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CALCULATION OF TURBULENT EXPANSION PROCESSES

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CALCULATION OF TURBULENT EXPANSION PROCESSES*

By Walter Tollmien

On the basis of certain formulas recently established by L. Prandtl for the turbulent interchange of momentum in stationary flows (reference 1), various cases of "free turbulence" - that is, of flows without boundary walls - are treated in the present report. Prandtl puts the apparent shearing stress introduced by the turbulent momentum interchange

$$\tau_{xy} = \rho l^2 \left|\frac{du}{dy}\right| \frac{du}{dy} \tag{1}$$

where

u average velocity in x direction

y coordinate at right angle to x

l mixing length

The underlying reasoning is as follows: The fluid bodies entering right and left through a fluid layer with the time average value of the velocity u, at turbulence, have the average velocity $u + l\,\frac{du}{dy}$ or $u - l\,\frac{du}{dy}$, while the transversely directed mixing velocity is $l\left|\frac{du}{dy}\right|$, discounting a constant of proportionality included in the more or less accurately known l of formula (1); l is no constant - at a wall $l = 0$. The previously cited report by Prandtl (reference 1) contains a lucid foundation for formula (1).

The present report deals first with the mixing of an air stream of uniform velocity with the adjacent still air, then

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*"Berechnung turbulenter Ausbreitungsvorgänge." Reprint from Zeitschrift für angewandte Mathematik und Mechanik, vol. 6, 1926, pp. 1-12.

with the expansion or diffusion of an air jet in the surrounding air space. Experience indicates that the width of the mixing zone increases linearly with x, if x is the distance from the point where the mixing starts. This fact is taken into account by the formula

$$l = cx \qquad (2)$$

The constant of proportionality c can as yet be determined only by comparison with experience; it is the only empirical constant of the theory. In many instances it will be expedient to introduce $\eta = y/x$ as a second coordinate.

1. MIXING OF HOMOGENEOUS AIR STREAM WITH THE ADJACENT STILL AIR

(Two-dimensional problem of the free jet boundary)

By reason of the limiting conditions for the average velocity the formula is preferably expressed with

$$u = f(y/x) = f(\eta) \qquad (3)$$

Then the stream function is

$$\psi = \int f\left(\frac{y}{x}\right) dy \qquad (4)$$

$$= x \int f(\eta)\, d\eta = xF(\eta)$$

hence

$$v = -\frac{\partial\psi}{\partial x} = -F(\eta) + \eta\, F'(\eta)$$

Quantity τ is put according to formulas (1) and (2)

$$\frac{\tau}{\rho} = c^2\, x^2 \left|\frac{du}{dy}\right| \frac{du}{dy}$$

The following boundary conditions exist: At the first boundary η_1 (homogeneous air stream), u = constant or by introduction of a suitable scale $u = 1$; that is,

$$F'(\eta_1) = 1 \tag{5}$$

furthermore

$$\frac{\partial u}{\partial \eta} = 0$$

a condition by which the continuous connection is secured - that is,

$$F''(\eta_1) = 0 \tag{6}$$

$$v(\eta_1) = 0$$

that is,

$$F(\eta_1) = \eta_1 \tag{7}$$

at the second boundary η_2 (still air) must be $u = 0$; that is,

$$F'(\eta_2) = 0 \tag{8}$$

and, to assure continuous connection $\frac{\partial u}{\partial \eta} = 0$; that is,

$$F''(\eta_2) = 0 \tag{9}$$

Since the pressure, in first approximation, can be assumed to be constant, the equation of motion reads

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial \tau_{xy}}{\partial y}$$

Counting y and η from the still toward the moving air, gives, after introduction of the formulas*, the equation of motion:

$$FF'' + 2c^2 F''F''' = 0 \tag{10}$$

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*It is readily apparent at this point, that the formula $u = f(\eta)$, necessarily requires an l proportional to x.

which is solved by $F'' = 0$ or $F + 2c^2 F''' = 0$. It affords uniform velocity in the one case, variable velocity in the other. The latter solution obviously applies between η_1 and η_2, the former, outside of these limits. In the boundary points the solutions coincide with discontinuity in F''. In order to determine the velocity distribution in the mixing zone, the differential equation of the third order

$$F + 2c^2 F''' = 0 \qquad (11)$$

must be solved. For the time being, a new scale for η is advisable, so that formula (11) simplifies to

$$F + F''' = 0 \qquad (11a)$$

The result then is

$$F = c_1\, e^{-\eta} + c_2\, e^{\eta/2} \cos \frac{\sqrt{3}}{2}\eta + c_3\, e^{\eta/2} \sin \frac{\sqrt{3}}{2}\eta$$

The five boundary conditions define the constants of integration c_1, c_2, c_3, and, in addition, the still unknown boundary points η_1 and η_2 themselves.

