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Definition of Contravariant Velocity Components

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I. Introduction

Written in generalized curvilinear coordinates $\xi = \xi(x, y), \eta = \eta(x, y)$ the 2-D inviscid Navier-Stokes equations are

$$\vartheta_\tau Q + \vartheta_\xi F + \vartheta_\eta G = 0 \tag{1}$$

$$Q = J^{-1}\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ e \end{pmatrix}, F = J^{-1}\begin{pmatrix} \rho U \\ \rho u U + \xi_x p \\ \rho v U + \xi_x p \\ U(e+p) \end{pmatrix}, G = J^{-1}\begin{pmatrix} \rho V \\ \rho u V + \eta_x p \\ \rho v V + \eta_x p \\ V(e+p) \end{pmatrix},$$

where, ρ, u, v, p, e are conventional physical properties, and

$$U = \xi_x u + \xi_y v, \; V = \eta_x u + \eta_y v.$$

Compared to the Navier-Stokes equations written in Cartesian coordinates (x, y), except with the additional inclusion of transformation Jacobian J, (U, V) are the replacements of velocity (u, v), and hence are called "contravariant velocity", or "contravariant velocity components" based on their associated with the contravariant vector.

The term "contravariant velocity component" is used uniformly in computational fluid dynamics (CFD) community and publications, in general without further statements or explanations, for instances, Refs. 1 - 2. In Ref. 3, while applied for body surface, it stated that

> " - - - the contravariant velocities are the decomposition of velocity vector into components along the ξ coordinate line, U, and along the η coordinate line, V."

Contrarily in Ref. 4, it stated (for 3-D case) that

> "The contravariant velocity components U, V, W are in directions normal to constant ξ, η, ζ surfaces, respectively."

The above two statements mainly contradict each other the directions of contravariant velocity components. As we shall see, neither of these statements is strictly correct. The contravariant velocity components are neither the decomposed physical components of velocity vector nor in directions normal to constant ξ, η, ζ surfaces, respectively. The objective of this article is to clarify what are the contravariant velocity components and to further explain their physical implications.

To avoid confusion, we will start with defining terms based on the algebra of vectors. The term *component* of a vector will be defined. With the definition of covaraiant and contravariant base vector systems for a given coordinates, the general mathematical term "contravariant component" will be discussed and its counter term "covariant component" will also be explained. For simplicity and clarity, we will use several 2-D linear cases as examples. Here in the 2-D examples, the (x^1, x^2) coordinates are interchangeable with (ξ, η) coordinates.

II. Algebra of Vectors

In an n-dimensional vector space a set of n linearly independent vectors $\mathbf{e}_1, \mathbf{e}_2, ---, \mathbf{e}_n$ is called a *base vector system.* Any vector $\mathbf{A}$ in the space can be expressed as a unique combination of the base vectors $\mathbf{A} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + --- + c_n\mathbf{e}_n$, or using the summation convention, $\mathbf{A} = c_i\mathbf{e}_i$. The vectors $c_1\mathbf{e}_1, c_2\mathbf{e}_2, ---, c_n\mathbf{e}_n$ are *component vectors* of vector $\mathbf{A}$ in the $\mathbf{e}_1, \mathbf{e}_2, ---$, and $\mathbf{e}_n$ directions, respectively. The coefficients, $c_1, c_2, ---$, and c_n, are called *components* of vector $\mathbf{A}$ in the $\mathbf{e}_1, \mathbf{e}_2, ---$, and $\mathbf{e}_n$ directions, respectively, with respect to the base $(\mathbf{e}_1, \mathbf{e}_2, ---, \mathbf{e}_n)$.

For example, in Cartesian coordinates, a vector $\mathbf{A} = 5\mathbf{i} + 7\mathbf{j}$ implies

$$\mathbf{e}_1 = \mathbf{i},\ \mathbf{e}_2 = \mathbf{j},\quad c_1 = 5,\ c_2 = 7$$

rewrite $\mathbf{A} = 10(\mathbf{i} - \mathbf{j})/2 + 12\mathbf{j}$, then that implies

$$\mathbf{e}_1 = (\mathbf{i} - \mathbf{j})/2,\ \mathbf{e}_2 = \mathbf{j},\quad c_1 = 10,\ c_2 = 12 \tag{2}$$

rewrite $\mathbf{A} = -\ (2\mathbf{i}) + 7\ (\mathbf{i} + \mathbf{j})$, then that implies

$$\mathbf{e}_1 = 2\mathbf{i},\ \mathbf{e}_2 = \mathbf{i} + \mathbf{j},\quad c_1 = -1,\ c_2 = 7 \tag{3}$$

As illustrated above, the invariant vector $\mathbf{A}$ can be expressed in different base vector systems which may or may not be orthogonal and normalized. While the "component vectors" are physical vectors, the magnitude of the components of a vector not only depend on the directions but also depend on the magnitude of the base vector system. Obviously, without the prescription of "base vectors", the "components" of a vector have no meaning.

III. Contravariant and Covariant Base Vectors

Let x^i be the coordinates of a point and $\mathbf{r}$ be a position vector. As shown in Fig. 1, a covariant base vector, $\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial x^i}$, is tangent to its corresponding coordinate line, x^i, and a contravariant base vector, $\mathbf{g}^i = \nabla x^i$, is normal to the other coordinate lines x^j (or the surface of $x^i = constant$), $j \neq i$. For any given coordinates x^i, its contravariant and covariant base vectors, $\mathbf{g}^i$ and $\mathbf{g}_i$ are uniquely determined. They have the reciprocal relation,

$$\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j = \begin{cases} 0\ if\ i \neq j \\ 1\ if\ i = j \end{cases} \tag{4}$$

where δ^i_j is the Kronecker delta. Based on the reciprocal relation, Eq. (4), we have

$$|\ \mathbf{g}^i\ | = \frac{1}{|\ \mathbf{g}_i\ | \cos(\mathbf{g}^i, \mathbf{g}_i)},\quad (no\ summation\ in\ i).$$

If they are orthogonal, then $|\ \mathbf{g}^i\ | = \frac{1}{|\mathbf{g}_i|}$ and $\mathbf{g}^i$ and $\mathbf{g}_i$ are in the same direction. Note that, the sketches in Fig. 1 (and Fig. 2) only give a general indication of the directions of $\mathbf{g}^i$ and $\mathbf{g}_i$ and do not imply any thing about their magnitudes.

