Deriving Laws From Ordering Relations

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"But Farmer Hoggett knew that little ideas that tickled and nagged and refused to go away should never be ignored for within them lie the seeds of destiny."

-from the movie Babe based on the novel of the same title written by Dick King-Smith
Outline

Cox and Probability
  Review of Cox’s Derivation

Order
  The Importance of Order
  Posets, Lattices and Algebras
  Valuations on a Lattice

Origins
  Geometry
  Probability Theory
  Quantum Mechanics

A New Methodology
  The Role of Order in Science
Cox and Probability
Cox’s Contribution

Cox generalized implication among logical statements to degrees of implication represented by real numbers.
Associativity of the Conjunction

Consistency with associativity of the conjunction

\[ p(b \square c \square d \mid a) = p((b \square c) \square d \mid a) \]
\[ = p(b \square (c \square d) \mid a) \]

Results in the Product Rule

\[ p(b \square c \mid a) = p(b \mid a) p(c \mid a \square b) \]
Complementation

Consistency with complementation

\[ p(b \mid a) = p(\sim (\sim b) \mid a) \]

Results in the Sum Rule

\[ p(b \mid a) + p(\sim b \mid a) = 1 \]
Commutativity of the Conjunction

Consistency with the commutativity of the conjunction

\[ p(a \land b \mid h) = p(b \land a \mid h) \]

Results in Bayes Theorem

\[ p(a \mid b \land h) = \frac{p(b \mid a \land h) \cdot p(a \mid h)}{p(b \mid h)} \]
Inferential Calculus

The inferential calculus (probability theory) derives directly from consistency with

**Associativity**

**Commutativity**

**Complementation**

These are **BASIC** mathematical ideas, not restricted to inference.
Other Trails
Up the Mountain
Dr. Aczel’s major contribution has been his thorough investigation of the functional equations central to this development.

Ray Smith and Gary Erickson investigated all possible forms of the associativity equations.

Anton Garrett derived the sum and product rules using consistency with the NAND operation.

Certainly I am missing other contributions…
Ariel Caticha derived the sum rule from consistency with associativity of the disjunction

\[ p(a \sqcup b \mid h) = p(a \mid h) + p(b \mid h) \]

when \( a \) and \( b \) are logically independent.

The product rule can then be derived from consistency with distributivity.

\[ p(a \sqcap b \mid h) = p(a \mid b \sqcap h) p(b \mid h) \]
What is the Big Deal?

Because the Sum and Product Rules are not JUST associated with Boolean algebras.

They are associated with Distributive Algebras!
Order
Ordering Relations

To sets of objects, one can often impose additional structure, such as a binary ordering relation denoted by \( a \sqsubseteq b \), which satisfies for all \( a, b, c \) (Birkhoff 1967):

- **P1.** For all \( a \), \( a \sqsubseteq a \). \hspace{1cm} \text{(Reflexive)}
- **P2.** If \( a \sqsubseteq b \) and \( b \sqsubseteq a \), then \( a = b \) \hspace{1cm} \text{(Antisymmetry)}
- **P3.** If \( a \sqsubseteq b \) and \( b \sqsubseteq c \), then \( a \sqsubseteq c \) \hspace{1cm} \text{(Transitivity)}

Now \( a \sqsubseteq b \) is read “\( b \) contains \( a \)” or “\( b \) includes \( a \)”

If \( a \sqsubseteq b \) and \( a \neq b \), one can write \( a < b \) and read “\( a \) is less than \( b \)” or “\( a \) is properly contained in \( b \)”.

If \( a < b \) but \( a < x < b \) is not true for any \( x \) in the poset \( P \), then we say that “\( b \) covers \( a \)”, written \( a \prec b \).
Posets

Together a set and an ordering relation are called a partially ordered set, or a poset.

The set of Natural numbers \( \{1, 2, 3, 4, 5\} \) with the binary ordering relation “is less than or equal to” \( \sqsubseteq \) is a poset.

