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Flutter, Postflutter, and Control of a Supersonic Wing Section

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A number of issues related to the flutter and postflutter of two-dimensional supersonic lifting surfaces are addressed. Among them there are the 1) investigation of the implications of the nonlinear unsteady aerodynamics and structural nonlinearities on the stable/unstable character of the limit cycle and 2) study of the implications of the incorporation of a control capability on both the flutter boundary and the postflutter behavior. To this end, a powerful methodology based on the Lyapunov first quantity is implemented. Such a treatment of the problem enables one to get a better understanding of the various factors involved in the nonlinear aeroelastic problem, including the stable and unstable limit cycle. In addition, it constitutes a first step toward a more general investigation of nonlinear aeroelastic phenomena of three-dimensional lifting surfaces.

Nomenclature

\[ \begin{align*}
  a_o & = \text{speed of sound} \\
  b & = \text{semichord length} \\
  c_b, c_a & = \text{linear viscous damping coefficients} \\
  h, \xi & = \text{plunging displacement and its dimensionless counterpart (} \equiv h/b \text{), respectively} \\
  I_a, K_a & = \text{mass moment of inertia per unit span} \\
  L_a, M_a, m_a & = \text{linear stiffness coefficients in plunging and pitching, respectively} \\
  M_C, f_1, f_2 & = \text{nonlinear moment control; linear and nonlinear control gains} \\
  M_\infty, \lambda & = \text{undisturbed flight Mach number, } U_\infty/a_o, \text{ and its normalized counterpart, } M_\infty/(\mu \lambda a/r_o), \text{ respectively} \\
  m & = \text{airfoil mass per unit span} \\
  p_\infty, \rho_\infty, a_\infty & = \text{pressure, density, and speed of sound of the undisturbed flow, respectively} \\
  q & = \text{dynamic pressure, } \frac{1}{2} \rho_\infty U_\infty^2 \\
  r_a & = \text{dimensionless radius of gyration with respect to the elastic axis, } \sqrt{(I_a/m b^2)} \\
  S_o, \chi_o & = \text{static unbalance about the elastic axis and its dimensionless counterpart, } S_o/m b, \text{ respectively} \\
  t, \tau & = \text{time variable and its dimensionless counterpart, } U_\infty t/m b, \text{ respectively} \\
  U_\infty, V & = \text{freestream speed and its dimensionless counterpart, } U_\infty/b a_o, \text{ respectively} \\
  \psi_1, \psi_2 & = \text{linear and nonlinear normalized control gains, } f_1/K_a \text{ and } f_2/K_a, \text{ respectively} \\
  \omega_a, \omega_a, \dot{\omega} & = \text{uncoupled frequencies of the linearized aeroelastic system counterpart in plunging } \sqrt{(K_a/m)} \text{, pitching } \sqrt{(K_a/I_a)} \text{, and frequency ratio } (\omega_a/\omega_a), \text{ respectively}
\end{align*} \]

Superscripts

\[ \begin{align*}
  * & = \text{speed and frequency of flutter} \\
  \chi_F & = \text{downwash velocity normal to the lifting surface} \\
  \gamma, \zeta, \xi & = \text{transverse displacement} \\
  \phi, \chi, \psi & = \text{axial elastic position measured from the leading edge (positive aft) and its dimensionless counterpart, } x_{EA}/b, \text{ respectively} \\
  \alpha, \gamma, \delta, \epsilon & = \text{twist angle about the pitch axis} \\
  \psi & = \text{aerodynamic correction factor} \\
  \delta_S, \delta_A, \delta_C & = \text{tracking quantities identifying the structural, aerodynamic, and nonlinear control terms, respectively} \\
  \zeta, \xi & = \text{damping ratios in plunging and pitching, } c_b/2 \rho_\infty a_o \text{ and } c_a/2 I_a a_\infty, \text{ respectively} \\
  \mu & = \text{isentropic gas coefficient} \\
  \kappa & = \text{dimensionless mass ratio, } m A p b^2 \\
  \chi & = \text{flight Mach number, } U_\infty/a_o, \text{ and its normalized counterpart, } M_\infty/(\mu \lambda a/r_o), \text{ respectively} \\
  \alpha, \omega, \dot{\omega} & = \text{uncoupled frequencies of the linearized aeroelastic system counterpart in plunging } \sqrt{(K_a/m)} \text{, pitching } \sqrt{(K_a/I_a)} \text{, and frequency ratio } (\omega_a/\omega_a), \text{ respectively}
\end{align*} \]

Introduction

HIGH-SPEED and high-performance combat aircraft perform aggressive maneuvers that can result in significant reductions in the flutter speed. Moreover, the tendency to increase structural flexibility and maximum operating speed increases the likelihood of flutter within the aircraft operational envelope. This can jeopardize aircraft performance and dramatically affect its survivability. To prevent such events from occurring, two principal issues need to be addressed: 1) increase of the flutter speed without weight penalties and 2) investigation of the possibilities of converting the unstable limit cycle into a stable limit cycle. The successful accomplishment of the second issue will permit the crossing of the flutter boundary without danger of a catastrophic failure. In such a case, however, structural fatigue becomes a concern.

