Stanford University
Doron Levy
NASA Ames/Stanford University
Steve Bryson

Hamilton-Jacobi Equations
Efficient High Order Central
Schemes for Multi-Dimensional
Outline

Introduction

1st and 2nd order methods

High-order methods

Conclusions
Encounter high-dimenisonal spaces

Applications in control theory, optics, ...
Numerical Methods for HJ

Our Goal: high-order, efficient, central

* Flux limiters, WENO, Central methods

* Adapt techniques from conservation laws

* Known to converge to viscosity solutions

* Complicated by non-smoothness of solutions

* Numerical Methods for HJ

dimension

methods that scale well to high
Existing Work

- Propagation
- Reduce dissipation by estimating local speed of
  Minmod limiter on 2nd derivative
- Semi-discrete
  Kurganov and Tadmor - 1st and 2nd order
  Provided 1st order convergence
- Minmod flux limiter on 1st derivative
  Lin and Tadmor - 1st and 2nd order staggered
- Central Schemes
  Jiang and Peng - high-order WENO methods
  Osher and Shu - high-order ENO methods
- Upwind Schemes
Good for systems and high dimensions
Avoid solving Riemann problems
Steps: reconstruct, evolve, represent

Evolve where data is smooth

The Central Philosophy
Assumes \( HE \in C \) order midpoint quadrature
- Use Taylor expansion for mid-values in 2nd

\[
\left[ \left( \frac{xV}{\frac{t-i}{\omega}(\phi V)} \right) H + \left( \frac{xV}{\frac{t+i}{\omega}(\phi V)} \right) H \right] \frac{\tau}{1V} - \left( \frac{\xi-i}{\omega}(\phi V) - \frac{\xi+i}{\omega}(\phi V) \right) \frac{\tau}{1} + \omega \phi = i + \omega \phi
\]

1st-order method:
- Evolve at evolution points using quadrature
- Same work as Lin-Tadmor in 2D
- Based on Lin-Tadmor and Kurganov-Tadmor onto original grid points
- Limit the second derivatives and reproject

First and Second Order
\[
\frac{u \wedge + u}{1} = \nu
\]

- Equidistant from simplex boundaries
- Optimal Evolution Points
- Singularities along simplex
- Singularities along simplex along diagonal
- Partition space into \( \mathbb{R}^n \)
At each point $x$, a polynomial reconstruction via polynomial of order $n$
\[(\lambda + x)\nu \frac{\tau}{I} \cos \phi = (0, x, 0, 0, 0)\phi \]  
\[0 = \tau \left(1 + \phi + \phi^2\right) + \phi^3\]  
\[0 = \tau \left(1 + \phi^2\right) + \phi^3\]  

Convex H Example
\[ ((\lambda + x)u \frac{z}{T})\cos - = (0, \lambda 'x)\phi \]
\[ (x\lambda)\cos - = (0, x)\phi \]
\[ 0 = (1 + \phi + \phi)\cos - \phi \]
\[ 0 = (1 + \phi)\cos - \phi \]

Non-Convex H Example
2D Example

\[
\begin{cases}
\phi(x,y,0) = \sin(x) + \cos(y) \\
0 = \phi_x + \phi_y
\end{cases}
\]
Convergence Rates
Higher Order

Strategy:

- Involves upwind WENO reconstruction of derivatives for each RK4 step
- Evolution of Simpson's formula
- Central WENO reconstruction
- SSP RK4 for higher order
suppressing oscillatory interpolants

to attain high order in smooth regions, while

\[ \frac{d(S + 3)}{f(x)} = f(x), \quad \frac{2\alpha + 4\xi}{f(x)} = f(x) \]

where

\[(\nabla \phi + 'x)^2 \phi \psi + (\nabla \phi + 'x)^2 \phi \psi = (\nabla \phi + 'x)^2 \phi \psi \]

So set

\[ (v + 1)\frac{v}{I} = \xi, \quad (v - 1)\frac{v}{I} = \xi \]

\[ (\xi(xv)) + (\nabla \phi + 'x)\phi = (\nabla \phi + 'x)\phi \]

\[ (\xi(xv)) + (v \phi + 'x)\phi = v(\phi + 'x)\phi \]

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3rd-order example

High-order 1D Interpolants
5th-order 1D Results
via upwind interpolation

In all cases, reconstruct derivatives

Interpolate along diagonal

Direction-by-direction

2D interpolation

Three options for reconstruction

High-order 2D Reconstruction
High-order 2D stencils

Use WENO for third order combination to suppress oscillations.

Combination covers 10 points required evolution point.

Stencils enclose 3rd-order example.
Interpolations

 Iterate n steps, each with n

 In n-D:

 Evolution point

 2: average coordinate interpolations to

 1: interpolate values along coordinate axes

 In 2-D:

 Direction-by-Direction Strategy
Interpolation have similar quality

Full 2D and direction by direction

3rd-order Results
Using SSP RK4 for mid-values, Simpson's method for time evolution, Jiang and Peng upwind estimation of derivatives from reconstruction. Direction-by-direction CWENO 5th-order 2D.
5th-order 2D Results
Scaling to N Dimensions

Direction by direction will scale better

to high dimension than fully
dimensional interpolation

What about upwind? Requires
estimation of the maximum of the
gradient of H at each point

- Significant computational burden
Conclusions

- Scale well to high dimensions
- No need to estimate numerical fluxes
- Central methods for HJ equations based on
- Developed efficient high-order