Low Density Parity Check Codes: Bandwidth Efficient Channel Coding

Wai Fong\textsuperscript{1}, Shu Lin\textsuperscript{2}, Gary Maki\textsuperscript{3} and Pen-Shu Yeh\textsuperscript{1}

\textsuperscript{1}Code 567, Goddard Space Flight Center, Greenbelt, MD 20771
\textsuperscript{2}University of California at Davis, Davis, CA 95616
\textsuperscript{3}Center for Advanced Microelectronics and Biological Research, U. of Idaho, Post Falls, ID 83854

\textit{Abstract}—Low Density Parity Check (LDPC) Codes provide near-Shannon Capacity performance for NASA Missions. These codes have high coding rates $R=0.82$ and $0.875$ with moderate code lengths, $n=4096$ and $8176$. Their decoders have inherently parallel structures which allows for high-speed implementation. Two codes based on Euclidean Geometry (EG) were selected for flight ASIC implementation. These codes are cyclic and quasi-cyclic in nature and therefore have a simple encoder structure. This results in power and size benefits. These codes also have a large minimum distance as much as $d_{\min} \approx 65$ giving them powerful error correcting capabilities and error floors less than $10^{-9}$ BER. This paper will present development of the LDPC flight encoder and decoder, its applications and status.

I. INTRODUCTION

Largely neglected for 35 years, LDPC Codes has recently been undergoing a rediscovery. Invented by Gallager in 1965 [1], it's near-Shannon Capacity performance was first documented by Mackay in 1996 [2]. These are large block codes with a very sparse (small density of "ones") parity check matrix. Current research follows on two paths: computer generated codes (generally unstructured and semi-random) and geometric codes (highly structured).

II. COMPUTER GENERATED LDPC CODES

Computer generated LDPC codes can be divided into two categories: Regular and Irregular. Regular LDPC codes have the property every column has a constant number of "ones" and every row has a constant number of "ones", though the number of column "ones" are generally not same as the number of row "ones". Irregular LDPC codes can have varying number of columns "ones" and row "ones".

A. Regular LDPC Codes

The properties of regular LDPC codes can be summarized as follows: the number of "ones" of each row and column are constant (because of this property Gallager codes are now classified as Regular LDPC codes) but small compared to the entire matrix and the number of "ones" in common between any two columns or two rows are small, preferably one. This last property allows for iterative decoding to converge to or close to the Shannon limit and provides for good minimum distance. Code synthesis is achieved through computer generation as each new column is randomly synthesized and tested to ensure compliance with the properties outlined above. With soft-in and soft-out (SISO) iterative decoding Gallager LDPC codes perform close to the Shannon capacity limit [2].

B. Irregular LDPC Codes

Irregular LDPC codes were introduced by Luby et al. [3] These are similar to the Gallager codes except that there is variance in the number of "ones" of each row and column. Like Gallager codes, irregular codes are computer generated and are decoded using the same iterative technique. They do outperform Regular codes in terms of distance to Shannon capacity at low signal to noise ratios. However, due to their small to moderate minimum distances both regular and irregular computer synthesized codes can have error floors around $10^{-9}$ BER or larger.

As a result of the semi-random nature of computer generated LDPC codes, there is the issue of the encoder complexity (on the order of $n^2$, where $n$ is the length of the codewords in bits) since there are no algebraic or geometric relationships within the generator matrix. And as with any random-like code, a very large memory must be used to generate the code. The implementation of which can be resource consuming and can impact the space, power and weight for a spacecraft.

III. GEOMETRY-BASED LDPC CODES

Geometry based LDPC codes were developed by Kou, Lin and Fossorier [4]. These are a class of Regular codes based on parallel lines of finite geometry. Due to their geometric structure some of these codes are cyclic and quasi-cyclic in nature, have large minimum distances, require less iterations to achieve close to Shannon limit decoding performance and have no low error floor >$10^{-9}$ BER. They also have many decoding options: iterative decoding with belief propagation (IDBP), bit-flipping (BF), weighted BF, majority-logic (ML), and weighted ML decoding. Their encoding complexity are on the order of $n$ and their encoding designs consist of a simple chain of shift registers.