Before addressing these issues, the search for the aeroelastic instability of lifting surfaces encompasses two basic problems. One of these, based on the linearized aeroelastic equations, allows determination of the flutter boundary. The second one, based on the nonlinear approach to the aeroelastic problem, allows determination
of the character of the flutter boundary. In this sense, the flutter boundary can feature either benign or catastrophic behavior.

Because of the necessity of avoiding flutter and/or flutter-related airplane performance restrictions, it appears that determination of both the flutter boundary and of its character, that is, catastrophic or benign, and the possibility of controlling both of these present considerable practical importance. The goal of the control is to expand the flight envelope without weight penalties by increasing the flutter speed and to convert the catastrophic flutter into benign flutter.

The concept of catastrophic and benign types of flutter can be found in the specialized literature under different connotations that depend on the particular approach of the problem. The terminology of benign or catastrophic flutter\(^1\) is synonymous with that of stable and unstable limit-cycle oscillation (LCO),\(^1\) also referred to in the literature as supercritical and subcritical Hopf bifurcation\(^1\) (also Refs. 7, 13, and 14), respectively. The various terminologies related to the character of the flutter boundary and a few sources where these can be found are shown in Table 1. These terminologies are used throughout the paper.

In this study, the issues related to both the increase of the flutter speed and the character of the flutter boundary, as well as of their control, will be addressed. In the aeroelastic governing equations, the various nonlinear effects on which basis is possible to analyze the character of the flutter boundary will be incorporated. An active control methodology capable of expanding the flutter boundary and of converting the unstable LCO into a stable LCO will be implemented. The nonlinearities to be included in the aeroelastic model can be structural, that is, arising from the kinematical equations,\(^2\) physical, that is, those involving the constitutive equations,\(^2-4,7,9,10\) or aerodynamic appearing in the unsteady aerodynamic equations\(^4,7,9,11\). Their contribution can be beneficial (benign flutter boundary) or detrimental (catastrophic flutter boundary). A discussion of this issue in the context of the panel flutter may be found in Refs. 2–5.

The nature of the LCO, which provides important information on the behavior of the aeroelastic system in the vicinity of the flutter boundary, can be examined by the nature of the Hopf bifurcation\(^1\) of the associated nonlinear aeroelastic system.\(^3,13,14\) Figure 1 presents several pertinent scenarios; \(V = V_F\) defines the flutter boundary that can be determined via a linearized analysis. The nonlinear approach to the problem enables one to determine the aeroelastic behavior in the vicinity of the flutter boundary. As a result of the nonlinear analysis, one can determine the aeroelastic behavior for a flight speed lower than the flutter speed \(V_F\) (curve 1), that is, for \(V < V_F\), where a subcritical aeroelastic response is experienced. For \(V > V_F\), the system can exhibit either a stable LCO (supercritical Hopf bifurcation\(^1\) (H-B), curve 2), or an unstable LCO (subcritical H-B, curve 3).

In this paper, a general approach to the problem of the stability of the LCO of supersonic/hypersonic two-dimensional lifting surfaces is addressed. This methodology enables one to accomplish a parametric study over a large number of parameters that characterize the aeroelastic system.\(^1\) Literature dealing with the problem of the determination of the flutter boundary of a supersonic/hypersonic wing section and on the nature of the LCO in the presence of both structural and aerodynamic nonlinearities is quite scarce.\(^2,4\)

