Unstructured Adaptive Meshes: Bad for your Memory?

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Motivation

- Do we care problems with irregular dynamical memory access? YES.
  - Problems with localized error source benefit from adaptive nonuniform meshes
- Do we need this benchmark? YES.
  - Certain machines perform poorly on such problems
  - Parallel implementation may provide further performance improvement but is difficult:
    - load balancing / data (re)distribution
    - data dependence
    - false and true data sharing

Heat Transfer Problem

- Mathematical model
  \[
  \frac{\partial T}{\partial t} + V \cdot \nabla T = \epsilon \nabla^2 T + S(x,t)
  \]
- Time splitting
  Convective: \( \frac{T^{n+1} - T^n}{\Delta t} = -V \cdot \nabla T^n + S(x,t^n) \) 4th order R-K
  Diffusive: \( \frac{T^{n+1} - T^n}{\Delta t} = \epsilon \nabla^2 T^{n+1} \) Euler implicit

Application Selection

- Representative of problem class relevant to scientific computing community
- Simple without sacrificing credibility and effectiveness
  - Stabilized heat transfer problem
  - Can be load balanced for range of processor sets with little communication and remapping
  - Spectral Element Method (Patera)
  - Have irregular, dynamic memory accesses feature.
  - Adaptive Nonconforming Mesh

Heat Source Term

\[
s(x,t) = \begin{cases} 
  \beta \cos\left( \pi \frac{x - x_0 - vt}{\alpha} \right) + 1 & \text{if } \|x - x_0 - vt\| \leq \alpha \\
  0 & \text{if } \|x - x_0 - vt\| > \alpha 
\end{cases}
\]
**Spectral Element Method**

- High-order weighted residual technique which combines
  - Geometrical flexibility of finite element method
  - High accuracy and rapid convergence of spectral method
- Variational form (GLL Quadrature)
  \[
  (\overline{T}^{\text{ext}} - \overline{F}, \varphi) = (\overline{B} \overline{T}^{\text{ext}}, \varphi) 
  \]
- High order function expansion
  \[
  T_n(x, y, z) = \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} T_i^j h_i(x) h_j(y) h_k(z), \quad x, y, z \rightarrow \xi, \eta, \zeta \in [-1, 1] 
  \]

**Base functions**

- Element $T_n(x, y, z) = \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} T_i^j h_i(x) h_j(y) h_k(z), x, y, z \rightarrow \xi, \eta, \zeta \in [-1, 1]$

**Elemental Discrete Equations**

\[
\frac{\partial}{\partial \xi} \left( \sum \sum \sum \frac{\partial}{\partial \xi} \right) \left( \begin{array}{c} \varphi_i \varphi_j \varphi_k \end{array} \right) + \frac{\partial}{\partial \eta} \left( \sum \sum \sum \frac{\partial}{\partial \eta} \right) \left( \begin{array}{c} \varphi_i \varphi_j \varphi_k \end{array} \right) + \frac{\partial}{\partial \zeta} \left( \sum \sum \sum \frac{\partial}{\partial \zeta} \right) \left( \begin{array}{c} \varphi_i \varphi_j \varphi_k \end{array} \right) \varphi_{\text{ext}} = \left( \sum \sum \sum \frac{\partial}{\partial \xi} \right) \left( \begin{array}{c} \varphi_i \varphi_j \varphi_k \end{array} \right) \varphi_{\text{ext}} 
\]

\[
\frac{\partial}{\partial \xi} \left( \sum \sum \sum \frac{\partial}{\partial \xi} \right) \left( \begin{array}{c} \varphi_i \varphi_j \varphi_k \end{array} \right) + \frac{\partial}{\partial \eta} \left( \sum \sum \sum \frac{\partial}{\partial \eta} \right) \left( \begin{array}{c} \varphi_i \varphi_j \varphi_k \end{array} \right) + \frac{\partial}{\partial \zeta} \left( \sum \sum \sum \frac{\partial}{\partial \zeta} \right) \left( \begin{array}{c} \varphi_i \varphi_j \varphi_k \end{array} \right) \varphi_{\text{ext}} = \left( \sum \sum \sum \frac{\partial}{\partial \xi} \right) \left( \begin{array}{c} \varphi_i \varphi_j \varphi_k \end{array} \right) \varphi_{\text{ext}} 
\]