It is clear that:

\[
\begin{align*}
2 \sqsubseteq 2 & \quad \text{(Reflexive)} \\
\text{As } 2 \sqsubseteq 2 \text{ and } 2 \sqsubseteq 2, \text{ then } 2 &= 2 & \text{(Antisymmetry)} \\
\text{As } 2 \sqsubseteq 3 \text{ and } 3 \sqsubseteq 4, \text{ then } 2 &\not\sqsubseteq 4 & \text{(Transitivity)}
\end{align*}
\]

Also \( 2 < 3 \) as \( 2 \sqsubseteq 3 \) but \( 2 \neq 3 \)

And \( 2 < 3 \) as \( 2 < 3 \) but there is no Natural number \( x \) such that \( 2 < x < 3 \)
The covering relation can be used to visualize the structure of a poset.

Whenever \( a \sqsubseteq b \), draw \( b \) above \( a \):

\[
5 \\
4 \\
3 \\
2 \\
1
\]
The covering relation can be used to visualize the structure of a poset.

Whenever $a \sqsubseteq b$ draw $b$ above $a$.

And whenever $a \prec b$ connect the elements with a line:

This poset forms a chain.
There are times where for a given ordering relation, it is not true that \( a \sqsubseteq b \) or \( b \sqsubseteq a \).

We then write \( a \parallel b \) read “\( a \) is incomparable to \( b \)”.

Perhaps for the ordering relation “\( \text{is healthier than} \)” we have
The diagram corresponding to a poset of three incomparable elements is a picture with the elements placed side-by-side.

This is called an antichain.
Consider the powerset of the set $S = \{ a, b, c \}$

This is the set of all possible subsets of $S$:

$$P(S) = \left\{ \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \right\}$$

A natural ordering is the relation “is a subset of”, \[\sqsubseteq\]

$$P = \left( \left\{ \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \right\}, \sqsubseteq \right)$$
First we note that \( \emptyset \varsubsetneq \{a\} \), from which we also see that: \( \emptyset \prec \{a\} \)

So we draw \( \{a\} \) above \( \emptyset \) and connect them with a line.
Completing the First Level

\[ P = \left( \left\{ \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \right\}, \square \right) \]

It is also true that \( \emptyset \prec \{b\} \) and \( \emptyset \prec \{c\} \) so we draw them above \( \emptyset \) as well and connect them with lines.

However, \( \{a\} \parallel \{b\} \) as neither one is the subset of the other.

In addition, \( \{a\} \parallel \{c\} \) and \( \{b\} \parallel \{c\} \). So we draw them on the same level and do not connect them.
The Second Level

\[ P = \left( \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \}, \emptyset \right) \]

Now we note that \{a\} is covered by two elements \{a, b\} and \{a, c\}.
These elements also cover \( \{b\} \) and \( \{c\} \)
Completing the Second Level

\[ P = \left( \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \} \right) \]

Now \( \{b, c\} \) also covers \( \{b\} \) and \( \{c\} \), but these top elements are also incomparable.
The Third Level

\[ P = \left( \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}, \square \right) \]

Finally \( \{a, b, c\} \) covers all three two-element subsets.
The Powerset of \{a, b, c\}

\[ P = \left( \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}, \square \right) \]
Lattices
A lattice is a poset $P$ where every pair of elements $x$ and $y$ has a least upper bound called the **join** $x \lor y$

a greatest lower bound called the **meet** $x \land y$

The green elements are upper bounds of the blue circled pair. The green circled element is their least upper bound or their join.

Similarly

$\{a, b\} \lor \{b, c\} = \{b\}$

$\{b\} \land \{c\} = \{b, c\}$
Lattice Identities

The Lattice Identities

L1. \( x \sqcap x = x, \quad x \sqcup x = x \) \hspace{1cm} \text{Idempotent}

L2. \( x \sqcup y = y \sqcup x, \quad x \sqcap y = y \sqcap x \) \hspace{1cm} \text{Commutative}

L3. \( x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup z, \quad x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap z \) \hspace{1cm} \text{Associative}

L4. \( x \sqcap (x \sqcup y) = x \quad (x \sqcup y) = x \) \hspace{1cm} \text{Absorption}

If \( x \sqcap y \) the meet and join follow the Consistency Relations

C1. \( x \sqcup y = x \) \quad (x \text{ is the greatest lower bound of } x \text{ and } y)

C2. \( x \sqcap y = y \) \quad (y \text{ is the least upper bound of } x \text{ and } y)
The dual lattice can be obtained by reversing the ordering relation.

