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Hamilton-Jacobi Equations
Schemes for Multi-Dimensional
Efficient High Order Central
Outline

- Introduction
- First and second order methods
- High-order methods
- Conclusions
Encounter high-dimensional spaces

Applications in control theory, optics, ... 

Viscosity Solution (Crandall, Lions, Evans)
Smooth initial data
Evolves discontinuous derivatives even from

Where we assume $H$ is at least continuous

Equations of the form

$0 = (x \phi)' H + \phi$ 

Hamilton-Jacobi Equations
Our goal: high-order, efficient, central methods that scale well to high dimension.

Flux limiters, WENO, Central methods

Adapt techniques from conservation laws

Known to converge to viscosity solution solutions

Complicated by non-smoothness of

Numerical Methods for HJ Equations
Existing Work

- Propagation
- Reduce dissipation by estimating local speed of
- Minmod limiter on 2nd derivative
- Semi-discrete
- Kurganov and Tadmor - 1st and 2nd order
- Proved 1st order convergence
- Minmod flux limiter on 1st derivative
- Lin and Tadmor - 1st and 2nd order staggered
- Central Schemes
- Jiang and Peng - high-order WENO methods
- Osher and Shu - high-order ENO methods
- Upwind Schemes
Good for systems and high dimensions
Avoid solving Riemann problems
Steps: reconstruct, evolve, replicate

Evolve where data is smooth

The Central Philosophy
Assume $H \in C$.

Order midpoint quadrature

- Use Taylor expansion for mid-values in 2nd.

\[
\left[ \left( \frac{xV}{\tilde{t} - I} \phi \nabla \right) H + \left( \frac{xV}{\tilde{t} + I} \phi \nabla \right) H \right] \frac{\tilde{t}}{I} - \left( \frac{\tilde{t} - I}{\tilde{t}} \phi \nabla \right) \frac{\tilde{t}}{I} + \phi = i \phi
\]

1st-order method:

Evolve at evolution points using quadrature

- Same work as Lin-Tadmor in 2D.
- Based on Lin-Tadmor and Kurganov-Tadmor onto original grid points
- Limit the second derivatives and reproject

First and Second Order
\[ \frac{u \wedge + u}{1} = p \]

- Equidistant from simplex boundaries
- Optimal Evolution Points
- Singularities along simplex
- Diagonal
- Simplices along + and -
- Partition space into \( \mathbb{R}^n \)
Reproject: $a^0_{1+w}(\phi \beta)_{\mathcal{A}} x \mathcal{U} V - (w \phi + \hat{w} \phi) \frac{\tau}{I} = 1_{1+w} \phi$

Where $\left(\frac{z}{1+w}(\phi \Delta)\right) H V \mathcal{L} + \frac{\tau}{I V} \left(\frac{z}{1+w}(\phi \Delta)\right) = \frac{z}{1+w}(\phi \beta)$

$\left[\frac{p_{\phi} d_{\phi}^{x} \nabla \mathcal{A}}{p_{\phi} d_{\phi}^{x} \nabla \mathcal{A}} + \frac{p_{\phi} d_{\phi}^{x} \nabla \mathcal{A}}{p_{\phi} d_{\phi}^{x} \nabla \mathcal{A}}\right] = \frac{\tau}{1+w} \left(\frac{z}{1+w}(\phi \Delta)\right) H V \mathcal{L} + \frac{\tau}{I V} \left(\frac{z}{1+w}(\phi \Delta)\right) = \frac{\tau}{1+w}(\phi \beta)$

At each point $x$

At evolution points: at each point $x$

Where $J$ is the min-mod limited derivative

$\left((x^0 - (\phi \beta))(x^0 - (\phi \beta))\right) \frac{z}{w \phi} \nabla \mathcal{A} \sum_{u=1}^{w} \frac{\tau}{1+w} \left(\frac{z}{1+w}(\phi \Delta)\right) H V \mathcal{L} + \frac{\tau}{I V} \left(\frac{z}{1+w}(\phi \Delta)\right) = \frac{\tau}{1+w}(\phi \beta) + (w \phi - (\phi \beta))(w \phi - (\phi \beta)) \frac{z}{w \phi} \nabla \mathcal{A} \sum_{u=1}^{w} \nabla + (\phi \beta) = (w \phi - (\phi \beta)) \frac{z}{w \phi}$