Actually, we have $d\mathbf{r} = \mathbf{g}_i dx^i$ and the differential of arc length ds is determined from $ds^2 = d\mathbf{r} \cdot d\mathbf{r}$; therefore

$$ds^2 = \mathbf{g}_i \cdot \mathbf{g}_j \; dx^i dx^j = g_{ij} \; dx^i dx^j$$

Here, the quantities g_{ij} are components of a covariant tensor of rank two called the metric tensor. Similarly, We can define a contravariant tensor $g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j$. Obviously, g^{ij} is conjugate or reciprocal tensor of g_{ij}, i.e.

$$g^{ik} g_{kj} = \delta^i_j$$

$$(\mathbf{g}^i \cdot \mathbf{g}_j = g^{ik} \mathbf{g}_k \cdot \mathbf{g}_j = g^{ik} g_{kj} = \delta^i_j)$$

g^{ik} and g_{kj} are two fundamental tensors. Let $g = |g_{ij}|$ and $G = |g^{ij}|$ are determinants of g_{ij} and g^{ij}, then $\sqrt{g} = 1/\sqrt{G}$ where $\sqrt{g}$ is the Jacobian of coordinate transformation from old to new and $\sqrt{G}$ is the Jacobian of coordinate transformation from new to old.

IV. Contravariant and Covariant Components of a Vector

A given vector $\mathbf{A}$ can be expressed in contravariant and covariant base vector systems as

$$\mathbf{A} = A^i \mathbf{g}_i = A_i \mathbf{g}^i \tag{5}$$

or

$$\mathbf{A} = A^1 \mathbf{g}_1 + A^2 \mathbf{g}_2 + A^3 \mathbf{g}_3 \; - - - + A^n \mathbf{g}_n \tag{6}$$

$$\mathbf{A} = A_1 \mathbf{g}^1 + A_2 \mathbf{g}^2 + A_3 \mathbf{g}^3 \; - - - + A_n \mathbf{g}^n \tag{7}$$

Here a super-script index is used for "contravariant" and a subscript index is used for "covariant". According to the definitions in Sect. 1, $A^1\mathbf{g}_1, A^2\mathbf{g}_2, - - -, A^n\mathbf{g}_n$ are component vectors of vector $\mathbf{A}$ in the directions of $\mathbf{g}_1, \mathbf{g}_2, - - -, \mathbf{g}_n$, respectively. Similarly, $A_1\mathbf{g}^1, A_2\mathbf{g}^2, - - -, A_n\mathbf{g}^n$ are component vectors of vector $\mathbf{A}$ in the directions of $\mathbf{g}^1, \mathbf{g}^2, - - -, \mathbf{g}^n$, respectively. Accordingly, A^i are components of vector $\mathbf{A}$ in the directions of $\mathbf{g}_i$, with respect to the base vector system $(\mathbf{g}_1, \mathbf{g}_2, - - -, \mathbf{g}_n)$. Similarly, A_i are components of vector $\mathbf{A}$ in the directions of $\mathbf{g}^i$, with respect to the base vector system $(\mathbf{g}^1, \mathbf{g}^2, - - -, \mathbf{g}^n)$. As described in Sect. I, in general, the base vector system can be arbitrary (as long as the base system is complete). However, in Eq. (5), $\mathbf{g}_i$ and $\mathbf{g}^i$ are two uniquely defined base vector systems for a given coordinates x^i. $\mathbf{g}_i$ is the covariant base vector system and $\mathbf{g}^i$ is the contravariant base vector system for a given coordinates. Hence, we call

> A^i are contravariant components of vector $\mathbf{A}$ in the directions of $\mathbf{g}_i$, with respect to the covariant base vector system $(\mathbf{g}_1, \mathbf{g}_2, - - -, \mathbf{g}_n)$.
>
> A_i are covariant components of vector $\mathbf{A}$ in the directions of $\mathbf{g}^i$, with respect to the contravariant base vector system $(\mathbf{g}^1, \mathbf{g}^2, - - -, \mathbf{g}^n)$.

For simplicity, we call A^i contravariant components of vector $\mathbf{A}$ and A_i covariant components of vector $\mathbf{A}$.

When a vector is expressed in covariant and contravariant base vector systems, as shown in Eq. (5), its corresponding "components" are formed as contravariant and covariant vectors, respectively. They follow the coordinate transformation rule. Indeed, in Eqs. (6) and (7), $A^i = (A^1, A^2, - - -, A^n)$ is a contravariant vector or a contravariant tensor of rank one, and $A_i = (A_1, A_2, - - -, A_n)$ is a covariant vector, or a covariant tensor of rank one. Un-ambiguously, A^1, A^2 - - - etc. are the components of the contravariant tensor A^i and A_1, A_2 - - - etc. are the components of the covariant tensor A_i. Therefore, what we call "contravariant components" of a given vector are really components of a contravariant vector and, vis versa, what we call "covariant components" of a given vector are really components of a covariant vector.

One can show that, from Eqs. (4) and (5),

$$A^i = \mathbf{A} \cdot \mathbf{g}^i \ and \ \ A_i = \mathbf{A} \cdot \mathbf{g}_i \tag{8}$$

Contravariant and covariant tensors, A^i, A_i can be transformed each other through the two fundamental tensors such as

$$A^i = g^{ij} \ A_j, \quad and \quad A_i = g_{ij} \ A^j$$

Eq. (8) is useful for evaluation and commonly leads to a mis-interpretation that A^i is the component of vector $\mathbf{A}$ in the direction of $\mathbf{g}^i$ and A_i the components of vector $\mathbf{A}$ in the direction of $\mathbf{g}_i$. This mis-interpretation stems from the common use of a dot-product with a unit vector, or "projection", to calculate the component of a vector in the direction of that unit vector. (In projection, it implies that a vector is decomposed into two component vectors; one parallel to and another perpendicular to a given direction.) In Eq. (8), neither the contravariant base vectors nor the covariant base vectors are normalized, so that Eq. (8) is not a process of projection. Indeed, following the definition of Eq. (5), there should be no confusion and mis-interpretation as mentioned above.