One possible drawback to Geometry codes is that the decoders are could be larger due to fact that they be more dense (larger number of "ones") than Irregular codes. Thus Ge-
A. Code Construction

$C_1$ is constructed based on the lines and points of the 2-dimensional Euclidean geometry, $\text{EG}(2, 2^6)$, over the Galois field $\text{GF}(2^6)$. The geometry $\text{EG}(2, 2^6)$ consists of 4096 points and 4160 lines. One of the points is the origin of the geometry. Each line consists of 64 points. Let $L$ be a line not passing through the origin of the geometry. The incidence vector of line $L$ is defined as a 4095-tuple over $\text{GF}(2)$,

$$v_L = (v_1, v_2, \ldots, v_{64}).$$

(1)

where $v_i = 1$ if and only if the $i$th non-origin point of $\text{EG}(2, 2^6)$ is on $L$, otherwise $v_i = 0$. Then the parity-check matrix $H_1$ of $C_1$ is a 4095 x 4095 square circulant (a cyclic matrix) with column and row weights of 64. The rows of $H_1$ are simply the incidence vectors of the 4095 lines in $\text{EG}(2, 2^6)$ not passing through the origin. This parity-check matrix can be easily generated by simply cyclically shifting the incidence vector of any line $L$ not passing through the origin 4095 times.

B. Decoding Methods

$C_1$ can be decoded with various methods, ranging from low to high complexity and reasonably good to very good error performance. These decoding methods are: one-step ML, BF, weighted BF, weighted ML, and IDBP. In general, the higher the decoder complexity, the better the performance. (A more comprehensive description of these decoders can be found in [4,6,7].)

These decoders can be classified as either hard decision or soft decision and iterative or one-shot. ML and BF decoders are hard-decision decoders which can be simply implemented with logic gates and require only logic operations. But ML is one-shot, while BF is an iterative approach.

The IDBP decoder is a soft decision (SISO), iterative decoding which requires real-number computations. The weighted BF decoder is an iterative reliability-based decoding method which also requires some real number computations so it can be viewed as a cross between a soft and hard design decoder. ML decoding is the simplest decoding but should give the least coding gain over the uncoded system compared with the other decoding methods. However it can be implemented to operate at a very high speed, for instance 10-40 Gbps. There is also a weighted ML decoder that should give an improvement over the straight ML approach whose approach is similar to the approach used for weighted BF. This is another hard decision decoder that has elements of soft decision only this is a one-shot approach. BF and ML are similar in complexity and so are weighted BF and weighted ML. However, BF approaches have a longer decoding delay than the ML approaches due to their iterative nature. The complexity of the weighted decoders are higher than their non-weighted counterparts but they should provide better error performance.
C. Simulated Bit Error Rate (BER) Performance

Figs. 1, 2 and 3 illustrate the simulated code $C_1$ performance based on BPSK over an additive white Gaussian noise (AWGN) channel.

Fig. 1 shows the error performance of bit and frame (block) error rates above $10^{-6}$. Fig. 2 shows the error performance of $C_1$ above BER of $10^{-10}$. At the BER of $10^{-4}$, the code with ML decoding achieves a 5.5 dB coding gain over the uncoded BPSK, while with IDBP, it achieves almost 8 dB coding gain over the uncoded BPSK. ML, weighted ML, BF, weighted BF and IDBP performance range from worst to best in that order. No error floor occurs above $10^{-8}$ BER for the iterative decoding. Note that the IDBP decoder, as expected, provides the best BER performance with the largest decoding complexity. However, it requires a longer decoding time due to the iterative decoding process. To achieve high-speed decoding, a parallel/pipeline decoding process is needed.

