NASA TN D-685

TECHNICAL NOTE

D-685

AN EXPLICIT LINEAR FILTERING SOLUTION FOR

THE OPTIMIZATION OF GUIDANCE SYSTEMS

WITH STATISTICAL INPUTS

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON February 1961

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

TECHNICAL NOTE D-685

AN EXPLICIT LINEAR FILTERING SOLUTION FOR THE OPTIMIZATION OF GUIDANCE SYSTEMS WITH STATISTICAL INPUTS

By Elwood C. Stewart

SUMMARY

The determination of optimum filtering characteristics for guidance system design is generally a tedious process which cannot usually be carried out in general terms. In this report a simple explicit solution is given which is applicable to many different types of problems. It is shown to be applicable to problems which involve optimization of constant-coefficient guidance systems and time-varying homing type systems for several stationary and nonstationary inputs. The solution is also applicable to off-design performance, that is, the evaluation of system performance for inputs for which the system was not specifically optimized. The solution is given in generalized form in terms of the minimum theoretical error, the optimum transfer functions, and the optimum transient response. The effects of input signal, contaminating noise, and limitations on the response are included. From the results given, it is possible in an interception problem, for example, to rapidly assess the effects on minimum theoretical error of such factors as target noise and missile acceleration. It is also possible to answer important questions regarding the effect of type of target maneuver on optimum performance.

INTRODUCTION

There are a number of statistical theories which have been derived in recent years for the purpose of determining optimum system design and optimum theoretical performance. Three problems are invariably encountered in the application of these theories. The first problem is that the solutions generally involve long and tedious computations. Moreover, explicit solutions can rarely be obtained because of the complexity of the equations and the many factors involved. For this reason solutions are usually carried out numerically for specific cases. Such a procedure contributes little to a basic understanding of the problem and makes it difficult to draw general conclusions as to the relation of the guidance and control task to both the best theoretical performance which can be achieved and to the optimum system design. Such considerations are important, for

example, in evaluating the effect on theoretical minimum error of such factors as limited vehicle maneuverability, the amount of noise, etc.

A second problem is concerned with off-design performance, that is, the evaluation of system performance for inputs for which the system was not specifically designed. In general the actual input to a system will be different than the design input. For example, the actual noise level might be different than design noise level, or the actual and design signal inputs might even be different processes. For this and other cases it is important to evaluate the deterioration in error for these off-design conditions.

A third problem in system optimization concerns the choice of signal component of the input, which in the interception problem is the target motion. Since the target may move in many different ways, it is important to consider two aspects: (1) the effects of type of signal characteristic for which the system is optimized, and (2) the effect on performance of subjecting the system to signal inputs for which the system was not specifically designed.

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This report will be concerned with the above three problems. The first two sections are concerned principally with the first problem. In the first section will be derived explicitly a simple but approximate solution to the filtering problem. Because of the length of this section a resumé is given at the end. In the second section will be shown the applicability of this solution to many different problems of interest; such problems involve the optimization of constant-coefficient guidance systems and time-varying homing systems, for stationary and nonstationary signal inputs. The third section is concerned with the off-design problem (the second problem). The last section is an example concerned with the third problem, the effect of type of signal input on the performance of optimum and nonoptimum systems.

LIST OF IMPORTANT SYMBOLS

a_T acceleration of target, ft/sec^2

A^2 mean-square acceleration, ft/sec^2

c steady-state output defined in equation (4)

E^2 mean-square error, ft^2

H_{co} optimum compensating network transfer function

H_f fixed network transfer function

M_A quantity related to c_A (see eqs. (B62))

N noise magnitude or zero frequency spectral density, ft^2/radian/sec

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P	weighting function
Q	quantity related to c_ϵ (see eqs. (B21) and (B24))
r	restricted quantity
R^2	mean-square restricted quantity
s	variable in the Laplace transform
T	time at collision, sec
u_o	unit impulse
x	dimensionless frequency, ω/β (see eq. (28))
y	output in adjoint diagram sketch (c)
Y_o	optimum closed-loop transfer function
α	constant multiplier in φ (see eq. (23))
β	input parameter, 1/sec (see eq. (28))
γ	vehicle parameter, sec (see eq. (37))
ϵ	error, ft
η	dimensionless parameter in optimum transfer function, $\gamma\beta$
λ_k	the kth determinant
ν	input dimensionless parameter, ξ/β
ξ	input parameter in frequency characteristic of ψ_s, 1/sec (see eq. (21))
ρ	Lagrangian multiplier, sec^4
σ	constant multiplier in ψ_s, ft^2/sec^3 (see eq. (21))
φ	frequency factor of input
ψ	frequency factor of input analogous to spectral density
ω	angular frequency, radians/sec
$\overline{(\ )}$	complex conjugate of ( )
$\widetilde{(\ )}$	ensemble average of ( )