V. 2-D Examples and Continuity Equation

Now in 2-D, let a position vector $\mathbf{r}(\xi, \ \eta)$ with coordinate system $(\xi, \ \eta)$, and

$$\mathbf{A} = u\mathbf{i} + v\mathbf{j} = A^1\mathbf{g}_1 + A^2\mathbf{g}_2 = A_1\mathbf{g}^1 + A_2\mathbf{g}^2$$

$$\mathbf{g}^1 = \nabla\xi = \xi_x \ \mathbf{i} + \xi_y \ \mathbf{j}$$

$$\mathbf{g}^2 = \nabla\eta = \eta_x\mathbf{i} + \eta_y\mathbf{j}$$

$$\mathbf{g}_1 = \frac{\partial \mathbf{r}}{\partial \xi} = x_\xi \ \mathbf{i} + y_\xi \ \mathbf{j}$$

$$\mathbf{g}_2 = \frac{\partial \mathbf{r}}{\partial \eta} = x_\eta\mathbf{i} + y_\eta\mathbf{j}$$

$$A^1 = (\mathbf{A} \cdot \nabla\xi) = \xi_x u + \xi_y v$$

$$A^2 = (\mathbf{A} \cdot \nabla\eta) = \eta_x u + \eta_y v$$

$$A_1 = (\mathbf{A} \cdot \mathbf{g}_1) = x_\xi u + y_\xi v$$

$$A_2 = (\mathbf{A} \cdot \mathbf{g}_2) = x_\eta u + y_\eta v$$

Here A^1 is the contravariant component of $\mathbf{A}$ in the direction of ξ and A^2 is the contravariant component of $\mathbf{A}$ in the direction of η, with respect to the base vector system $(\frac{\partial \mathbf{r}}{\partial \xi}, \frac{\partial \mathbf{r}}{\partial \eta})$. A_1 is the covariant component of $\mathbf{A}$ in the direction of $\nabla\xi$ and A_2 is the covariant component of $\mathbf{A}$ in the direction of $\nabla\eta$, with respect to the base vector system $(\nabla\xi, \nabla\eta)$.

As one can see, neither contravariant nor covariant components are physical quantities. Their relations to physical quantities can be illustrated, in 2-D, as Fig. 2. In Fig. 2, the vector $\mathbf{A} = \underline{OC}$ is expressed by physical component vectors $\underline{OB}$ and $\underline{OD}$ in the directions tangent to corresponding coordinate lines, $\mathbf{g}_1$, $\mathbf{g}_2$, or component vectors $\underline{OE}$ and $\underline{OF}$ in directions normal to the other coordinate lines, $\mathbf{g}^1$, $\mathbf{g}^2$, respectively.

Note, in Fig. 2, $OBCD$ is a parallelogram formed by $\underline{BC} \parallel \mathbf{g}_2$ and $\underline{CD} \parallel \mathbf{g}_1$, and $OECF$ is a parallelogram formed by $\underline{EC} \parallel \mathbf{g}^2$ and $\underline{CF} \parallel \mathbf{g}^1$. By the definitions of contravariant and covariant base vectors, we have

$$\underline{BC} \perp \mathbf{g}^1, \quad \underline{CD} \perp \mathbf{g}^2, \quad \underline{EC} \perp \mathbf{g}_1, \quad \underline{CF} \perp \mathbf{g}_2.$$

Here, OG and OH are projections of vector $\underline{OC}$ in directions normal to the other coordinate lines, $\mathbf{g}^1$ and $\mathbf{g}^2$, respectively. Note that, except in orthogonal coordinates, OG and OH are not "components" of vector $\underline{OC}$. ($\underline{OG} + \underline{OH} \neq \underline{OC}$, in general.) It is $\underline{OG}$ and $\underline{GC}$ or $\underline{OH}$ and $\underline{HC}$ that are component vectors of vector $\underline{OC}$; i.e. $\underline{OG} + \underline{GC} = \underline{OH} + \underline{HC} = \underline{OC}$ as shown in the sketch.

Here we would like to illustrate the above discussions with several examples. For the example A, let a vector $\mathbf{A} = 5\mathbf{i} + 7\mathbf{j}$, with coordinates (ξ, η) such as

$$\xi = 2x, \quad \eta = y$$

$$\mathbf{r}(\xi, \eta) = \xi/2\mathbf{i} + \eta\mathbf{j}$$

$$\mathbf{g}^1 = \nabla\xi = \xi_x\mathbf{i} + \xi_y\mathbf{j} = 2\mathbf{i}, \quad \mathbf{g}^2 = \nabla\eta = \eta_x\mathbf{i} + \eta_y\mathbf{j} = \mathbf{j}$$

$$\mathbf{g}_1 = \frac{\partial \mathbf{r}}{\partial \xi} = 1/2\mathbf{i}, \quad \mathbf{g}_2 = \frac{\partial \mathbf{r}}{\partial \eta} = \mathbf{j}$$

Therefore, the contravariant components are

$$A^1 = \mathbf{A} \cdot \mathbf{g}^1 = 10, \quad A^2 = \mathbf{A} \cdot \mathbf{g}^2 = 7$$

and the covariant components are

$$A_1 = \mathbf{A} \cdot \mathbf{g}_1 = 5/2, \quad A_2 = \mathbf{A} \cdot \mathbf{g}^2 = 7.$$

Fig.3 shows the sketch of this example, with magnitude and direction. Note that the new coordinates (ξ, η) are same as the (x, y) Cartesian coordinates, except ξ is scaled by $2x$. While $\mathbf{g}_2$ and $\mathbf{g}^2$ are identical with in both magnitude and direction, $\mathbf{g}_1$ and $\mathbf{g}^1$ are in the same direction as x-coordinate, $\mathbf{i}$, but different in magnitude, and the magnitudes of A^1 and A_1 are scaled accordingly.