- Global Discrete Equations

\[
\sum \sum \sum \left( \begin{array}{c} \varphi_i \varphi_j \varphi_k \end{array} \right) \varphi_{\text{ext}} = \left( \sum \sum \sum \frac{\partial}{\partial \xi} \right) \left( \begin{array}{c} \varphi_i \varphi_j \varphi_k \end{array} \right) \varphi_{\text{ext}} - \frac{\partial}{\partial \xi} \left( \sum \sum \sum \frac{\partial}{\partial \xi} \right) \left( \begin{array}{c} \varphi_i \varphi_j \varphi_k \end{array} \right) \varphi_{\text{ext}} 
\]

**Nonconforming mesh**

- Why nonconforming: local area refinement
- What is nonconforming
  - Problem raised by nonconforming mesh: Continuity across element boundary

**Mortar Element Method**

- Introduces a new mortar trace space:
  - Preserve local structure
  - Decouple the local/global computation
  - Efficient for parallel computation
  - Degrees of freedom are located in
    - Element interior
    - Mortar elements
Mortar Elements

- Mortar element shared by element 3 & 4

N=4

Mortar Element Method

- Solution on elements
  \[ T_i(x, y, z) = \sum_{l=1}^{N} \sum_{j=1}^{N} T_l(x) h_j(y) A_{ij} h_i(z), \quad x, y, z \rightarrow r, s, t \in [-1,1] \]

- Solution on mortars
  \[ \psi(x, y, z) = \sum_{l=1}^{N} \phi_{l} h_{l}(y) h_{l}(z), \quad x, y, z \rightarrow \hat{r}, \hat{s}, \hat{t} \in [-1,1] \]

Mapping for Nonconforming Faces

- Mapping direction from element to mortar.

Mortar collocation points on nonconforming edges

Continuity across nonconforming element interfaces

\( C^0 \) continuity is replaced by two conditions:

1. **Vertex condition**: the solution on an element vertex equals to the solution at the corresponding mortar point.
2. **L2 condition**: the solution difference between an element face and its related mortar elements is minimized in an integral sense.

\[ \int_{r_{l+1}} (T_1 - \psi) \psi ds = 0 \]

\[ \forall l = 1, \ldots, 4 \quad \forall \psi \in P_{l+1}(T^1) \]

Discrete Equations

- Conforming
  \[ A T_{h}^{T+1} = B T_{h}^{T+1} \]

- Nonconforming
  \[ \Theta^T \Theta T_{h}^{T+1} = \Theta^T B T_{h}^{T+1} \]

Where \( \Theta \) refers to Global transformation matrix assembled using local transformation matrix \( Q \)

- Symmetrical
- Positive definite

Solved by CG with a Diagonal Preconditioner.
Mesh Adaptation Procedure

- Perform adaptation every $m$ time steps
- Refine elements close to high error region: elements have overlap with the heat source
- Coarsen the grid elsewhere if possible

Mesh Adaptation Restrictions

- The maximal refinement levels/the minimal element size
- Neighboring elements can not differ by more than one level of adaptation.

Sample problem

Adaptation in 3-D (h-type)

Time stepping procedure

- Move the source
- Convection (RK4)
- Diffusion (PCG)
- Step mod $m = 0$?
- Yes
- Mesh adaptation
- Solution interpolation
- No

Initial & Boundary condition

- Initial grid $[0,1]^3$
- Initial temperature $T=0$
- Initial heat source location $(0.30, 0.28, 0.28)$
- Heat source strength $\beta = 10$
- Heat source movement / Velocity field $\mathbf{v} = (1,1,1)$
- Boundary condition: $T=0$ @ all faces
Problem parameters and Verification

\[ \int_{\Omega} d\Omega \quad \text{At the last time step} \]

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<th>Class</th>
<th>( R_{\text{max}} )</th>
<th>( \Delta t )</th>
<th>( \alpha )</th>
<th>Adaptation interval</th>
<th>Step size</th>
<th>( # ) of elements</th>
<th>( \int_{\Omega} d\Omega )</th>
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Current Status

- Benchmark Design: [http://www.nas.nasa.gov/Software/npb](http://www.nas.nasa.gov/Software/npb)
- Sequential implementation:
  - under construction
- Parallel implementation:
  - Space filling curve to handle the load balance

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