 1997: Weibull Maximum Likelihood Parameter
Krantz, T.L., et al., 2000: Surface Fatigue Lives of Case-Carburized Gears With an Improved Surface
John Wiley & Sons, New York, NY.
Townsend, Dennis P.; and Shimski, John, 1994: Evaluation of the EHL Film Thickness and Extreme
Pressure Additives on Gear Surface Fatigue Life. NASA TM–106663.
# Software for Statistical Analysis of Weibull Distributions With Application to Gear Fatigue Data: User Manual With Verification

## Abstract

The Weibull distribution has been widely adopted for the statistical description and inference of fatigue data. This document provides user instructions, examples, and verification for software to analyze gear fatigue test data. The software was developed presuming the data are adequately modeled using a two-parameter Weibull distribution. The calculations are based on likelihood methods, and the approach taken is valid for data that include type I censoring. The software was verified by reproducing results published by others.