A NONLINEAR, SIX-DEGREE OF FREEDOM, PRECISION FORMATION CONTROL ALGORITHM, BASED ON RESTRICTED THREE BODY DYNAMICS

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Precision Formation Flying is an enabling technology for a variety of proposed space-based observatories, including the Micro-Arcsecond X-ray Imaging Mission (MAXIM), the associated MAXIM Pathfinder mission, and the Stellar Imager. An essential element of the technology is the control algorithm. This paper discusses the development of a nonlinear, six-degree of freedom (6DOF) control algorithm for maintaining the relative position and attitude of a spacecraft within a formation. The translation dynamics are based on the equations of motion for the restricted three body problem. The control law guarantees the tracking error convergences to zero, based on a Lyapunov analysis. The simulation, modelled after the MAXIM Pathfinder mission, maintains the relative position and attitude of a Follower spacecraft with respect to a Leader spacecraft, stationed near the L2 libration point in the Sun-Earth system.

INTRODUCTION

Distributed Spacecraft Systems (DSS) represent the future architecture for constructing very large, space-based observatories. Widely varying science goals lead to an equal assortment of design concepts and requirements. Stellar Imager, the Micro-Arcsecond X-ray Imaging Mission (MAXIM), and the associated MAXIM Pathfinder, represent a subset of missions requiring precision formation flying. The spacecraft are expected to be stationed at the Earth-Sun L2 point, or in a Earth drift-away orbit about the Sun. In both cases the Restricted Three Body Problem (RTBP) provides a natural context for design and analysis of the formation control law. Mission success will depend on the development of many supporting technologies.

One essential technology element is the algorithm for both relative position and attitude control. Traditionally, spacecraft control algorithms treat the orbital dynamics and attitude dynamics as uncoupled. However, precision formation flying requires continuous low thrust to achieve a sub-millimeter mission design criteria. Continuous thruster action introduces a coupling action between orbital trajectory control and attitude control. With proper design the thrusters can serve as actuators to simultaneously control the spacecraft orbit and attitude trajectories. Implementation of this control strategy requires a six-degree of freedom (6DOF) control law. This paper proposes a 6DOF nonlinear control law to solve this problem building on previous work of the authors, which focused on the 3DOF, orbital control problem.4,5

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Refer to the Notation section at the end of this paper for definitions of mathematical symbols not provided as part of the discussion.

**PROBLEM STATEMENT**

Consider a two spacecraft formation stationed in the vicinity of L2 in the Earth/Moon - Sun rotating frame, as shown in Figure 1. The spacecraft designations are Leader and Follower. The Earth-Moon system is modelled as a combined mass located at the system center of mass. The position and relative motion of the Earth/Moon system and the Sun define the rotating frame. The Leader maintains a planned ballistic trajectory with periodic station-keeping maneuvers, and a predetermined attitude trajectory. Attitude control on the Leader is accomplished with reaction wheels to avoid perturbing its orbit trajectory. The Follower tracks a specified separation trajectory, $x_d$, relative to the Leader, and a predetermined attitude trajectory, $q_d$. Measurement data provides the relative position and velocity between the two spacecraft. Each spacecraft is equipped to measure attitude, referenced to an inertial coordinate frame. The Follower is equipped with thrusters to serve as actuators for both translation and attitude maneuvers.

The problem:

- Design a control algorithm for the Follower to track a desired trajectory, $x_d$ and $q_d$.
- Demonstrate the tracking error approaches zero, at least asymptotically.
- Characterize the stability properties, i.e. local versus global stability.

**SYSTEM DYNAMICS AND KINEMATICS**

This section provides a brief review of both the dynamics and kinematics of translation and rotation for a spacecraft.

**Translation**

The following analysis is extracted from Refs. 4 and 5. Given a spacecraft stationed near the Sun-Earth/Moon L2 point, the principle environmental forces are gravity and solar pressure. These
forces, combined with thruster action, drive the spacecraft dynamics. The principal gravitational
sources are the Sun and the Earth-Moon system. In comparison, the mutual gravitational interaction
between the spacecraft is insignificant. Based on the reference vectors shown in Figure 1, the Leader
dynamics (per unit mass) are given by Eq. (1). The Follower dynamics per unit mass are given by
Eq. (2).

\[
\begin{align*}
\mathbf{F}_L &= -\mu_m \frac{r_{EL}}{|r_{EL}|^3} - \mu_s \frac{r_{SL}}{|r_{SL}|^3} + f_{\text{solar}, L} + f_{\text{pert}, L} + u_{\text{thrust}, L} \\
\mathbf{F}_F &= -\mu_m \frac{r_{EF}}{|r_{EF}|^3} - \mu_s \frac{r_{SF}}{|r_{SF}|^3} + f_{\text{solar}, F} + f_{\text{pert}, F} + u_{\text{thrust}, F}
\end{align*}
\]

