Abstract: Small but macroscopic particles - chondrules, higher temperature mineral inclusions, metal grains, and their like - dominate the fabric of primitive meteorites. The properties of these constituents, and their relationship to the fine dust grains which surround them, suggest that they led an extended existence in a gaseous protoplanetary nebula prior to their incorporation into their parent primitive bodies. In this paper we explore in some detail the velocities acquired by such particles in a turbulent nebula. We treat velocities in inertial space (relevant to diffusion), velocities relative to the gas and entrained microscopic dust (relevant to accretion of dust rims), and velocities relative to each other (relevant to collisions). We extend previous work by presenting explicit, closed-form solutions for the magnitude and size dependence of these velocities in this important particle size regime, and compare these expressions with new numerical calculations. The magnitude and size dependence of these velocities have immediate applications to chondrule and CAI rimming by fine dust, and to their diffusion in the nebula, which we explore separately.

1 Background

The fabric of the most primitive meteorites undoubtedly contains many clues as to their origin. While most chondrites are samples of surfaces that have been well worked over by impacts and stirring ("regolith breccias"), the dominance of chondrules and like-sized objects remains clear. How it came about that most chondrite parent bodies are so dominated by particles with such a well-defined range of physical, chemical, and petrographic properties remains one of the big puzzles of meteoritics. Since there are relatively few examples of anything larger than 0.1-10 mm size particles in most primitive planetesimals, the way such particles interact with the gaseous nebula is of prime importance.

Fe-Mg-Si-O mineral chondrules, which solidified from a melt, constitute 30-80% of primitive meteorites. There are a number of extant hypotheses for the formation of the chondrules. Most workers in the field believe that chondrules are formed by either localized or nebula scale energetic events operating on freely floating precursors of comparable mass, at some location or locations in the protoplanetary nebula. However, some still maintain they are made in or on primitive bodies, or in collisions between them. In a hybrid scenario, some suggest they are formed in shock waves generated by already-formed planetesimals, and thus that they are a secondary phenomenon to primary accretion of planetesimals. See eg. Grossman (1989), Grossman et al. (1989), Boss (1996), Connolly and Love (1999), and Jones et al. (2000) for reviews of hypotheses on this long-controversial and perennially fascinating subject.
Another meteorite constituent of great interest are the mineral grains called Ca-Al-rich refractory inclusions (CAIs) - so called because their constituent minerals condense out of nebula gas at a much higher temperature than do chondrules. These objects are widely believed to be direct nebula condensates, and have a complex subsequent thermal history which has some similarities to that of chondrules and some differences. There is some indication from radioisotope ages that CAIs might be $\sim 10^6$ years older than the chondrules, but this remains slightly controversial. They make up 1-10% of primitive meteorites depending on type, and their size distribution is broader than that of the chondrules. How these high-temperature minerals find themselves intimately mixed with lower-temperature minerals remains a puzzle.

It remains unresolved at this time whether the nebula gas was turbulent or laminar during the chondrule era. In previous papers, we have suggested that some of the observed properties of chondrules themselves - their typical size and size distribution - can be associated with, and easily explained by, the effects of weak nebula turbulence (Cuzzi et al 1996, 2001). Nevertheless, a consistent end-to-end scenario for formation of primitive bodies in this environment, and relying on these processes, is not yet in hand. In this paper, we focus on the velocity evolution of this specific class of particles in a weakly turbulent nebula as a step towards developing a more complete scenario that operates to produce primitive bodies in a similar way across a variety of environments. The velocity evolution is critical for our understanding of several important aspects of chondrules and chondrites: (a) the radial distribution and redistribution or transport of chondrules and/or CAIs, once formed, before their accumulation into parent bodies; (b) The presence of fine grained rims on chondrules, CAIs, and other coarse particles in primitive chondrites (Metzler and Bischoff 1996, Brearley and Jones 1998); and (c) collision rates and velocities between chondrule-sized particles. The main goal of this paper is to provide a theoretical framework within which we can better understand mm-to-cm-size particle evolution in general. We accomplish this in sections 1 (analytical theory) and 2 (supporting numerical calculations). In another paper we apply these results to diffusion and dust rimming (Cuzzi 2002b).

1.1 Particle Velocities in Turbulence

Astrophysical modeling of the basic physics of particle behavior in fluid flows, laminar or turbulent, tends to begin and end with the classic papers by Whipple (1973), Adachi et al. (1976), Weidenschilling (1977, 1980), and Völk et al. (1980, henceforth VJMR; also Völk et al. 1978), with important recent updates by Markiewicz et al. (1991; henceforth MMV). In the fluid dynamics literature, however, the study of particle motions in fluid flows has both a long history, and a robust ongoing presence. This history is nicely summarized by Meek and Jones (1973). More recent work in the fluids literature is noted in various relevant places below. VJMR first developed a useful formalism for calculating the dispersion velocities $V_p$ (relative to inertial space) and collision velocities $V_{pp}$ (relative to each other) of particles in a turbulent nebula. They circumvented the thorny problem of “essential nonlinearity” (cf. Meek and Jones 1973) by translating clever physical insights into mathematics and adopting a velocity autocorrelation function approach, which we discuss in more detail below. While it serves an important internal role in their solutions, neither VJMR nor MMV say much about the relative velocity between particles and gas, $V_{pg}$. Yet, $V_{pg}$ is the determinant quantity for
accretion of rims of fine dust grains by small, macroscopic objects (Paque and Cuzzi 1997, Cuzzi et al. 1998, Morfill et al. 1998). Our goal in this paper is to quantify $V_p$, $V_{pg}$, and $V_{pp}$ for such particles in a way that extends and focuses the formulation of VJMR and MMV, and which allows insights to be gained into the history of chondrules and like-sized particles in the protoplanetary nebula.

In this paper, we determine velocities of all three kinds - $V_p$, $V_{pg}$, and $V_{pp}$ - with emphasis on particles having stopping times $t_s$ comparable to the overturn time $t_\eta$ of Kolmogorov scale eddies. Particles in this size regime have behavior more complex than tiny "dust" grains, which are essentially trapped to the gas flow on all scales. In particular, particles with $t_s = t_\eta$ are subject to "preferential concentration" by large factors in turbulence, and based on some of its apparent fingerprints in the meteorite record, we have suggested a link between this process, chondrules, and primary accretion. Specifically, we refer to the fact that the typical size and the shape of the size distribution of chondrules are readily explained by turbulent concentration (Cuzzi et al. 1996, 2001). In a parallel paper (Cuzzi 2002b) we explore the possibility that the functional form of $V_{pg}$ might reveal another fingerprint of turbulent concentration, and that turbulence might help us understand the puzzling mix of CAIs and chondrules in the same meteorites.

Particles are aerodynamically classified by their Stokes number $St$, the ratio of their stopping time $t_s$ to the overturn time of some characteristic eddy. We will make use of Stokes numbers defined relative to two different eddy overturn timescales: the Stokes number relative to the largest, or integral scale eddy time $t_L$: $St_L = t_s/t_L$, and that defined relative to the smallest, or Kolmogorov scale eddy time $t_\eta$: $St_\eta = t_s/t_\eta$. The overturn time of the largest scale eddy $t_L$ is generally regarded as the local orbit period. Preferentially concentrated particles (chondrules, we believe) have $St_\eta = 1$ and $St_L \ll 1$. For these particles, which are smaller than the gas molecular mean free path, the stopping time $t_s = \rho_s/c_p$, where $r$ is particle radius, $\rho_s$ is particle material density, $c$ is the nebula sound speed, and $\rho_g$ is the nebula gas density (Weidenschilling 1977). That is, $t_s$ and thus both $St_L$ and $St_\eta$ are linearly proportional to particle radius.

1.2 Previous work; the autocorrelation function

We briefly review and simplify the notation of VJMR and MMV. VJMR assumed a fully developed inertial range of turbulence with some largest, or integral scale $L$ and zero smallest scale. MMV also adopted the Kolmogorov energy spectrum (as shall we) but correctly pointed out that turbulence ceases for scales smaller than the Kolmogorov or inner scale $\eta$. Especially for small particles in the chondrule-and-CAI size range, MMV point out that this has important implications for $V_p$ and $V_{pp}$, and we will show that the implications are important for $V_{pg}$ as well. In a Kolmogorov spectrum, an inertial range of turbulent gas kinetic energy extends from the largest or integral scale $l = L$ to the smallest or Kolmogorov scale $l = \eta$. Following VJMR, we work in the spatial frequency regime, where $k(l) = 2\pi/l$ and $E(k) = E_L(k/k_L)^{-5/3}$ for the Kolmogorov spectrum (note our $E(k)$ is a true energy, and is half of VJMR's $P(k)$). Then $v(k) = (2kE(k))^{1/2}$ and $t(k) = 1/(kv(k)) = t_L(k/k_L)^{-2/3}$. As did MMV, we assume $E(k) = 0$ for $k > k_\eta$ (no turbulent energy at scales smaller than the Kolmogorov scale). The mean square turbulent (fluctuating) gas velocity is $V_g$; thus the typical turbulent kinetic energy per unit gas mass is $V_g^2/2$, providing the normalization
The turbulent gas motions induce fluctuating velocities in the particle population, leading to diffusion \( V_p \), mutual collisions \( V_{pp} \), and motion relative to the local gas \( V_{pg} \).

VJMR derive \( V_p \) formally by a backwards time integration of the instantaneous acceleration (their equations 5 and 6):

\[
V_p(t) = t_s^{-1} \int_0^t \exp\left(-\frac{(t - t')}{t_s}\right) V_g(t') dt'
\]

where \( V_g(t') \) represents the fluctuating gas velocity history along a particle trajectory (formally unknown at this point). They proceed by approximating \( V_g(t') \) as an integral over all (independently acting) spatial frequencies \( k \) with eddy timescales \( t_k \), and approximate the contributions as coming from two classes of eddies: "class 1" eddies, with overturn times long enough \( (t_k > t_s) \) that particles are always in equilibrium within them, and are primarily just advected by their (temporally fluctuating) motions, and "class 3" eddies with overturn times too short \( (t_k < t_s) \) for the particle to come to equilibrium as it passes through them. Intermediate, or what might be "class 2" eddies are not treated separately, but simply absorbed into the classes on either side. Different simplifications are allowed for each class. The boundary between eddy classes 1 and 3 is \( k^* \), where \( t_{k^*} = t_s \). VJMR show that the class 3 (small, fast) eddies are negligible for velocity components \( V_p \) and \( V_{pg} \), but dominate the contributions to \( V_{pp} \). We will make use of these results below.

VJMR first obtain the product \( \langle V_p(t) V_p(t) \rangle = \langle V_p^2 \rangle \) by integrating backwards over two separate time histories. They introduce the gas velocity autocorrelation function for gas velocities (in their equation 16) \( R(t, t'; k) = \exp\left(-|t - t'|/t_k\right) \) Even though they don't make this distinction, the autocorrelation function to be used in this way is properly that determined along a particle trajectory (Batchelor 1948, Hinze 1975, Squires and Eaton 1990, Elghobashi 1991), and is thus a function of \( t_s \) in general. However, for \( St_L \ll 1 \), and at this stage of our knowledge, this distinction is not significant (Squires 1990).

