Closed-Loop Endoatmospheric Ascent Guidance

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Extended Abstract

This paper will present a complete formulation of the optimal control problem for atmospheric ascent of rocket powered launch vehicles subject to usual load constraints and final condition constraints. We shall demonstrate that the classical finite difference method for two-point-boundary-value-problems (TPBVP) is suited for solving the ascent trajectory optimization problem in real time, therefore closed-loop optimal endoatmospheric ascent guidance becomes feasible. Numerical simulations with a the vehicle data of a reusable launch vehicle will be provided.

1. Ascent Guidance Problem Formulation

The equations of motion of the RLV in an inertial coordinate system can be expressed as

$$\dot{\mathbf{r}} = \mathbf{V} \tag{1}$$

$$\dot{\mathbf{V}} = \mathbf{g}(\mathbf{r}) + \mathbf{A}/m(t) + T\mathbf{1}_b/m(t) + \mathbf{N}/m(t) \tag{2}$$

where $\mathbf{r}$ and $\mathbf{V}$ are inertial position and velocity vectors; $\mathbf{g}$ the gravitational acceleration; T the thrust magnitude; $\mathbf{A}$ and $\mathbf{N}$ are aerodynamic axial and normal forces, respectively; $\mathbf{1}_b$ the unit vector defining the RLV body longitudinal axis; m is the mass of the RLV. The magnitudes of the aerodynamic forces and thrust are modeled by

$$A = \frac{1}{2}\rho V_r^2 S_{ref} C_A(\text{Mach}, \alpha) \tag{3}$$

$$N = \frac{1}{2}\rho V_r^2 S_{ref} C_N(\text{Mach}, \alpha) \tag{4}$$

$$T = T_{vac} + \Delta T(r) \tag{5}$$

where ρ is the atmospheric density, and V_r is the magnitude of the Earth-relative velocity $\mathbf{V}_r = \mathbf{V} - \bar{\omega}_E \times \mathbf{r}$ with $\bar{\omega}_E$ being the Earth angular rotation rate vector. The axial and

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normal aerodynamic coefficients C_A and C_N are functions of angle of attack α and Mach number. They are usually expressed in analytical forms by curve-fitting tabulated data. The reference area S_{ref} is a constant. The vacuum thrust T_{vac} may be time varying when the thrust needs to be throttled down to meet axial acceleration constraint. The thrust loss inside the atmosphere $\Delta T \leq 0$ is a function of altitude through the dependence of ΔT on ambient pressure. Mass $m(t)$ is an explicit function of time. The mass flow rate will be reduced by the same percentage as the thrust when the thrust is throttled down.

If the RLV symmetric plane is assumed to be always the plane formed by the body-axis $\mathbf{1}_b$ and the Earth-relative velocity vector $\mathbf{V}_r$, we can further have

$$\mathbf{A} = -A\mathbf{1}_b, \quad \mathbf{N} = N\mathbf{1}_n \tag{6}$$

with the unit vector of the body normal axis $\mathbf{1}_n$ defined by

$$\mathbf{1}_n = \mathbf{1}_b \times \frac{(\mathbf{1}_b \times \mathbf{1}_{V_r})}{\|\mathbf{1}_b \times \mathbf{1}_{V_r}\|} \tag{7}$$

where $\mathbf{1}_{V_r} = \mathbf{V}_r / V_r$. The angle of attack is then

$$\cos\alpha = \mathbf{1}_b^T \mathbf{1}_{V_r} \tag{8}$$

The following expression for $\mathbf{1}_n$ is valid for both $\alpha > 0$ and $\alpha < 0$

$$\mathbf{1}_n = \mathbf{1}_b \times \frac{(\mathbf{1}_b \times \mathbf{1}_{V_r})}{\sin\alpha} \tag{9}$$

Note that in this formulation the sideslip angle $\beta \equiv 0$ when there is no wind.