This flips the lattice upside-down and exchanges meets and joins.
The **greatest element** is called the **top** and is symbolized by $1$, $I$, or $T$. So that $T \geq x$ for all $x$ in $L$.

The **least element** is called the **bottom** and is symbolized by $\emptyset$ or $\bot$. So that $x \geq \emptyset$ for all $x$ in $L$. 
A **Distributive Lattice** possesses structure additional to L1-4.

It also satisfies the following identity for all elements $x, y, z$.

\[
x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)
\]

\[
x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)
\]

Note that these two equations are related by duality as the dual of a distributive lattice is a distributive lattice.
Complemented Distributive Lattices

There is a special case of a distributive lattice that possesses an interesting property where each element is associated with one other element called its complement.

The complement has these properties

B1. \( x \sim x = \bot \quad \sim x = \top \)

B2. \( \sim (\sim x) = x \)

B3. \( \sim (x \sqcup y) = \sim x \sim y \quad \sim (x \sqcap y) = \sim x \sim \sim y \)
Associated with every lattice is an algebra.

Thus a lattice can be expressed either in terms of its elements and its ordering relation

$$\langle L ; \square \rangle$$

or in terms of its algebra

$$\langle L ; \square, \quad, \sim \rangle$$

or perhaps $$\langle L ; \square, \quad, \sim \rangle$$
Origins
Probability from Order
At this point the algebra associated with the complemented distributive lattice should look familiar - **Boolean algebra**.

More commonly, the poset is a set of assertions and the ordering relation is “implies” \( \Top \implies \Top \).
Boolean Lattices

The elements that cover $\emptyset$ are called atoms. In a Boolean lattice the atoms are the mutually exclusive assertions. All other elements are joins of the atoms.
Join-Irreducible Elements

An important subset of a lattice is the set of elements that cannot be written as a join of elements. They are the **join-irreducible elements**.

In a Boolean lattice, these are the atoms, which by themselves form an **antichain**.

One can use this property to identify a Boolean lattice.
The powerset (the set of all subsets) of a set forms a Boolean lattice under the ordering relation $\subseteq$. Note that the atoms form an antichain.

The complement of a set $S$ is the set $T \setminus S$. 
Deductive inference is easy.

For all $x \neq y$

If $x \nmid y$ then $x \nmid y$.

If $x \nmid y$ then $y \not\parallel x$.

If $x \parallel y$ then $x \not\parallel y$ and $y \not\parallel x$.

Note that:
The absurdity $\Box (\emptyset)$ implies everything.
The truism $T$ is implied by everything.
Deduction is nice and all, but sometimes I know that one of a set of possibilities is true.

And I want to know to what degree that knowledge implies a more simple hypothesis.

But since \( a \geq b \geq c \), it is only true that \( a \equiv a \equiv b \equiv c \).

Not vice versa!
To generalize to degrees of implication, we introduce a real-valued function that takes two lattice elements to a real number:

\[ p : L \times L \to \mathbb{R} \]

In particular we write this as:

\[ p(a \mid T) \]

Since one of \(a, b,\) or \(c\) is true, the truism can be considered to be our prior information, in part.
This function can be written in two different ways.

Typically when the premise is the truism, we can write this function as a function that takes a single lattice element to a real number:

\[ p(a) \]

And write the function as

\[ p(a \mid h) \]

in other cases.
Following the Rules

To be consistent with the Boolean lattice structure, this new measure must follow the rules: L1-4, D1, and B1-3.

The key ones we use are

L3  Associativity of $\sqcap$

\[ x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z \]

D1  Distributivity

\[ x (y \sqcap z) = (x y) \sqcap (y z) \]

L2  Commutativity of $\sqcup$

\[ x \sqcup y = y \sqcup x \]
Valuations
A valuation on a lattice is defined as a function that takes a lattice element to a commutative ring.

\[ \nu : L \rightarrow A \]

where A is an element of a commutative ring.