Example B, let the the same vector $\mathbf{A} = 5\mathbf{i} + 7\mathbf{j}$, with new coordinate lines (ξ, η) so as

$$\xi = 2x, \ \eta = x + y$$

$$\mathbf{r}(\xi,\eta) = \xi/2\mathbf{i} + (\eta - \xi/2)\mathbf{j}$$

$$\mathbf{g}^1 = 2\ \mathbf{i},\ \mathbf{g}^2 = (\mathbf{i} + \mathbf{j})$$

$$\mathbf{g}_1 = (\mathbf{i} - \mathbf{j})/2,\ \mathbf{g}_2 = \mathbf{j}$$

Therefore, the contravariant components are

$$A^1 = 10, \quad A^2 = 12$$

and the covariant components are

$$A_1 = -1, \quad A_2 = 7.$$

These are identical as Eqs. (2) and (3) in Sect. 1, written as

$$\mathbf{A} = A^1\mathbf{g}_1 + A^2\mathbf{g}_2 = 10(\mathbf{i} - \mathbf{j})/2 + 12\mathbf{j}$$

$$\mathbf{A} = A_1\mathbf{g}^1 + A_2\mathbf{g}^2 = -(2\mathrm{i}) + 7(\mathbf{i} + \mathbf{j})$$

Fig. 4a shows detailed sketches of the base vectors, with magnitude and direction, and relations with (x, y). Fig. 4b shows the vector $\mathbf{A}$ expressed in its contravariant and covariant base vector systems. Since the magnitudes of base vectors are not normalized, the marked number are scaled to each base vectors accordingly.

It is interesting to note that if vector $\mathbf{A}$ is decomposed into two component vectors so that one is parallel to $\mathbf{g}^1$ and another is normal to $\mathbf{g}^1$, then $\mathbf{A} = 5/\sqrt{2}\ \frac{(\mathbf{i}-\mathbf{j})}{\sqrt{2}} + 7/2\ \frac{(\mathbf{i}+\mathbf{j})}{\sqrt{2}}$. These are the results of projection on to unit vectors $(\mathbf{i} - \mathbf{j})/\sqrt{2}$and $(\mathbf{i} + \mathbf{j})/\sqrt{2}$, and are quite different from the expression in contravariant or covariant base vector system.

Example C, let a new coordinate system be $\xi = (x - y)/2,\ \eta = y$, and hence $\mathbf{r}(\xi,\eta) = (2\xi + \eta)\mathbf{i} + \eta\mathbf{j}$. Therefore, the contravariant and covariant base vectors are

$$\mathbf{g}^1 = (\mathbf{i} - \mathbf{j})/2,\ \mathbf{g}^2 = \mathbf{j}$$

$$\mathbf{g}_1 = 2\mathbf{i},\ \mathbf{g}_2 = (\mathbf{i} + \mathbf{j})$$

Hence, again as Eqs. (2) and (3),

$$\mathbf{A} = A_1\mathbf{g}^1 + A_2\mathbf{g}^2 = 10(\mathbf{i} - \mathbf{j})/2 + 12\mathbf{j}$$

$$\mathbf{A} = A^1\mathbf{g}_1 + A^2\mathbf{g}_2 = -(2\mathrm{i}) + 7(\mathbf{i} + \mathbf{j})$$

Fig. 5 shows the contravariant and covariant base vectors for this new coordinate system. Note that the coordinate system in Example C is the conjugate or reciprocal of the the coordinate system in Example B. The contravariant base vectors in Example C are covariant base vectors in Example B and the covariant base vectors in Example C are contravariant base vectors in Example B. Clearly, the vector $\mathbf{A}$ is an invariant and the specification of coordinate system is essential for vector decomposition.

In fluid dynamics, the continuity equation can be written in integral and differential forms as

$$\frac{\partial}{\partial t}\int_v \rho\ dv + \int_S \rho\mathbf{W} \cdot ndS = 0,$$

$$\frac{\partial}{\partial t}\rho(g)^{\frac{1}{2}} + \frac{\partial}{\partial \xi}\rho\mathbf{W}\cdot(g)^{\frac{1}{2}}\mathbf{g}^1 + \frac{\partial}{\partial \eta}\rho\mathbf{W}\cdot(g)^{\frac{1}{2}}\mathbf{g}^2 = 0, \tag{9}$$

$$\frac{\partial}{\partial t}\rho(g)^{\frac{1}{2}} + \frac{\partial}{\partial \xi}\rho(g)^{\frac{1}{2}}U + \frac{\partial}{\partial \eta}\rho(g)^{\frac{1}{2}}V = 0.$$

Here, ρ is density, $\mathbf{W} = u\mathbf{i} + v\mathbf{j}$ the velocity vector, and $U = \mathbf{W}\cdot\mathbf{g}^1$ and $V = \mathbf{W}\cdot\mathbf{g}^2$ are contravariant velocity components in the directions of covariant base vectors $\mathbf{g}_1$ and $\mathbf{g}_2$, respectively. Clearly, here $\sqrt{g}$ is the same as J^{-1} in Eq. (1), the Jacobian of transformation from old to new coordinates (or $\sqrt{g}^{-1} = \sqrt{G} = J$). In a finite volume formation, $\sqrt{g}$ is the volume of the grid cell and $(g)^{\frac{1}{2}}\mathbf{g}^1$ and $(g)^{\frac{1}{2}}\mathbf{g}^2$ are surface normal in the directions of $\mathbf{g}^1$ and $\mathbf{g}^2$, respectively. The concept of mass conservation can be interpreted accordingly. As Eq. (9), while the contravariant velocity (U, V) is closely associated with the dot-product of velocity with surface normal, there should not be confusion that the contravariant velocity components, U, V, are velocity components in the directions of normal to the corresponding surfaces.

Finally, it is important to take note that the contravariant velocity components, (U, V), of the the velocity vector $\mathbf{W}$ do not have the physical dimension of $\mathbf{W}$ (velocity) itself. In a discretized numerical grid domain, contravariant velocity components, (U, V), have dimension of velocity divided by a length scale. Conversely, covariant velocity components have dimension of velocity multiplied by a length scale. They magnitudes are scaled by their corresponding base vectors accordingly.