The launch conditions for $\mathbf{r}_0$ and $\mathbf{V}_0$ are specified. The ascent guidance problem is to find the desired body-axis orientation $\mathbf{1}_b(t)$ at each instant which determines the thrust direction and aerodynamic forces during atmospheric portion of the ascent. The final conditions will be the engine-cutoff conditions which ensure insertion into the required orbit. These orbital insertion conditions can in general be written as k, $0 < k \leq 6$, algebraic end conditions

$$\Psi(\mathbf{r}(t_f), \mathbf{V}(t_f)) = 0, \quad \Psi \in R^k \tag{10}$$

In addition, there will be path constraints which limit the aerodynamic load and thrust acceleration of the RLV. These are expressed in terms of inequality constraints

$$S(\mathbf{r}, \mathbf{V}, t) \leq 0 \tag{11}$$

The mathematical tool used to solve this problem is the optimal control theory. In this setting a performance index is desired to be minimized, usually (but not necessarily) the burn time of the rocket engine because it is directly related to the propellant usage. Denote the performance index by

$$J = \phi(\mathbf{r}_f, \mathbf{V}_f, t_f)$$

The necessary conditions for the optimal control $\mathbf{1}_b^*$ are given by defining the Hamiltonian

$$H = \mathbf{p}_r^T \mathbf{V} + \mathbf{p}_V^T [\mathbf{g} + (T - A)\mathbf{1}_b/m + N\mathbf{1}_n/m] + \lambda^T S(\mathbf{r}, \mathbf{V}, t) + \mu(\mathbf{1}_b^T \mathbf{1}_b - 1)$$

where μ is a scalar multiplier and λ a vector multiplier of appropriate dimension. Then, the so-called costate equations and optimality condition for the optimal $\mathbf{1}_b^*$ are[1]

$$\dot{\mathbf{p}}_r = -\frac{\partial H}{\partial \mathbf{r}} \tag{12}$$

$$\dot{\mathbf{p}}_V = -\frac{\partial H}{\partial \mathbf{V}} \tag{13}$$

$$H(\mathbf{p}_r, \mathbf{p}_V, \mathbf{r}, \mathbf{V}, \mathbf{1}_b^*, t) = \max_{\mathbf{1}_b} H(\mathbf{p}_r, \mathbf{p}_V, \mathbf{r}, \mathbf{V}, \mathbf{1}_b, t) \tag{14}$$

And the optimal solution satisfies the terminal constraints (10) and the transversality conditions

$$\mathbf{p}_r(t_f) = -\frac{\partial \phi(\mathbf{r}_f, \mathbf{V}_f, t_f)}{\partial \mathbf{r}_f} + \left(\frac{\partial \Psi}{\partial \mathbf{r}_f}\right)^T \nu \tag{15}$$

$$\mathbf{p}_V(t_f) = -\frac{\partial \phi(\mathbf{r}_f, \mathbf{V}_f, t_f)}{\partial \mathbf{V}_f} + \left(\frac{\partial \Psi}{\partial \mathbf{V}_f}\right)^T \nu \tag{16}$$

$$H(\mathbf{p}_r, \mathbf{p}_V, \mathbf{r}^*, \mathbf{V}^*, \mathbf{1}_b^*, t)|_{t_f} = \frac{\partial \phi}{\partial t_f} \tag{17}$$

where $\nu \in R^k$ is a constant multiplier vector. The first two conditions can be rearranged to yield $6-k$ independent conditions involving only $\mathbf{p}_f = (\mathbf{p}_{r_f}^T\ \mathbf{p}_{V_f}^T)^T$ and $\mathbf{x}_f = (\mathbf{r}_f^T\ \mathbf{V}_f^T)^T$. The general approach will be first finding the $6-k$ linear independent solutions of the homogeneous system

$$\left(\frac{\partial \Psi}{\partial \mathbf{x}_f}\right) \xi = 0$$

Let $\xi_i(\mathbf{x}_f) \in R^6$, $i = 1, ..., 6-k$ be such solutions. Note that ξ_i's are functions of $\mathbf{x}_f$. Transversality conditions (15) and (16) are then equivalent to

$$(\mathbf{p}_f + \frac{\partial \phi}{\partial \mathbf{x}_f})^T \xi_i \triangleq \Gamma_i(\mathbf{p}_f, \mathbf{x}_f) = 0,\ i = 1, \ldots, 6-k. \tag{18}$$

For a given problem, these conditions in above equation can often times be obtained more conveniently by using the terminal constraints (10) and taking dot products of Eqs. (15) and (16) with appropriate vectors related to the final state $\mathbf{x}_f$.