Subscripts

s	signal
r	restricted quantity
a	acceleration
n	noise
ϵ	error

EXPLICIT FILTERING SOLUTION

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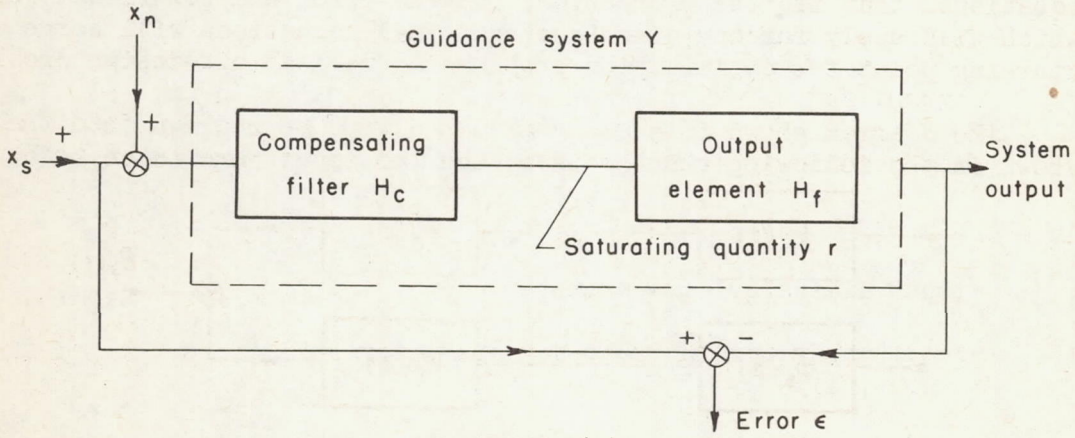
System Description

The type of guidance system with which we will be concerned has been described previously in references 1 and 2, but for completeness a brief summary will be desirable here. A great many guidance and interception problems can be represented by the block diagram shown in sketch (a). This representation can be shown to apply, for example, to certain space



Sketch (a)

navigation and missile problems which are time-invariant, or even to time-varying navigation problems, as will be shown later. It does not apply however, to guidance systems which operate part of the time as an open-loop system, such as certain interception weapons systems, fire control systems, etc. The two inputs to the guidance system are the signal x_s, which contains the true information about the motion of the target to be intercepted, and the noise x_n, which enters unavoidably with the desired signal. The outputs of interest are two in number. The first is the error ϵ which is a measure of how far the system output deviates from the desired signal part of the input. The second is the saturating quantity r. Although there may be many quantities subject to saturation in a guidance system, it has been shown (see ref. 1) that only the most

critical quantity need be considered. In the interception problem this quantity is the thrust producing quantity, such as the deflection of a gimbaled rocket engine or of a control surface of a missile. In order that this quantity will appear explicitly, the guidance system is split into a compensating filter and an output element. The output element, which represents the dynamics of the vehicle, is assumed to be the fixed element because it is relatively unalterable compared to the remainder of the system. The other element is the compensating network which the designer is free to choose.

The above representation for the guidance system does not represent the actual physical form of any particular system; that is, any linear missile guidance system of the type discussed above regardless of its form or the number of feedback loops can be put into the equivalent form shown. Conversely, for any given characteristics of the compensating and fixed filters, there are any number of corresponding physical systems which can be constructed.

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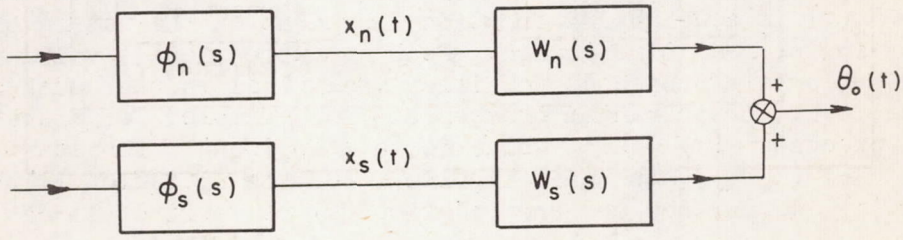
Performance Equations

The purpose of this section will be to present the exact performance equations, that is, the expressions for the error and restricted quantities which will apply for any given system. These equations will serve as a starting point for optimization problems as well as off-design problems.