The relative motion of the Follower with respect to the Leader is computed as the difference of Eqs.
(1) and (2).

\[
\mathbf{x} = \mathbf{F}_F - \mathbf{F}_L = -\mu_m \frac{r_{EF}}{|r_{EF}|^3} - \mu_s \frac{r_{SF}}{|r_{SF}|^3} + f_{\text{solar}, F} + f_{\text{pert}, F} + u_{\text{thrust}, F} \\
- \left( -\mu_m \frac{r_{EL}}{|r_{EL}|^3} - \mu_s \frac{r_{SL}}{|r_{SL}|^3} + f_{\text{solar}, L} + f_{\text{pert}, L} + u_{\text{thrust}, L} \right)
\]

\[
\mathbf{x} = -\mu_m \left( \frac{r_{EF}}{|r_{EF}|^3} - \frac{r_{EL}}{|r_{EL}|^3} \right) - \mu_s \left( \frac{r_{SF}}{|r_{SF}|^3} - \frac{r_{SL}}{|r_{SL}|^3} \right)
+ \Delta f_{\text{solar}} + \Delta f_{\text{pert}} + u_{\text{thrust}, F} - u_{\text{thrust}, L}
\]

The terms are arranged to resolve the main gravitational forces into three components along the
vectors, \(\mathbf{x}, r_{EL},\) and \(r_{SL}.\) The typical formation under consideration represents a space-based
observatory. Therefore, the desired, \(x_d,\) is assumed constant in magnitude and direction with respect
to inertial space. On the short time scale for control the spacecraft position will remain essentially
fixed in the rotating frame, Figure 1. Therefore, the gravitational component along \(\mathbf{x},\) remains
constant in inertial space, except during reorientation. The other components remain constant in
the rotating frame of the RTBP. Theoretically, the magnitude and direction of these terms could be
computed based on the position of the spacecraft relative to the Earth, Sun and other gravitational
bodies, allowing direct compensation for the gravitational gradient between spacecraft in the control
design. However, in practice precise position data is not available for spacecraft stationed at L2, or a
heliocentric drift away orbit. Therefore, knowledge of the magnitude and direction of these terms is
considered unknown. So, with \(\Theta_1\) and \(\Theta_2\) representing the unknown constant vectors in the inertial
and RTBP frames, respectively, Eq. (3) can be expressed as:

\[
\mathbf{x} = -I_3 \ast \Theta_1 - A_{\mathbf{r}} \ast \Theta_2 + \Delta f_{\text{solar}} + \Delta f_{\text{pert}} + u_{\text{thrust}, F} - u_{\text{thrust}, L}
\]

As a further simplification, the perturbing forces due to solar pressure and other gravitational
sources are also modelled as unknown, constant components, one inertially fixed, the other fixed in
the RTBP frame. The assumption is reasonable. Solar pressure acts along the spacecraft to sun
line, which is fixed in the RTBP rotating frame. As with the Sun and Earth/Moon, the contribution
of other gravitational sources can be resolved into two components, one in each of the inertial and
RTBP frames. These components also remain essentially constant under the stated assumptions.
Therefore, the terms, \(\Delta f_{\text{solar}}\) and \(\Delta f_{\text{pert}}\), are absorbed into the terms, \(\Theta_1\) and \(\Theta_2.\) Then, Eq. (4)
is rewritten as:
The equation for rotational dynamics of a rigid spacecraft without reaction wheels is given by:

\[ \dot{H} \otimes \omega - S([H \otimes \omega]) \otimes \omega = \tau \]  
(6)

where: \( S([H \otimes \omega]) \) is the skew symmetric matrix formed by the vector, \([H \otimes \omega]\).

In general, the rotational kinematics are given by:

\[ \dot{q} = \frac{1}{2} \left[ \eta \otimes I_3 + S(\varepsilon) \right] \otimes \omega \]  
(7)

### CONTROL LAW DESIGN

The proposed control law design combines algorithms for control of translation from Ref. 5, and rotation from Ref. 6.