The calculation is suitably arranged as follows: Put

$$\bar{\eta} = \eta - \eta_1 \quad \text{or} \quad \eta = \bar{\eta} + \eta_1$$

so that

$$F = d_1\, e^{-\bar{\eta}} + d_2\, e^{\bar{\eta}/2} \cos \frac{\sqrt{3}}{2}\bar{\eta} + d_3\, e^{\bar{\eta}/2} \sin \frac{\sqrt{3}}{2}\bar{\eta}$$

The boundary conditions are:

$$F(\bar{\eta}) = \eta_1, \quad F'(\bar{\eta}) = 1, \quad F''(\bar{\eta}) = 0 \quad \text{for} \quad \bar{\eta} = 0, \quad F'(\bar{\eta}) = 0,$$

$$F''(\bar{\eta}) = 0 \quad \text{for} \quad \bar{\eta}_2 \qquad (12a\text{-}e)$$

From (12a to c), d_1, d_2, and d_3 can be linearly expressed in η_1, equation (12d) yields η_1, expressed by $\overline{\eta}_2$, and (12e) finally gives a transcendental equation for η_2, solvable by successive approximation. It follows that

$$\overline{\eta}_2 = -3.02, \quad \eta_1 = 0.981, \quad \eta_2 = -2.04, \quad d_1 = -0.0062,$$

$$d_2 = 0.987, \quad d_3 = 0.577$$

With this F and F' are defined as function of η. For comparison with experience, the original scale must be used - that is, the reduced η, employed thus far, must be multiplied by $\sqrt[3]{2c^2}$, and the reduced F' by U, the velocity of the homogeneous air stream. The curves of the velocities and of the streamlines are given in the table and in figures 2 to 4. The streamlines are plotted for equidistant values of the stream function. The streamline emanating from $x = 0$ is a straight line with angle of inclination

$$-\tan^{-1} (0.19 \sqrt[3]{2c^2})$$

For comparison, a Göttingen measurement (reference 2) of the dynamic pressure distribution with an automatic pressure recorder at the edge of the nozzle of the big tunnel was employed. The distance from the nozzle edge was 112 centimeters, the dynamic pressure of the undisturbed jet $q = 56$ kilograms per square meter. It can be presumed that the assumptions of the two-dimensional problem hold good for the size of the nozzle. In figure 5 the calculated dynamic pressure is shown as a dashed line over the measured dynamic pressure. The unknown constant of proportionality c follows from the conversion factor for η, which is $\sqrt[3]{2c^2} = 0.0845$. The width of the mixing zone is

$$b = \sqrt[3]{2c^2} \times 3.02x = 0.0845 \times 3.02x = 0.255x$$

giving a mixing length of

$$l = 0.0174\,x = 0.0682\,b$$

The relative smallness of l is unusual. The agreement between the theoretical and the measured average velocity distribution is very good.

2. JET EXPANSION AS TWO-DIMENSIONAL PROBLEM

Visualize a wall with a narrow slot, which for the study may be regarded as being linear, through which a jet of air is discharged and mixes with the surrounding still air. Assuming, for the first, that the pressure in the jet is the same as outside, the application of the momentum theorem affords a ready separation of the variables - that is, x and η. By reason of the constant pressure the momentum in x direction must be

$$\int_{-\infty}^{+\infty} u^2 \, dy = \text{constant}$$

Putting $u = \varphi(x)\, f(\eta)$ results in

$$\varphi^2(x) \times \int_{-\infty}^{+\infty} f^2(\eta)\, d\eta = \text{constant}$$

Consequently $\varphi(x) = \frac{1}{\sqrt{x}}$

$$u = \frac{1}{\sqrt{x}} f(\eta) \tag{13a}$$

$$\psi = \int \frac{1}{\sqrt{x}} f(\eta) dy = \sqrt{x} \int f(\eta) d\eta = \sqrt{x}\, F(\eta) \tag{13b}$$

$$v = -\frac{1}{2\sqrt{x}} F(\eta) + \frac{1}{\sqrt{x}} F'(\eta)\, \eta \tag{13c}$$

The equation of motion can be set up again, which now, however, can be immediately integrated once. This intermediate integral can equally be obtained direct from the momentum theorem. By marking off a control surface conforming to figure 6, the impulse entering through the lower boundary is $\rho u v$, while on the other side the impulse variation

$$\rho \frac{\partial}{\partial x} \int_{\infty}^{y} u^2 \, dy$$ occurs. The turbulent shearing stress

$$\tau = \rho c^2 x^2 \left|\frac{\partial u}{\partial y}\right| \frac{\partial u}{\partial y}$$

acts as outside force, hence the relation exists

$$uv + \frac{\partial}{\partial x} \int_{\infty}^{y} u^2 \, dy = \frac{\tau}{\rho}$$

From this follows the equation for $F(\eta)$:

$$2c^2 F''^2 = F F' \tag{14}$$

(valid for positive η, reflected velocity distribution for negative η). With a suitable scale for η the differential equation is simplified to

$$F''^2 = F F' \tag{14a}$$

The order of this differential equation can be lowered by introducing $z = \ln F$; that is, $F = e^z$ as new dependent variable. Then, $(z'' + z'^2)^2 = z'$ whence, after putting $z' = Z$, finally follows the differential equation of the first order:

$$Z' = -Z^2 - \sqrt{Z}$$

The solution of the original equation then requires only squaring and removal of the logarithms.