### Nonlinear Model of the Wing Section

#### Incorporating Active Control

The aeroelastic governing equations of controlled wing section featuring plunging and twisting degrees of freedom, elastically constrained by a linear translational spring and a nonlinear torsional spring exposed to a supersonic/hypersonic flow field are\(^18\)

\[
\begin{align*}
\dot{m}h(t) + c_hh(t) + K_hh(t) &= L_s(t) \\
S_h(t) + L_\alpha(t) + c_\alpha(t) + M_s &= M_\alpha(t) - M_C
\end{align*}
\]

where \(h(t)\) is the plunging displacement (positive downward), \(\alpha(t)\) is the pitch angle (positive nose up), and the superposed dots denote differentiation with respect to time \(t\). Moreover, in Eq. (2)

\[
M_s = K_s\alpha(t) + \delta_1K_\alpha\alpha(t)
\]

represents the overall nonlinear restoring moment that involves both the linear and the nonlinear stiffness coefficients, \(K_s\) and \(K_\alpha\), respectively. The tracer \(\delta_1\) in Eq. (3) can take the value 1 or 0 depending on whether the nonlinearity is included or ignored, respectively. Within a linear model (\(\delta_1 = 0\)), \(M_s\) reduces to \(K_s\alpha(t)\). The nonlinear coefficient \(K_\alpha\) in Eq. (3) can assume positive or negative values. Positive values of \(K_\alpha\) account for hard structural nonlinearities, whereas negative values of \(K_\alpha\) account for soft structural nonlinearities. (Notice that this nonlinearity appears in the present case in the equation relating the restoring moment with the pitch angle and that it has the character of a constitutive equation. For this reason, it would be more appropriate to refer to these as physical nonlinearities.)

The active nonlinear control can be represented in terms of the moment \(M_C\) in a similar functional form as

\[
M_C = f_1\alpha(t) + \delta_2f_2\alpha^2(t)
\]

In Eq. (4) \(f_1\) and \(f_2\) are the linear and nonlinear control gains, respectively. Within a linear active control methodology, the tracer assumes the value \(\delta_2 = 0\). Reducing the aeroelastic equations to dimensionless form, we define the parameter \(B\) that represents a measure of the degree of the structural nonlinearity of the system and two normalized linear and nonlinear control gain parameters \(\psi_1\) and \(\psi_2\), respectively, as

\[
\begin{align*}
B &= K_\alpha/K_s \\
\psi_1 &= f_1/K_s \\
\psi_2 &= f_2/K_s
\end{align*}
\]

Corresponding to \(B < 0\) or \(B > 0\), the structural nonlinearities are soft or hard, respectively, whereas for \(B = 0\), the system is structurally linear.

The nonlinear unsteady aerodynamic lift and moment, from piston theory aerodynamics (PTA), defines pressure on the upper and lower faces of the lifting surface as

\[
p(x, t) = p_\infty[1 + v_2(x - 1)/2d_\infty]^{2/(-1)}
\]

where \(d_\infty = \kappa p_\infty/\rho_\infty\); \(v_2\) is the...
downwash velocity normal to the lifting surface expressed as \( \alpha = (\partial \alpha/\partial t + U_\infty \partial \alpha/\partial x) \\) and \( \rho \) is the transversal displacement of the two-dimensional lifting surface, \( u(t) = h(t) + \alpha(t)x - x_{EA} \); and sign \( z \) is the sign distribution that assumes the values 1 or -1 for \( z > 0 \) and \( z < 0 \), respectively. In addition, \( x_{EA} = bx_{0} \) is the streamwise position of the pitch axis measured from the leading edge (positive aft). Retaining in the binomial expansions of \( p(x, t) \), the terms up to and including \( (v_{u}/a_{\infty})^{3} \), yields the pressure formula for the PTA in the third-order approximation \[ p/p_\infty = 1 + \kappa \gamma v_{u}/a_{\infty} + \kappa (k + 1)/4(y_{v}/a_{\infty})^{2} + \kappa (k + 1)/12(y_{v}/a_{\infty})^{3} \] (6) The aerodynamic correction factor \( \gamma = M_\infty \sqrt{(M_\infty^{2} - 1)} \) enables one to extend the applicability of the PTA to the low supersonic flight-speed range. Equation (6) is valid as long as the transformations through compression and expansion are considered to be isentropic (low-intensity waves). On the other hand, a more general pressure expression, obtained from the theory of oblique shock waves (SWT), that is valid over the entire supersonic/hypersonic range was obtained in Refs. 21 and 22, and it was used in aeroelastic analyses in Refs. 2-4. It is given by \[ p/p_\infty = 1 + \kappa \gamma v_{u}/a_{\infty} + \kappa (k + 1)/4(y_{v}/a_{\infty})^{2} + \kappa (k + 1)/32(y_{v}/a_{\infty})^{3} \] (7) With the exception of the cubic terms, Eqs. (6) and (7) resemble each other. This is explained by the entropy variation appearing to Eq. (6), Eq. (7) encompasses additional features in the sense of 1) taking into account shock losses that occur in the comparison of results showing the unstable and stable LCOs using attack (\( \alpha < 30 \text{ deg} \)) and Mach numbers \( \text{M} \geq 1.3 \) (Refs. 21 and 22), and it was used in aeroelastic analyses to extend the applicability of the PTA to the low supersonic range.