This has been investigated in great detail by a small group of mathematicians led by Gian-Carlo Rota.
The function we defined for probability is a valuation.

\[ v : L \models A \]
\[ p : L \models L \models \]

\[ p(a \mid \top) \equiv p(a) \]
Why is this Useful?

There is a theorem that all valuations can be uniquely determined from the valuations on the join-irreducible elements of the lattice

AND

their assignments are arbitrary!

Thus, by assigning the prior probabilities

\[ p(a) \quad p(b) \quad p(c) \]

the probabilities of any other pair of elements in the lattice is determined.
Assigning Priors is Hard

One can now see why assigning priors is difficult.

There is NO structure in the Boolean algebra of assertions that can guide us in these assignments.

We must employ other principles to assign them.
Prior Probabilities

Symmetry, constraints and consistency with other aspects of the problem can be used to assign prior probabilities.

Order-theoretic principles dictate the remaining probabilities.
Probability Theory from Order
Thanks to the efforts of Ed Jaynes, Myron Tribus and others, I am able to wave my hands and say that I can derive much of physics from order-theoretic principles.

Understanding Maximum Entropy remains a challenge.
Geometry from Order
Many geometric laws can be derived from order-theoretic considerations.

Geometric objects can be ordered, conjoined and disjoined often resulting in a distributive lattice structure.

Valuations are assigned, which are invariant with respect to Euclidean translations and rotations.
These Valuations have a Basis!

For three-dimensional Euclidean geometry all invariant valuations can be written as a linear combination of 4 basis valuations.

\[ V = \text{volume} \]
\[ A = \text{surface area} \]
\[ W = \text{mean width} \]
\[ c = \text{Euler characteristic} \]

\[ c = aV + bA + cW + d\hat{c} \]
Euler Characteristic

The Euler characteristic is a valuation.

\[ \square = F \square E + V \]

For a 3D tetrahedron it is found by

\[ \square(\text{tetra}) = 4 \square 6 + 4 = 2 \]
Euler Characteristic

For a cube, it is

\[ \bigbox = F \bigbox E + V \]

\[ \bigbox(cube) = 6 \bigbox 12 + 8 = 2 \]
Corresponding Lattice
Lattice Structure of a Cube

cube

faces

edges

vertices

a b c d e f g h

rank 3
rank 2
rank 1
rank 0
Mobius Functions

A Mobius function for a partially ordered set \( P \) is a function that satisfies:

\[
\begin{align*}
\mu(x, x) &= 1, & x \sqsubseteq P \\
\prod_{x \sqsubseteq z \sqsubseteq y} \mu(x, z) &= 0 & x < y \\
\mu(x, y) &= 0 & y > x
\end{align*}
\]

These functions are important for inverting other functions on the lattice, as we will see later.

This is also related to the Euler characteristic in distributive lattices.
Mobius functions allow us to compute valuations on elements higher in the lattice based on linear combinations of valuations of elements lower in the lattice.

For many familiar structures this leads to Rota’s **inclusion-exclusion principle** where when summing we add at one level and subtract at the next and so on.

We saw this with the Euler characteristic

\[
\bigcirc = F \bigcirc E + V
\]

And we’ll see it also on the next slide…
Joining Parallelotopes

\[ \nu(P_1 \quad P_2) = \nu(P_1) + \nu(P_2) \square \nu(P_1 \square P_2) \]
Quantum Mechanics from Order
Ariel has also developed a very interesting derivation of quantum mechanics using Cox’s method applied to experimental setups rather than logical statements.