When none of the inequality constraints in (11) are active, the multiplier vector λ in the expression of H is zero. In such a case the condition (14) is equivalent to

$$\frac{\partial H}{\partial \mathbf{1}_b} = 0$$

After much differentiation and algebraic operation, this condition can be shown to lead to

$$\mathbf{1}_b^* = c_1 \mathbf{p}_V + c_2 \mathbf{V}_r \tag{19}$$

where c_i's are scalar functions of the state and costate. Therefore we conclude that the optimal body-axis lies in the plane formed by the primer vector $\mathbf{p}_V$ and relative velocity vector $\mathbf{V}_r$. Note that as the atmospheric density decreases (approaching vacuum flight), $c_2 \to 0$ and $c_1 > 0$. The optimal thrust vector becomes aligned solely with the primer vector $\mathbf{p}_V$.[2]

The condition (19) suggests that the search for the optimal body-axis orientation can be reduced to a one-dimensional search in the plane of $\mathbf{p}_V$ and $\mathbf{V}_r$. Let Φ be the angle between the vectors $\mathbf{p}_V$ and $\mathbf{V}_r$. Then it is straightforward to see that $\mathbf{1}_b^T \mathbf{1}_{p_V} = \cos(\Phi - \alpha)$ and $\mathbf{1}_n^T \mathbf{1}_{p_V} = \sin(\Phi - \alpha)$. Using these two relations in the expression of H, it is clear that maximizing H with respect to $\mathbf{1}_b$ as in Eq. (14) is equivalent to $\partial H/\partial\alpha = 0$, which in turn results in

$$\tan(\Phi - \alpha)(T - A + N_\alpha) - (A_\alpha + N) = 0 \tag{20}$$

with $N_\alpha = \partial N/\partial\alpha$ and $A_\alpha = \partial A/\partial\alpha$. The above equation needs to be solved numerically for α (note that A, N, A_α and N_α are still functions of α). Once α is found, we have[3]

$$\mathbf{1}_b^* = \left(\frac{\sin\alpha}{\sin\Phi}\right)\mathbf{1}_{p_V} + \left[\frac{\cos\alpha - \cos\Phi\cos(\Phi - \alpha)}{\sin^2\Phi}\right]\mathbf{1}_{V_r} \tag{21}$$

with $\mathbf{1}_{p_V} = \mathbf{p}_V/p_V$.

It should be stressed that for on-board guidance applications, it is imperative that great care be taken to determine the sign of Φ in Eqs. (20) and (21). Not only the value and sign of α depend on the sign of Φ, but more importantly is the physical implication of the sign change of Φ. Since the body y-axis can be defined by

$$\mathbf{1}_y = \frac{\mathbf{1}_{V_r} \times \mathbf{1}_{p_V}}{\sin\Phi} \tag{22}$$

When the vector $\mathbf{1}_{V_r} \times \mathbf{1}_{p_V}$ changes direction by 180 degrees, the sign of Φ should change. If the correct sign of Φ is identified when it becomes negative, the y-body axis will have an instantaneous change of direction by 180 degrees. This would require an undesirable (may be even unfeasible) 180-degree roll of the vehicle.

The typical path constraints (11) in an ascent guidance problem include $\alpha\bar{q}$, α, $\bar{q}$, and acceleration limits. It can be shown that the condition (19) with different c_1 and c_2 still holds true when these constraints become active. In these cases α is determined by the constraints rather than Eq. (20). The corresponding body axis $\mathbf{1}_b$ is still found according to Eq. (21). Hence the necessary conditions for the optimal control problem constitute a two-point-boundary-value problem (TPBVP) involving the state and costate equations (1-2)and (12-13), given initial launch conditions $\mathbf{r}_0$ and $\mathbf{V}_0$, and the final conditions (10) and (18).