In fact, in addition to their magnitudes, the dimensions of contravariant and covariant components are controlled by the dimensions of the covariant and contravariant base vectors, respectively. In Eq. (4) the contravariant and covariant base vectors are reciprocal not only in magnitude but also in dimension. This can be clearly illustrated by writing continuity equation in the cylindrical coordinates as

$$\xi = r = \sqrt{(x^2 + y^2)},\ \eta = \theta = tan^{-1}x/y$$

$$\mathbf{g}_1 = sin\theta\ i + cos\theta\ j,\ \mathbf{g}_2 = r\ cos\theta\ i - r\ sin\theta\ j$$

$$\mathbf{g}^1 = sin\theta\ i + cos\theta\ j,\ \mathbf{g}^2 = 1/r\ cos\theta\ i - 1/r\ sin\theta\ j$$

$$U = u\ sin\theta + v\ cos\theta,\ V = 1/r(u\ cos\theta - v\ sin\theta)$$

$$\frac{\partial}{\partial t}\rho/r + \frac{\partial}{\partial r}\rho/r(u\ sin\theta + v\ cos\theta) + \frac{\partial}{\partial \theta}\rho/r^2(u\ cos\theta - v\ sin\theta) = 0.$$

Obviously, in the above, the dimension of contravariant velocity component U is different from that of contravariant velocity component V.

VI. Concluding Remarks

In this paper we have reviewed the basics of tensor analysis in an attempt to clarify some misconception regard contravariant and covariant vector components as used in fluid dynamics. We have indicated that contravariant components are components of a given vector expressed as a unique combination of the covariant base vector system and vis versa, the covariant components are components of a vector expressed with the contravariant base vector system. Mathematically, expressing a vector with a combination of base vector is a decomposition process for a specific base vector system. Hence, the contravariant velocity components are decomposed *components* of velocity vector along the directions of coordinate lines, with respect to the covariant base vector system. However, the contravariant (and covariant) components are not physical quantities. Their magnitudes and dimensions are controlled by their corresponding covariant (and contravariant) base vectors.

VI. References

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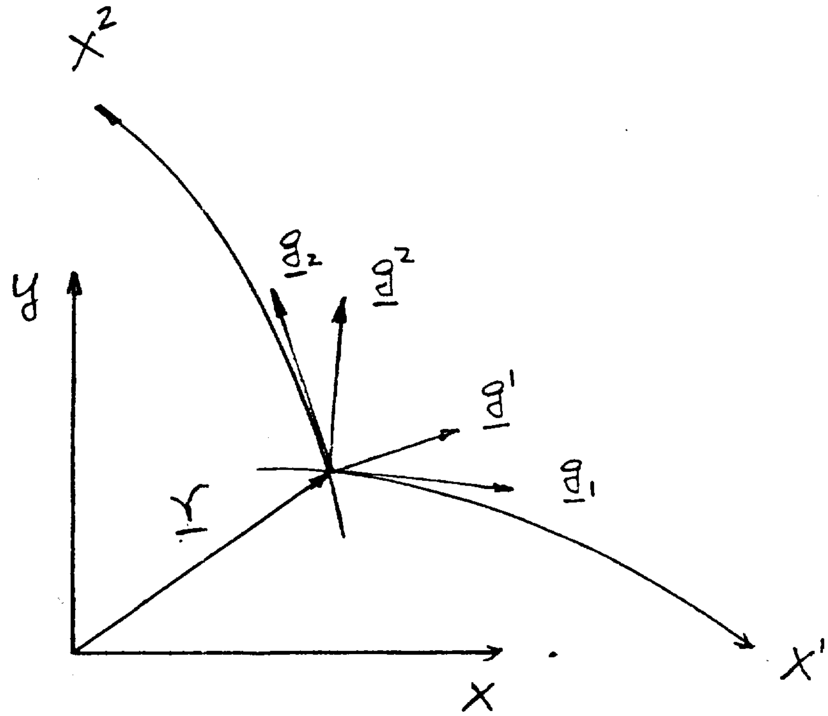
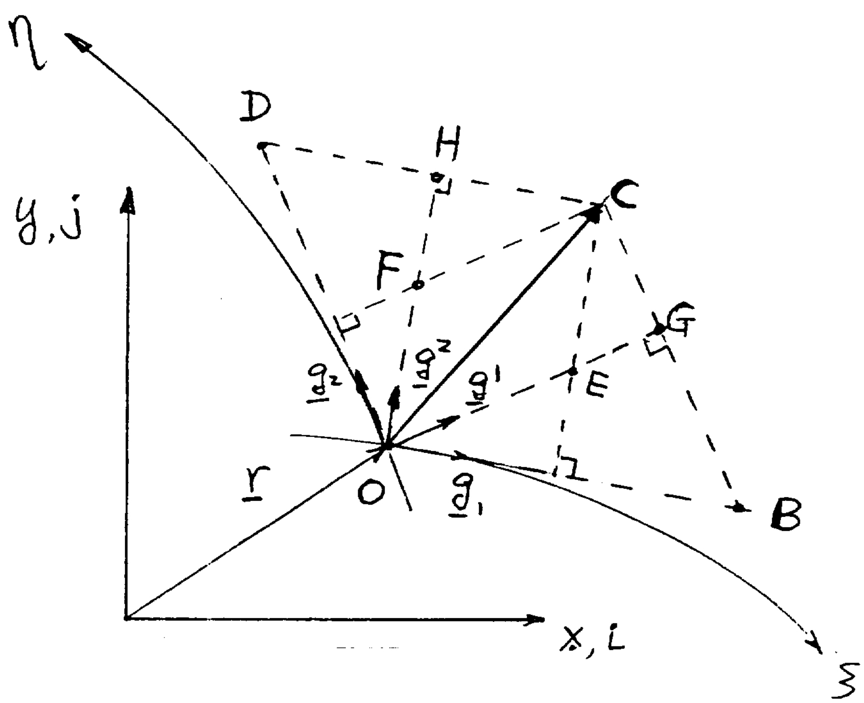