The concept of consistency with the order-theoretic structure is central here as well.
A particle moves from $x_i$ to $x_f$

$$[x_f, x_i]$$
A particle moves from $x_i$ to $x_1$ and then from $x_1$ to $x_f$.

\[ [x_f, x_1, x_i] \]
A particle goes from $x_i$ to $x_f$ via $x_1$ or $x'_1$

\[ [x_f, (x_1, x'), x_i] \]
We can look at this experimental setup as
We can look at this experimental setup as being a combination of two setups…

\[
[\ x_f , x_1 \ ] \square [\ x_1 , x_i \ ] = [\ x_f , x_1 , x_i \ ]
\]
The Join Operation

This is a different way to combine setups

\[
[x_f, x_1, x_i] = [x_f, (x_1, x\Box), x_i]
\]
The meet is associative, as

\[
[x_f, x_2] \sqcup [x_2, x_1] \sqcup [x_1, x_i] = [x_f, x_2] \sqcup ([x_2, x_1] \sqcup [x_1, x_i]) = ([x_f, x_2] \sqcup [x_2, x_1]) \sqcup [x_1, x_i] = [x_f, x_2, x_1, x_i]
\]
However, experimental setups are not commutative under the meet operation!

This is because it makes no sense for a particle to go from $x_1$ to $x_f$ and then from $x_i$ to $x_1$!
The join is also associative, as

\[ [x_f, x_i, x_i] = [x_f, (x_i, x_i), x_i] = \]

\[ = [x_f, x_i] = [x_f, x_i, x_i] = [x_f, x_1, x_i] \]

As long as each of these joins is allowed
Joins are commutative

\[ [x_f, x \square x_i] = [x_f, x \square x_i] = [x_f, x \square x_i] = [x_f, x \square x_i] \]
What is interesting about setups is that because not all meets and joins exist, setups do not form a lattice structure.

They do form a poset however.

As the measure we will define is not probability, Ariel represented it with $\square(a)$ rather than $p(a | i)$

So lets continue…
Sum and Product Rules Again

Caticha showed that the

**Sum Rule** is derived from **Associativity of the Join**.

\[ \Box (a \lor b) = \Box (a) + \Box (b) \]

**Product Rule** from **Distributivity**.

\[ \Box (a \land b) = \Box (a) \Box (b) \]
If we let the valuations on this poset take on complex values, we have quantum mechanical amplitudes.

Caticha showed that one can then easily derive Schrodinger’s Equation.

Feynman Path Integrals are simply analogous to marginalizations.
Probability and QM

Quantum Mechanics is NOT Probability Theory

Probability is a degree of implication defined on a partially ordered set of logical statements.

Amplitudes are an analogous measure defined on the partially ordered set of experimental setups.

This is exciting as it suggests that other analogous measures can be constructed for other partially ordered sets, leading to new laws!
Take Home Message
Symmetry + Order = Laws
I would like to thank Ariel Caticha and Carlos Rodríguez for their discussions, which have enlightened and inspired me.

I would like to thank Robert Fry for introducing me to this fascinating area of study.
Deriving Laws From Ordering Relations

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We look at the conjunction of two assertions implied by a premise

$$(a \Box b \Box c)$$

and take as an axiom that this is a function of

$$(a \Box b) \quad \text{and} \quad (a \Box b \Box c)$$

so that

$$(a \Box b \Box c) = F[(a \Box b), (a \Box b \Box c)]$$
Conjunctions

We now conjoin an additional assertion

\[(a \Box b \Box c \Box d) = (a \Box (b \Box c) \Box d)\]

\[= F[(a \Box b \Box c), (a \Box b \Box c \Box d)]\]

Letting

\[x = (a \Box b)\quad y = (a \Box b \Box c)\quad z = (a \Box b \Box c \Box d)\]

We have

\[(a \Box b \Box c \Box d) = F[F[x, y] z]\]
Associativity of the Conjunction

We could have grouped the assertions differently

\[(a \square b \square c \square d) = (a \square b \square (c \square d))\]
\[= F[(a \square b), (a \square b \square c \square d)]\]
\[= F[(a \square b), F[(a \square b \square c), (a \square b \square c \square d)]]\]
\[= F[x, F[y, z]]\]

This gives us a functional equation

\[F[F[x, y]z] = F[x, F[y, z]]\]
The Product Rule

Functional Equation

\[ F[F[x, y]z] = F[x, F[y, z]] \]