The following conditions must be satisfied: For $\eta = 0$ (center of jet), $v = 0$ - that is, $F = e^z = 0$. Since $u \sim F' = z'e^z$ is not to disappear for $\eta = 0$, z' must be of the same order of ∞ for $\eta = 0$ as e^z is of 0. Hence, for $\eta = 0$ by suitable scale determination,

$$F = 0 \tag{15a}$$

$$F' = 1 \tag{15b}$$

Thus there are afforded two conditions through which the z, that satisfies an equation of the second order, is completely defined.

The boundary point η_r itself then follows from the condition

$$u = 0; \quad \text{that is,} \quad z' = Z = 0 \tag{16}$$

for the boundary η_r.

Integration of equation (14) gives

$$\eta = C - \frac{2}{3}\left\{\ln\left(\sqrt{Z} + 1\right) - \ln\left[\left(Z - \sqrt{Z} + 1\right)^{1/2}\right] + \sqrt{3}\tan^{-1}\frac{2\sqrt{Z}-1}{\sqrt{3}}\right\} \tag{17}$$

The constant of integration C follows from the condition $z' = Z = \infty$ for $\eta = 0$:

$$0 = C - \frac{2}{3}\sqrt{3}\,\frac{\pi}{2} \quad \text{at} \quad C = \frac{\pi}{\sqrt{3}}$$

The condition (16) $Z = 0$ for η_r yields

$$\eta_r = C - \frac{2}{3}\sqrt{3}\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = \frac{4}{3}\frac{\pi}{\sqrt{3}} = 2.412$$

Compliance with equation (15) is predicated on a study of the behavior of equation (14) for $\eta = 0$, $Z = \infty$. As the solution (17) in this range is inconvenient, a new form of solution which applies for $\eta = 0$; $Z = \infty$ is derived.

For $Z \to \infty$, obviously

$$\frac{dZ}{d\eta} \to -Z^2, \quad \text{that is,} \quad Z = z' \to \frac{1}{\eta}$$

hence

$$z \to \ln\eta + c_1, \quad \text{so that} \quad F' = z' e^z = e^{c_1}; \quad \text{for } \eta = 0.$$

Thus the last constant of integration follows from equation

(15b) as $c_i = 0$; and the asymptotic approximation for $\eta = 0$ follows at

$$z' = \frac{1}{\eta} - 0.4\sqrt{\eta} + 0.01\,\eta^2 \ldots, \quad z = \ln \eta - \frac{0.8}{3}\eta^{3/2} + \frac{0.01}{3}\eta^3 \ldots \tag{18}$$

The quality of the asymptotic approximation (18) is easily appraised by a comparison with the exact solution (17) in a zone in which both forms of solution are appropriate.

The method is as follows: Compute $\eta(Z)$ and hence $Z(\eta) = z'(\eta)$ by equation (17), thus obtaining $z(\eta)$, possibly by graphical integration, where in the region about $\eta = 0$, the previously determined asymptotic approximation (18) is taken into account and $z' = \infty$. Then the desired functions $F = e^z$ and $F' = z'e^z$ are obtained by removal of logarithms and multiplication.

The solution F = constant joins the just-derived solution with a discontinuity in F'' toward the outside. In the center ($\eta = 0$), F' acts as $1 - 0.4\,\eta^{3/2}$, which entails a disappearance of the radius of curvature.* For comparison with experience, it is necessary to revert from the reduced to the original quantities as shown in the table. The conversion factor for η is $\sqrt[3]{2c^2}$; the letter s, in the table, signifies a characteristic distance from the gap, where the speed in the center of the jet equals U_s. According to (13a) the speed at jet center distant x from the gap is then

$$U_m(x) = U_s\sqrt{\frac{s}{x}}\,.$$

(See table.)

3. JET EXPANSION AS ROTATIONALLY SYMMETRICAL PROBLEM

The corresponding rotationally symmetrical problem, in which a jet of air discharges from a very narrow hole in a wall, is treated in exactly the same manner as the two-

* Prandtl has given a refinement of the theory by which the disappearance of the radius of curvature in the center can be avoided. But, since it would lead too far afield, it is not discussed here.

dimensional problem. First, the variables x and η are easily separated again. For, on assuming that the pressure in the jet is constant,

$$2\pi \int_{-\infty}^{+\infty} u^2 y \, dy = \text{constant}$$

whence for u

$$u = \frac{1}{x} f(\eta)$$

Putting

$$\int f(\eta) \, \eta \, d\eta = F(\eta)$$

affords

$$u = \frac{F'}{x\eta}, \quad v = \frac{F'}{x} - \frac{F}{x\eta}$$

The differential equation for F is again obtained by integration of the equation of motion or by a second application of the impulse theorem in analogy to figure 6:

$$c^2 \left(F'' - \frac{F'}{\eta}\right)^2 = F F' \tag{19}$$

With the introduction of a suitable scale for η, the differential equation is simplified to

$$\left(F'' - \frac{F'}{\eta}\right)^2 = F F' \tag{19a}$$

By substitution:

$$z = \ln F, \quad F = e^z$$

there is afforded

$$\left(z'' + z'^2 - \frac{z'}{\eta}\right)^2 = z'$$

and lastly, after introducing $Z = z'$, the differential equation of the first order

$$Z' = \frac{z}{\eta} - Z^2 - \sqrt{Z} \tag{20}$$

In addition, the following conditions hold for $\eta = 0$: u may not disappear, while $v = 0$; that is, $F(0) = e^{z(0)} = 0$, while $\frac{F'}{\eta} = \frac{z'e^z}{\eta}$ remains finite and becomes equal to unity by appropriate regularization.