Subsequently, MMV suggested a more general, even if ad hoc, functional form for \( R(t, t'; k) \):

\[
R(t, t'; k) = \left(1 + \frac{|t - t'|}{t_k}\right)^n e^{-|t - t'|/t_k},
\]

with \( n = 0 \) or 1. They note that the \( n = 1 \) case has more plausible physical behavior (zero slope) near \( t = t' \) than the \( n = 0 \) (pure exponential) form assumed by VJMR.

### 1.2.1 New results regarding the form of the autocorrelation function, and the value of \( n \):

The selection of \( n = (0, 1) \) determines the form of the fluid velocity autocorrelation function \( R(t, t'; k) \). Squires (1990) measured this function directly in his direct numerical simulations of turbulence, by following fluid motions along the trajectories of a number of particles with different \( St_L \). In figure 1 we compare the results of Squires (1990) with the predictions based on the \( n = 0 \) and \( n = 1 \) expressions of MMV for \( R(t, t'; k) \). Note that, since MMV
express their autocorrelation function as a function of \(k\), it must be integrated over an energy spectrum to compare with the numerical results of Squires (1990). Because Squires (1990) only calculated a 1-D autocorrelation function (i.e., using only one velocity component), we integrated the \(R(t,t';k)\) of MMV over a 1D energy spectrum (essentially, one-third of the total \(E(k)\)) (see also Squires and Eaton 1991). It is clear from figure 1 that \(n = 1\) is the better choice. This has important implications, primarily for \(V_{pg}\) and \(\langle \hat{p} \rangle\). In section 2, we directly compare \(V_p\) and \(V_{pg}\) calculated in full 3D turbulence using the two alternate autocorrelation functions, and again reach the same conclusion.

1.3 Particle random velocities relative to inertial space

After some algebra, VJMR derive an expression (their equation 18) for the mean square particle fluctuating velocity \(V_p\), of which we need only the large, slow (class 1) eddy contribution since the small eddy contribution is negligible for \(St_\eta = 1\) particles (we will henceforth drop the \(< >\) notation on \(V_p, V_{pg}, V_{pg}\), and \(\langle \hat{p} \rangle\), and will merely recall that all are statistical expectation values based on extensive temporal or spatial averaging). Because of our emphasis on particles with \(St_\eta = 1\), we also replace the upper limit of VJMR's class 1 integral \((k^*)\) with the Kolmogorov scale \(k_\eta\). This simplification is, in fact, actually fairly good over the entire range of \(St_L \ll 1\), precisely because the contribution of eddies on smaller scales than \(k^*\) (the class 3 eddies) is negligible. That is, the upper limit can be extended from \(k^*\) to \(k_\eta\) in general for mathematical simplicity without incurring significant error. Mathematically,
the upper limit could even be extended to infinity (e.g., Völk et al. 1980), but the important role of the Reynolds number and of the Kolmogorov scale is then lost. Thus,

$$V_p^2 \approx 2 \int_{k_L}^{k_g} E(k) \frac{t_k}{t_k + t_s} dk. \quad (4)$$

Similarly, the generalized MMV expression for $V_p^2$ (their equation 6) can be simplified to

$$V_p^2 \approx 2 \int_{k_L}^{k_g} E(k) \left( 1 - \left( \frac{t_s}{t_k + t_s} \right)^{n+1} \right) dk = 2 \int_{k_L}^{k_g} E(k) \left( 1 - \left( \frac{1}{1 + t_k/t_s} \right)^{n+1} \right) dk \quad (5)$$

for the particle size regime of interest here. As did VJMR, MMV note that the second integral of their equation (6) - the class 3 eddy contribution - is negligible for small particles, so we retain only the first integral of their equation (6). We again simplify the upper limit of integration in the remaining integral for the nominal $St_\eta \approx 1$ case where $k^* \approx k_\eta >> k_L$. We validate this by comparing our results with those of MMV (section 1.7).

The result for $V_p^2$ was plotted, but not stated explicitly, by VJMR and MMV (figure 1 in both papers), and explicitly derived by Cuzzi et al. (1993; Appendix B): $V_p^2 = V_g^2/(1 + St_L)$. It is simple to see why $V_p^2 \approx V_g^2$ in the limit $St_L << 1$ and certainly for $St_\eta \approx 1$, since $t_s << t_k$ in equations (4) or (5) for nearly all $k$ and overwhelmingly all $E(k)$. This limit is appropriate for chondrule-and-CAI-sized particles even in the presence of their small vertical settling velocity - they diffuse nearly as well as a gas molecule, and do not “settle to the midplane” in even a very weakly turbulent nebula (Dubrulle et al. 1995, Cuzzi et al. 1996). The implications are discussed in section 3. However, $V_p^2$ and $V_g^2$ are not exactly equal, resulting in a small, but very important, relative energy of motion $V_{pg}^2$, giving the velocity with which particles move through the gas and encounter tiny (micron-sized) dust grains.

1.4 Particle velocities relative to the gas

The average relative velocity magnitude between a particle and the turbulent gas is $V_{pg}$. VJMR make use only of the spatial frequency components of this quantity, which they refer to as $V_{rel}(k)$ (their equation 15). Practically speaking, however, a particle will instantaneously sense all eddy contributions as one $V_{pg}$; we obtain this by merely integrating VJMR equation (15) over $k$. Considering only the part of the expression relevant for $St_\eta \approx 1$ (that for $k^* > k_L$), neglecting any systematic velocity, and again letting $k^* \approx k_\eta >> k_L$, the second integral vanishes and we obtain

$$V_{pg}^2 \approx 2 \int_{k_L}^{k_g} E(k) \left( \frac{t_s}{t_k + t_s} \right) dk. \quad (6)$$

For this $n = 0$ case treated by VJMR, it can be easily verified using equations (4) and (6) that

$$V_{pg}^2 + V_p^2 = 2 \int_{k_L}^{k_g} E(k) dk = V_g^2. \quad (7)$$

However, this useful result is true independent of $n$. It may also be obtained by Fourier transform solution of the forcing equations in temporal frequency ($\omega$) space, where the
energy spectrum of gas velocity fluctuations $E_g(\omega)$, particle velocity fluctuations $E_p(\omega)$, and relative velocity fluctuations $E_{pg}(\omega)$ are related by

$$E_p(\omega) = E_g(\omega)/(1 + t_s^2 \omega^2) \quad \text{and} \quad E_{pg}(\omega) = t_s^2 \omega^2 E_p(\omega). \quad (8)$$

This approach can be traced to Csanady (1963); it is also described by Hinze (1975, chapter 5), Meek and Jones (1973), and Squires (1990, sections 4.2 and 4.5.1). The $E_p$ solution was also derived in this way by Cuzzi et al. (1993, Appendix B). It is also clear then that $E_{pg}(\omega) + E_p(\omega) = E_g(\omega)$, essentially the same result as equation (7) above. Finally, we have directly verified equation (7) in our numerical simulations.

Using this general relationship, we can extend the results of MMV to obtain $V_{p}^2$ for their more generalized gas velocity autocorrelation functions (they only present results for $V_p^2$). That is, using equations (1) and (5),

$$V_{pg}^2 = V_g^2 - V_p^2 = 2 \int_{k_L}^{k_U} E(k) \left(1 + \frac{t_s}{t_s + t_k} \right)^{n+1} dk = 2 \int_{k_L}^{k_U} E(k) \left(1 + \frac{1}{1 + t_k/t_s} \right)^{n+1} dk. \quad (9)$$

We will use equations (5) and (9), with assumed inertial range expressions for $E(k)$, to derive analytical expressions for $V_p$ and $V_{pg}$ of hypothetically “chondrule-like” (i.e., $St_\eta \approx 1$) particles as functions of their size and the turbulent Reynolds number.

### 1.5 Relative velocities between particles of similar sizes

Expressions for $V_{pp}$ (VJMR Appendix C and equation 19; MMV equations 7 and 8) are more cumbersome, but respond nicely to certain simplifying assumptions. The full expression for $V_{pp}$ for two particles of equal size is (changing notation slightly from MMV equation 9, and allowing for a finite Kolmogorov scale):

$$V_{pp}^2 = 4 \int_{k_*}^{k_U} E(k) \left(1 - \frac{t_s}{t_s + t_k} \right) \left[ g(\chi) + \frac{n t_s h(\chi)}{t_s + t_k} \right] dk, \quad (10)$$

where $g(\chi) = \tan^{-1}(\chi)/\chi$ and $h(\chi) = 1/(1 + \chi^2)$. The parameter $\chi$ of VJMR and MMV is small in our regime of interest:

$$\chi = \frac{V_{rel}(k) t_s (t_K)}{t_s + t_k} = \frac{V_{rel}(k) t_s}{v(k)(t_s + t_k)} \approx \frac{V_{rel}(k)}{2v(k)} < 1, \quad (11)$$

since in the very limited range of $k$ over which the integral is done, $t_s \approx t_k$. In fact $\chi << 1$ over most of the integral where $t_s << t_k$, so the functions $g(\chi)$ and $h(\chi)$ are $\approx 1$ or perhaps as small as a fraction of order unity; thus

$$V_{pp}^2 \approx 4 \int_{k_*}^{k_U} E(k) \left[1 - \left(\frac{t_s}{t_s + t_k} \right)^{n+1} \right] dk = 4 \int_{k_*}^{k_U} E(k) \left[1 - \left(\frac{1}{1 + t_k/t_s} \right)^{n+1} \right] dk. \quad (12)$$

The integrand is identical to that for $V_p^2$, but the integral has different limits which make it clear that only the eddies faster than $t_s$ can perturb identical particles into having incoherent relative velocities.

$^1$In the above equation, the mathematical generalization of $V_{pg}$ by VJMR and MMV to its $k$-th components $V_{rel}(k)$ momentarily reappears. However, it is true in general, at any spatial frequency, that the particle-gas relative velocity is less than, or at most equal to, the gas velocity itself.
1.6 Scaling relations

Recall that for the gas,

\[ t_k = l(k)/v(k) = (L/V_g)(k/k_L)^{-2/3} = t_L(k/k_L)^{-2/3} \]  

(Cuzzi et al. 2001). In equation (13) we have made the usual identification of \( V_g \) with the largest scale eddy \( L \). For the particles,

\[ \frac{t_s}{t_L} = St_L = (k_s/k_L)^{-2/3} \] (14)

and

\[ \frac{t_s}{t_k} = \frac{k_s}{k_L} = \frac{t_s}{t_L} = (k_s/k_L)^{-2/3} = St_L(k/k_L)^{-2/3} \] (15)

Note that if we restrict our attention to particles with \( St_\eta = t_s/t_\eta \approx 1 \), then their Stokes number referred to the integral scale automatically becomes

\[ St_L = t_s/t_L = t_\eta/t_L = (k_\eta/k_L)^{-2/3} = (Re^{3/4})^{-2/3} = Re^{-1/2}. \] (16)

The last substitution of \( (k_\eta/k_L) = Re^{3/4} \), where \( Re = LV_g/\nu \) is the flow Reynolds number, with \( \nu \) being the molecular kinematic viscosity, is a direct consequence of the definitions of the Kolmogorov scale, the energy dissipation rate, and the Reynolds number (Tennekes and Lumley 1972). This relation can be obtained without any reference at all to the Kolmogorov spectrum but merely using scaling arguments relating to \( t_L \) and \( t_\eta \). \( Re \) is related to astrophysical "\( \alpha \)"-models of the protoplanetary nebula by \( Re = \alpha cH/\nu \) with \( c \) = sound speed and \( H \) = nebula vertical scale height (Cuzzi et al. 2001).