Fig. 1 Sketch of Contravariant and Covariant Base Vectors, $\mathbf{g}^i$ and $\mathbf{g}_i$, (indication of direction only).



$$\mathbf{A} = \underline{OC} = \underline{OB} + \underline{OD} = \underline{OE} + \underline{OF}$$

$$\underline{OB} = (\mathbf{A} \cdot \mathbf{g}^1)\mathbf{g}_1 = A^1\ \mathbf{g}_1$$

$$\underline{OD} = (\mathbf{A} \cdot \mathbf{g}^2)\mathbf{g}_2 = A^2\ \mathbf{g}_2$$

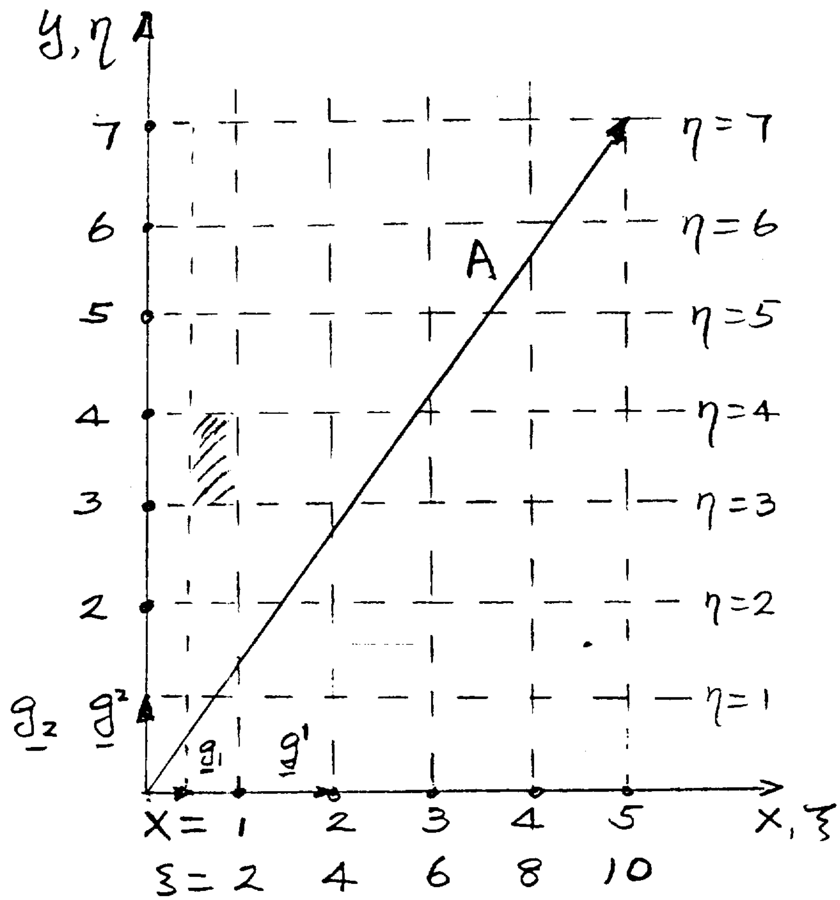
$$\underline{OE} = (\mathbf{A} \cdot \mathbf{g}_1)\mathbf{g}^1 = A_1\ \mathbf{g}^1$$

$$\underline{OF} = (\mathbf{A} \cdot \mathbf{g}_2)\mathbf{g}^2 = A_2\ \mathbf{g}^2$$

$$OG = (\mathbf{A} \cdot \mathbf{g}^1)/|\mathbf{g}^1| = A^1/|\mathbf{g}^1|$$

$$OH = (\mathbf{A} \cdot \mathbf{g}^2)/|\mathbf{g}^2| = A^2/|\mathbf{g}^2|$$

Fig. 2 The relations among its physical quantities of a given vector **A** to contravariant and covariant components