Now a series development of $Z(\eta)$ for $\eta = 0$ can be applied in such a way that these conditions are satisfied; z must be negative ∞ for $\eta = 0$, in order that $e^z = 0$, which is like $\ln \eta^2$; because F'/η then assumes precisely a finite value. The result is the following development in powers of $\eta^{3/2}$:

$$Z = \frac{2}{\eta} + a\sqrt{\eta} + b\eta^2 + c\eta^{7/2} + d\eta^5 + e\eta^{13/2}. \; . \; . \tag{21}$$

The coefficients are obtained by introduction of this formula in the differential equation and comparison of equal powers:

$$a = -\frac{2}{7}\sqrt{2}, \quad b = -\frac{1}{245}, \quad c = \frac{\sqrt{2}}{1715}, \quad d = \frac{37}{240100}, \quad e = 0.000014 \; . \; . \; .$$

The convergence is poor on approaching the boundary point $\eta_r(Z=0)$, but a development particularly suitable near η_r is as follows: Put

$$\bar{\eta} = \eta_r - \eta$$

and

$$Z = \bar{a}\,\bar{\eta}^2 + \bar{b}\,\bar{\eta}^3 + \bar{c}\,\bar{\eta}^4 + \bar{d}\,\bar{\eta}^5 + \bar{e}\,\bar{\eta}^6 + \bar{f}\,\bar{\eta}^7 \; . \; . \; .$$

and obtain

$$\bar{a} = \frac{1}{4}, \quad \bar{b} = -\frac{1}{8\,\eta_r}, \quad \bar{c} = -\frac{3}{64\,\eta_r^{2}}, \quad \bar{d} = \frac{1}{64} - \frac{3}{128\,\eta_r^{3}},$$

$$\bar{e} = -\frac{19}{256 \times 5\,\eta_r} - \frac{133}{256 \times 40\,\eta_r^{4}}, \quad \bar{f} = -\frac{0.00278}{\eta_r^{2}} + \frac{\dots}{\eta_r^{5}} \; . \; . \; .$$

The unknown constant of integration η_r is obtained by making the values for Z, as known from the two developments, agree in a certain junction point. It results in $\eta_r = 3.4$.

Quantity F'/η acts like $1 - 0.202\,\eta^{3/2}$ in the center; the outward junction in F again takes place with a discontinuity in F'''. The conversion factor from the reduced to the actual quantity η is $\sqrt[3]{c^2}$; s signifies in the table a characteristic distance from the discharge hole for which the speed in the center of the jet is U_s.

The computed velocities were compared with Göttingen test data. (See reference 2.) The diameter of the discharge nozzle was 137 millimeters. The velocity distributions at 100 centimeters and 150 centimeters distance from the nozzle edge were used for the comparison. This nozzle distance a may not be put equal to x, in view of the point discharge orifice assumed in the present calculation; x is rather computed from a by addition of a constant quantity e which results - for example - from the fact that for greater a, for which the comparison with these calculations is solely permissible, the central velocity decreases as $1/x$. In the present case $e = 26$ centimeters. Figure 13 shows the theoretical and the experimental dynamic pressure for $a = 100$ centimeters, it amounts to 104 kilograms per square meter at the discharge orifice; the agreement of the average values is good, aside from a certain asymmetry of the jet which must have had different reasons. From the conversion factor for η follows

$$\sqrt[3]{c^2} = 0.063$$

The radius r of the jet is

$$r = \sqrt[3]{c^2}\ 3.4\ x = 0.063 \times 3.4\ x = 0.214\ x$$

The mixing distance l is $= cx = 0.0158\ x = 0.0729\ r$.

Zimm (reference 3) has made corresponding experimental investigations at considerably lower speed. His findings would yield

$$\sqrt[3]{c^2} = 0.080$$

with a dynamic pressure of 5.1 kilograms per square meter in the discharge orifice. According to it, a slight increase in mixing path by decreasing Reynolds number is likely.

4. PREDICTION OF PRESSURE DIFFERENCE

So far, all cases had been premised on constant pressure. This first approximation can be improved by analysis of the pressure differences due to impulse variation on the basis of the computed speeds and stresses. For the first step, start, say, with the second equation of motion, which in the first two cases reads

$$u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} = \frac{1}{\rho}\left(\frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y}\right) - \frac{1}{\rho}\,\frac{\partial p}{\partial y}$$

and

$$u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} = \frac{1}{\rho}\left(\frac{\partial \tau}{\partial x} + \frac{1}{y}\,\frac{\partial(\sigma_y y)}{\partial y} - \frac{\sigma_t}{y}\right) - \frac{1}{\rho}\,\frac{\partial p}{\partial y}$$

in the rotationally symmetrical case; σ_y and σ_t are normal stresses, respectively effective in y direction or at right angles to y and x. Then, integrate with respect to y:

$$\Big[v^2\Big]_0^y + \frac{\partial}{\partial x}\int_0^y u\,v\,dy - \frac{1}{\rho}\,\frac{\partial}{\partial x}\int_0^y \tau\,dy - \frac{1}{\rho}\Big[\sigma_y\Big]_0^y = -\frac{1}{\rho}\Big[p\Big]_0^y \quad . . . (22)$$