As a particular solution we can take \( F[x, y] = xy \) which gives

\[ (a \boxtimes b \boxtimes c) = (a \boxtimes b)(a \boxtimes b \boxtimes c) \]

and can be written in a more familiar form by changing notation

\[ (b \boxtimes c | a) = (b | a)(c | a \boxtimes b) \]
The Product Rule

In general however the solution is

$$F[x, y] = G^{-1}[G[x]G[y]]$$

where $G$ is an arbitrary function.

$$G[b \cap c | a] = G[b | a]G[c | a \cap b]$$

We could call $G$ probability!
The degree to which a premise implies a statement determines the degree to which it implies its contradictory.

\[(a \oplus \sim b) = f[(a \oplus b)]\]

So

\[(a \oplus \sim (\sim b)) = f[(a \oplus \sim b)]\]

\[(a \oplus \sim (\sim b)) = f[f[(a \oplus b)]]\]

\[(a \oplus b) = f[f[(a \oplus b)]]\]

\[f[f[x]] = x\]
The Sum Rule

Another functional equation

\[ f[f[x]] = x \]

A particular solution \( f(x) = 1 \square x \) is which gives

\[ (a \boxplus b) + (a \boxplus \sim b) = 1 \]
\[ (b | a) + (\sim b | a) = 1 \]

In general

\[ g(b | a) + g(\sim b | a) = C \]
The solution to the first functional equation puts some constraints on the second

\[
G[b \sqcap c | a] = G[b | a]G[c | a \sqcup b] \\
g[b | a] + g[\sim b | a] = C
\]

The final general solution is

\[
G[x] = g[x] = x^r
\]

and we have

\[
(b \sqcap c | a)^r = (b | a)^r (c | a \sqcup b)^r \\
(b | a)^r + (\sim b | a)^r = C
\]
The Rules of Probability

Setting \( r = C = 1 \) and writing the function \( g(x) = G(x) \) as \( p(x) \)
we recover the familiar **sum** and **product** rules of probability

\[
p(b \sqcap c \mid a) = p(b \mid a) \ p(c \mid a \sqcap b)
\]

\[
p(b \mid a) + p(\sim b \mid a) = 1
\]

Note that probability is necessarily conditional!
And we never needed the concept of frequencies of events!

The utility of this formalism becomes readily apparent when the
implicant is an assertion representing a premise and the implicate
is an assertion or proposition representing a hypothesis

\[
p(\text{hypothesis} \mid \text{premise}) \equiv (\text{premise} \sqcap \text{hypothesis})
\]
The symmetry of the conjunction of assertions

\[ a \square b \equiv b \square a \]

means that under implication

\[ (h \square a \square b) \equiv (h \square b \square a) \]

also written as

\[ p(a \square b \mid h) \equiv p(b \square a \mid h) \]

which means we can write

\[ p(a \square b \mid h) = p(a \mid b \square h) p(b \mid h) = p(b \mid a \square h) p(a \mid h) \]
Bayes’ Theorem

\[ p(\text{model} | \text{data}, I) = \frac{p(\text{model} | I) \cdot p(\text{data} | \text{model}, I)}{p(\text{data} | I)} \]

- Posterior Probability
- Prior Probability
- Likelihood
- Evidence
Bayes’ Theorem is a Learning Rule

\[ p(\text{model} \mid \text{data}, I) = p(\text{model} \mid I) \frac{p(\text{data} \mid \text{model}, I)}{p(\text{data} \mid I)} \]

- Prior Knowledge
- Improved State of Knowledge
- Data Dependent Term
In short we have the following calculus:

**Product Rule**

\[ p(x \square y \mid I) = p(y \mid x \square I) \ p(x \mid I) \]

associativity of \( \square \)

**Sum Rule**

\[ p(x \mid I) + p(\sim x \mid I) = 1 \]

complements \( x = \sim (\sim x) \)

**Bayes Theorem**

\[ p(y \mid x \square I) = p(y \mid I) \frac{p(x \mid y \square I)}{p(x \mid I)} \]

commutativity of \( \square \)