Hamilton-Jacobi Equations

Efficient High Order Central Schemes for Multi-Dimensional

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Outline

- Introduction
- 1st and 2nd order methods
- Higher order methods
- Conclusions
Encounter high-dimensional spaces.

Applications in control theory, optics, ...

Viscosity solution (Crandall, Lions, Evans)

Smooth initial data evolves discontinuous derivatives even from

Where we assume $H$ is at least continuous

Equations of the form $0 = (x \phi)' + \phi$

Hamilton-Jacobi Equations
Numerical Methods for HJ

- Numerical Methods for HJ Eqns
  - Complicated by non-smoothness of solutions
    - Known to converge to viscosity solution (Souganitas)
    - Adapt techniques from conservation laws
      - Flux limiters, WENO, Central methods
  - Our Goal: *high-order, efficient, central methods that scale well to high dimension*
Existing Work

- Propagation
- Reduce dissipation by estimating local speed of
- Minmod limiter on 2nd derivative
- Semi-discrete
- Kurganov and Tadmor - 1st and 2nd order
- Proved 1st order convergence
- Minmod flux limiter on 1st derivative
- Lin and Tadmor - 1st and 2nd order staggered
- Central Schemes
- Jiang and Peng - high-order WENO methods
- Osher and Shu - high-order ENO methods
- Upwind Schemes
The Central Philosophy

Avoid solving Riemann problems. Steps: reconstruct, evolve, represent.

Evolve where data is smooth.
Assumes $H \in C^1$

Order midpoint quadrature

Use Taylor expansion for mid-values in 2nd-

\[
\left[ \left( \frac{x V}{\frac{\tau}{2}} \right) H + \left( \frac{x V}{\frac{\tau}{2+1}} \right) H \right] \frac{\tau}{2} - \left( \frac{\tau}{2} \right) (\phi V) - \left( \frac{\tau}{2+1} \right) (\phi V) \frac{\tau}{2} + \frac{\tau}{1} + \phi = \frac{\tau}{2+1} \phi
\]

1st-order method:

Evolve at evolution points using quadrature

Evolve at evolution points using quadrature

Same work as Lin-Tadmor in 2D

Based on Lin-Tadmor and Kurganov-Tadmor

onto original grid points

Limit the second derivatives and reproject

First and Second Order
\[
\frac{u^\lambda + u}{1} = p
\]

Equidistant from simplex
Optimal Evolution Points
Boundaries
Singularity along simplex

diagonal
- simplices along + and
- Partition space into

Evolution in \( \mathbb{R}^n \)
\[ t \left( \frac{(w \times \partial) \mathcal{A} \cdot \nabla u}{t} \right)^{\frac{\mathcal{A}}{t}} - \left( \frac{\partial \phi}{t} + \frac{\partial \phi}{\mathcal{A} \cdot \nabla} \right)^{\mathcal{A} \cdot \nabla u} = \left( \frac{\partial \phi}{t} + \frac{\partial \phi}{\mathcal{A} \cdot \nabla} \right)^{\mathcal{A} \cdot \nabla u} \]

**Reproject:**

\[
\left( \begin{array}{c}
\pi(w \times \partial) \\
\pi(w \times \partial) \\
\vdots \\
\pi(w \times \partial)
\end{array} \right) \left( \begin{array}{c}
\pi(w \times \partial) \\
\pi(w \times \partial) \\
\vdots \\
\pi(w \times \partial)
\end{array} \right) = \pi(w \times \partial)
\]

**Where**

\[
\left( \frac{\pi(w \times \partial)}{t} \mathcal{A} \cdot \nabla u \right)^{\frac{\mathcal{A}}{t}} - \left( \frac{\partial \phi}{t} + \frac{\partial \phi}{\mathcal{A} \cdot \nabla} \right)^{\mathcal{A} \cdot \nabla u} = \left( \frac{\partial \phi}{t} + \frac{\partial \phi}{\mathcal{A} \cdot \nabla} \right)^{\mathcal{A} \cdot \nabla u}
\]

\[
\frac{\partial \phi}{t} \cdot \frac{\mathcal{A} \cdot \nabla u}{t} \left( \begin{array}{c}
\frac{\mathcal{A} \cdot \nabla u}{t} \\
\frac{\mathcal{A} \cdot \nabla u}{t} \\
\vdots \\
\frac{\mathcal{A} \cdot \nabla u}{t}
\end{array} \right) = \left( \frac{\partial \phi}{t} + \frac{\partial \phi}{\mathcal{A} \cdot \nabla} \right)^{\mathcal{A} \cdot \nabla u}
\]

**At each point**

\[
\frac{\partial \phi}{t} \cdot \frac{\mathcal{A} \cdot \nabla u}{t} \left( \begin{array}{c}
\frac{\mathcal{A} \cdot \nabla u}{t} \\
\frac{\mathcal{A} \cdot \nabla u}{t} \\
\vdots \\
\frac{\mathcal{A} \cdot \nabla u}{t}
\end{array} \right) = \left( \frac{\partial \phi}{t} + \frac{\partial \phi}{\mathcal{A} \cdot \nabla} \right)^{\mathcal{A} \cdot \nabla u}
\]