and

$$\Big[v^2\Big]_0^y + \frac{\partial}{\partial x}\int_0^y u\,v\,dy + \int_0^y \frac{v^2}{y}\,dy - \frac{1}{\rho}\,\frac{\partial}{\partial x}\int_0^y \tau\,dy - \frac{1}{\rho}\Big[\sigma_y\Big]_0^y$$
$$- \frac{1}{\rho}\int_0^y \frac{\sigma_y}{y}\,dy + \frac{1}{\rho}\int_0^y \frac{\sigma_t}{y}\,dy = -\frac{1}{\rho}\Big[p\Big]_0^y \qquad (23)$$

which is equivalent to applying the impulse theorem. If, as heretofore, the normal stress, in this case σ_y and σ_t, are discounted, there is obtained

$$\Big[2FF'\,\eta - F^2\Big]_{\eta_2}^{\eta_1} - 2\int_{\eta_2}^{\eta_1} F'^2\,\eta\,d\eta = \frac{1}{\rho}\Big[p\Big]_{y_2}^{1}$$

for the free jet boundary,

$$\frac{1}{x}\left[F F' \eta - \frac{F^2}{4}\right]_0^{\eta} = \frac{1}{\rho}\left[p\right]_0^{y}$$

for the two-dimensional jet expansion

and
$$\left[\frac{2 F F'}{x^2 \eta}\right]_0^{\eta} + \int_0^{\eta} \frac{-F^2}{x^2 \eta^3} d\eta = \frac{1}{\rho}\left[p\right]_0^{y}$$

for axially symmetrical jet expansion. With p_r denoting the pressure at the jet boundary, p_m the pressure at jet center and of the homogeneous air stream, respectively, particularization of the above formulas yields

$$\frac{p_m - p_r}{\rho} = 0.410(2c^2)^{2/3}U^2 \quad \text{and} \quad \frac{p_m - p_r}{\rho} = 0.248(2c^2)^{2/3}U_m^2(x)$$

and

$$\frac{p_m - p_r}{\rho} = -0.316(c^2)^{2/3}\, U_m^2(x)$$

Quantity U indicates the speed of the homogeneous air stream, and $U_m(x)$ the central speed at x. In the first and third case, c has been determined, giving

$$p_m - p_r = 0.00584\, \frac{\rho\, U^2}{2}$$

and
$$p_m - p_r = -0.0025\, \frac{\rho\, U_m^2(x)}{2}$$

It is apparent that the thus computed pressure differences, being small, do not cause a substantial modification of the velocities.

When computing the pressure difference with respect to still air, it should be borne in mind that at the jet boundary a negative pressure equal to the dynamic pressure of the radial inflow speed prevails. With p_0 as the pressure in still air

$$p_m - p_o = 0.338(2c^2)^{2/3} \rho U^2 = 0.00482 \frac{\rho U^2}{2}$$ [1]

$$p_m - p_o = 0.124(2c^2)^{2/3} \rho U_m^2(x).$$

$$p_m - p_o = -0.372(c^2)^{2/3} \rho U_m^2(x) = -0.00295 \frac{\rho U_m^2}{2} (x)$$

Hence there is positive pressure within the jet in the two-dimensional cases, but negative pressure in the axially symmetrical case. This surprising result, which also is at variance with a rough impulse consideration, points to a defect in the theory. The necessary extension will be given in the following.

5. EXTENDED THEOREM FOR THE APPARENT STRESSES

The theorem applied up to now to the stresses introduced by the turbulent impulse exchange

$$\tau = l^2 \left| \frac{\partial u}{\partial y} \right| \frac{\partial u}{\partial y}, \quad \sigma_x = \sigma_y = \sigma_t = 0$$

is no more than a first approximation. In any case, it can be easily proved that $\partial u/\partial y$ in the cases in point is great with respect to $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$, hence the theorem for the mixing speed $l \left| \frac{\partial u}{\partial y} \right|$ caused by the speed difference is good. So, in a natural generalization of the previous theorem, the stress tensor is put equal to

$$l^2 \left| \frac{\partial u}{\partial y} \right| (\nabla \underline{v} + \underline{v} \nabla)$$

($\nabla \underline{v}$ = affinor of $\underline{v}$; $\underline{v}\nabla$ is the conjugate affinor.)

[1] This relation is important for the calibration of pitot tubes in a jet discharging from a nozzle.