The error floor of an LDPC code depends very much on its minimum distance. Code $C_1$ has a very large minimum distance = 65. For a code with a large minimum distance, either there is no error floor or the error floor occurs at a very low BER. Therefore, there should be no error floor at all or an error floor below $10^{-10}$ BER. If there is no error floor above BER of $10^{-10}$, $C_1$ with IDBP would achieve at least a 9 dB coding gain over the uncoded BPSK.

Figs. 1 and 2 also show that the code with IDBP also has good frame error performance. Frame error performance also depends very much on the minimum distance of the code. Computer generated LDPC codes, in general, do not have large minimum distance, and hence their frame error performance in general has error floor at a higher frame error rate, for instance $10^{-4}$.

Though not simulated, it is possible to build a two-stage IDBP/ML decoder where two IDBP soft decision iterations are followed by an ML decoder to shorten the decoding time. This hybrid decoder not only shortens the decoding delay but also reduces the computational complexity with a small performance degradation compared to pure IDBP decoding.

D. Simulated Iterative Performance

Another special feature of this code is that the IDBP decoder converges very fast as shown in Fig. 3. The performance gap between 2 iterations and 100 iterations is within 0.5 dB, while the gap between 5 iterations and 100 iterations is less than 0.2 dB. Therefore, the decoding can be terminated after 5 iterations with only a 0.2 dB loss. This reduces the
decoding computational complexity and speeds up the decoding process (or shorten the decoding delay) significantly.

Alternatively, as mentioned before, the decoding can be terminated after 2 iterations followed with ML decoding (two-stage iterative/ML decoding).

It is worthwhile to note that Computer generated LDPC codes, in general, converge at a much slower rate and take at least 20 iterations to be close to the performance of 100 iterations.

E. Measured Performance

Fig. 4 shows the results of the Field Programmable Gate Array (FPGA) implementation of the EG (4095, 3367) encoder and an FPGA ML decoder operating at 400 Mbps. These are Xilinx FPGA implementations which were tested with a BPSK modulation and AWGN channel. The graph verifies the simulated performance found in Fig. 1. It is also shown that there is no error floor down to $10^{-10}$ BER. These results give confidence that the IDBP decoder will perform very close to simulations.

VI. EG(8176, 7156)

This code, denoted $C_2$, is a (8176, 7156) Euclidean geometry LDPC code with rate $R = 0.875$. It is not cyclic but quasi-cyclic [6,7]. Every cyclic shift of 4 bits of a codeword is also a codeword. Its encoding can also be implemented with shift-registers.

A. Code Construction

$C_2$ is constructed based on the 3-dimensional Euclidean geometry $EG(3, 2^3)$ over the field $GF(2^3)$. This geometry consists of 512 points and 4672 lines. Each line consists of 8 points. For each point, there are 73 lines intersecting at it. The incidence vectors of the 4599 lines, not passing through the origin of the geometry, can be partitioned into 9 cyclic classes, $Q_1, Q_2, \ldots, Q_9$, each class consists of 511 incidence vectors. Each cyclic class $Q_i$ can be obtained by cyclically shifting any vector in $Q_i$ 511 times. For each cyclic Class $Q_i$, a 511 x 511 square circulant matrix $A_i$ is formed whose rows are simply the incidence vectors of $Q_i$. Both the column and row weights of $A_i$ are 8. Splitting each column of $Q_i$ into 4 columns of the same length with its weight uniformly distributed into the 4 new columns as suggested by Kou, Lin and Fossorier [4], $Q_i$ can be partitioned into four 511 x 511 square circulant matrices, $A_i^{(1)}, A_i^{(2)}, A_i^{(3)}, A_i^{(4)}$. Each circulant $A_i^{(j)}$ has column and row weights of 2. Using these 4 circulants, form a 511x2044 matrix,

$$G_2 = [A_i^{(1)}, A_i^{(2)}, A_i^{(3)}, A_i^{(4)}]$$  

The column and row weights of $G_i$ are 2 and 8, respectively. Then the parity check matrix $H_2$ of $C_2$ is given below:

$$H_2 = \begin{bmatrix} G_1 & G_2 & G_3 & G_4 \\ G_5 & G_6 & G_7 & G_8 \end{bmatrix}$$  

The column and row weights of $H_2$ are 4 and 32, respectively. $C_2$ is simply the null space of $H_2$.