**At evolution points:**

\[
\left( \frac{\partial \phi}{t} \right)^{\mathcal{A} \cdot \nabla u} \left( \begin{array}{c}
\frac{\mathcal{A} \cdot \nabla u}{t} \\
\frac{\mathcal{A} \cdot \nabla u}{t} \\
\vdots \\
\frac{\mathcal{A} \cdot \nabla u}{t}
\end{array} \right) = \left( \frac{\partial \phi}{t} \right)^{\mathcal{A} \cdot \nabla u}
\]

**Where**

\[
\frac{\partial \phi}{t} \cdot \frac{\mathcal{A} \cdot \nabla u}{t} \left( \begin{array}{c}
\frac{\mathcal{A} \cdot \nabla u}{t} \\
\frac{\mathcal{A} \cdot \nabla u}{t} \\
\vdots \\
\frac{\mathcal{A} \cdot \nabla u}{t}
\end{array} \right) = \left( \frac{\partial \phi}{t} \right)^{\mathcal{A} \cdot \nabla u}
\]

**Reconstruct via polynomial**

2nd-Order Generalization to \( \mathbb{R}^n \)
\[
(\lambda + x) \mu \frac{\gamma}{1} \cos - = (0, x) \phi \\
0 = \gamma \left( I + \gamma \phi + \gamma \phi \right) \frac{\gamma}{1} \phi + ' \phi
\]

Example
\[
\begin{align*}
((\kappa + x) \nu \frac{z}{T}) \cos \theta &= (0', \kappa' x) \phi \\
0 &= (I + \kappa \phi + x \phi) \cos \theta
\end{align*}
\]

Non-Convex H Example
$$\begin{align*}
(\lambda)\cos + (x)\sin &= (0, \lambda, x) \phi \\
0 &= \epsilon \phi^x \phi + i \phi
\end{align*}$$

2D Example
Convergence Rates
Higher Order

- Strategy:
- Central WENO for Reconstruction
- Simpson's formula SSP RK4 for
- Evolution
- Involves upwind WENO Reconstruction of
derivatives for each RK4 step.
High-order 1D Interpolants

3rd-order example

\[ \varphi_1(x_i + a\Delta x) = \left( -\frac{1}{2} a + \frac{1}{2} a^2 \right) \varphi_{i-1} + (1 - a^2) \varphi_i + \left( \frac{1}{2} a + \frac{1}{2} a^2 \right) \varphi_{i+1} = \varphi(x_i + ah) + O((\Delta x)^3) \]

\[ \varphi_2(x_i + a\Delta x) = \left( 1 - \frac{3}{2} a + \frac{1}{2} a^2 \right) \varphi_i + (2a - a^2) \varphi_{i+1} + \left( -\frac{1}{2} a + \frac{1}{2} a^2 \right) \varphi_{i+2} = \varphi(x_i + ah) + O((\Delta x)^3) \]

\[ \varphi_2(x_i + a\Delta x) = c_1 \varphi_1(x_i + a\Delta x) + c_2 \varphi_2(x_i + a\Delta x) = \varphi(x_i + a\Delta x) + O((\Delta x)^4) \]

\[ c_1 = \frac{1}{3}(2-a), \quad c_2 = \frac{1}{3}(1+a) \]

So set \[ \varphi_w^\pm(x_i \pm a\Delta x) = w_1 \varphi_1^\pm(x_i \pm a\Delta x) + w_2 \varphi_2^\pm(x_i \pm a\Delta x) \]

where \[ w_j = \frac{\alpha_j}{\alpha_1 + \alpha_2}, \quad \alpha_j = \frac{c_j}{(\varepsilon + S)^p} \]

are defined to attain high order in smooth regions while suppressing oscillatory interpolants.
5th-order 1D Results
via upwind interpolation

In all cases, reconstruct derivatives

Interpolate along diagonal

Interpolate by direction

2D interpolation

Three options for reconstruction

Higher-order 2D Reconstruction
High-order 2D stencils

with oscillations suppress stencils combination to
use WENO for third order 10 points required
Combination covers evolution point stencils enclose third-order example
Interpolations

\[ \text{Interpolate each with } n \text{ steps, each with } n \]

\[ \text{In n-D: } \]

Evolution point

\[ \mathbb{1}: \text{Interpolate values along coordinate axes} \]

\[ \mathbb{2}: \text{Average coordinate interpolations to} \]

Direction-by-Direction Strategy
Interpolating have similar quality
FULL 2D and direction by direction
3rd-order Results
using SSP RK4 for mid-values
Simpson's method for time evolution,
Jiang and Peng
Upwind estimation of derivatives from
reconstruction
Direction-by-direction CWENO
5th-order 2D
5th-order 2D Results
Significant computational burden

gradient at each point

estimation of the maximum of the

What about upwind? Requires
dimensional interpolation

to high dimension than fully

Direction by direction will scale better

Scaling to N Dimensions
Conclusions

- Scale well to high dimensions
- No need to estimate numerical fluxes
- Central methods for HL equations based on developed efficient high-order