Unstructured Adaptive Meshes: Bad for your Memory?
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Motivation
- Do we care problems with irregular dynamical memory access? 
  YES.
  Problems with localized error source benefit from adaptive nonuniform meshes
- Do we need this benchmark? 
  YES.
  Certain machines perform poorly on such problems
  Parallel implementation may provide further performance improvement but is difficult:
  - load balancing / data (re)distribution
  - data dependence
  - false and true data sharing

Background
- NAS Parallel Benchmarks (NPB, 1991)
  http://www.nasa.gov/Software/NPB

<table>
<thead>
<tr>
<th>Benchmark</th>
<th>CG</th>
<th>SP</th>
<th>SI</th>
<th>MA</th>
<th>PT</th>
<th>LU</th>
<th>SP</th>
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</table>

- Lack in the area of irregular and dynamically changing memory access

Application Selection
- Representative of problem class relevant to scientific computing community
- Simple without sacrificing credibility and effectiveness
  - Stylized heat transfer problem
  - Can be load balanced for range of processor sets with little communication and remapping
  - Spectral Element Method (Patera)
  - Have irregular, dynamic memory accesses feature.
    - Adaptive Nonconforming Mesh

Heat Transfer Problem
- Mathematical model
  \[ \frac{\partial T}{\partial t} + V \cdot \nabla T = \varepsilon \nabla^2 T + S(x,t) \]
- Time splitting
  \[ \frac{T^{n+1} - T^n}{\Delta t} = -V \cdot \nabla T^n + S(x,t^n) \]
  4th order R-K
  \[ \frac{T^{n+1} - T^n}{\Delta t} = \varepsilon \nabla^2 T^{n+1} \]

Heat Source Term
\[ s(x,t) = \begin{cases} 
\beta \left( \cos \left( \frac{\pi}{\alpha} \frac{x - x_0 - vt}{\alpha} \right) + 1 \right) & \text{if } \|x - x_0 - vt\| \leq \alpha \\
0 & \text{if } \|x - x_0 - vt\| > \alpha 
\end{cases} \]
Spectral Element Method

- High-order weighted residual technique which combines
  - Geometrical flexibility of finite element method
  - High accuracy and rapid convergence of spectral method
- Variational form (GLL Quadrature)
  \[
  (T^m, v) = (\mathbf{G}_{m} \mathbf{G}_{m}^{T}, v)
  \]
- High order function expansion
  \[
  T_i(x, y, z) = \sum \sum \sum \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} h_i(r) h_j(s) h_k(t), -1 < x, y, z < 1
  \]

Base functions

\[
T_i(x, y, z) = \sum \sum h_i(r) h_j(s) h_k(t), -1 < x, y, z < 1
\]

Elemental Discrete Equations

\[
\sum b_{i}^{j} \sum b_{i}^{k} \sum f_{i}^{k} = \frac{\partial}{\partial \xi_{i}} h(x, y, z)
\]

Global Discrete Equations

\[
\sum \sum \sum p_{i}^{j} \sum p_{i}^{k} \sum f_{i}^{k} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \mathbf{G}_{m} \mathbf{G}_{m}^{T} \mathbf{G}_{m} \mathbf{G}_{m}^{T}
\]

Nonconforming mesh

- Why nonconforming: local area refinement
- What is nonconforming
- Problem raised by nonconforming mesh: Continuity across element boundary

Mortar Element Method

Introduces a new mortar trace space:
- Preserve local structure
- Decouple the local/global computation
- Efficient for parallel computation
- Degrees of freedom are located in
  - Element interior
  - Mortar elements

\[
\text{Stiffness summation}
\]

\[
\sum \mathbf{A} \mathbf{T} = \sum \mathbf{B} \mathbf{T} \rightarrow AT = BT
\]
Mortar Elements

- Mortar Element Method
  - Solution on elements
    \[ T_h(x, y, z) = \sum_{\alpha, \beta} \sum_{k, m} \phi_{k,m}^\alpha h_{k,m}(\alpha)(\beta), \quad x, y, z \rightarrow r, s, t \in [-1,1] \]
  - Solution on mortars
    \[ \phi(x, y, z) = \sum_{\alpha, \beta} \phi_{k,m}^\alpha h_{k,m}(\alpha)(\beta), \quad x, y, z \rightarrow \tilde{p}, \tilde{q}, \tilde{r} \in [-1,1] \]

- Continuity across nonconforming element interfaces
  - \( C^0 \) continuity is replaced by two conditions:
    1. Vertex condition: the solution on an element vertex equals to the solution at the corresponding mortar point.
    2. \( L^2 \) condition: the solution difference between an element face and its related mortar elements is minimized in an integral sense.
    \[ \int_{\Gamma} (T \left|_{\Gamma} \right. - \phi) v ds = 0 \]
    \[ \forall I = 1, ..., N, \forall v \in P_{h,2}(\Gamma) \]

- Discrete Equations
  - Conforming
    \[ A^{T_h}_{h+1} = B^{T_h}_{h+1} \]
  - Nonconforming
    \[ \Theta^{T_h}_{h+1} = \Theta^{T_h}_{h+1} \]

Where \( \Theta \) refers to Global transformation matrix assembled using local transformation matrix Q.
- Symmetrical
- Positive definite
Solved by CG with a Diagonal Preconditioner
Mesh Adaptation Procedure
- Perform adaptation every \( m \) time steps
- Refine elements close to high error region: elements have overlap with the heat source
- Coarsen the grid elsewhere if possible

Mesh Adaptation Restrictions
- The maximal refinement levels/the minimal element size
- Neighboring elements can not differ by more than one level of adaptation.

Sample problem

Adaptation in 3-D (h-type)

Time stepping procedure

Initial & Boundary condition
- Initial grid \([0,1]^3\)
- Initial temperature \( T=0 \)
- Initial heat source location \((0.30,0.28,0.28)\)
- Heat source strength \( \beta = 10 \)
- Heat source movement / Velocity field \( \mathbf{v} = (1,1,1) \)
- Boundary condition: \( T=0 @ \) all faces
Problem parameters and Verification

\[ \int_T d\Omega \] At the last time step

<table>
<thead>
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<th>Class</th>
<th>( R_{\text{max}} )</th>
<th>( \Delta t )</th>
<th>( \alpha )</th>
<th>Adaptation interval</th>
<th>Steps</th>
<th># of elements</th>
<th>( \int_T d\Omega )</th>
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Current Status

- Benchmark Design: http://www.nas.nasa.gov/Software/NPB
- Sequential implementation:
  - under construction
- Parallel implementation:
  - Space filling curve to handle the load balance

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