The final expressions can be cast in compact form as \[ L_{a}(t) = \int_{0}^{2\pi} \delta p(x - x_{EA}) \, dx \] (9a) \[ M_{a}(t) = - \int_{0}^{2\pi} \delta p(x - x_{EA}) \, dx \] (9b) The final expressions can be cast in compact form as \[ M_{a}(t) = (bU_\infty \rho_\infty/3M_\infty)y \left( \frac{1}{2}U_{\infty}^{2} (b - x_{EA}) \alpha(t) \right) \] (10a) \[ + \delta_{a} M_{\infty}^{2} U_{\infty} (b - x_{EA}) \left( 1 + \kappa \right) \gamma^{2} a^{2} \right) + 4 \left( b - x_{EA} \right) \hat{h}(t) + \left( 4b^{2} - 6bx_{EA} + 3x_{EA}^{2} \right) \hat{\alpha}(t) \right) \] (10b) where \( \delta_{a} \) is a tracer that is set equal to 1 if the aerodynamic nonlinearity is included or set equal to 0 if the aerodynamic nonlinearity is ignored. As a result, the governing equations (1) and (2) considered in conjunction with Eqs. (10a) and (10b) feature inertial and aerodynamic coupling. Using the dimensionless time \( \tau = U_\infty/x_{EA} \), the system of governing equations can be expressed as \[ \dot{\xi}(\tau) + x_{a} \alpha^{\prime}(\tau) + 2 \left( \dot{\omega}_{a}/V \right) \xi^{\prime}(\tau) + \left( \dot{\omega}_{a}/V \right) \dot{\xi}(\tau) = L_{a}(\tau) \] (11) \[ \left( x_{a}/r_{a}^{2} \right) \ddot{x}(\tau) + \alpha^{\prime}(\tau) + \left( 2 \dot{\omega}_{a}/V \right) \alpha^{\prime}(\tau) + 1/V^{2} \alpha(\tau) \] + \left( 1/V^{2} \right) \dot{B} \alpha(\tau) = m_{a}(\tau) - \psi_{1}/V^{2} \dot{\alpha}(\tau) - \psi_{2}/V^{2} \alpha^{\prime}(\tau) \] (12) In these equations, the dimensionless lift increment and moment are represented as \[ L_{a}(\tau) = - \left( \gamma / 12 \mu \right) M_{\infty} \left[ 12 \alpha(\tau) + \delta_{a} M_{\infty}^{2} (1 + \kappa) \gamma \alpha^{3}(\tau) \right] \] (13a) \[ m_{a}(\tau) = \left( \gamma / 12 \mu \right) M_{\infty} \left[ 12 \alpha(\tau) - \delta_{a} M_{\infty}^{2} (1 + \kappa) \gamma \alpha^{3}(\tau) \right] \] (13b) where \( \xi = h/b \) is the dimensionless plunging displacement and the primes denote differentiation with respect to dimensionless time \( \tau \).
When the procedure developed by Bautin and Lyapunov is used, pertinent conditions defining the character of the flutter boundary (benign or catastrophic), can be determined. These conditions are expressed in terms of the sign of the Lyapunov first quantity \( L(V_f) \) determined on the flutter boundary. Specifically, the inequalities \( L(V_f) < 0 \) and \( L(V_f) > 0 \) define the benign (supercritical) and catastrophic (subcritical) nature of the flutter boundary. The application of Bautin’s procedure requires that the characteristic equations obtained on the flutter boundary exhibit either one root or two roots that are purely imaginary. These conditions are equivalent to the H-B theorem.12

The Lyapunov first quantity \( L(V_f) \) corresponding to the nonlinear flutter of the wing section in a supersonic/hypersonic flowfield is derived next and is used to determine the conditions that characterize the nature of the flutter boundary.

The governing equations (11) and (12) are converted to a system of four differential equations in the form \( 2-4^{24} \)

\[
\frac{dx_j}{dt} = \sum_{m=1}^{4} a_{j}^{(m)} x_m + P_j(x_1, x_2, x_3, x_4), \quad j = 1, 4
\]  

(14)