The diagram shown in sketch (a) can always be redrawn into the form shown in the following sketch. Here the two inputs are taken to be



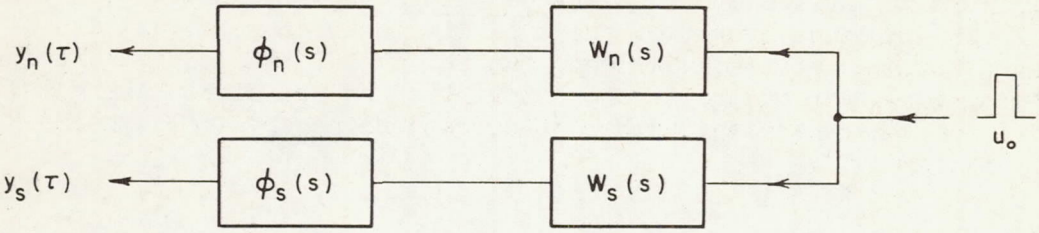
Sketch (b)

impulsive in character. Such a representation is a natural one both because nearly all inputs can be represented by a series of impulses and because the adjoint theory to be used later requires impulse type inputs. Each of these two inputs is then modified by a frequency factor $\varphi(s)$ before entering the system transfer function $W(s)$. The actual signal input, $x_s(t)$, to $W_s(s)$ may be an analytic signal, such as a step, etc., or some kind of random process. However, the noise input, $x_n(t)$, to $W_n(s)$ will be restricted to be a stationary random process, which occurs so often in practice. The output, $\theta_o(t)$, may represent either of the two outputs of interest, the error or the restricted quantity, if W_s and W_n are properly defined. The mean-square ensemble average, $\theta_o^2(T)$, at a

particular time T is given by the following expression, assuming no correlation between the two inputs:

$$\overline{\theta_o^2(T)} = \overline{\theta_s^2(T)} + \overline{\theta_n^2(T)} \tag{1}$$

Each of these components can be evaluated by use of the corresponding adjoint diagram (see refs. 3 and 4 for a description of the adjoint theory). The adjoint system corresponding to sketch (b) is shown below where it can be seen that now there are two outputs, y_n and y_s, and only one input, u_o,



Sketch (c)

which is always an impulse. For the moment let us confine our interest to only one of these components, the kth. The mean-squared ensemble value of the output $\theta_k(t)$ at a particular time T can be expressed in terms of the adjoint response $y_k(\tau)$ by the following general expression

$$\overline{\theta_k^2(T)} = \int_0^T y_k^2(\tau) P_k(\tau)\, d\tau \tag{2}$$

where $P_k(\tau)$ is a weighting function dependent on the nature of the input process in the real time domain. Such an expression is valid for linear systems which are subjected to either analytical inputs, such as a step, impulse, etc., or to random processes. For example, for a stationary random process, $P(\tau) = 2\pi$[1] while for a step input which starts at t_1, $P(\tau) = u_o[\tau-(T-t_1)]$, the unit impulse. For the situation illustrated in sketch (c), we may now use the relation (2) to rewrite (1) as

$$\overline{\theta_o^2(T)} = \int_0^T y_s^2(\tau) P_s(\tau)\, d\tau + \int_0^T y_n^2(\tau) P_n(\tau)\, d\tau \tag{3}$$

It often happens that because of the nature of the input frequency function $\varphi_s(s)$ and the system $W_s(s)$, the output $y_s(\tau)$ has a steady-state value c. Thus it will be found convenient to let

$$y_t(\tau) = y_s(\tau) - c \tag{4}$$

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[1]The 2π is due to definitions of spectral density (see ref. 4, p. 9).

Since in the cases in which we will be interested $P_k(\tau)$ is independent of τ, equation (3) becomes

$$\overline{\theta_o^2(T)} = P_s \int_o^T y_t^2(\tau)d\tau + 2P_s c \int_o^T y_t(\tau)d\tau + P_s c^2 T + P_n \int_o^T y_n^2(\tau)d\tau \quad (5)$$

By the definition in equation (4), we know that $y_t \to 0$ as $T \to \infty$. Furthermore, in the usual case, $y(\tau)$ becomes so small for reasonable values of T that $\int_T^\infty y_t^2(\tau)d\tau \ll \int_o^T y_t^2(\tau)d\tau$. Physically, this inequality corresponds to the assumption that the system response time is less than the interval of interest T. Thus by increasing the upper limits to ∞ and by transforming to the frequency domain[2], we can re-express equation (5) as