2. Finite Difference Approach for Atmospheric Ascent Guidance

2.1 Methodology

Finite difference is one of the several techniques commonly used for TPBVPs. It tends to be more robust and less sensitive with respect to initial guesses as compared to shooting methods. Suppose that the TPBVP at hand from the necessary conditions of the optimal control problem in the preceding section is put in the form of

$$\dot{\mathbf{y}} = f(t, \mathbf{y}) \tag{23}$$

$$B_0(\mathbf{y}_0) = 0, \qquad B_f(\mathbf{y}_f) = 0 \tag{24}$$

where $\mathbf{y} = (\mathbf{x}^T\ \mathbf{p}^T)^T \in R^{2n}$ with $n = 6$. Let t_f be the specified final time. The $2n$ boundary conditions in Eq. (24) are from the given launch conditions, terminal constraints and transversality conditions. Note that the control $\mathbf{1}_b$ has been expressed as a function of the $\mathbf{y}$ in the system equations through the optimality condition (21). To find the solution to the TPBVP, divide the time interval $t_f - t_0$ in to N subinterval of the same length $h = (t_f - t_0)/M$. Let $\mathbf{y}_k = \mathbf{y}(t_0 + kh)$ be the value of the solution at the node $t_k = t_0 + kh$, $k = 0, \ldots, M$. At the middle point between t_{k-1} and t_k, denoted by $t_{k-1/2} = t_k - h/2$, the differential equations (23) are approximated by central finite difference:

$$\frac{1}{h}(\mathbf{y}_k - \mathbf{y}_{k-1}) = f(t_{k-1/2}, \frac{\mathbf{y}_k - \mathbf{y}_{k-1}}{2}) \tag{25}$$

Or equivalently,

$$E_k(\mathbf{y}_k, \mathbf{y}_{k-1}) = \mathbf{y}_k - \mathbf{y}_{k-1} - hf(t_{k-1/2}, \frac{\mathbf{y}_k - \mathbf{y}_{k-1}}{2}) = 0,\ k = 1, \ldots, M-1. \tag{26}$$

In addition, the boundary conditions are denoted by

$$E_0(\mathbf{y}_0) = B_0(\mathbf{y}_0) = 0 \tag{27}$$

$$E_M(\mathbf{y}_M) = B_f(\mathbf{y}_f) = 0 \tag{28}$$

Treat $\mathbf{Y} = (\mathbf{y}_0^T\ \mathbf{y}_1^T \cdots \mathbf{y}_M^T)^T \in R^{2n(M+1)}$ as the unknowns. The equal number of conditions are

$$\mathbf{E}(\mathbf{Y}) = 0 \tag{29}$$

where $\mathbf{E} = (E_0\ E_1 \ldots E_M)$. Now the problem becomes a root-finding problem for a system of nonlinear algebraic equations. It has been rigorously established that under certain conditions on smoothness and the boundary conditions, the following holds true[4]

1. The original TPBVP and the finite difference problem have unique solution;
2. The solution of the above finite difference problem $\mathbf{y}_k$ is a second-order approximation to the solution of the TPBVP $\mathbf{y}^*(t)$ at each t_k, i.e.,

   $$\|\mathbf{y}^*(t_k) - \mathbf{y}_k\| = \mathcal{O}(h^2)$$

   where $\lim_{h\to 0} \mathcal{O}(h^2)/h^2 < \infty$.

For ascent guidance applications, since the time-to-go $t_f - t_0$ is decreasing, the accuracy of the finite difference solution will be higher and higher as h becomes smaller even for moderate number of nodes.

2.2 Algorithm

The modified Newton method is probably the best suited algorithm for solving the problem (29). Starting from an initial guess $\mathbf{Y}_0$, the search direction $\mathbf{d}_j$ in the j-th iteration is determined by solving the linear algebraic equations

$$\left[\frac{\partial \mathbf{E}(\mathbf{Y}_{j-1})}{\partial \mathbf{Y}}\right] \mathbf{d}_j = -\mathbf{E}(\mathbf{Y}_{j-1}),\ j = 1, \dots \tag{30}$$

Then the update is given by

$$\mathbf{Y}_j = \mathbf{Y}_{j-1} + \sigma_j \mathbf{d}_j, \quad 0 < \sigma_j \le 1, \tag{31}$$

The step size parameter σ_j is determined by the following criterion

$$\sigma_j = \max_{0 \le i} \left\{ \frac{1}{2^i} \mid \mathbf{E}^T(\mathbf{Y}_{j-1} + \frac{1}{2^i}\mathbf{d}_j)\mathbf{E}(\mathbf{Y}_{j-1} + \frac{1}{2^i}\mathbf{d}_j) < \mathbf{E}^T(Y_{j-1})\mathbf{E}(Y_{j-1}) \right\} \tag{32}$$

In other words, starting from $\sigma_j = 1$, σ_j is halved repeatedly if necessary till the above condition is satisfied.[5] Convergence is achieved when $\|\mathbf{E}(\mathbf{Y}_j)\| \le \varepsilon$ for some preselected small $\varepsilon > 0$. The possible additional function evaluations required in checking the above step size condition pose negligible computational burden because function evaluations are not expensive in this setting. The result on the other hand is a much more robust algorithm, especially when the initial guesses are not close to the final solution.