The functions \( P_j(x_1, x_2, x_3, x_4) \) include the structural, aerodynamic, and nonlinear control terms that can be cast as

\[
P_j(x_1, x_2, x_3, x_4) = \sum_{i=1}^{4} a_{j}^{(i)} x_i^2 + 2 \sum_{i,j=1 \atop i \neq j}^{4} a_{j}^{(ij)} x_i x_j + \sum_{i=1}^{4} a_{j}^{(i)} x_i^3
\]

\[+ 3 \sum_{i=1}^{4} a_{j}^{(iii)} x_i^3 + 6 \sum_{i=1}^{4} a_{j}^{(ii)} x_i x_j x_k + 6 \sum_{i=1}^{4} a_{j}^{(i)} x_i^2 x_l + 6 \sum_{i=1}^{4} a_{j}^{(i)} x_i x_j x_k x_l + 6 \sum_{i=1}^{4} a_{j}^{(i)} x_i^2 x_j x_k x_l (15)

For the present case, Eqs. (14) and (15) reduce to a state-space form:

\[
\frac{dx_1}{dt} = x_3
\]

(16a)

\[
\frac{dx_2}{dt} = x_4
\]

(16b)

\[
\frac{dx_3}{dt} = a_3^{(1)} x_1(t) + a_3^{(2)} x_2(t) + a_3^{(3)} x_3(t) + a_3^{(4)} x_4(t)
\]

(16c)

\[
\frac{dx_4}{dt} = a_4^{(1)} x_1(t) + a_4^{(2)} x_2(t) + a_4^{(3)} x_3(t) + a_4^{(4)} x_4(t)
\]

(16d)

where

\[
\xi = x_1, \quad \alpha = x_2, \quad \xi' = x_3, \quad \alpha' = x_4
\]

(16c)

The linear active control appears in the coefficients \( a_3^{(2)} \) and \( a_4^{(2)} \), whereas nonlinear control terms are included in the terms accompanying the tracer \( \delta \). The coefficients of Eqs. (16) are given in Appendix A. When the streamwise position of the pitch axis coincides with that of the midchord, \( x_0 = 1 \), the nonlinear aerodynamic terms become negligible as the aerodynamic pitching moment vanishes.

When the solution of Eqs. (16) is considered in the form \( x_j = A_j \exp(\omega t) \), the characteristic equation corresponding to the linearized system obtained on the flutter boundary is

\[
\omega^2 + \rho^2 \omega \cos \psi + \omega \cos \psi + r + s = 0
\]  

(17)

where

\[
p = \{ V \gamma [4 + 3x_0^2 + 6x_0(x_0 - 1) - 6x_0] + 3\rho^2 (V \gamma + 2M_{\infty} \omega \delta \xi + M_{\infty} \mu \omega \xi) / M_{\infty} \nu \nu V (r_0^2 - \chi_0^2) \}
\]

(18a)

\[
q = \{ 3M_{\infty} \mu \nu \nu \nu [M_{\infty} \mu (1 + \omega \delta + \psi_0) + 2c_\nu (V \gamma + 2M_{\infty} \omega \xi)]
\]

\[
+ V \gamma (V (V + 3M_{\infty} \mu - 3M_{\infty} \mu \xi)
\]

\[+ 8M_{\infty} \mu \omega \xi + 6M_{\infty} \mu \omega x_0 \xi]
\]

\[- 3M_{\infty} \mu (V + 4\omega \xi)) / \]  

(18b)

Notice that this expression is general and includes the relationship between the flutter speed and the flutter frequency parameters evaluated on \( \Re = 0 \) in terms of the basic geometrical and flight parameters. In the particular case in which the structural damping ratios
in plunging and pitching are ignored and \( y \Rightarrow 1 \), the expressions of flutter frequency \( \chi_F \) and flutter speed \( V_F \) reduce to

\[
\chi_F = \left( \frac{\omega_0}{\omega} \right)_F = \left[ \frac{r_a^2}{\chi_F} (1 - x_0)^2 + \frac{1}{2} \right]^{-1}
\]

and

\[
V_F = \frac{U_F}{b_{oa}} = \frac{\mu M}{\chi_F} \sqrt{x_F^2 - (\hat{\omega}^2 \chi_F - 1)r_a^2 (\chi_F - 1)}
\]

respectively.

Equations (21) and (22) represent the dimensionless flutter frequency and flutter speed of the actively controlled system. These constitute the extension of the equations obtained by Ashley and Zartarian to include active control.

In Fig. 3, the dependence of the dimensionless flutter speed as a function of the Mach number for selected values of the feedback gain \( \psi_1 \) is presented. It clearly appears that, with the increase of the linear control gain, an increase of the flutter speed is experienced. Moreover, as values of \( \psi_1 \) are increased, the efficiency of the active control is increased.