The stresses to be newly added here, are, in general, neglected, except $\sigma_y = 2 l^2 \left|\frac{\partial u}{\partial y}\right| \frac{\partial v}{\partial y}$ and $\sigma_t = 2 l^2 \left|\frac{\partial u}{\partial y}\right| \frac{v}{y}$ which are used for calculating the pressure differences. This portion cancels in the model problem worked out for the two-dimensional case because of the employed boundaries, but not for the axially symmetrical case. Here the pressure differences are augmented by the integral

$$\int_0^y \frac{\sigma_y - \sigma_t}{y} \, dy$$

so that

$$p_m - p_r = +0.151(c^2)^{2/3} \, \rho \, U_m^2(x) = +0.0012 \, \frac{\rho \, U_m^2(x)}{2}$$

and

$$p_m - p_o = 0.095(c^2)^{2/3} \, \rho \, U_m^2(x) = 0.00075 \, \frac{\rho \, U_m^2(x)}{2}$$

that is, positive pressure within the jet, as in the other cases.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

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Table I

Free jet boundary				Flat jet				Round jet			
$\frac{\eta}{\sqrt[3]{2c^2}}$	$\frac{F'}{U}$	$\frac{F}{U\sqrt[3]{2c^2}}$	$\frac{v}{U\sqrt[3]{2c^2}}$	$\frac{\eta}{\sqrt[3]{2c^2}}$	$\frac{F'}{U_s\cdot\sqrt{s}}$	$\frac{F}{U_s\cdot\sqrt{s}\sqrt[3]{2c^2}}$	$\frac{v}{U_s\cdot\sqrt{s}\sqrt[3]{2c^2}}$	$\frac{\eta}{\sqrt[3]{c^2}}$	$\frac{F'}{U_s\cdot s\eta}$	$\frac{F}{U_s\cdot s\eta\cdot\sqrt[3]{c^2}}$	$\frac{v}{U_s\cdot s\sqrt[3]{c^2}}$
0,981	1	0,981	0	0	1	0	0	0	1	0	0
0,731	0,969	0,732	−0,022	0,05	0,995	0,050	0,025	0,0625	0,995	0 031	0,031
0,481	0,895	0,500	−0,069	0,1	0,979	0,099	0,049	0,125	0,977	0,062	0,060
0,231	0,791	0,286	−0,104	0,15	0,962	0,148	0,070	0,25	0,941	0,121	0,114
−0,019	0,668	0,104	−0,117	0,2	0,940	0,195	0,091	0,375	0,895	0,176	0,160
−0,269	0,538	−0,047	−0,096	0,4	0,842	0,374	0,150	0,5	0,843	0,227	0,194
−0,519	0,411	−0,166	−0,047	0,6	0,721	0,530	0,168	0,625	0,789	0,274	0,219
−0,769	0,296	−0,254	+0,027	0,8	0.604	0,664	0,151	0,75	0,727	0,314	0,230
−1,019	0,193	−0,312	+0,115	1	0,474	0,766	0,091	1	0,606	0,381	0,225
−1,269	0,112	−0,351	+0,209	1,2	0,357	0,850	0,003	1,25	0,487	0,427	0,182
−1,519	0,049	−0,371	+0,297	1,4	0,254	0,910	−0,099	1,5	0,376	0,455	0,109
−1,769	0,012	−0,379	+0,358	1,6	0,165	0,951	−0,212	1,75	0,283	0,467	0,028
−2,019	0	−0,379	+0,379	1,8	0,095	0,976	−0,318	2	0,198	0,462	−0,066
−2,039	0	−0,379	+0,379	2	0,046	0,989	−0,402	2,25	0,130	0,452	−0,159
				2,2	0,013	0,995	−0,469	2,5	0,077	0.430	−0,237
				2,4	0	0,996	−0,498	2,75	0,039	0,405	−0,298
								3	0,014	0,376	−0,334
								3,25	0,002	0,350	−0,344
								3,4	0	0,335	−0,335

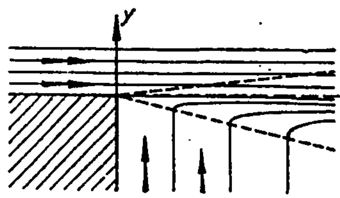

Figure 1

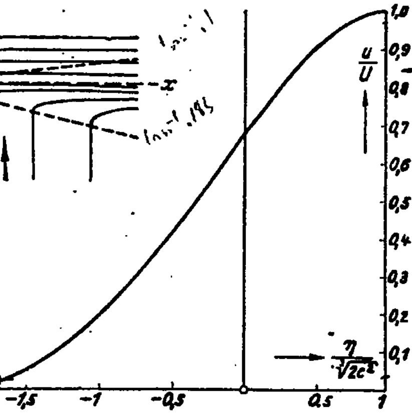

Figure 2.- Free-jet boundary.

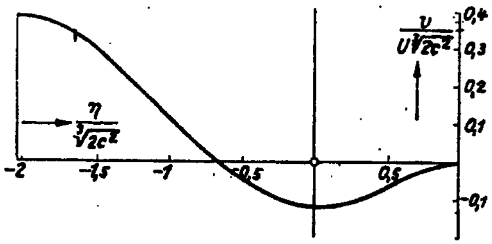

Figure 3.- Free-jet boundary.

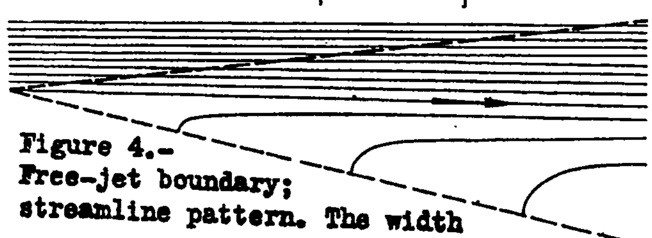

Figure 4.- Free-jet boundary; streamline pattern. The width is raised $\frac{1}{9\sqrt[3]{2c^2}}$ $(=1,32)=$ times.