B. Simulated Performance

The bit and frame error performance of $C_2$ with IDBP is shown in Fig. 5. Both the BER and FER curves have a sharp waterfall characteristic. The BER performs very well all the way down to the BER of $4 \times 10^{-10}$ without any error floor while the FER exhibits no error floor down to $10^{-6}$. At the $10^{-9}$, the BER achieves more than 9 dB coding gain over the uncoded BPSK and performs only 1.2 dB from the Shannon limit. If there is no error floor above BER of $10^{-9}$ then $C_2$ with iterative decoding should provide 10 dB coding gain over the uncoded BPSK. Decoding of $C_2$ also converges very fast as shown in Fig. 6. At the BER of $10^{-8}$, the performance gap between 10 iterations and 100 iterations is about 0.2 dB, while the performance gap between 20 iterations and 100 iterations is within 0.05 dB. To shorten the decoding delay and reduce the computational complexity, the maximum number of iterations can be set to 10 with only a 0.2 dB performance loss.
The exact minimum distance of $C_2$ is unknown. But since there is no error floor down to the BER of almost $10^{-5}$, its minimum distance should be relatively large. Experiments indicate that the minimum distance of $C_2$ is at least 7.

VII. PERFORMANCE COMPARISON WITH CURRENT STANDARDS

Most near-Earth missions currently follow the CCSDS recommendation of concatenating a $(1/2, 7)$ convolutional code (CC) with a $(255, 233)$ Reed-Solomon (RS) code. Although this coding technique provides good coding gain, a heavy penalty is placed on the bandwidth efficiency. With a code rate of 0.44, an overhead of 0.56 or 56% of the bandwidth is consumed by the parity check bits. Bandwidth constrained missions will sometimes chose to use RS coding alone to reduce the parity check overhead to 12%. However this comes with a large cost in additional power at the transmitter.

Fig. 7 compares the BER performance of EG LDPC codes $C_1$ and $C_2$ with RS and CC/RS. Note that at $10^{-5}$ BER both $C_1$ and $C_2$ outperform straight RS by 2.5 dB. This is a substantial improvement. $C_1$ and $C_2$ also comes within about 1 dB of CC/RS at $10^{-3}$ BER. This signifies that these EG codes can replace CC/RS with only a 1 dB penalty in $E_b/N_0$. And thus the complexity of the communication system is greatly simplified by the replacement of two codes (CC/RS) with one (EG) and thereby doubling the bandwidth efficiency at the same time. Notice the closeness of the $C_1$ and $C_2$ performance curves with their corresponding capacity limit. $C_1$ is within 1.5 dB away while $C_2$ is within 1 dB away.

VIII. SCHEDULE AND STATUS

There are two parallel development tracks: 1. Flight ASIC encoder along with the ASIC decoder and 2. the FPGA encoders and decoders. As of the beginning of June 2003, $C_1$ has been designed for the flight ASIC and the FPGA. The flight encoder design has simulated to greater than 1 Gbps operation. It’s fabrication is scheduled to be completed by the first quarter of 2004. As mentioned before, the FPGA encoder as well as the FPGA MGL decoder has been tested at 400 Mbps. Currently, an FPGA IDBP decoder has been designed and is being optimized for operating speed. It’s testing will be completed by the end of August 2003. The FPGA effort for $C_2$ will begin immediately following this. An FPGA encoder and decoder of $C_2$ will probably be completed by end of 2003. Also the ASIC decoders for $C_1$ and $C_2$ are...
undergoing an architectural study. Their fabrication won’t be completed until first quarter 2005.

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REFERENCES