The expression of the Lyapunov first quantity is given in closed form in Refs. 2 and 4. For the present case, the Lyapunov first quantity is expressed in terms of the coefficients \( A^{(i)}_{ii} \) as (Appendix B)

\[
L(V_F) = (3\pi/4c)(A^{(1)}_{111} + A^{(2)}_{222} + A^{(3)}_{122} + A^{(4)}_{112} + A^{(2)}_{212})
\]

The terms in the parentheses of Eq. (23) are expressed via the coefficients \( A^{(i)}_{ii} \) appearing in Eqs. (15) as

\[
A^{(i)}_{11} = (1/\Delta_0) \left[ \hat{g}_2(\chi_F) \frac{r_a^2 x_0}{V_F \chi_F} + \hat{g}_4(\chi_F) \frac{r_a^2 x_0}{V_F \chi_F} \right]
\]

whereas the coefficients of the system of Eq. (24) are

\[
A^{(3)}_{222} = -A^{(3)}_{212} = -A^{(3)}_{122} = (A^{(3)}_{112} + A^{(3)}_{212} + A^{(3)}_{222})
\]

The dimensionless structural, aerodynamic, and control nonlinearities are defined as

\[
S_{NL} = \frac{B}{V_F (r_a^2 - \chi_a)}
\]

\[
A_{NL} = \frac{M_{oa}(1 + \kappa)Y^3}{12\mu (r_a^2 - \chi_a)}
\]

\[
C_{NL} = \frac{\psi_2}{V_F (r_a^2 - \chi_a)}
\]

where \( V_F \) is the flutter speed. Upon defining \( \delta_p = |(\kappa)_p|^{-1} \) and having in view that, according to Eqs. (19b) \( c_o \Rightarrow 0 \), the Lyapunov first quantity reduces to

\[
L(V_F) = \left( \frac{A^{(3)}_{222} + A^{(3)}_{212} + A^{(3)}_{222}}{\chi} \right)^2 + \left( \frac{A^{(3)}_{222} + A^{(3)}_{212} + A^{(3)}_{222}}{\chi} \right)^2
\]

\[
+ \left( \frac{A^{(3)}_{222} + A^{(3)}_{212} + A^{(3)}_{222}}{\chi} \right)^2 + \left( \frac{A^{(3)}_{222} + A^{(3)}_{212} + A^{(3)}_{222}}{\chi} \right)^2
\]

(27)

When the expressions of \( S_{NL}, A_{NL} \), and \( C_{NL} \) [Eqs. (26)] are used and the procedure devised in Refs. 2-4 is applied, the benign or catastrophic character of the flutter boundary defined as \( L(V_F) < 0 \) or \( L(V_F) > 0 \), respectively. These conditions can be restated as

\[
V_F < V_r
\]

or

\[
V_F > V_r
\]

where

\[
V_r^2 = A_1/A_2
\]

In Eq. (28) the parameter \( A_1 \) includes the structural nonlinearities and the nonlinear control gain parameter, whereas \( A_2 \) includes the aerodynamic nonlinearities. Their expressions are provided in Appendix C.

In the absence of the nonlinear control and for \( B < 0 \) (soft structural nonlinearities), \( V_r \) is negative. The relation \( V_F > V_r \) corresponds to the catastrophic flutter boundary (unstable LCO) and occurs for any supersonic flight Mach number. For this case, an unstable LCO is experienced even in the presence of the linear control. On the other hand, for \( B > 0 \) (hard structural nonlinearities), the transition from benign to catastrophic flutter (from stable to the unstable LCO) occurs at an increased flight Mach number in the presence of the linear control.

As a special case, for \( x_0 = 1 \), Eq. (23) reduces to the following form:

\[
L(V_F) = \left( \frac{\delta_s S_{NL} + \delta_c C_{NL}}{\chi} \right) \left[ (3\chi_x - 3\chi_a)\alpha_{21}^3 + (3\chi_x - 3\chi_a)\alpha_{22}^3 + (3\chi_x - 3\chi_a)\alpha_{22}^3 \right]
\]

(29)

In this case, a decrease of the influence of the aerodynamic nonlinearities on the aeroelastic system is experienced.

Stability in the Presence of Active Control

To enhance understanding of the effect of the active control on the character of the flutter boundary, some explanations are provided in Figs. 4a and 4b. In Fig. 4a, the intersection point between the two curves \( V_F \) and \( V_r \) separates the benign flutter boundary (stable LCO) characterized by \( V_F < V_r \) from the catastrophic flutter boundary (unstable LCO) defined by \( V_F > V_r \). For these cases, a change in the sign of the Lyapunov first quantity \( L(V_F) \) occurs (Fig. 4b). In this context, the following four possible scenarios are distinguished: 1) for \( V < V_F \), as time unfolds, a decay of the motion amplitude
sponds to Fig. 4 Generic representation of boundary, reveals that aerodynamic quantity the namic nonlinearities are half-plane to the center is experienced, indicative of the presence of only Lyapunov first quantity for cases in which the soft structural nonlinearities only, the opposite situation is experienced. It is also shown that in the case of high Mach numbers, the neglect of nonlinear aerodynamic terms yields inadvertent predictions related to the character of the flutter boundary. Moreover, when the aerodynamic nonlinearities are discarded ($\delta_A = 0$) for any flight Mach number, the flutter boundary is benign or catastrophic, depending on whether hard ($B > 0$) or soft ($B < 0$) structural nonlinearities are present, respectively.