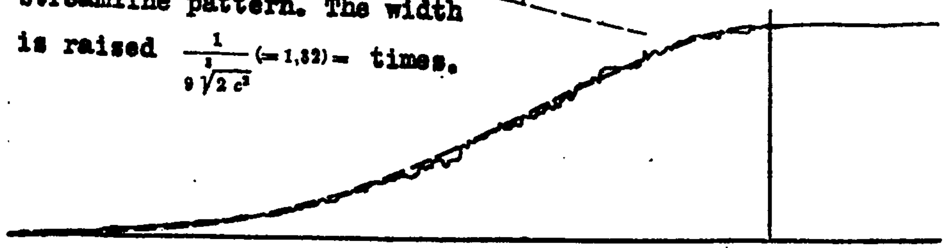

Figure 5.- Theoretical (dashes) over recorded dynamic pressure.

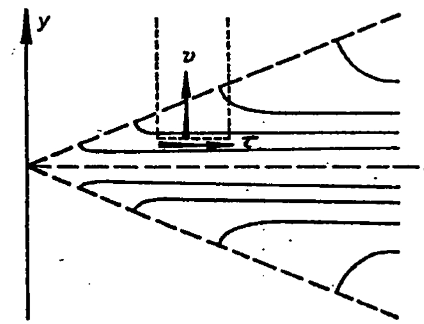

Figure 6.- Control surface.

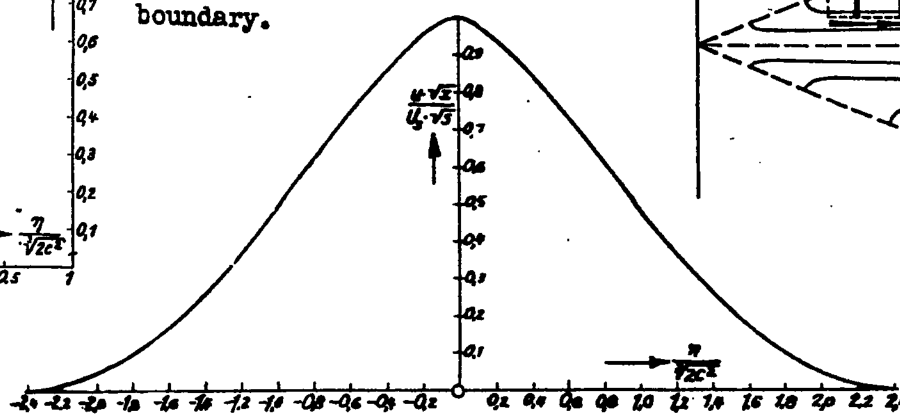

Figure 7.- Two-dimensional jet.

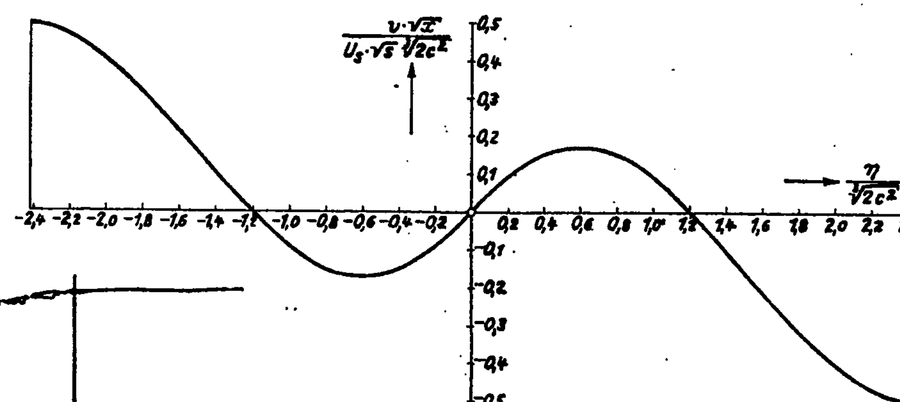

Figure 8.- Two-dimensional jet.

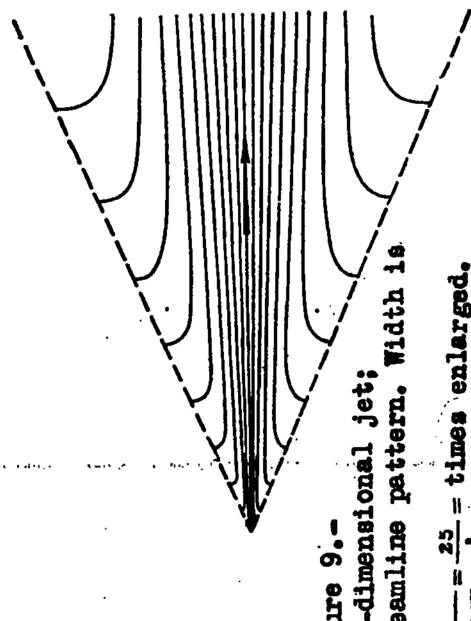

Figure 9.- Two-dimensional jet; streamline pattern. Width is $\frac{25}{1.44\sqrt[3]{2c^2}} = \frac{25}{3} =$ times enlarged.