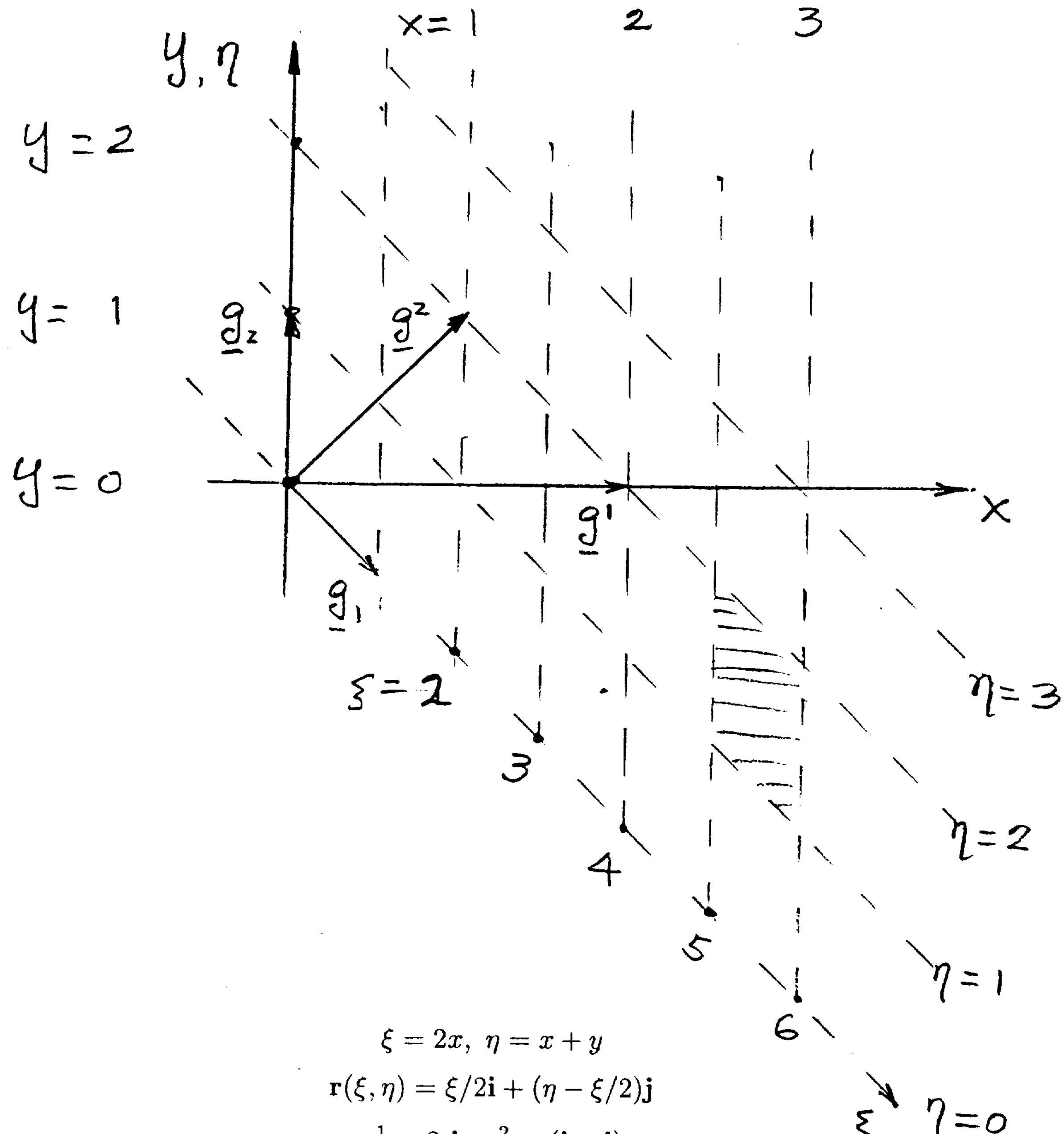
$$\xi = 2x, \quad \eta = y$$

$$\mathbf{r}(\xi, \eta) = \xi/2\mathbf{i} + \eta\mathbf{j}$$

$$\mathbf{g}^1 = \nabla\xi = \xi_x\mathbf{i} + \xi_y\mathbf{j} = 2\mathbf{i}, \quad \mathbf{g}^2 = \nabla\eta = \eta_x\mathbf{i} + \eta_y\mathbf{j} = \mathbf{j}$$

$$\mathbf{g}_1 = \frac{\partial\mathbf{r}}{\partial\xi} = 1/2\mathbf{i}, \quad \mathbf{g}_2 = \frac{\partial\mathbf{r}}{\partial\eta} = \mathbf{j}, \ \sqrt{g} = 1/2$$

Fig. 3 The Contravariant and Covariant Base Vectors in the Coordinates of $\xi = 2x$ and $\eta = y$, with Vector $\mathbf{A} = 5\mathbf{i} + 7\mathbf{j}$