At the first glance, solving the linear system (30) may seem to be a formidable task because the dimension of the problem ($2n(M+1)$) which can easily be over 1000. However, a close inspection reveals that the Jacobian matrix in (30) has a special sparse pattern due to the dependence of $\mathbf{E}_k$ on only $\mathbf{y}_k$ and $\mathbf{y}_{k-1}$ (cf. Eq. (26)). Therefore a very efficient algorithm, both in speed and storage, based on Gauss elimination and back substitution can be devised to solve the system (30). More details in implementation of such an algorithm are well documented in.[6]

The evaluation of the Jacobian $\partial \mathbf{E}/\partial \mathbf{Y}$ can certainly be done analytically. But we believe that simple forward finite differencing is more advantageous in this case. This is because unlike in a case where integrations of the differential equations are involved for each function evaluation, the function evaluation is fast. Using analytical Jacobian offers no computational speed benefits. On the other hand, analytical Jacobian will make the computer code significantly more complicated because second-order partial derivatives of the right hand sides of the state equations are needed. Also, when some of the path constraints in Eq. (11) become active, the multiplier λ will become functions of state and costate, adding more complexity to the analytical Jacobian.

When using the finite difference algorithm, initial guess for the state as well as the costate are required at t_k, $k = 0, 1, ...M$. The initial guess for $\mathbf{x}(t)$ can be obtained by linearly interpolate the given $\mathbf{x}_0$ and an estimated $\mathbf{x}(t_f)$ which is relatively easy to make because of the physical meaning of $\mathbf{x}(t_f)$. For the costate $\mathbf{p}(t)$, a reasonable initial guess can be obtained from the costate history obtained by solving the problem with zero atmospheric density (vacuum flight). For vacuum flight, the finite difference algorithm converges in many cases with even constant guess for $\mathbf{p}(t)$. Another possibility is to use the costate and state solutions from the analytical vacuum guidance algorithm which is described in the next section. The atmospheric solution is then obtained by applying homotopic continuation on the atmospheric density with

$$\hat{\rho} = \eta\rho, \quad 0 \leq \eta \leq 1 \tag{33}$$

For each η, $\hat{\rho}$ is used in place of atmospheric density in the aerodynamic forces terms. The homotopic parameter η starts at 0 (vacuum solution), and gradually increase to unity for full atmospheric solution.

3. Analytical Vacuum Ascent Guidance

While the finite difference approach in the preceding sections applies equally well to vacuum ascent guidance, a simpler and even better analytical approach exists. This approach combines a number of elegant results in optimal vacuum trajectory studies over the past three decades, and is summarized in Ref.[7] The key ingredients are the linear gravity simplification and the closed-form solution of the costate equation and nearly closed-form solution of the sate equation. At the beginning of each guidance update cycle, let $\mathbf{r}_0$ be the position vector. The gravity acceleration $\mathbf{g}$ is then approximated by

$$\mathbf{g} = -\frac{\mu_E}{r_0^3}\mathbf{r} = -\omega^2\mathbf{r} \tag{34}$$

where μ_E is the gravitational parameter of the Earth. In an ascent guidance problem, the correct direction of the gravity is more important than the accuracy of its magnitude. This approximation preserves the change of direction of the gravitational acceleration with $\mathbf{r}$. The magnitude of $\mathbf{g}$ will be slightly different from that of a Newtonian central gravity field. But when r_0 is continuously updated by the radius at beginning of each guidance cycle, the effect of this difference will be negligible.