1.7 Final expressions for \( V_{pg} \) and \( V_{pp} \)

Substituting the scaling relations from above for \( t_s/t_k \), equation (9) for \( V_{pg} \) becomes

\[ V_{pg}^2 = 2 \int_{k_L}^{k_s} E(k) \left( \frac{1}{1 + t_k/t_s} \right)^{n+1} dk = 2 \int_{k_L}^{k_s} E(k) \left( \frac{St_L}{St_L + (k/k_L)^{-2/3}} \right)^{n+1} dk. \] (17)

We use the normalization (equation 1) to write \( E(k) = (V_g^2/3k_L)(k/k_L)^{-5/3} \), and change integration variable to \( x = k/k_L \), leaving

\[ V_{pg}^2 = \frac{2V_g^2}{3} \int_1^{Re^{3/4}} \left( \frac{St_L}{St_L + x^{-2/3}} \right)^{n+1} x^{-5/3} dx. \] (18)

where in the upper limit we have substituted \( k_\eta/k_L = Re^{3/4} \) from the scaling relations. Closed form solutions for equation (18) can be obtained for \( n = 0 \) or \( 1 \). For example, for \( n = 1 \) the result of the integral is

\[ V_{pg}^2 = V_g^2 \left[ \frac{St_L}{1 + St_L^{2/3}} \right]_{Re^{3/4}}^1 \]

\[ = V_g^2 \left[ \frac{St_L^2(Re^{1/2} - 1)}{(St_L + 1)(St_L Re^{1/2} + 1)} \right]. \] (19)

\(^2\)Let the energy dissipation rate be \( \epsilon \). Then \( \epsilon = V_g^2/t_L = V_g^3/L \) where the first expression defines \( t_L \) and the last expression defines \( L \). Also \( t_\eta = (v/\epsilon)^{1/2} \) and \( \eta = (v^3/\epsilon)^{1/4} \). Solving gives \( t_L/t_\eta = Re^{1/2} \) and \( \eta/L = Re^{3/4} \).
For \( n = 0 \) the result of the integral is:

\[
V_{pg}^2 = V_g^2 \left[ St_L \ln \left( \frac{Re^{1/2}(1 + St_L)}{Re^{1/2}St_L + 1} \right) \right]. \tag{20}
\]

These results make it quite easy to predict both the magnitude and the \( St \) dependence of \( V_{pg} \) for arbitrary nebula turbulent intensity.

We solve equation (12) for \( V_{pp} \) in a similar fashion to the solution for \( V_{pg} \) above, to obtain

for \( n = 1 \):

\[
V_{pp}^2 = \frac{4V_g^2}{3} \int_{k(t_*)/k_L}^{k_{**}/k_L} \left( \frac{2St_L x^{-7/3} + x^{-9/3}}{St_L^2 + 2St_L x^{-2/3} + x^{-4/3}} \right) dx. \tag{21}
\]

As before, the upper integration limit is \( k_{**}/k_L = Re^{3/4} \). For the lower limit, \( k_{**}/k_L = k(t_*)/k_L = (t_*/t_L)^{-3/2} = St_L^{-3/2} \) from the scaling relations. The closed form analytic solution of this integral is:

\[
V_{pp}^2 = 2V_g^2 \left[ \frac{x^{-2/3}}{1 + St_L x^{2/3}} \right]_{Re^{3/4}}^{St_L^{-3/2}} = 2V_g^2 \left[ \frac{St_L}{2} - \frac{1}{St_L Re + Re^{1/2}} \right]. \tag{22}
\]

The \( n = 0 \) form of the solution is somewhat less useful, and we note it without expanding it as it will not be used further.

\[
V_{pp}^2 = 2V_g^2 \left[ St_L \ln \left( \frac{1 + St_L x^{2/3}}{x^{2/3}} \right) - \frac{1}{x^{2/3}} \right]_{St_L^{-3/2}}^{Re^{3/4}}.
\]

1.8 Detailed comparisons with the models of Markiewicz et al.

In addition to developing the analytical expressions discussed and applied in the paper, we also developed a detailed numerical model following the prescriptions of MMV exactly (but with a generalized turbulent energy spectrum). This was needed both to evaluate their theoretical approach in the context of our numerical simulations of turbulence (section 2), which have a non-Kolmogorov spectrum and low Reynolds number compared to nebula applications, and to assess the validity of our analytical approximations. The numerical model of MMV is no longer in active use (W. Markiewicz, personal communication 2002), so we digitized their \( V_{pp} \) results (their figure 5) to facilitate comparisons. As seen in figure 4, our full numerical model for \( V_{pp} \) (solid curves) agrees very well with their results for \( V_{pp} \) (long dashed curves). In figure 2 we also show our results for \( V_{pg} \), not presented by VJMR or MMV, as obtained by integrating MMV equation 4 over all spatial frequencies. Note that we, and MMV, both use the appropriate form of \( R(t, t'; k) \) (i.e., that for the correct choice of \( n \); section 1.2) for these calculations.

The most striking feature of the results, first noted by MMV, is that \( V_{pp} \) very quickly falls to zero for particles with \( St_{\eta} < 1 \) (i.e. \( St_L < Re^{-1/2} \), as shown in the scaling relations of section 1.6 above) because there is no more energy in faster eddies to provide relative velocities to such particles. This does not happen to \( V_{pg} \), because eddies on all scales contribute. Also note that \( V_p \) and \( V_{pp} \) decrease for large particles \( (St_L > 1) \), as fewer eddies can effectively
Figure 2: Comparison of our numerical version of the full MMV model for $n = 0$ (light curves) and $n = 1$ (heavy curves), along with digitized results for $V_{pp}$ from MMV (dashed curves, their figure 5, for $n = 1$). Three different $Re$ are shown: (a) $10^4$, (b) $10^7$, and (c) $10^9$. The dash-dot curves are for $V_p$, which has the same shape for all three $Re$. $V_{pg}$ is shown in the two sets of dotted curves and $V_{pp}$ in the two sets of solid curves. Note that the $n = 0$ values of $V_{pg}$ (light dotted curves) are considerably (3-4 times) higher than the preferred $n = 1$ values (heavy dotted curves), and the $St_L$-dependence of $V_{pg}$, for $n = 0$, never gets much above 0.5, whereas for $n = 1$ a linear dependence is seen for $St_L < Re^{-1/2}$. As in figures 4 and 5, vertical hash marks correspond to $St_L = Re^{-1/2}$ for the three values of $Re$. 
couple to particles with such long stopping times. Naturally, \( V'_{pg} \) simply approaches \( V_g \) for these large particles.

Upon comparing our original analytical results (equations 19 and 22) with our full numerical model and the MMV results, we found some small quantitative discrepancies at the order unity level, as might be expected. The responsible approximations were easily identified. First, we approximated the boundary between class 1 and class 3 eddies by \( t_s = t(k^*) \) rather than the more complete equation 9 of VJMR and equation 4 of MMV, which obtains the relevant eddy frequency in the moving frame of the particle and involves \( V_{rel}(k) \). Comparison of the two criteria revealed that, to a very good approximation, the criterion \( t_s = t(k^*) \) gives a value of \( k^* \) that is too large by a factor close to 2 (figure 3). So, after this “calibration”, we merely decrease the lower limit of integration in our equation (22) by a factor of 2. Second, even after this correction, our values of \( V_{pp} \) are about 20\% high. This is easily ascribed to our approximation that \( g(x) \) and \( h(x) \) are equal to unity throughout the entire range of \( k \); in fact, they are tens of percent smaller than unity over some part of this range, depending on the value of \( St_L \). Empirically, this is corrected by multiplying our analytical expression for \( V_{pp} \) by a constant factor of 0.8. With these two simple adjustments, each correcting a known oversimplification, our analytical expression for \( V_{pp} \) achieves very good agreement with the MMV results, and with our own full numerical model, over the relevant range of \( St_L \leq 0.1 \) or so. There appears to be no reason to make such refinements to our analytical expression for \( V'_{pg} \) (equation 19), because our approximations are better justified and the agreement with MMV acceptable.

### 1.9 Numerical refinements to the model

With insights gained from comparison of our numerical and analytical models, we have made two small adjustments to equation (22) for \( V_{pp} \) which correct for two of our approximations. Equation (22) is multiplied by a factor of 0.8, and the upper integration limit \( (St_L^{-3/2}) \) is divided by two, so the first term in the final expression changes from \( St_L/2 \) to \( St_L/1.03 \approx St_L \). The approximations entering into our expression for \( V_{pp} \) are better, so no correction is applied. The final equation for \( V_{pp} \) is then

\[
V_{pp}^2 = 1.6V_g^2 \left[ St_L - \frac{1}{St_L Re + Re^{1/2}} \right].
\]

The results of equations (19) and (23) (the preferred and adjusted \( n = 1 \) forms), normalized by \( V_g \), are shown in figure 4 for the same three values of \( Re \) as in MMV, and in closeup form in figure 5.

As shown by MMV (their figure 2), and as seen previously in our figure 2, the falloff of \( V_{pp} \) is extremely steep for \( St_L < 1 \) (i.e. \( St_L < Re^{-1/2} \) as shown in the scaling relations of section 1.6 above) because there is no more energy in faster eddies to provide relative velocities to such particles.

### 1.10 Simplification of analytically determined velocity expressions:

Equations (19) and (23) - for the preferred \( n = 1 \) case - are readily simplified in different limits of interest. It is simply shown by retaining leading terms that equation (19) for \( V_{pg} \)
Figure 3: Correction of our approximation $k^* \approx St_L^{-3/2}$ by a factor of two (dash-dot line) which brings it into excellent agreement (in our range of validity $St_L < 0.1$) with the exact numerical solution for $k^*$, shown for $Re = 10^4, 10^7, \text{ and } 10^9$, computed using the full VJMR/MMV expression. Only very close to $St_\eta = 1$ does our approximation deviate slightly; notice the tiny tail at $St_L = 6 \times 10^{-3}, k^* = 10^9$, which is the Kolmogorov scale for $Re = 10^4$. 
Figure 4: $V_{pg}(St_L)$ (dotted; equation 19) and $V_{pp}(St_L)$ (solid; equation 23) for $Re = 10^4$ (a), $10^7$ (b), and $10^9$ (c). The digitized results of MMV (their figure 5) for $V_{pp}$, for the same three values of $Re$, are shown by the dashed lines. Our $V_{pp}$ expression is invalid for $St_L > 0.1$ or so (see text).

Figure 5: A closeup plot of $V_{pg}$ (dotted), $V_{pp}$ (solid), and the digitized MMV results (their figure 5) for $V_{pp}$ (dashed, see Appendix) all for the $Re = 10^7$ case. The dash-dot line has slope $1/2$. The vertical short dashed line indicates $St_\eta = 1$, where $St_L = Re^{-1/2}$; here, $V_{pg} \propto St_L^{0.75}$. 