### Translation

The control design is based on nonlinear adaptive theory developed for robotic applications. Under the assumption that Eq. (5) represents the relative dynamics, Ref. 4 presents an adaptive control strategy that provides globally stable, perfect tracking of a desired smooth trajectory for a spacecraft with a known mass. The analysis assumes perfect knowledge of the motion of the Follower with respect to the Leader. Terms used in the following discussion are defined as:

- \( m_F \) - Mass of Follower
- \( \hat{m}_F \) - Estimated Mass of Follower
- \( x_d \) - Desired spacecraft separation vector
- \( \Gamma_T \) - \([ I_3 \ I_{A_1}^T \]
- \( \Theta_T \) - \([ \Theta_1 \ \Theta_2 ]\)
- \( \hat{\Theta}_T \) - Estimate value of \( \Theta_T \)
- \( \ddot{x}^r \) - \((x_d - A_\alpha* (x - x_d))\), reference velocity
- \( s_T \) - \((\ddot{x} - \ddot{x}_d) + \Lambda_T * (x - x_d) = \ddot{x}^r - \ddot{x}, \) error metric
- \( \alpha \) - Design parameter, adaptive gain, positive constant
- \( K_T \) - Design parameter, symmetric, uniformly positive definite matrix
- \( \Lambda_T \) - Design parameter, constant, symmetric, positive definite matrix

The proposed control law for differential thrust per unit mass is:

\[ u_{\text{thrust},F} - u_{\text{thrust},L} = \ddot{x}^r + \left[ I_3 \ | \ A_1^T \right] \left[ \begin{array}{c} \dot{\Theta}_1 \\ \dot{\Theta}_2 \end{array} \right] - K_T * s_T = \ddot{x}^r + \Gamma_T * \dot{\Theta}_T - K_T * s_T \]  
(8)

\[ \dot{\Theta}_T = -\gamma * \Gamma_T^T * s_T \]  
(9)

For a spacecraft of known mass, Eqs. (5) and (8) combine to form the expression for the tracking error dynamics:
The error dynamics, Eq. (10), combined with the adaptive rule, Eq. (9), results in exponentially stability of the tracking error. The proof is provided by Ref. 4.

With the additional restriction that the desired acceleration, \( \ddot{x}_d = 0 \), ref. 5 extends this result, demonstrating that the above algorithm is still valid for a spacecraft with uncertain mass. For a spacecraft of unknown/estimated mass, Eqs. (5) and (8) combine to form the expression for the tracking error dynamics:

\[
\ddot{x} = -\Gamma_x \cdot \Theta_T + \frac{m_T}{m_p} (\dot{x} + \Gamma_x \cdot \dot{\Theta}_T - K_x \cdot s_T')
\]

(11)

Where the error metric is redefined as, \( s_T' = (\ddot{x} - \dot{x}_d) + \sigma \cdot \Lambda \cdot (x - x_d) \), with \( \sigma = \frac{m_T}{m_p} \). Further, define \( \tilde{\Theta}_T = \Theta_T - \Theta_T \). Then Eq. (11) is rewritten as:

\[
(\ddot{x} - \ddot{x}_d) + \sigma \cdot \Lambda \cdot (\dot{x} - \dot{x}_d) = \Gamma_x \cdot (\sigma \cdot \tilde{\Theta}_T - \Theta_T) - (1 - \sigma) \cdot \ddot{x}_d - K_x \cdot s_T'
\]

(12)

Under the stated assumption, \( \ddot{x}_d = 0 \), the error dynamics are simplified as:

\[
\dot{s}_T' = \Gamma_x \cdot \tilde{\Theta}_T - K_x \cdot s_T'
\]

(13)

Coupled with the following modified adaptive rule the equations assume the same form as Eqs. (8) and (9).

\[
\dot{\Theta}_T = -\gamma \cdot \Gamma_x^T \cdot s_T
\]

(14)

Exponential stability of the tracking error is directly inferred from the Lyapunov proof presented in Ref. 4. Also, \( \tilde{\Theta}_T \rightarrow 0 \), which implies \( \tilde{\Theta}_T \rightarrow \frac{1}{\sigma} \Theta_T \), versus \( \Theta_T \).

As a final note, the modified adaptive rule and error metric, \( s_T' \), facilitate the proof of exponential stability. As the value of \( \sigma \) is unknown, it cannot be directly factored into the algorithm design. However, the control law provided in Eq. (8), combined with the adaptive rule, Eq. (9), are sufficient to generate the desired result. Effectively, the term, \( (\sigma \cdot \Lambda) \), is treated as a single design parameter for defining \( s_T' \).

**Rotation**

Ref. 6 presents a passivity-based attitude control algorithm for a rigid spacecraft. The formulation for a spacecraft with gas jet actuators is applied to this design. Terms used in the following discussion are defined as:

- \( \omega_d \) - Desired Spacecraft Angular Rate
- \( q_d = [e_d \eta_d]^T \) - Desired Spacecraft Attitude Quaternion
- \( \bar{q} = [\bar{e} \bar{\eta}]^T = q \otimes q_d^{-1} \), error quaternion
- \( \omega' = (\omega_d - \lambda_x \cdot \bar{e}) \), reference angular rate
- \( s_R = (\omega - \omega') \), angular rate error metric
- \( K_R \) - Design parameter, symmetric, uniformly positive definite matrix
- \( \Lambda_R \) - Design parameter, constant, symmetric, positive definite matrix
Based on the rotational dynamics, Eq. (6), the control law is defined as:

$$\tau = H_r \cdot \dot{\omega} - S([H_r \cdot \omega]) \cdot \omega - K_r \cdot \omega_r$$