Let $g_0 = \mu_E/r_0^2$ be the magnitude of the gravity acceleration at r_0. We normalize the equations of motion (1) and (2) with unit distance r_0, unit time $\sqrt{r_0/g_0}$, and unit velocity $\sqrt{r_0 g_0}$. The dimensionless equations of motion, with $\mathbf{A} = \mathbf{N} = 0$ for vacuum flight, become

$$\dot{\mathbf{r}} = \mathbf{V} \tag{35}$$

$$\dot{\mathbf{V}} = -\mathbf{r} + T(\tau)\mathbf{1}_b \tag{36}$$

where the Schuler frequency ω has become unity in the normalized time, and $T(t) = T_{vac}/m(\tau)g_0$ with τ as the normalized time. Note that this normalization is done in each guidance cycle with the r_0 being the radius at the beginning of that cycle. The Hamiltonian now is

$$H = \mathbf{p}_r^T\mathbf{V} + \mathbf{p}_V^T[-\mathbf{r} + T(\tau)\mathbf{1}_b] + \mu(\mathbf{1}_b^T\mathbf{1}_b - 1) \tag{37}$$

The optimality condition from $\partial H/\partial \mathbf{1}_b = 0$ yields

$$\mathbf{1}_b^* = -\frac{T(\tau)}{2\mu}\mathbf{p}_V \tag{38}$$

The sufficient condition for the optimality condition (14) in this case is $\partial^2 H/\partial \mathbf{1}_b^2 = 2\mu I_3 < 0$ with I_3 being an 3×3 identity matrix. We have $\mu < 0$, hence the well-known result that the optimal thrust direction must be aligned with that of $\mathbf{p}_V$. The costate equations (12) and (13) now become

$$\dot{\mathbf{p}}_r = \mathbf{p}_V \tag{39}$$
$$\dot{\mathbf{p}}_V = -\mathbf{p}_r \tag{40}$$

The costate equations have closed-form solution of

$$\begin{bmatrix} \mathbf{p}_V(\tau) \\ -\mathbf{p}_r(\tau) \end{bmatrix} = \begin{bmatrix} \cos\tau I_3 & \sin\tau I_3 \\ -\sin\tau I_3 & \cos\tau I_3 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{V_0} \\ -\mathbf{p}_{r_0} \end{bmatrix} \triangleq \Omega(\tau) \begin{bmatrix} \mathbf{p}_{V_0} \\ -\mathbf{p}_{r_0} \end{bmatrix} \tag{41}$$

where $\mathbf{p}_{V_0}$ and $\mathbf{p}_{r_0}$ are the (unknown) initial conditions for the costate. Define

$$\mathbf{I}_c(\tau) = \int_0^\tau \mathbf{1}_{p_V}(\zeta)\cos\zeta T(\zeta)d\zeta \triangleq \int_0^\tau \dot{\mathbf{I}}_c(\zeta)d\zeta \tag{42}$$
$$\mathbf{I}_s\tau) = \int_0^\tau \mathbf{1}_{p_V}(\zeta)\sin\zeta T(\zeta)d\zeta \triangleq \int_0^\tau \dot{\mathbf{I}}_s(\zeta)d\zeta \tag{43}$$

It can be easily verified that the state equations have the solution of

$$\begin{bmatrix} \mathbf{r}(\tau) \\ \mathbf{V}(\tau) \end{bmatrix} = \Omega(\tau) \begin{bmatrix} \mathbf{r}_0 \\ \mathbf{V}_0 \end{bmatrix} + \Gamma(\tau) \begin{bmatrix} \mathbf{I}_c(\tau) \\ \mathbf{I}_s(\tau) \end{bmatrix} \tag{44}$$

where

$$\Gamma(\tau) = \begin{bmatrix} \sin\tau I_3 & -\cos\tau I_3 \\ \cos\tau I_3 & \sin\tau I_3 \end{bmatrix} \tag{45}$$

The integrals $\mathbf{I}_c$ and $\mathbf{I}_s$ can be evaluated by a numerical quadrature scheme. Calise *et al* use the Simpson's rule in Ref.[7] We decide to use the Milne's rule because it only increases computation by a small margin yet offers a combined four-orders of magnitude higher accuracy for a 200-second burn. Let $\delta = \tau_{togo}/4$ where τ_{togo} is the dimensionless time-to-go till MECO, yet to be determined. With Milne's rule, we have

$$\mathbf{I}_i(\tau_{togo}) = \frac{\tau_{togo}}{90}\left[7\dot{\mathbf{I}}_i(0) + 32\dot{\mathbf{I}}_i(\delta) + 12\dot{\mathbf{I}}_i(2\delta) + 32\dot{\mathbf{I}}_i(3\delta) + 7\dot{\mathbf{I}}_i(4\delta)\right],\ i = c, s \tag{46}$$