The influence of the hard structural nonlinearities, in conjunction with the aerodynamic nonlinearities ($\delta_A = 1$), for the controlled/uncontrolled system is presented in Fig. 6. The dotted lines identify the cases in which hard structural nonlinearity are included ($B = 50$), whereas the solid lines identify the cases in which the structural nonlinearities are ignored ($B = 0$). The control acts in both situations toward the stabilization of the system. Also, the unstable LCO that occurs when only the aerodynamic nonlinearities are considered can be converted to a stable LCO.

Figures 7 and 8 show that soft structural nonlinearities ($B = -10$) result in a catastrophic flutter boundary and that via a combined...
active control \((\psi_1 = 0.5, \psi_2 = 100\psi_1)\) the unstable LCO can become stable (Fig. 7). Moreover, it clearly appears that, when soft structural \((B < 0)\) and aerodynamic nonlinearities are present, the linear active control \((\forall \psi_1 > 0, \psi_2 = 0)\) cannot change the character of the flutter boundary (Fig. 8).

From Eq. (23), which defines the Lyapunov first quantity, the benign flutter boundary is expressed in closed form. When Eqs. (28a) and (28b) are used, the character of the flutter boundary is examined and has been plotted in Figs. 9–11. Each of Figs. 9–11 displays in the plane \((V, \lambda_{flight})\) the benign and catastrophic characters of the flutter boundary for the actively controlled wing section where \(\lambda = M_{\infty}/(\mu X \alpha'_{a})\). The corresponding Lyapunov first quantity is shown in Figs. 12–14 in the plane \((L, \lambda_{flight})\). In Figs. 9–15, the aerodynamic and hard structural nonlinearities have been included, \(\delta_4 = 1\) and \(\delta_5 = 1, B = 50\). In Figs. 9–15, the transition from catastrophic to benign flutter is shown. Figures 9–15 help one

\[ \psi_1 = 0.5, \psi_2 = 100\psi_1 \]

\[ \lambda = M_{\infty}/(\mu X \alpha'_{a}) \]

\[ \delta_4 = 1, \delta_5 = 1, B = 50 \]

\[ (V, \lambda_{flight}) \]

\[ (L, \lambda_{flight}) \]

\[ \psi_1 = 0.5, \psi_2 = 100\psi_1 \]

\[ M_{\infty}/(\mu X \alpha'_{a}) \]

\[ \delta_4 = 1, \delta_5 = 1, B = 50 \]

Fig. 9 Stable and unstable LCOs in the presence of linear control.

Fig. 10 Stable and unstable LCOs in the presence of nonlinear control.

Fig. 11 Stable and unstable LCOs in the presence of linear and nonlinear controls.

Fig. 12 Influence of the linear control on the Lyapunov first quantity \(L\).

Fig. 13 Influence of the nonlinear control on the Lyapunov first quantity \(L\).

Fig. 14 Influence of the linear and nonlinear controls on the Lyapunov first quantity \(L\).

Fig. 15 Stable and unstable LCOs with and without control in presence of both nonlinearities; comparison between SWT and PTA prediction.
to understand the behavior of the aeroelastic system in the presence of linear and nonlinear active control.

The stable and unstable characters of the LCOs via the use of the two different aerodynamic theories (PTA and SWT) are presented in Fig. 15. From Fig. 15 and Table 2, one can infer that the transition from the stable LCO to the unstable LCO occurs at slightly lower Mach numbers (less than 3%) for the PTA as compared to those predicted by the SWT. This implies that the PTA provides conservative results as compared to the SWT.

Conclusions

An original LCO analysis was presented. In contradistinction with the ones generally used in the literature, where the character of the flutter boundary is determined from the path variation amplitude of displacement quantities, in the present approach this information is obtained via the Lyapunov first quantity.