(15)

Implementation of this algorithm requires knowledge of the spacecraft moment of inertia, $H_r$, which is limited by measurement errors and mass property variations. Therefore, an adaptive control strategy, designed to estimate the mass properties, is preferred. This is accomplished by reformulating Eq. (15) as:

$$\tau = \Gamma_r (q, \omega, \omega_r, \omega^r) \cdot \Theta_r - K_r \cdot \omega_r$$

(16)

Where:

$$\Gamma_r (q, \omega, \omega_r, \omega^r) \cdot \Theta_r = H_r \cdot \dot{\omega} - S([H_r \cdot \omega]) \cdot \omega$$

(17)

The control, Eq. (16), combined with the adaptive rule, Eq. (18), provides global convergence of the tracking error to zero.

$$\dot{\Theta}_r = -\Lambda_r \cdot \Gamma_r (q, \omega, \omega_r, \omega^r)^T \cdot \omega_r$$

(18)

Simultaneous implementation of the above control algorithms requires coordinated thruster action to generate the desired translation and rotation control inputs. The relationship between the thruster output and the net control inputs is expressed as:

$$bU = R(q) \cdot (u_{\text{thr.}}, -u_{\text{thr.},\text{L}})$$

(19)

where:

- $bU$ – Combined, translation and rotation control in body coordinates
- $R(q)$ – Transformation from inertial to body coordinates
- $B$ – Control sensitivity matrix
- $F$ – Thruster output, $[f_1, f_2, \ldots, f_n]^T$

With proper placement of the thrusters the control sensitivity matrix, $B$, will have a pseudo inverse. Then, the thruster commands are computed as:

$$F = B^T (B \cdot B^T)^{-1} \cdot bU$$

(20)

This implementation for computing thruster commands generates the desired thrust and torque for translation and rotation control. Therefore, the convergence and stability properties of the previously discussed algorithms are retained, and further analysis/proof is not required.
The performance of the control design is demonstrated through a simulation based on the MAXIM Pathfinder mission. The spacecraft are initialized in an orbit about the L2 point in the Earth/Moon-Sun system. The leader spacecraft remains on a ballistic trajectory, $u_{\text{bran}, \text{L}} = 0$. The scenario starts with the Follower at the same position and attitude as the Leader, i.e. initial deployment. The Follower tracks a command trajectory to a 200 km separation, Fig. 2, while simultaneously tracking a 90 degree slew maneuver, Fig. 3. The simulation is implemented in MATLAB®.

The successful controller performance is evidenced by the tracking error, shown in Figs. 4 and 5. The controller gains are not designed for optimal performance.

Figure 2: Desired Relative Separation between Follower and Leader

Figure 3: Euler Angle of Desired Attitude, $q_d$, of Follower with Respect to Inertial
CONCLUSIONS AND FUTURE WORK

This paper presents a successful adaptive, nonlinear 6DOF formation control strategy for spacecraft experiencing slow changes in environmental forces, typical for an orbit about L2. Adaptation is employed to estimate the spacecraft mass properties. Stellar Imager, MAXIM and MAXIM Pathfinder are typical missions with potential for employing this strategy. The development assumes the availability of perfect measurement data. Further thruster performance is ideal. Future work will consider issues associated with corrupted measurement data, and issues associated with thruster alignment and performance, and coupled stability issues with an observer/estimator.
NOTATION

Symbols for additional terms are defined in the following table. All vectors are resolved in inertial coordinates, unless otherwise stated.

- $f_{solar}$ – Force exerted on spacecraft due to solar pressure
- $f_{pert}$ – Force exerted on spacecraft due to other perturbations
- $u_{thrust}$ – Force exerted on spacecraft due to thrusters
- $I_3$ – Identity Matrix, 3x3
- $A_{ri}$ – Rotation Matrix from Inertial to Earth/Moon - Sun Rotating Frame
- $\mu_{em}$ – Gravitational parameter of the Earth/Moon
- $\mu_s$ – Gravitational parameter of the Sun
- $H_n$ – Spacecraft Moment of Inertia
- $\omega$ – Spacecraft Angular Rate
- $\tau$ – External Torque on Spacecraft
- $q = [e \eta]^T$ – Spacecraft Attitude Quaternion
- $e$ – Vector Component of Spacecraft Attitude Quaternion
- $\eta$ – Scalar Component of Spacecraft Attitude Quaternion
- $S(v)$ – Skew-symmetric matrix of vector, $v$. $S(v) \bullet w$, equivalent to cross product, $v \times w$.

REFERENCES