Reconstruct via polynomial

2nd-Order Generalization to $\mathbb{R}^n$
\[
\begin{align*}
((\lambda + x)w^2)\cos - &= (0', x)\phi \\
0 &= z \left( I + \phi + \phi^2 \right) \frac{T}{I} + \phi
\end{align*}
\]

Convex H Example
Non-Convex H Example
\[(\lambda \cos \alpha + x \sin \alpha) = (0, \lambda^2 x) \phi \]

\[
0 = \frac{\partial \phi}{\partial x}^\phi + \frac{\partial \phi}{\partial y}^\phi
\]

**2D Example**
Convergence Rates
Higher Order

▲ Strategy:
▲ Central WENO for reconstructions
▲ Simpson's formula/SSP RK4 for evolution
▲ Involves upwind WENO reconstruction of derivatives for each RK4 step
Suppressing oscillatory interpolants to attain high order in smooth regions while the are defined

\[
\frac{d(S + 3)}{d \phi} = f_0, \quad \frac{d(x + \epsilon)}{d \phi} = f_\infty
\]

where

\[
(x \nabla + \epsilon \phi) \phi \mu + (x \nabla + \epsilon \phi) \nu \phi \mu = (x \nabla + \epsilon \phi) \mu \phi
\]

so set

\[
(v + \epsilon) \xi = \epsilon \phi, (v - \epsilon) \xi = \epsilon \phi
\]

\[
\left(\epsilon \nabla \right) f + (x \nabla + \epsilon \phi) f = (x \nabla + \epsilon \phi) \phi f + (x \nabla + \epsilon \phi) \phi f = (x \nabla + \epsilon \phi) \phi f
\]

\[
\left(\epsilon \nabla \right) f + (y \nabla + \epsilon \phi) f = \epsilon \phi \left(\frac{\zeta}{\xi} + \frac{\zeta}{\xi} + \epsilon \phi \left(\frac{\zeta}{\xi} - \frac{\zeta}{\xi} - \epsilon \phi \left(\frac{\zeta}{\xi} + \frac{\zeta}{\xi} - \epsilon \phi \right)ight) = (x \nabla + \epsilon \phi) \phi f
\]

\[
\left(\epsilon \nabla \right) f + (y \nabla + \epsilon \phi) f = \epsilon \phi \left(\frac{\zeta}{\xi} + \frac{\zeta}{\xi} + \epsilon \phi \left(\frac{\zeta}{\xi} - \frac{\zeta}{\xi} - \epsilon \phi \left(\frac{\zeta}{\xi} + \frac{\zeta}{\xi} - \epsilon \phi \right)ight) = (x \nabla + \epsilon \phi) \phi f
\]

3rd-order example

High-order 1D Interpolants
5th-order 1D Results
Via upwinding interpolation

In all cases, reconstruct derivatives

Interpolate along diagonal
direction-by-direction

2D interpolation

Three options for reconstruction

High-order 2D Reconstruction
High-order 2D stencils

with oscillations
suppress stencils
combination
to
Use WENO
for third order
10 points required
Combination covers
Evolution point
Stencils enclose
3rd-order example
Interpolations

 Iterate n steps, each with n

 In n-D:

 Evolution point

 2: average coordinate interpolations to

 1: interpolate values along coordinate axes

 In 2D:

 Direction-by-Direction Strategy
Interpolation have similar quality
Full 2D and direction by direction
3rd-order Results
Direction-by-direction CWENO

5th-order 2D

Upwind estimation of derivatives from

Jiang and Peng

Simpson's method for time evolution,

using SSP RK4 for mid-values
5th-order 2D Results
Significant computational burden at each point gradient of \( H \) at each point estimation of the maximum of the dimensional interpolation requires. What about upwind? Requires to high dimensional than fully direction by direction will scale better.

Scaling to \( N \) Dimensions
Conclusions

Scale well to high dimensions

No need to estimate numerical fluxes

Central methods for HL equations based on

Developed efficient high-order