13
results in three separate regimes: $V_{pg} \approx V_{g}$ for $St_{L} > 1$, $V_{pg} \propto St_{L}^{1/2}$ for $Re^{-1/2} < St_{L} << 1$, and $V_{pg} \propto St_{L} Re^{1/4}$ for $St_{L} < Re^{-1/2}$. This is confirmed by inspection of figures 2 and 4. In the special case of $St_{n} = 1$, or $St_{L} = Re^{-1/2}$, equation (19) reduces directly to

$$V_{pg}(St_{n} = 1) = V_{g} \frac{Re^{-1/4}}{\sqrt{2}} = ca^{1/4} \left( \frac{\nu}{4cH} \right)^{1/4},$$

(24)

where we have substituted $V_{g} = ca^{1/2}$ (Cuzzi et al. 2001). This $Re$-dependence, which also applies for $St_{n} < 1$ in general, quite naturally explains a result we obtained empirically from our numerical models over a range of $Re$ much smaller than nebula values, namely that $V_{pg}/V_{g} \propto Re^{-1/4}$ (Cuzzi et al. 1998). By contrast, it is similarly shown from equation (20) that the $St_{L}$-dependence of $V_{pg}$ for the older $n = 0$ case continues the $St_{L}^{1/2}$ dependence to arbitrarily small $St_{L}$.

These results are also consistent with arguments in Cuzzi et al. (1993, Appendix B; A. Dobrovolskis, personal communication). Expand and time-average the instantaneous quantity $<(V_{p} - V_{g})^{2}>$ to obtain $<V_{pg}V_{pg}> = <V_{p}V_{p}> + <V_{g}V_{g}> - 2 <V_{p}V_{g}>/(1 + St_{L})$, leading to $V_{pg} = (St_{L}/(1 + St_{L}))^{1/2} V_{g}$, which reaches the same limits as equation (19) except for particles with $t_{s} \leq t_{n}$, or $St_{n} \leq 1$, because the integral in its derivation (equation B11 of Cuzzi et al. 1993) extends to infinite eddy frequency.

Thus, unless $t_{s} \leq t_{n}$ ($St_{L} < Re^{-1/2}$), the particle-gas relative velocity in turbulence is generally proportional to $\sqrt{St_{L}}$ for small $St_{L}$. The steeper dependence of $V_{pg}$ on $St_{L}$ and $St_{n}$ is restricted (in turbulence) to particles with $St_{n} \leq 1$. That is, evidence for a more nearly linear dependence of $V_{pg}$ on $r$, if the environment was turbulent, would imply that the particles in question were $St_{n} \leq 1$ particles. This new result derives directly from the use of the $n = 1$ gas velocity autocorrelation function. The primary qualitative change is in the particle size dependence of $V_{pg}$ for particles with $St_{n} \leq 1$. We address the significance of this in more detail in a forthcoming paper (Cuzzi 2002b).

Finally, using equation (23) for $V_{pp}$, we get

$$V_{pp}(St_{n} = 1) = \sqrt{0.8} V_{g} Re^{-1/4} = 1.26 V_{pg},$$

(25)

where we used equation (24) for $V_{pg}$.

## 2 Comparison with numerical results

In this section we compare numerical results from our full 3D Lagrangian particle-gas model (Hogan et al. 1999) with full numerical calculations using our implementation of MMV (sections 1.8-1.9). We present particle velocities relative to the computational box ($V_{p}$), and relative to the local fluid velocity ($V_{pg}$), as obtained from our simulations. These velocities are defined as RMS spatial averages over all particles in a single snapshot, or $V = <(V_{x} - <V_{x}>)^{2}> + <(V_{y} - <V_{y}>)^{2}> + <(V_{z} - <V_{z}>)^{2}>^{1/2}$, where $V$ represents $V_{p}$ or $V_{pg}$ at the location of each particle, and $<>$ is the averaging operator $<...> = \sum_{i=1}^{N_{p}} (...) / N_{p}$, where $N_{p}$ is the number of particles in a single snapshot. Of course, $<V>$ is very close to zero for both these quantities since there is no mean flow in our simulations.
Figure 6: $V_p$ vs. $St_L$ obtained from our direct simulations compared with MMV predictions for models $n = 0$ (solid line) and $n = 1$ (dashed line). All velocities have been normalized by the RMS fluid velocity $V_g$. Results are shown for three different $Re$; the $St_L$ values for each point are defined relative to a large eddy time based on energy dissipation, which varies with $Re$ for our numerical calculations. When they are defined relative to a constant large eddy time, as are our analytical models and the MMV models shown in figure 2, points and models for all $Re$ collapse onto the same curve as seen in figure 2. The $n = 1$ MMV prediction is clearly a better fit to the numerically simulated velocities, regardless of the choice of normalization timescale.

This spatial averaging approach is equivalent to the temporal averaging implicit in the MMV model, because of the ergodic principle that equates temporal and spatial averaging under suitable conditions. In our case, the conditions are satisfied because our integral length scale $L$ is small compared to the spatial period of the computational domain, for all $Re$.

The case of $V_{pp}$ is more complicated, as the results depend on the proximity region chosen for "neighboring" particles. For the most useful comparisons with the predictions of MMV and VJMR, and with the expected uses of this quantity in mind, the region over which particle neighbors are selected should be as small as possible - less than $\eta$ certainly - and here we run into sampling errors. Perhaps most important, the deviation of our model energy spectrum from a Kolmogorov spectrum is significant (e.g. Squires and Eaton 1990), and $V_{pp}$ is much more sensitive to the details of the high-spatial-frequency end of the energy spectrum than either $V_p$ or $V_{pg}$. Since the main purpose of these calculations is to verify numerically the preference for the $n = 1$ autocorrelation function in an independent way from the direct comparison shown in figure 1, and because this case is already well made by the $V_p$ and $V_{pg}$ plots, we present no comparisons for $V_{pp}$.

Figures 6 - 9 show that the $n = 1$ autocorrelation function provides a much better
Figure 7: $V_p$, as in figure 6, but plotted against $St_\eta$.

Figure 8: $V_{pg}$ obtained from our direct simulations, compared with predictions of the MMV models with $n = 0$ (solid line) and $n = 1$ (dashed line) vs $St_L$. All curves have been normalized by the RMS fluid velocity $V_g$.
fit to both $V_p$ and $V_{pg}$ than the $n = 0$ version. For $V_{pg}$, the fits of the MMV theory to our simulations are less perfect than for $V_p$. We can see several possible explanations for this. For instance, the mathematically simple form adopted for the $n = 1$ autocorrelation function is not a perfect fit to the actual numerically determined one (figure 1), by about the correct fractional amount. Also, we have emphasized that the correct velocity autocorrelation function to use is that along a particle trajectory (Meek and Jones 1973), and this function is actually somewhat size dependent even over the range $St_\eta \sim 1$ (see, eg., Squires 1990, figure 4-23). Finally, because of the deviation of our turbulent kinetic energy spectrum from an inertial range, some of the definitions of eddy times used in the MMV theory might be inappropriate. It would not be surprising for $V_{pg}$ to be more sensitive to these small deviations than $V_p$ (compare figures 6 and 8, or 7 and 9). In spite of the small deviations in $V_{pg}$, the combination of the direct comparisons of the autocorrelation functions themselves (figure 1), and the comparison of the velocities derived using them (figures 6-9) makes it clear that the $n = 1$ autocorrelation function is the best choice.

### 3 Summary and conclusions

We present theoretical and numerical results which describe the turbulence-driven velocities of particles in the $St_L << 1$ size regime which might characterize chondrules and similar sized particles. We numerically verify the general approach of VJMR as modified by MMV, and verify in two different ways the intuitive preference of MMV for an $n = 1$ gas velocity autocorrelation function - at least along the trajectories of $St_\eta \approx 1$ particles. We find theoretically that the $n = 1$ autocorrelation function leads to a particle-gas relative velocity function that approaches linear dependence on particle size for particles in the $St_\eta \approx 1$
regime, and becomes and remains linear for arbitrarily small sizes. This is quite a different result than predicted by the original VJMR $n = 0$ expressions. We derive simple analytic expressions for $V_p$, $V_p\delta$, and $V_{pp}$ (the latter, for comparable size particles only) for arbitrary levels of nebula intensity, as characterized by its Reynolds number $Re$ or its corresponding "$\alpha$". In a separate paper (Cuzzi 2002b) we will present some implications of these results for meteoritics.

Acknowledgements: We are grateful to Sandy Davis for helpful discussions on matters mathematical, to Robert Last for computational and graphics support, and to Wojtek Markiewicz for helpful discussions concerning MMV. We thank Ignacio Mosqueira, Steve Desch, Pat Cassen, and Tony Dobrovolskis for careful reading of the manuscript and helpful comments. This research was supported by Grants to JNC from the Planetary Geology and Geophysics Program and the Origins of Solar Systems Program.
Table I. List of symbols

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
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<tbody>
<tr>
<td>$c$</td>
<td>gas molecule thermal speed</td>
</tr>
<tr>
<td>$C$</td>
<td>particle concentration factor</td>
</tr>
<tr>
<td>$E(k)$</td>
<td>turbulent kinetic energy at wavenumber $k$</td>
</tr>
<tr>
<td>$H$</td>
<td>nebula vertical scale height</td>
</tr>
<tr>
<td>$k$</td>
<td>eddy wavenumber</td>
</tr>
<tr>
<td>$k_L$</td>
<td>wavenumber of largest eddy</td>
</tr>
<tr>
<td>$k_\eta$</td>
<td>wavenumber of Kolmogorov scale eddy</td>
</tr>
<tr>
<td>$L$</td>
<td>integral or largest scale in turbulent energy spectrum</td>
</tr>
<tr>
<td>$r$</td>
<td>particle radius</td>
</tr>
<tr>
<td>$R$</td>
<td>gas velocity autocorrelation function</td>
</tr>
<tr>
<td>$Re$</td>
<td>flow Reynolds number</td>
</tr>
<tr>
<td>$St_L$</td>
<td>Stokes number relative to largest eddy</td>
</tr>
<tr>
<td>$St_\eta$</td>
<td>Stokes number relative to Kolmogorov scale eddy</td>
</tr>
<tr>
<td>$t_s$</td>
<td>stopping time of particle due to gas drag</td>
</tr>
<tr>
<td>$t_k$</td>
<td>overturn time of eddy with wavenumber $k$</td>
</tr>
<tr>
<td>$t_L$</td>
<td>overturn time of largest eddy</td>
</tr>
<tr>
<td>$t_\eta$</td>
<td>overturn time of Kolmogorov scale eddy</td>
</tr>
<tr>
<td>$V_g$</td>
<td>gas turbulent velocity (large eddy)</td>
</tr>
<tr>
<td>$V_p$</td>
<td>particle random velocity in inertial space</td>
</tr>
<tr>
<td>$V_{pg}$</td>
<td>relative velocity between particles and gas</td>
</tr>
<tr>
<td>$V_{pp}$</td>
<td>relative velocity between particles</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>nebula viscosity parameter; $Re = \alpha c H / \nu$</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>dissipation of turbulent kinetic energy</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Kolmogorov scale</td>
</tr>
<tr>
<td>$\nu$</td>
<td>molecular kinematic viscosity</td>
</tr>
<tr>
<td>$\nu_T$</td>
<td>turbulent kinematic viscosity</td>
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<td>$\omega$</td>
<td>eddy temporal frequency</td>
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<tr>
<td>$\rho_g$</td>
<td>gas mass density</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>solid particle mass density</td>
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4 References


Meek, C. C. and B. G. Jones (1973) Studies of the behavior of heavy particles in a turbulent fluid flow; J. Atm. Sciences, 30, 239-244


Blowing in the Wind: I. Velocities of Chondrule-sized Particles in a Turbulent Protoplanetary Nebula
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Submitted to Icarus, September 17, 2002

Abstract: Small but macroscopic particles - chondrules, higher temperature mineral inclusions, metal grains, and their like - dominate the fabric of primitive meteorites. The properties of these constituents, and their relationship to the fine dust grains which surround them, suggest that they led an extended existence in a gaseous protoplanetary nebula prior to their incorporation into their parent primitive bodies. In this paper we explore in some detail the velocities acquired by such particles in a turbulent nebula. We treat velocities in inertial space (relevant to diffusion), velocities relative to the gas and entrained microscopic dust (relevant to accretion of dust rims), and velocities relative to each other (relevant to collisions). We extend previous work by presenting explicit, closed-form solutions for the magnitude and size dependence of these velocities in this important particle size regime, and compare these expressions with new numerical calculations. The magnitude and size dependence of these velocities have immediate applications to chondrule and CAI rimming by fine dust, and to their diffusion in the nebula, which we explore separately.

