Unstructured Adaptive Meshes: Bad for your Memory?

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Motivation

• Do we care problems with irregular dynamical memory access? YES.
  Problems with localized error source benefit from adaptive nonuniform meshes
• Do we need this benchmark? YES.
  Certain machines perform poorly on such problems
  Parallel implementation may provide further performance improvement but is difficult:
  load balancing / data (re)distribution
  data dependence
  false and true data sharing

Application Selection

• Representative of problem class relevant to scientific computing community
• Simple without sacrificing credibility and effectiveness
  Spectral Element Method (Patera)
  Have irregular, dynamic memory accesses feature.
  Adaptive Nonconforming Mesh

Heat Transfer Problem

• Mathematical model
  \[ \frac{\partial T}{\partial t} + V \cdot \nabla T = \varepsilon \nabla^2 T + S(x,t) \]
• Time splitting
  Convectio\n  \[ \frac{T^{n+1} - T^n}{\Delta t} = -V \cdot \nabla T^n + S(x,t^n) \quad 4th \text{ order R-K} \]
  Diffusion
  \[ \frac{T^{n+1} - T^n}{\Delta t} = \varepsilon \nabla^2 T^{n+1} \quad \text{Euler implicit} \]

Heat Source Term

\[ s(x,t) = \begin{cases} \cos \left( \frac{\pi}{\alpha} \left| x - x_0 - vt \right| \right) + 1 & \text{if } \left| x - x_0 - vt \right| \leq \alpha \\ 0 & \text{if } \left| x - x_0 - vt \right| > \alpha \end{cases} \]
Spectral Element Method
- High-order weighted residual technique which combines
  - Geometrical flexibility of finite element method
  - High accuracy and rapid convergence of spectral method
- Variational form (GLL Quadrature)
  \[
  (T^n - \frac{\mathbf{F}}{\Delta t}, \phi) = \left( \mathbf{B} T^n, \phi \right)
  \]
- High order function expansion

\[
T_n(x, y, z) = \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} T_i(x) h_j(y) h_k(z), \quad x, y, z \rightarrow \xi, \eta, \zeta [-1,1]
\]

Base functions
- \( h_i(\xi) = \begin{cases} 
1 & \text{if } \xi = \xi_i \\
\frac{1}{N(N+1)I_i} (1-\xi)^2 I_i(\xi) & \text{if } \xi = \xi_i \\
0 & \text{otherwise} \end{cases} \) for \( \xi \in [-1,1] \)
- \( I_i \) is the \( i \)th GLL collocation point

Elemental Discrete Equations
- \( e_j = \frac{\partial}{\partial \xi} (\xi)(\xi) \quad \forall i = \{1, \ldots, K\} \quad \forall i, j, k = \{0, \ldots, N\} \)

Global Discrete Equations
- stifness summation

Nonconforming mesh
- Why nonconforming: local area refinement
- What is nonconforming
- Problem raised by nonconforming mesh:
  Continuity across element boundary

Mortar Element Method
- Introduces a new mortar trace space:
  - Preserve local structure
  - Decouple the local/global computation
  - Efficient for parallel computation
  - Degrees of freedom are located in
    - Element interior
    - Mortar elements
Mortar Elements

- Mortar element shared by element 3 & 4

N=4

mortar elements

Mortar Element Method

- Solution on elements
  \[ T_\theta(x, y, z) = \sum_{i=0}^{N} \sum_{j=0}^{N} a_{ij}(r) \phi_j(x) \phi_i(y), \quad x, y, z \rightarrow r, \beta, \delta \in [-1, 1] \]

- Solution on mortars
  \[ \phi(x, y, z) = \sum_{\text{mortar } \delta} \phi_{\alpha} \phi_\beta \psi_\delta, \quad x, y, z \rightarrow \bar{x}, \bar{y} \in [-1, 1] \]

Mortar collocation points on nonconforming edges

Continuity across nonconforming element interfaces

C^0 continuity is replaced by two conditions:
1. **Vertex condition**: the solution on an element vertex equals to the solution at the corresponding mortar point.
2. **L^2 condition**: the solution difference between an element face and its related mortar elements is minimized in an integral sense.

\[ \int_{\Gamma} (T_{\theta} - \phi) \psi ds = 0 \]
\[ \forall I = 1, \ldots, 4 \quad \forall \psi \in P_{\delta}\{T_{\theta}\} \]

Discrete Equations

- Conforming
  \[ A T^{\delta+1}_h = B T^{\delta+1}_h \]

- Nonconforming
  \[ \Theta^T \Theta T^{\delta+1}_h = T^{\delta+1}_h \]

Where \( \Theta \) refers to Global transformation matrix assembled using local transformation matrix Q.

- Symmetrical
- Positive definite

Solved by CG with a Diagonal Preconditioner

Mapping for Nonconforming Faces

- Mapping direction
- Intermediate mortar elements
- Q^T for reverse mapping from element to mortar

QT: For reverse mapping from element to mortar
**Mesh Adaptation Procedure**
- Perform adaptation every \( m \) time steps
- Refine elements close to high error region: elements have overlap with the heat source
- Coarsen the grid elsewhere if possible

**Mesh Adaptation Restrictions**
- The maximal refinement levels/the minimal element size
- Neighboring elements can not differ by more than one level of adaptation.

**Sample problem**

**Adaptation in 3-D (h-type)**

**Time stepping procedure**

**Initial & Boundary condition**
- Initial grid \([0,1]^3\)
- Initial temperature \( T=0 \)
- Initial heat source location \((0.30,0.28,0.28)\)
- Heat source strength \( \beta = 10 \)
- Heat source movement / Velocity field \( v = (1,1,1) \)
- Boundary condition: \( T=0 \) @ all faces
Problem parameters and Verification

\[ \int T d\Omega \] at the last time step

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<th>( \Delta t )</th>
<th>( \alpha )</th>
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Current Status

- Benchmark Design: http://www.nas.nasa.gov/Software/NPB
- Sequential implementation:
  - under construction
- Parallel implementation:
  - Space filling curve to handle the load balance

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