Note that we have left T_{vac} inside the integrals (as a part of $T(\tau)$) because T_{vac} will be time-varying (throttled down) when a thrust acceleration limit is imposed and becomes active. With the thrust integrals given in (46) and the costate given in (41) as function of its initial conditions, the state is found in closed-form from Eq. (45). Therefore the final state and costate are explicitly functions of $\mathbf{p}_{V_0}$, $\mathbf{p}_{r_0}$ and τ_{togo}. Consequently, the total 6 terminal conditions (10) and (18) are functions of the 7 unknowns $\mathbf{p}_{V_0}$, $\mathbf{p}_{r_0}$ and τ_{togo}. The 7th condition is from condition (17). For minimum-time problem, the costate can be scaled by arbitrary positive constant without changing any necessary conditions for the optimal control problem. And it will be shown that for Keplerian orbit insertion as the terminal conditions in (10), the condition (17) is automatically satisfied. Therefore the problem can actually be reduced to a six-unknown problem by requiring, for instance, that $\|\mathbf{p}_0\| = \|(\mathbf{p}_{V_0}^T\ \mathbf{p}_{r_0}^T)\| = 1$. One of the six components of $\mathbf{p}_0$ is determined by the other five. But this still leaves us the ambiguity of determining the sign of that component of $\mathbf{p}_0$. We opt to avoid this problem by still treating the problem as a seven-unknown problem, and adding a trivial condition

$$\|\mathbf{p}(\tau_f)\| = 1 \tag{47}$$

From the costate equations (39) and (40), it can be shown by simple differentiation that

$$\frac{d\|\mathbf{p}(\tau)\|}{d\tau} = 0 \tag{48}$$

Therefore $\|\mathbf{p}(\tau)\|$ =constant. The condition (47) thus will always be trivially satisfied if we scale $\mathbf{p}_0$ to have $\|\mathbf{p}_0\| = 1$ (and we must, in order for the condition (47) to be met).

For minimum-time problem where $J = \phi = t_f$, the condition (17) be comes $H|_{t_f} = 1$. This condition is equivalent to $H|_{t_f} > 0$ because the costate $\mathbf{p}$ can always be scaled by a positive constant to achieve $H|_{t_f} = 1$ if $H|_{t_f} > 0$ before scaling. But this condition is automatically satisfied for Keplerian orbit insertion if other necessary conditions are satisfied. For Keplerian orbit insertion, the terminal conditions (10) represent the conditions on orbital elements which are constant if the engine is cut off at this point. Therefore

$$\frac{d\Psi(\mathbf{x}_f)}{d\tau} = \frac{\partial\Psi(\mathbf{x}_f)}{\partial\mathbf{x}_f}\begin{pmatrix} \mathbf{V}_f \\ -\mathbf{r}_f \end{pmatrix} = 0 \tag{49}$$

where $-\mathbf{r}_f = \mathbf{g}(\mathbf{r}_f)$ by the linear gravity approximation. Premultiply the above equation with the constant multiplier vector ν in Eqs. (15-16) and use these transversality conditions:

$$\nu^T\frac{\partial\Psi(\mathbf{x}_f)}{\partial\mathbf{x}_f}\begin{pmatrix} \mathbf{V}_f \\ -\mathbf{r}_f \end{pmatrix} = \mathbf{p}_{r_f}^T\mathbf{V}_f - \mathbf{p}_{V_f}^T\mathbf{r}_f = 0 \tag{50}$$

Use the expression of the Hamiltonian (37), $\mathbf{1}_b = \mathbf{1}_{\mathbf{p}_V}$, and the above relationship, we arrive at

$$H|_{t_f} = T(\tau_f)\|\mathbf{p}_{V_f}\| > 0$$

In summary, the optimal vacuum ascent guidance problem becomes a root-finding problem with seven unknowns ($\mathbf{p}_0$ and τ_{togo}), six constraints (10) and (18) plus one "easy" constraint (47). Through the use of quadrature (46), all the final state $\mathbf{x}_f$ and costate $\mathbf{p}_f$ are explicit functions of the seven unknowns. The modified Newton method again works very well and the convergence occurs rapidly with almost any initial guesses that do not result in totally wrong initial thrust direction.

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