1 Background

The fabric of the most primitive meteorites undoubtedly contains many clues as to their origin. While most chondrites are samples of surfaces that have been well worked over by impacts and stirring ("regolith breccias"), the dominance of chondrules and like-sized objects remains clear. How it came about that most chondrite parent bodies are so dominated by particles with such a well-defined range of physical, chemical, and petrographic properties remains one of the big puzzles of meteoritics. Since there are relatively few examples of anything larger than 0.1-10 mm size particles in most primitive planetesimals, the way such particles interact with the gaseous nebula is of prime importance.

Fe-Mg-Si-O mineral chondrules, which solidified from a melt, constitute 30-80\% of primitive meteorites. There are a number of extant hypotheses for the formation of the chondrules. Most workers in the field believe that chondrules are formed by either localized or nebula scale energetic events operating on freely floating precursors of comparable mass, at some location or locations in the protoplanetary nebula. However, some still maintain they are made in or on primitive bodies, or in collisions between them. In a hybrid scenario, some suggest they are formed in shock waves generated by already-formed planetesimals, and thus that they are a secondary phenomenon to primary accretion of planetesimals. See eg. Grossman (1989), Grossman et al. (1989), Boss (1996), Connolly and Love (1999), and Jones et al. (2000) for reviews of hypotheses on this long-controversial and perennially fascinating subject.
Another meteorite constituent of great interest are the mineral grains called Ca-Al-rich refractory inclusions (CAIs) - so called because their constituent minerals condense out of nebula gas at a much higher temperature than do chondrules. These objects are widely believed to be direct nebula condensates, and have a complex subsequent thermal history which has some similarities to that of chondrules and some differences. There is some indication from radioisotope ages that CAIs might be \( \sim 10^6 \) years older than the chondrules, but this remains slightly controversial. They make up 1-10% of primitive meteorites depending on type, and their size distribution is broader than that of the chondrules. How these high-temperature minerals find themselves intimately mixed with lower-temperature minerals remains a puzzle.

It remains unresolved at this time whether the nebula gas was turbulent or laminar during the chondrule era. In previous papers, we have suggested that some of the observed properties of chondrules themselves - their typical size and size distribution - can be associated with, and easily explained by, the effects of weak nebula turbulence (Cuzzi et al 1996, 2001). Nevertheless, a consistent end-to-end scenario for formation of primitive bodies in this environment, and relying on these processes, is not yet in hand. In this paper, we focus on the velocity evolution of this specific class of particles in a weakly turbulent nebula as a step towards developing a more complete scenario that operates to produce primitive bodies in a similar way across a variety of environments. The velocity evolution is critical for our understanding of several important aspects of chondrules and chondrites: (a) the radial distribution and redistribution or transport of chondrules and/or CAIs, once formed, before their accumulation into parent bodies; (b) The presence of fine grained rims on chondrules, CAIs, and other coarse particles in primitive chondrites (Metzler and Bischoff 1996, Brearley and Jones 1998); and (c) collision rates and velocities between chondrule-sized particles. The main goal of this paper is to provide a theoretical framework within which we can better understand mm-to-cm-size particle evolution in general. We accomplish this in sections 1 (analytical theory) and 2 (supporting numerical calculations). In another paper we apply these results to diffusion and dust rimming (Cuzzi 2002b).

1.1 Particle Velocities in Turbulence

Astrophysical modeling of the basic physics of particle behavior in fluid flows, laminar or turbulent, tends to begin and end with the classic papers by Whipple (1973), Adachi et al. (1976), Weidenschilling (1977, 1980), and Völk et al. (1980, henceforth VJMR; also Völk et al. 1978), with important recent updates by Markiewicz et al. (1991; henceforth MMV). In the fluid dynamics literature, however, the study of particle motions in fluid flows has both a long history, and a robust ongoing presence. This history is nicely summarized by Meek and Jones (1973). More recent work in the fluids literature is noted in various relevant places below. VJMR first developed a useful formalism for calculating the dispersion velocities \( V_p \) (relative to inertial space) and collision velocities \( V_{pp} \) (relative to each other) of particles in a turbulent nebula. They circumvented the thorny problem of "essential nonlinearity" (cf. Meek and Jones 1973) by translating clever physical insights into mathematics and adopting a velocity autocorrelation function approach, which we discuss in more detail below. While it serves an important internal role in their solutions, neither VJMR nor MMV say much about the relative velocity between particles and gas, \( V_{pg} \). Yet, \( V_{pg} \) is the determinant quantity for
accretion of rims of fine dust grains by small, macroscopic objects (Paque and Cuzzi 1997, Cuzzi et al 1998, Morfill et al. 1998). Our goal in this paper is to quantify $V_p$, $V_{pg}$, and $V_{pp}$ for such particles in a way that extends and focusses the formulation of VJMR and MMV, and which allows insights to be gained into the history of chondrules and like-sized particles in the protoplanetary nebula.

In this paper, we determine velocities of all three kinds - $V_p$, $V_{pg}$, and $V_{pp}$ - with emphasis on particles having stopping times $t_s$ comparable to the overturn time $t_\eta$ of Kolmogorov scale eddies. Particles in this size regime have behavior more complex than tiny “dust” grains, which are essentially trapped to the gas flow on all scales. In particular, particles with $t_s = t_\eta$ are subject to “preferential concentration” by large factors in turbulence, and based on some of its apparent fingerprints in the meteorite record, we have suggested a link between this process, chondrules, and primary accretion. Specifically, we refer to the fact that the typical size and the shape of the size distribution of chondrules are readily explained by turbulent concentration (Cuzzi et al 1996, 2001). In a parallel paper (Cuzzi 2002b) we explore the possibility that the functional form of $V_{pg}$ might reveal another fingerprint of turbulent concentration, and that turbulence might help us understand the puzzling mix of CAIs and chondrules in the same meteorites.

Particles are aerodynamically classified by their Stokes number $St$, the ratio of their stopping time $t_s$ to the overturn time of some characteristic eddy. We will make use of Stokes numbers defined relative to two different eddy overturn timescales: the Stokes number relative to the largest, or integral scale eddy time $t_L$: $St_L = t_s/t_L$, and that defined relative to the smallest, or Kolmogorov scale eddy time $t_\eta$: $St_\eta = t_s/t_\eta$. The overturn time of the largest scale eddy $t_L$ is generally regarded as the local orbit period. Preferentially concentrated particles (chondrules, we believe) have $St_\eta = 1$ and $St_L << 1$. For these particles, which are smaller than the gas molecular mean free path, the stopping time $t_s = r\rho_s/c p_g$, where $r$ is particle radius, $\rho_s$ is particle material density, $c$ is the nebula sound speed, and $p_g$ is the nebula gas density (Weidenschilling 1977). That is, $t_s$ and thus both $St_L$ and $St_\eta$ are linearly proportional to particle radius.

1.2 Previous work; the autocorrelation function

We briefly review and simplify the notation of VJMR and MMV. VJMR assumed a fully developed inertial range of turbulence with some largest, or integral scale $L$ and zero smallest scale. MMV also adopted the Kolmogorov energy spectrum (as shall we) but correctly pointed out that turbulence ceases for scales smaller than the Kolmogorov or inner scale $\eta$. Especially for small particles in the chondrule-and-CAI size range, MMV point out that this has important implications for $V_p$ and $V_{pp}$, and we will show that the implications are important for $V_{pg}$ as well. In a Kolmogorov spectrum, an inertial range of turbulent gas kinetic energy extends from the largest or integral scale $l = L$ to the smallest or Kolmogorov scale $l = \eta$. Following VJMR, we work in the spatial frequency regime, where $k(l) = 2\pi/l$ and $E(k) = E_L(k/k_L)^{-5/3}$ for the Kolmogorov spectrum (note our $E(k)$ is a true energy, and is half of VJMR’s $P(k)$). Then $v(k) = (2kE(k))^{1/2}$ and $t(k) = 1/(kv(k)) = t_L(k/k_L)^{-2/3}$. As did MMV, we assume $E(k) = 0$ for $k > k_\eta$ (no turbulent energy at scales smaller than the Kolmogorov scale). The mean square turbulent (fluctuating) gas velocity is $V_g^2$; thus the typical turbulent kinetic energy per unit gas mass is $V_g^2/2$, providing the normalization
The turbulent gas motions induce fluctuating velocities in the particle population, leading to diffusion ($V_p$), mutual collisions ($V_{pp}$), and motion relative to the local gas ($V_{pg}$).

VJMR derive $V_p$ formally by a backwards time integration of the instantaneous acceleration (their equations 5 and 6):

$$V_p(t) = t_s^{-1} \int_0^t \exp\left(-\frac{(t - t')}{t_s}\right)V_g(t')dt'$$

where $V_g(t')$ represents the fluctuating gas velocity history along a particle trajectory (formally unknown at this point). They proceed by approximating $V_g(t')$ as an integral over all (independently acting) spatial frequencies $k$ with eddy timescales $t_k$, and approximate the contributions as coming from two classes of eddies: “class 1” eddies, with overturn times long enough ($t_k > t_s$) that particles are always in equilibrium within them, and are primarily just advected by their (temporally fluctuating) motions, and “class 3” eddies with overturn times too short ($t_k < t_s$) for the particle to come to equilibrium as it passes through them. Intermediate, or what might be “class 2” eddies are not treated separately, but simply absorbed into the classes on either side. Different simplifications are allowed for each class. The boundary between eddy classes 1 and 3 is $k^*$, where $t_{k*} = t_s$. VJMR show that the class 3 (small, fast) eddies are negligible for velocity components $V_p$ and $V_{pg}$, but dominate the contributions to $V_{pp}$. We will make use of these results below.

VJMR first obtain the product $\langle V_p(t)V_p(t) \rangle$ by integrating backwards over two separate time histories. They introduce the gas velocity autocorrelation function for gas velocities (in their equation 16) $R(t, t'; k) = \exp\left(-|t - t'|/t_k\right)$. While they don’t make the distinction, the autocorrelation function to be used in this way is properly that determined along a particle trajectory (Batchelor 1948, Hinze 1975, Squires and Eaton 1990, Elghobashi 1991), and is thus a function of $t_s$ in general. However, for $St_L << 1$, and at this stage of our knowledge, this distinction is not significant (Squires 1990).

Subsequently, MMV suggested a more general, even if ad hoc, functional form for $R(t, t'; k)$:

$$R(t, t'; k) = \left(1 + \frac{|t - t'|}{t_k}\right)^n e^{-|t - t'|/t_k},$$

with $n = 0$ or 1. They note that the $n = 1$ case has more plausible physical behavior (zero slope) near $t = t'$ than the $n = 0$ (pure exponential) form assumed by VJMR.