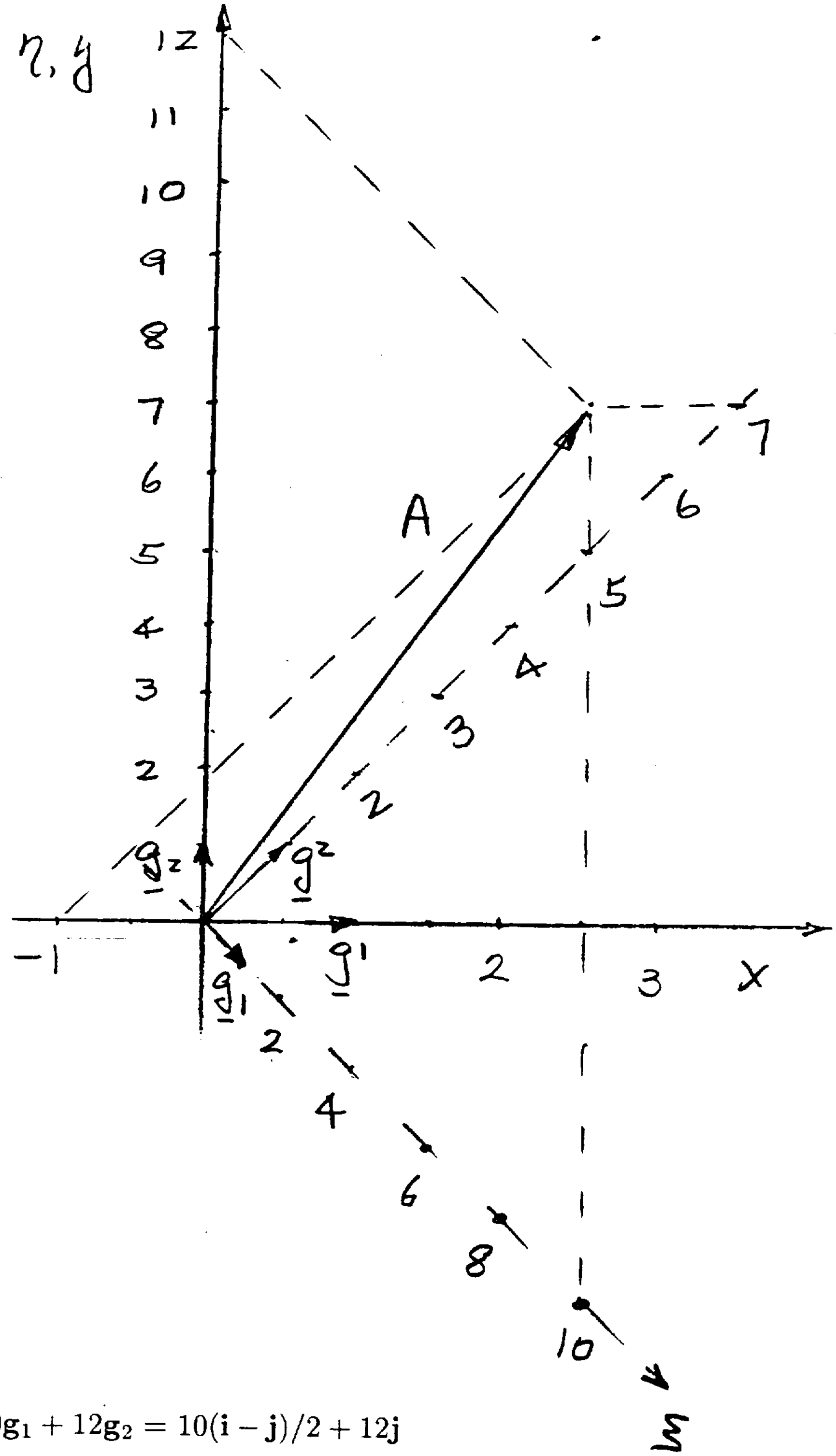
$$\xi = 2x,\ \eta = x + y$$

$$\mathbf{r}(\xi, \eta) = \xi/2\mathbf{i} + (\eta - \xi/2)\mathbf{j}$$

$$\mathbf{g}^1 = 2\ \mathbf{i},\ \mathbf{g}^2 = (\mathbf{i} + \mathbf{j})$$

$$\mathbf{g}_1 = (\mathbf{i} - \mathbf{j})/2,\ \mathbf{g}_2 = \mathbf{j},\ \sqrt{g} = 1/2$$

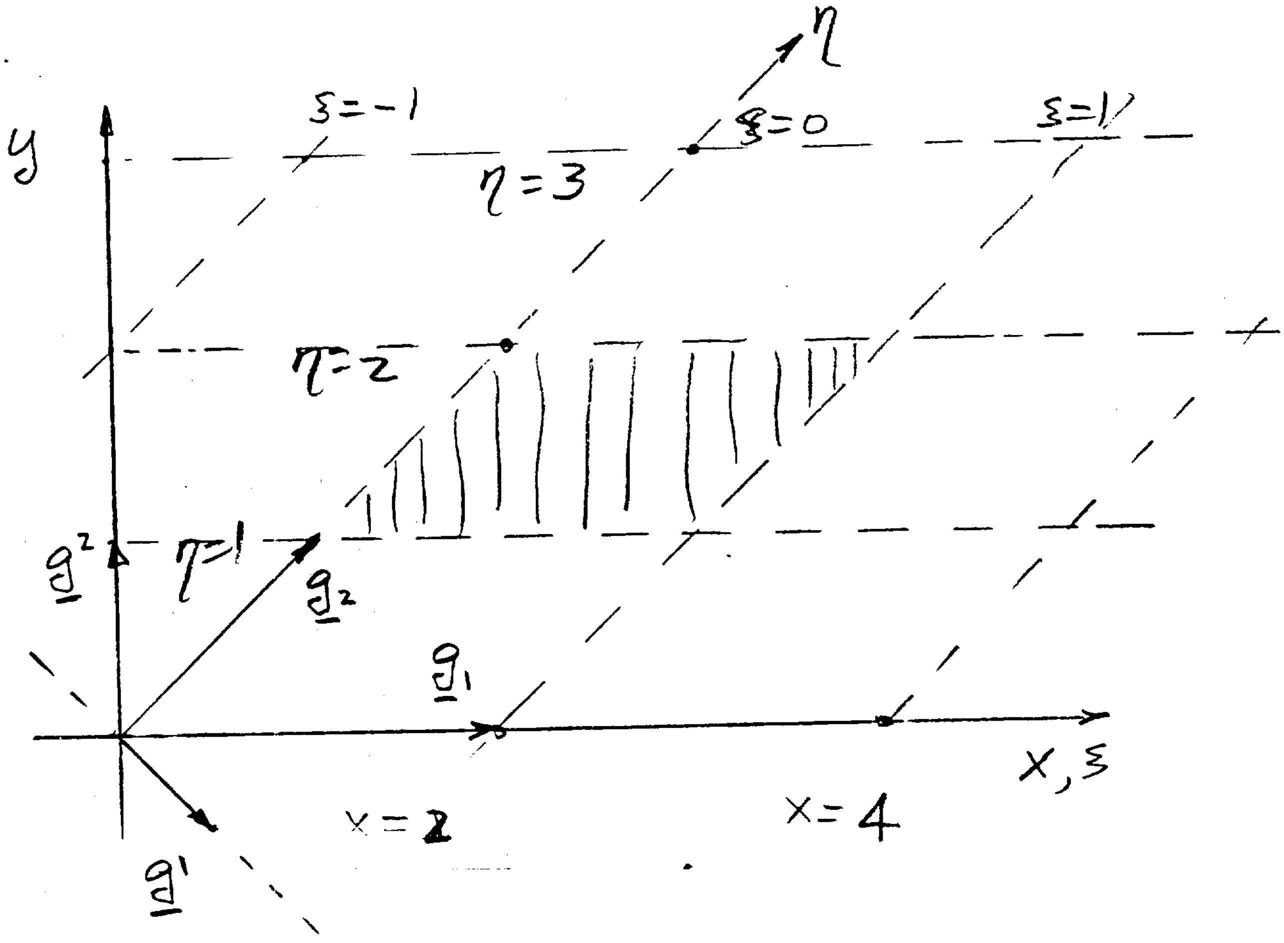
Fig. 4a The Contravariant and Covariant Base Vectors in the Coordinates of $\xi = 2x$ and $\eta = x + y$



$$\mathbf{A} == 10\mathbf{g}_1 + 12\mathbf{g}_2 = 10(\mathbf{i} - \mathbf{j})/2 + 12\mathbf{j}$$

$$\mathbf{A} = -\mathbf{g}^1 + 7\mathbf{g}^2 = -(2\mathbf{i}) + 7(\mathbf{i} + \mathbf{j})$$

Fig. 4b A Vector $\mathbf{A} = 5\mathbf{i} + 7\mathbf{j}$ Expressed with respect to Contravariant and Covariant Base Vector Systems in the Coordinates of $\xi = 2x$ and $\eta = x + y$



$\mathbf{g}^1 = (\mathbf{i}-\mathbf{j})/2,\ \mathbf{g}^2 = \mathbf{j}$

$\mathbf{g}_1 = 2\mathbf{i},\ \mathbf{g}_2 = (\mathbf{i}+\mathbf{j}),\ \sqrt{g} = 2$

Fig. 5 The Contravariant and Covariant Base Vectors in the Coordinates of $\xi = (x-y)/2$ and $\eta = y$