It was shown that in some circumstances, the aerodynamic and hard structural nonlinearities contribute in different ways to the determination of stable or unstable LCOs. It was shown that the hard structural nonlinearities result in a stable LCO, whereas the soft structural nonlinearities result in an unstable LCO. At high flight Mach numbers, the aerodynamic nonlinearities are primary contributors to the destabilization of the aeroelastic system. This implies that with increasing the hypersonic speed, when the aerodynamic nonlinearities become prevalent, an unstable LCO occurs irrespective of the presence of hard structural nonlinearities. On the other hand, soft structural nonlinearities ($B < 0$) contribute in the same sense, as the aerodynamic nonlinearities, to the unstable LCO. It is also shown that active control can be used to increase the flutter speed and to convert the catastrophic flutter boundary into a benign flutter boundary and/or to shift the transition between these two states toward higher flight Mach numbers.

The issue of generating the active control moment was not addressed. It is the authors' belief that this can be produced via a device operating similarly to a spring, whose linear and nonlinear characteristics can be controlled, but additional analysis are required to confirm this observation.

Appendix A: Coefficients Occurring in Eqs. (16)

The coefficients of the aeroelastic system represented in state-space form [Eqs. (16)]:

\[
\delta \phi_{a(3)} = \frac{M}{M_0} \chi \frac{r^2}{(r^2 - x^2)}
\]

\[
\delta \phi_{a(4)} = \frac{V}{M} \left( 3x^2 - 6x + 4 \right) x_0 + 3(x_0 - 1) x_0^2 + 6M_0 \mu_0 \chi \frac{r^2}{(r^2 - x^2)}
\]

where the adjoint of a matrix is defined as $\Delta_0 \left[ a_{ij} \right]^{-1}$ and $\Delta_0 = \|a_{ij}\|$ is the determinant of the matrix of coefficient $a_{ij}$. In addition, the parameters $a_{ij}$ are

\[
\alpha_{12} = \frac{a_{12}^{(2)} a_{14}^{(2)} + c a_{13}^{(2)} a_{14}^{(3)}}{a_{14}^{(3)}}
\]

\[
\alpha_{21} = \frac{a_{13}^{(3)} a_{14}^{(3)} - c^2 a_{14}^{(2)}}{a_{14}^{(3)}}
\]

\[
\alpha_{22} = \frac{a_{13}^{(3)} a_{14}^{(3)} + c a_{12}^{(3)} a_{14}^{(2)}}{a_{14}^{(3)}}
\]

Appendix B: Coefficients $A_{13}^{(i)}$ Intervening in Eq. (23)

The various elements of Eq. (24), evaluated on the instability boundary, are expressed as

\[
A_{11}^{(1)} = (1 + \Delta_0) \left( a_{13}^{(2)} a_{23}^{(3)} + a_{14}^{(3)} a_{24}^{(3)} \right)^{a_1}
\]

\[
A_{12}^{(2)} = (1 + \Delta_0) \left( a_{13}^{(3)} a_{23}^{(3)} + a_{14}^{(3)} a_{24}^{(3)} \right)^{a_2}
\]

\[
A_{11}^{(3)} = (1 + \Delta_0) \left( a_{13}^{(3)} a_{24}^{(3)} + a_{14}^{(3)} a_{22}^{(3)} \right) a_{12}^{a_2}
\]

\[
A_{12}^{(4)} = (1 + \Delta_0) \left( a_{13}^{(3)} a_{23}^{(3)} + a_{14}^{(3)} a_{24}^{(3)} \right) a_{12}^{a_2}
\]
Appendix C: Coefficients $A_1$ and $A_2$

Appearing in Eq. (28c)

$$A_1 = \frac{\delta_3 B + \delta_6 \psi_2 r^3}{r^2 - \chi_a} \left[ (3_{13} \chi_a - 3_{14} \alpha_2^2 + (3_{23} \chi_a - 3_{24} \alpha_2^3 \right)$$

$$+ (3_{33} \chi_a - 3_{34} \alpha_2^3 \alpha_2^2 + 3_{13} \chi_a - 3_{14} \alpha_2^3 \alpha_2^2 \right] \right] (C1)$$

$$A_2 = \frac{\delta_4 M_0 (1 + \kappa) r^3}{12 \pi (r^2 - \chi_a)} \left\{ 3_{13} \left[ -X_0 + \alpha_2^3 \right]$$

$$- 3_{14} (X_0 - 1 + \chi_a) \alpha_2^2 + 3_{24} (X_0 - 1 + \chi_a) \alpha_2^3$$

$$- 3_{24} (X_0 - 1 + \chi_a) \alpha_2^3 \alpha_2^2 + 3_{13} [X_0 - 1 + \chi_a + r_0^2]$$

$$- 3_{14} (X_0 - 1 + \chi_a) \alpha_2^3 \alpha_2^2 \right\} \right\} (C2)$$

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