1.2.1 New results regarding the form of the autocorrelation function, and the value of $n$:

The selection of $n = (0, 1)$ determines the form of the fluid velocity autocorrelation function $R(t, t'; k)$. Squires (1990) measured this function directly in his direct numerical simulations of turbulence, by following fluid motions along the trajectories of a number of particles with different $St_L$. In figure 1 we compare the results of Squires (1990) with the predictions based on the $n = 0$ and $n = 1$ expressions of MMV for $R(t, t'; k)$. Note that, since MMV
Figure 1: Autocorrelation function for gas velocities along the trajectory of a $St_\eta = 1$ particle, as computed directly from our simulations (dotted) and from the simulations of Squires (1990), and as calculated using the $n = 0$ and $n = 1$ models of MMV. Here, $\tau = t - t'$ and is normalized by the large eddy turnover time $T_L$. The $n = 1$ model is clearly the better choice.

express their autocorrelation function as a function of $k$, it must be integrated over an energy spectrum to compare with the numerical results of Squires (1990). Because Squires (1990) only calculated a 1-D autocorrelation function (i.e., using only one velocity component), we integrated the $R(t, t'; k)$ of MMV over a 1D energy spectrum (essentially, one-third of the total $E(k)$) (see also Squires and Eaton 1991). It is clear from figure 1 that $n = 1$ is the better choice. This has important implications, primarily for $V_{pp}$ and $V_{pg}$. In section 2, we directly compare $V_p$ and $V_{pg}$ calculated in full 3D turbulence using the two alternate autocorrelation functions, and again reach the same conclusion.

1.3 Particle random velocities relative to inertial space

After some algebra, VJMR derive an expression (their equation 18) for the mean square particle fluctuating velocity $V_p$, of which we need only the large, slow (class 1) eddy contribution since the small eddy contribution is negligible for $St_\eta = 1$ particles (we will henceforth drop the $<>$ notation on $V_p, V_g, V_{pg},$ and $V_{pp}$, and will merely recall that all are statistical expectation values based on extensive temporal or spatial averaging). Because of our emphasis on particles with $St_\eta = 1$, we also replace the upper limit of VJMR's class 1 integral ($k^*$) with the Kolmogorov scale $k_\eta$. This simplification is, in fact, actually fairly good over the entire range of $St_L << 1$, precisely because the contribution of eddies on smaller scales than $k^*$ (the class 3 eddies) is negligible. That is, the upper limit can be extended from $k^*$ to $k_\eta$ in general for mathematical simplicity without incurring significant error. Mathematically,
the upper limit could even be extended to infinity (e.g., Völk et al. 1980), but the important role of the Reynolds number and of the Kolmogorov scale is then lost. Thus,

$$V_p^2 \approx 2 \int_{k_L}^{k_n} E(k) \frac{t_k}{t_k + t_s} \, dk.$$  \hspace{1cm} (4)

Similarly, the generalized MMV expression for $V_p^2$ (their equation 6) can be simplified to

$$V_p^2 \approx 2 \int_{k_L}^{k_n} E(k) \left( 1 - \left( \frac{t_s}{t_k + t_s} \right)^{n+1} \right) \, dk = 2 \int_{k_L}^{k_n} E(k) \left( 1 - \left( \frac{1}{1 + t_k/t_s} \right)^{n+1} \right) \, dk$$  \hspace{1cm} (5)

for the particle size regime of interest here. As did VJMR, MMV note that the second integral of their equation (6) - the class 3 eddy contribution - is negligible for small particles, so we retain only the first integral of their equation (6). We again simplify the upper limit of integration in the remaining integral for the nominal $St_e \approx 1$ case where $k^* \approx k_n >> k_L$. We validate this by comparing our results with those of MMV (section 1.7).

The result for $V_p^2$ was plotted, but not stated explicitly, by VJMR and MMV (figure 1 in both papers), and explicitly derived by Cuzzi et al. (1993; Appendix B): $V_p^2 = V_g^2/(1 + St_L)$. It is simple to see why $V_p^2 \approx V_g^2$ in the limit $St_L << 1$ and certainly for $St_e \approx 1$, since $t_s << t_k$ in equations (4) or (5) for nearly all $k$ and overwhelmingly all $E(k)$. This limit is appropriate for chondrule-and-CAI-sized particles even in the presence of their small vertical settling velocity - they diffuse nearly as well as a gas molecule, and do not “settle to the midplane” in even a very weakly turbulent nebula (Dubrulle et al. 1995, Cuzzi et al. 1996). The implications are discussed in section 3. However, $V_p^2$ and $V_g^2$ are not exactly equal, resulting in a small, but very important, relative energy of motion $V_{pg}^2$, giving the velocity with which particles move through the gas and encounter tiny (micron-sized) dust grains.

### 1.4 Particle velocities relative to the gas

The average relative velocity magnitude between a particle and the turbulent gas is $V_{pg}$. VJMR make use only of the spatial frequency components of this quantity, which they refer to as $V_{rel}(k)$ (their equation 15). Practically speaking, however, a particle will instantaneously sense all eddy contributions as one $V_{pg}$; we obtain this by merely integrating VJMR equation (15) over $k$. Considering only the part of the expression relevant for $St_e \approx 1$ (that for $k^* > k_L$), neglecting any systematic velocity, and again letting $k^* \approx k_n >> k_L$, the second integral vanishes and we obtain

$$V_{pg}^2 \approx 2 \int_{k_L}^{k_n} E(k) \left( \frac{t_s}{t_k + t_s} \right) \, dk.$$  \hspace{1cm} (6)

For this $n = 0$ case treated by VJMR, it can be easily verified using equations (4) and (6) that

$$V_{pg}^2 + V_p^2 = 2 \int_{k_L}^{k_n} E(k) \, dk = V_g^2.$$  \hspace{1cm} (7)

However, this useful result is true independent of $n$. It may also be obtained by Fourier transform solution of the forcing equations in temporal frequency ($\omega$) space, where the
energy spectrum of gas velocity fluctuations \( E_g(\omega) \), particle velocity fluctuations \( E_p(\omega) \), and relative velocity fluctuations \( E_{pg}(\omega) \) are related by

\[
E_p(\omega) = E_g(\omega)/(1 + t_s^2 \omega^2) \quad \text{and} \quad E_{pg}(\omega) = t_s^2 \omega^2 E_p(\omega).
\]

This approach can be traced to Csanady (1963); it is also described by Hinze (1975, chapter 5), Meek and Jones (1973), and Squires (1990, sections 4.2 and 4.5.1). The \( E_p \) solution was also derived in this way by Cuzzi et al. (1993, Appendix B). It is also clear then that \( E_{pg}(\omega) + E_p(\omega) = E_g(\omega) \), essentially the same result as equation (7) above. Finally, we have directly verified equation (7) in our numerical simulations.

Using this general relationship, we can extend the results of MMV to obtain \( V_{pg}^2 \) for their more generalized gas velocity autocorrelation functions (they only present results for \( V_p^2 \)). That is, using equations (1) and (5),

\[
V_{pg}^2 = V_g^2 - V_p^2 = 2 \int_{k_L}^{k_L} E(k) \left( \frac{t_s}{t_s + t_k} \right)^{n+1} dk = 2 \int_{k_L}^{k_L} E(k) \left( \frac{1}{1 + t_k/t_s} \right)^{n+1} dk.
\]

We will use equations (5) and (9), with assumed inertial range expressions for \( E(k) \), to derive analytical expressions for \( V_p \) and \( V_{pg} \) of hypothetically "chondrule-like" (ie., \( St_\eta \approx 1 \)) particles as functions of their size and the turbulent Reynolds number.

### 1.5 Relative velocities between particles of similar sizes

Expressions for \( V_{pp} \) (VJMR Appendix C and equation 19; MMV equations 7 and 8) are more cumbersome, but respond nicely to certain simplifying assumptions. The full expression for \( V_{pp} \) for two particles of equal size is (changing notation slightly from MMV equation 9, and allowing for a finite Kolmogorov scale):

\[
V_{pp}^2 = 4 \int_{k_L}^{k_L} E(k) \left( 1 - \frac{t_s}{t_s + t_k} \right) \left[ g(\chi) + \frac{nt_s h(\chi)}{t_s + t_k} \right] dk;
\]

where \( g(\chi) = \tan^{-1}(\chi)/\chi \) and \( h(\chi) = 1/(1 + \chi^2) \). The parameter \( \chi \) of VJMR and MMV is small in our regime of interest:

\[
\chi = \frac{V_{rel}(k) t_s (kt_k)}{t_s + t_k} = \frac{V_{rel}(k) t_s}{v(k)(t_s + t_k)} \approx \frac{V_{rel}(k)}{2v(k)} < 1,
\]

since in the very limited range of \( k \) over which the integral is done, \( t_s \approx t_k \). In fact \( \chi \ll 1 \) over most of the integral where \( t_s \ll t_k \), so the functions \( g(\chi) \) and \( h(\chi) \) are \( \approx 1 \) or perhaps as small as a fraction of order unity; thus

\[
V_{pp}^2 \approx 4 \int_{k_L}^{k_L} E(k) \left[ 1 - \left( \frac{t_s}{t_s + t_k} \right)^{n+1} \right] dk = 4 \int_{k_L}^{k_L} E(k) \left[ 1 - \left( \frac{1}{1 + t_k/t_s} \right)^{n+1} \right] dk.
\]

The integrand is identical to that for \( V_p^2 \), but the integral has different limits which make it clear that only the eddies faster than \( t_s \) can perturb identical particles into having incoherent relative velocities.

---

\(^1\)In the above equation, the mathematical generalization of \( V_{pg} \) by VJMR and MMV to its \( k \)-th components \( V_{rel}(k) \) momentarily reappears. However, it is true in general, at any spatial frequency, that the particle-gas relative velocity is less than, or at most equal to, the gas velocity itself.
1.6 Scaling relations

Recall that for the gas,

\[ t_k = l(k)/v(k) = (L/V_g)(k/k_L)^{-2/3} = t_L(k/k_L)^{-2/3} \]  

(Cuzzi et al. 2001). In equation (13) we have made the usual identification of \( V_g \) with the largest scale eddy \( L \). For the particles,

\[ \frac{t_s}{t_L} = St_L = (k_s/k_L)^{-2/3} \]

(14)

and

\[ \frac{t_s}{t_k} = (k/k_s)^{2/3} = \frac{t_s}{t_L} (k/k_L)^{2/3} = St_L (k/k_L)^{2/3} \]

(15)

Note that if we restrict our attention to particles with \( St_\eta = t_s/t_\eta \approx 1 \), then their Stokes number referred to the integral scale automatically becomes

\[ St_L = t_s/t_L = t_\eta/t_L = (k_\eta/k_L)^{-2/3} = (Re_3^{1/4})^{-2/3} = Re^{-1/2}. \]  

The last substitution of \( (k_\eta/k_L) = Re_3^{1/4} \), where \( Re = LV_g/\nu \) is the flow Reynolds number, with \( \nu \) being the molecular kinematic viscosity, is a direct consequence of the definitions of the Kolmogorov scale, the energy dissipation rate, and the Reynolds number (Tennekes and Lumley 1972). This relation can be obtained without any reference at all to the Kolmogorov spectrum but merely using scaling arguments relating to \( t_L \) and \( t_\eta \). \(^2\) \( Re \) is related to astrophysical "\( \alpha \)-models of the protoplanetary nebula by \( Re = \alpha c H/\nu \) with \( c \) = sound speed and \( H = \) nebula vertical scale height (Cuzzi et al. 2001).