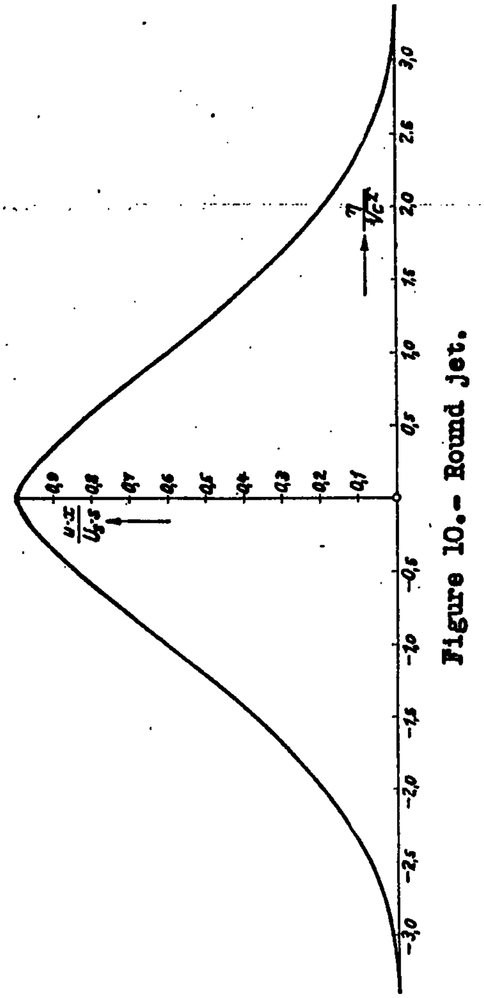

Figure 10.- Round jet.

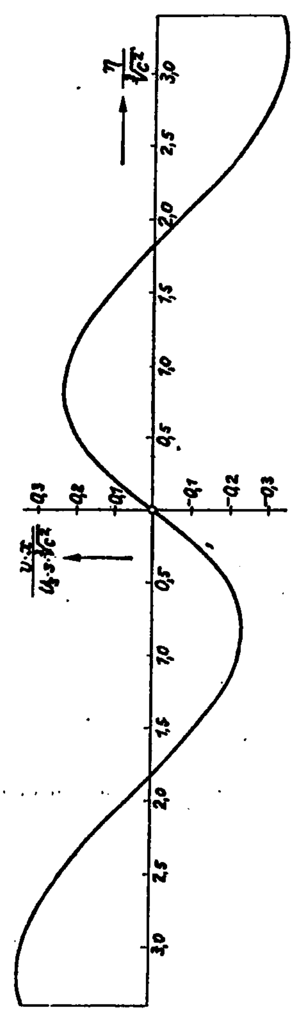

Figure 11.- Round jet.

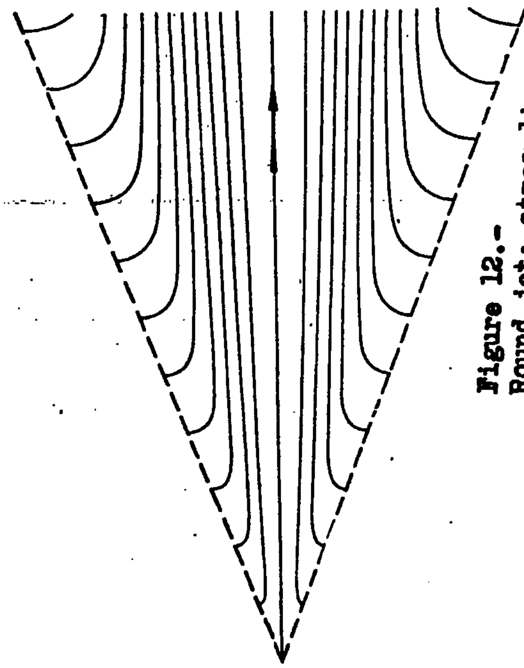

Figure 12.- Round jet; streamline pattern. Width is $\frac{1}{9\sqrt[3]{c^2}}$ (= 1,76) = times enlarged.

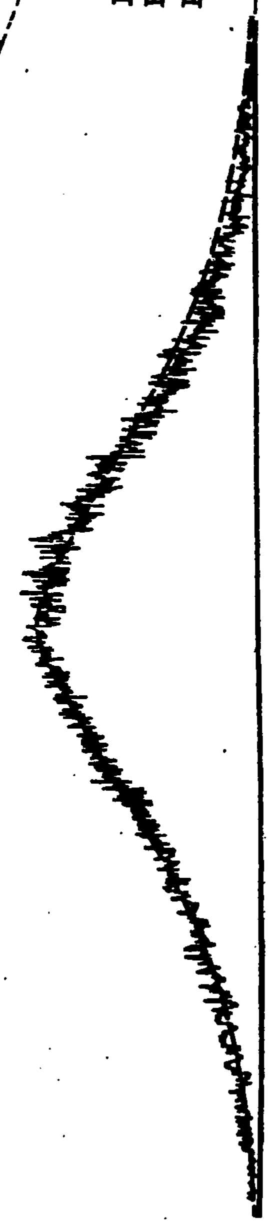

Figure 13.- Calculated and recorded dynamic pressures for a = 100 cm.