1.7 Final expressions for \( V_{pg} \) and \( V_{pp} \)

Substituting the scaling relations from above for \( t_s/t_k \), equation (9) for \( V_{pg} \) becomes

\[ V_{pg}^2 = 2 \int_{k_L}^{k_\eta} E(k) \left( \frac{1}{1 + t_k/t_s} \right)^{n+1} dk = 2 \int_{k_L}^{k_\eta} E(k) \left( \frac{St_L}{St_L + (k/k_L)^{-2/3}} \right)^{n+1} dk. \]  

(17)

We use the normalization (equation 1) to write \( E(k) = (V_g^2/3k_L)(k/k_L)^{-5/3} \), and change integration variable to \( x = k/k_L \), leaving

\[ V_{pg}^2 = \frac{2V_g^2}{3} \int_{1}^{Re_3^{1/4}} \left( \frac{St_L}{St_L + x^{-2/3}} \right)^{n+1} x^{-5/3} dx. \]  

(18)

where in the upper limit we have substituted \( k_\eta/k_L = Re_3^{1/4} \) from the scaling relations. Closed form solutions for equation (18) can be obtained for \( n = 0 \) or 1. For example, for \( n = 1 \) the result of the integral is

\[ V_{pg}^2 = \frac{V_g^2}{1 + \frac{St_L}{x^{2/3}}} \bigg|_{Re_3^{1/4}} = \frac{V_g^2}{1 + \left[ (St_L + 1)(St_LRe_3^{1/2} + 1) \right]}. \]  

(19)

\(^2\)Let the energy dissipation rate be \( \epsilon \). Then \( \epsilon = V_g^2/t_L = V_g^3/L \) where the first expression defines \( t_L \) and the last expression defines \( L \). Also \( t_\eta = (\nu/\epsilon)^{1/2} \) and \( \eta = (\nu^3/\epsilon)^{1/4} \). Solving gives \( t_L/t_\eta = Re_3^{1/2} \) and \( \eta/L = Re_3^{3/4} \).
For \( n = 0 \) the result of the integral is:

\[
V_{pg}^2 = V_g^2 \left[ St_L \ln \left( \frac{Re^{1/2}(1 + St_L)}{Re^{1/2}St_L + 1} \right) \right].
\]

(20)

These results make it quite easy to predict both the magnitude and the \( St_n \) dependence of \( V_{pg} \) for arbitrary nebula turbulent intensity.

We solve equation (12) for \( V_{pp} \) in a similar fashion to the solution for \( V_{pg} \) above, to obtain for \( n = 1 \):

\[
V_{pp}^2 = \frac{4V_g^2}{3} \int_{k_*/k_L}^{k_n/k_L} \left( \frac{2St_Lx^{-7/3} + x^{-9/3}}{St_L^2 + 2St_Lx^{-2/3} + x^{-4/3}} \right) dx.
\]

(21)

As before, the upper integration limit is \( k_n/k_L = Re^{3/4} \). For the lower limit, \( k^*/k_L = k(t_s)/k_L = (t_s/t_L)^{-3/2} = St_L^{-3/2} \) from the scaling relations. The closed form analytic solution of this integral is:

\[
V_{pp}^2 = 2V_g^2 \left[ St_L \ln \left( \frac{1 + St_Lx^{2/3}}{x^{2/3}} \right) - \frac{1}{x^{2/3}} \right]^{Re^{3/4}}_{St_L^{-3/2}}.
\]

(22)

The \( n = 0 \) form of the solution is somewhat less useful, and we note it without expanding it as it will not be used further.

\[
V_{pp}^2 = 2V_g^2 \left[ St_L \ln \left( \frac{1 + St_Lx^{2/3}}{x^{2/3}} \right) - \frac{1}{x^{2/3}} \right]^{Re^{3/4}}_{St_L^{-3/2}}.
\]

1.8 Detailed comparisons with the models of Markiewicz et al.

In addition to developing the analytical expressions discussed and applied in the paper, we also developed a detailed numerical model following the prescriptions of MMV exactly (but with a generalized turbulent energy spectrum). This was needed both to evaluate their theoretical approach in the context of our numerical simulations of turbulence (section 2), which have a non-Kolmogorov spectrum and low Reynolds number compared to nebula applications, and to assess the validity of our analytical approximations. The numerical model of MMV is no longer in active use (W. Markiewicz, personal communication 2002), so we digitized their \( V_{pp} \) results (their figure 5) to facilitate comparisons. As seen in figure 4, our full numerical model for \( V_{pp} \) (solid curves) agrees very well with their results for \( V_{pp} \) (long dashed curves). In figure 2 we also show our results for \( V_{pg} \), not presented by VJMR or MMV, as obtained by integrating MMV equation 4 over all spatial frequencies. Note that we, and MMV, both use the appropriate form of \( R(t, t'; k) \) (i.e., that for the correct choice of \( n \); section 1.2) for these calculations.

The most striking feature of the results, first noted by MMV, is that \( V_{pp} \) very quickly falls to zero for particles with \( St_\eta < 1 \) (i.e. \( St_L < Re^{-1/2} \), as shown in the scaling relations of section 1.6 above) because there is no more energy in faster eddies to provide relative velocities to such particles. This does not happen to \( V_{pg} \), because eddies on all scales contribute. Also note that \( V_p \) and \( V_{pp} \) decrease for large particles (\( St_L > 1 \)), as fewer eddies can effectively
Figure 2: Comparison of our numerical version of the full MMV model for $n = 0$ (light curves) and $n = 1$ (heavy curves), along with digitized results for $V_{pp}$ from MMV (dashed curves, their figure 5, for $n = 1$). Three different $Re$ are shown: (a) $10^4$, (b) $10^7$, and (c) $10^9$. The dash-dot curves are for $V_p$, which has the same shape for all three $Re$. $V_{pp}$ is shown in the two sets of dotted curves and $V_{pp}$ in the two sets of solid curves. Note that the $n = 0$ values of $V_{pp}$ (light dotted curves) are considerably (3-4 times) higher than the preferred $n = 1$ values (heavy dotted curves), and the $St_L$-dependence of $V_{pp}$, for $n = 0$, never gets much above 0.5, whereas for $n = 1$ a linear dependence is seen for $St_L < Re^{-1/2}$. As in figures 4 and 5, vertical hash marks correspond to $St_L = Re^{-1/2}$ for the three values of $Re$. 
couple to particles with such long stopping times. Naturally, $V_{pg}$ simply approaches $V_g$ for these large particles.

Upon comparing our original analytical results (equations 19 and 22) with our full numerical model and the MMV results, we found some small quantitative discrepancies at the order unity level, as might be expected. The responsible approximations were easily identified. First, we approximated the boundary between class 1 and class 3 eddies by $t_s = t(k^*)$ rather than the more complete equation 9 of VJMR and equation 4 of MMV, which obtains the relevant eddy frequency in the moving frame of the particle and involves $V_{rel}(k)$. Comparison of the two criteria revealed that, to a very good approximation, the criterion $t_s = t(k^*)$ gives a value of $k^*$ that is too large by a factor close to 2 (figure 3). So, after this “calibration”, we merely decrease the lower limit of integration in our equation (22) by a factor of 2. Second, even after this correction, our values of $V_{pp}$ are about 20% high. This is easily ascribed to our approximation that $g(x)$ and $h(x)$ are equal to unity throughout the entire range of $k$; in fact, they are tens of percent smaller than unity over some part of this range, depending on the value of $St_L$. Empirically, this is corrected by multiplying our analytical expression for $V_{pp}$ by a constant factor of 0.8. With these two simple adjustments, each correcting a known oversimplification, our analytical expression for $V_{pp}$ achieves very good agreement with the MMV results, and with our own full numerical model, over the relevant range of $St_L \sim 0.1$ or so. There appears to be no reason to make such refinements to our analytical expression for $V_{pg}$ (equation 19), because our approximations are better justified and the agreement with MMV acceptable.

1.9 Numerical refinements to the model

With insights gained from comparison of our numerical and analytical models, we have made two small adjustments to equation (22) for $V_{pp}$ which correct for two of our approximations. Equation (22) is multiplied by a factor of 0.8, and the upper integration limit $(St_L^{3/2})$ is divided by two, so the first term in the final expression changes from $St_L/2$ to $St_L/1.03 \approx St_L$. The approximations entering into our expression for $V_{pp}$ are better, so no correction is applied. The final equation for $V_{pp}$ is then

$$V_{pp}^2 = 1.6V_g^2 \left[ St_L - \frac{1}{St_L Re + Re^{1/2}} \right].$$

The results of equations (19) and (23) (the preferred and adjusted $n = 1$ forms), normalized by $V_g$, are shown in figure 4 for the same three values of $Re$ as in MMV, and in closeup form in figure 5.

As shown by MMV (their figure 2), and as seen previously in our figure 2, the falloff of $V_{pp}$ is extremely steep for $St_\eta < 1$ (i.e. $St_L < Re^{-1/2}$ as shown in the scaling relations of section 1.6 above) because there is no more energy in faster eddies to provide relative velocities to such particles.

1.10 Simplification of analytically determined velocity expressions:

Equations (19) and (23) - for the preferred $n = 1$ case - are readily simplified in different limits of interest. It is simply shown by retaining leading terms that equation (19) for $V_{pg}$
Figure 3: Correction of our approximation $k^* \approx St^{-3/2}$ by a factor of two (dash-dot line) which brings it into excellent agreement (in our range of validity $St_L < 0.1$) with the exact numerical solution for $k^*$, shown for $Re = 10^4$, $10^7$, and $10^9$, computed using the full VJMR/MMV expression. Only very close to $St_\eta = 1$ does our approximation deviate slightly; notice the tiny tail at $St_L = 6 \times 10^{-3}$, $k^* = 10^5$, which is the Kolmogorov scale for $Re = 10^4$. 
Figure 4: $V_{pg}(St_L)$ (dotted; equation 19) and $V_{pp}(St_L)$ (solid; equation 23) for $Re = 10^4$ (a), $10^7$ (b), and $10^9$ (c). The digitized results of MMV (their figure 5) for $V_{pp}$, for the same three values of $Re$, are shown by the dashed lines. Our $V_{pp}$ expression is invalid for $St_L > 0.1$ or so (see text).

Figure 5: A closeup plot of $V_{pg}$ (dotted), $V_{pp}$ (solid), and the digitized MMV results (their figure 5) for $V_{pp}$ (dashed, see Appendix) all for the $Re = 10^7$ case. The dash-dot line has slope $1/2$. The vertical short dashed line indicates $St_\eta = 1$, where $St_L = Re^{-1/2}$; here, $V_{pg} \propto St_L^{0.75}$. 

13
results in three separate regimes: \( V_{pg} \approx V_g \) for \( St_L > 1 \), \( V_{pg} \propto St_L^{1/2} \) for \( Re^{-1/2} < St_L << 1 \), and \( V_{pg} \propto St_L Re^{1/4} \) for \( St_L < Re^{-1/2} \). This is confirmed by inspection of figures 2 and 4. In the special case of \( St_\eta = 1 \), or \( St_L = Re^{-1/2} \), equation (19) reduces directly to

\[
V_{pg}(St_\eta = 1) = V_g \frac{Re^{-1/4}}{\sqrt{2}} = c \alpha^{1/4} \left( \frac{\nu}{4cH} \right)^{1/4},
\]  

(24)

where we have substituted \( V_g = c \alpha^{1/2} \) (Cuzzi et al. 2001). This \( Re \)-dependence, which also applies for \( St_\eta < 1 \) in general, quite naturally explains a result we obtained empirically from our numerical models over a range of \( Re \) much smaller than nebula values, namely that \( V_{pg}/V_g \propto Re^{-1/4} \) (Cuzzi et al. 1998). By contrast, it is similarly shown from equation (20) that the \( St_L \)-dependence of \( V_{pg} \) for the older \( n = 0 \) case continues the \( St_L/2 \) dependence to arbitrarily small \( St_L \).

These results are also consistent with arguments in Cuzzi et al. (1993, Appendix B; A. Dobrovolskis, personal communication). Expand and time-average the instantaneous quantity \( <(V_p - V_g)^2> \) to obtain \( <V_{pg}V_{pg}>=<V_pV_p> + <V_gV_g> - 2<V_pV_g> \). Substituting from Cuzzi et al. (1993, equation B11) we find \( <V_pV_p>=<V_gV_g>=<V_gV_g>/\left(1 + St_L\right) \), leading to \( V_{pg} = (St_L/(1 + St_L))^{1/2} V_g \), which reaches the same limits as equation (19) except for particles with \( t_s \leq t_\eta \), or \( St_\eta \leq 1 \), because the integral in its derivation (equation B11 of Cuzzi et al. 1993) extends to infinite eddy frequency.

Thus, unless \( t_s \leq t_\eta \) \( (St_L < Re^{-1/2}) \), the particle-gas relative velocity in turbulence is generally proportional to \( \sqrt{St_L} \) for small \( St_L \). The steeper dependence of \( V_{pg} \) on \( St_L \) and \( St_\eta \) is restricted (in turbulence) to particles with \( St_\eta \leq 1 \). That is, evidence for a more nearly linear dependence of \( V_{pg} \) on \( r \), if the environment was turbulent, would imply that the particles in question were \( St_\eta \leq 1 \) particles. This new result derives directly from the use of the \( n = 1 \) gas velocity autocorrelation function. The primary qualitative change is in the particle size dependence of \( V_{pg} \) for particles with \( St_\eta \leq 1 \). We address the significance of this in more detail in a forthcoming paper (Cuzzi 2002b).

Finally, using equation (23) for \( V_{pp} \), we get

\[
V_{pp}(St_\eta = 1) = \sqrt{0.8} V_g Re^{-1/4} = 1.26 V_{pg},
\]  

(25)

where we used equation (24) for \( V_{pg} \).

2 Comparison with numerical results

In this section we compare numerical results from our full 3D Lagrangian particle-gas model (Hogan et al. 1999) with full numerical calculations using our implementation of MMV (sections 1.8-1.9). We present particle velocities relative to the computational box \( (V_p) \), and relative to the local fluid velocity \( (V_{pg}) \), as obtained from our simulations. These velocities are defined as RMS spatial averages over all particles in a single snapshot, or \( V = (\langle (V_x - <V_x>)^2 \rangle + \langle (V_y - <V_y>)^2 \rangle + \langle (V_z - <V_z>)^2 \rangle)^{1/2} \), where \( V \) represents \( V_p \) or \( V_{pg} \) at the location of each particle, and \( \langle \rangle \) is the averaging operator \( \langle ... \rangle = \sum_{i=1}^{N_p} (...) / N_p \), where \( N_p \) is the number of particles in a single snapshot. Of course, \( <V> \) is very close to zero for both these quantities since there is no mean flow in our simulations.
Figure 6: $V_p$ vs. $St_L$ obtained from our direct simulations compared with MMV predictions for models $n = 0$ (solid line) and $n = 1$ (dashed line). All velocities have been normalized by the RMS fluid velocity $V_g$. Results are shown for three different $Re$; the $St_L$ values for each point are defined relative to a large eddy time based on energy dissipation\(^2\), which varies with $Re$ for our numerical calculations. When they are defined relative to a constant large eddy time, as are our analytical models and the MMV models shown in figure 2, points and models for all $Re$ collapse onto the same curve as seen in figure 2. The $n = 1$ MMV prediction is clearly a better fit to the numerically simulated velocities, regardless of the choice of normalization timescale.

This spatial averaging approach is equivalent to the temporal averaging implicit in the MMV model, because of the ergodic principle that equates temporal and spatial averaging under suitable conditions. In our case, the conditions are satisfied because our integral length scale $L$ is small compared to the spatial period of the computational domain, for all $Re$.

The case of $V_{pp}$ is more complicated, as the results depend on the proximity region chosen for “neighboring” particles. For the most useful comparisons with the predictions of MMV and VJMR, and with the expected uses of this quantity in mind, the region over which particle neighbors are selected should be as small as possible - less than $\eta$ certainly - and here we run into sampling errors. Perhaps most important, the deviation of our model energy spectrum from a Kolmogorov spectrum is significant (e.g. Squires and Eaton 1990), and $V_{pp}$ is much more sensitive to the details of the high-spatial-frequency end of the energy spectrum than either $V_p$ or $V_{pg}$. Since the main purpose of these calculations is to verify numerically the preference for the $n = 1$ autocorrelation function in an independent way from the direct comparison shown in figure 1, and because this case is already well made by the $V_p$ and $V_{pg}$ plots, we present no comparisons for $V_{pp}$.

Figures 6 - 9 show that the $n = 1$ autocorrelation function provides a much better
Figure 7: \( v_p \) as in figure 6, but plotted against \( St_\eta \).

Figure 8: \( v_{pg} \) obtained from our direct simulations, compared with predictions of the MMV models with \( n = 0 \) (solid line) and \( n = 1 \) (dashed line) vs \( St_L \). All curves have been normalized by the RMS fluid velocity \( V_p \).
fit to both $V_p$ and $V_{pg}$ than the $n = 0$ version. For $V_{pg}$, the fits of the MMV theory to our simulations are less perfect than for $V_p$. We can see several possible explanations for this. For instance, the mathematically simple form adopted for the $n = 1$ autocorrelation function is not a perfect fit to the actual numerically determined one (figure 1), by about the correct fractional amount. Also, we have emphasized that the correct velocity autocorrelation function to use is that along a particle trajectory (Meek and Jones 1973), and this function is actually somewhat size dependent even over the range $St_\eta \sim 1$ (see, eg., Squires 1990, figure 4-23). Finally, because of the deviation of our turbulent kinetic energy spectrum from an inertial range, some of the definitions of eddy times used in the MMV theory might be inappropriate. It would not be surprising for $V_{pg}$ to be more sensitive to these small deviations than $V_p$ (compare figures 6 and 8, or 7 and 9). In spite of the small deviations in $V_{pg}$, the combination of the direct comparisons of the autocorrelation functions themselves (figure 1), and the comparison of the velocities derived using them (figures 6-9) makes it clear that the $n = 1$ autocorrelation function is the best choice.

3 Summary and conclusions

We present theoretical and numerical results which describe the turbulence-driven velocities of particles in the $St_L << 1$ size regime which might characterize chondrules and similar sized particles. We numerically verify the general approach of VJMR as modified by MMV, and verify in two different ways the intuitive preference of MMV for an $n = 1$ gas velocity autocorrelation function - at least along the trajectories of $St_\eta \approx 1$ particles. We find theoretically that the $n = 1$ autocorrelation function leads to a particle-gas relative velocity function that approaches linear dependence on particle size for particles in the $St_\eta \approx 1$
regime, and becomes and remains linear for arbitrarily small sizes. This is quite a different result than predicted by the original VJMR $n = 0$ expressions. We derive simple analytic expressions for $V_p$, $V_{pg}$, and $V_{pp}$ (the latter, for comparable size particles only) for arbitrary levels of nebula intensity, as characterized by its Reynolds number $Re$ or its corresponding “$\alpha$”. In a separate paper (Cuzzi 2002b) we will present some implications of these results for meteoritics.

Acknowledgements: We are grateful to Sandy Davis for helpful discussions on matters mathematical, to Robert Last for computational and graphics support, and to Wojtek Markiewicz for helpful discussions concerning MMV. We thank Ignacio Mosqueira, Steve Desch, Pat Cassen, and Tony Dobrovolskis for careful reading of the manuscript and helpful comments. This research was supported by Grants to JNC from the Planetary Geology and Geophysics Program and the Origins of Solar Systems Program.
Table I. List of symbols

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>gas molecule thermal speed</td>
</tr>
<tr>
<td>$C$</td>
<td>particle concentration factor</td>
</tr>
<tr>
<td>$E(k)$</td>
<td>turbulent kinetic energy at wavenumber $k$</td>
</tr>
<tr>
<td>$H$</td>
<td>nebula vertical scale height</td>
</tr>
<tr>
<td>$k$</td>
<td>eddy wavenumber</td>
</tr>
<tr>
<td>$k_L$</td>
<td>wavenumber of largest eddy</td>
</tr>
<tr>
<td>$k_\eta$</td>
<td>wavenumber of Kolmogorov scale eddy</td>
</tr>
<tr>
<td>$L$</td>
<td>integral or largest scale in turbulent energy spectrum</td>
</tr>
<tr>
<td>$r$</td>
<td>particle radius</td>
</tr>
<tr>
<td>$R$</td>
<td>gas velocity autocorrelation function</td>
</tr>
<tr>
<td>$Re$</td>
<td>flow Reynolds number</td>
</tr>
<tr>
<td>$St_L$</td>
<td>Stokes number relative to largest eddy</td>
</tr>
<tr>
<td>$St_\eta$</td>
<td>Stokes number relative to Kolmogorov scale eddy</td>
</tr>
<tr>
<td>$t_s$</td>
<td>stopping time of particle due to gas drag</td>
</tr>
<tr>
<td>$t_k$</td>
<td>overturn time of eddy with wavenumber $k$</td>
</tr>
<tr>
<td>$t_L$</td>
<td>overturn time of largest eddy</td>
</tr>
<tr>
<td>$t_\eta$</td>
<td>overturn time of Kolmogorov scale eddy</td>
</tr>
<tr>
<td>$V_g$</td>
<td>gas turbulent velocity (large eddy)</td>
</tr>
<tr>
<td>$V_p$</td>
<td>particle random velocity in inertial space</td>
</tr>
<tr>
<td>$V_{pg}$</td>
<td>relative velocity between particles and gas</td>
</tr>
<tr>
<td>$V_{pp}$</td>
<td>relative velocity between particles</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>nebula viscosity parameter; $Re = \alpha c H / \nu$</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>dissipation of turbulent kinetic energy</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Kolmogorov scale</td>
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<tr>
<td>$\nu$</td>
<td>molecular kinematic viscosity</td>
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<tr>
<td>$\nu_T$</td>
<td>turbulent kinematic viscosity</td>
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<tr>
<td>$\omega$</td>
<td>eddy temporal frequency</td>
</tr>
<tr>
<td>$\rho_g$</td>
<td>gas mass density</td>
</tr>
<tr>
<td>$\rho_s$</td>
<td>solid particle mass density</td>
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</tbody>
</table>
4 References


Meek, C. C. and B. G. Jones (1973) Studies of the behavior of heavy particles in a turbulent fluid flow; J. Atm. Sciences, 30, 239-244