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$$\overline{\theta_o^2(T)} = \frac{P_s}{2\pi}\int_{-\infty}^{\infty} \left| Y_t(\omega) \right|^2 d\omega + 2P_s c \lim_{s\to o} Y_t(s) + P_s c^2 T$$

$$+ \frac{P_n}{2\pi}\int_{-\infty}^{\infty} \left| \varphi_n(\omega) W_n(\omega) \right|^2 d\omega \quad (6)$$

An expression for $Y_t(s)$ can be obtained from equation (4) and sketch (c):

$$Y_t(s) = \varphi_s(s) W_s(s) - \frac{c}{s} \quad (7)$$

where

$$c = \lim_{s\to o} s\varphi_s(s) W_s(s) \quad (8)$$

Thus,

$$\overline{\theta_o^2(T)} = \frac{P_s}{2\pi}\int_{-\infty}^{\infty} \left| \varphi_s(\omega) W_s(\omega) - \frac{c}{i\omega} \right|^2 d\omega + 2P_s c \lim_{s\to o} \left[\varphi_s(s) W_s(s) - \frac{c}{s} \right]$$

$$+ P_s c^2 T + \frac{P_n}{2\pi}\int_{-\infty}^{\infty} \left| \varphi_n(\omega) W_n(\omega) \right|^2 d\omega \quad (9)$$

Equation (9) gives the mean-square ensemble or time average of any quantity at time T in terms of the system transfer function and the input weighting and frequency functions.

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[2] Note that $2\pi \int_o^\infty |y(\tau)|^2 d\tau = \int_{-\infty}^{\infty} |Y(\omega)|^2 d\omega$.

Now the expressions for the error ϵ and restricted quantity r can be written. We introduce E^2 and R^2 to represent either mean-square time or ensemble averages of the error and restricted quantity, respectively. If sketch (a) is interpreted in terms of sketch (b), we can deduce the following.

$$\left.\begin{array}{ll} \text{For error } \epsilon: & W_s(s) = 1-Y(s) \\ & W_n(s) = Y(s) \\ \text{For restricted quantity } r: & W_s(s) = Y(s)/H_f(s) \\ & W_n(s) = Y(s)/H_f(s) \end{array}\right\} \tag{10}$$

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Therefore the error and restricted quantities become, from equation (9),

$$E^2 = \frac{P_s}{2\pi}\int_{-\infty}^{\infty}\left|\varphi_s(\omega)[1-Y(\omega)] - \frac{c_\epsilon}{i\omega}\right|^2 d\omega + 2P_s c_\epsilon \lim_{s\to 0}\left\{\varphi_s(s)[1-Y(s)] - \frac{c_\epsilon}{s}\right\}$$

$$+ P_s c_\epsilon^2 T + \frac{P_n}{2\pi}\int_{-\infty}^{\infty}|\varphi_n(\omega)Y(\omega)|^2 d\omega \tag{11}$$

$$R^2 = \frac{P_s}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\varphi_s(\omega)Y(\omega)}{H_f(\omega)} - \frac{c_r}{i\omega}\right|^2 d\omega + 2P_s c_r \lim_{s\to 0}\left\{\frac{\varphi_s(s)Y(s)}{H_f(s)} - \frac{c_r}{s}\right\} + P_s c_r^2 T$$

$$+ \frac{P_n}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\varphi_n(\omega)Y(\omega)}{H_f(\omega)}\right|^2 d\omega \tag{12}$$

These two expressions will be useful throughout the remainder of the report.

Optimum System Transfer Function

In this section will be given a simplified explicit solution for the optimum system based on certain simplifying approximations.

Exact equations.- The optimization problem is one of determining the optimum over-all system transfer functions which will minimize

$$E^2 + \rho R^2 \tag{13}$$

where ρ is a Lagrangian multiplier having the units of $(E/R)^2$, and E and R are given by equations (11) and (12). Such minimization could probably be carried out by brute force. However, anticipating the nature of the final solution can simplify the problem. After a little reflection one might expect that both the steady-state terms, c_ϵ and c_r, would be zero. We will assume that this is the case and after we have the answer, we can check back to verify the assumption. Equations (11) and (12) now become

$$E^2 = \int_{-\infty}^{\infty} |1-Y(\omega)|^2 \frac{P_s|\varphi_s(\omega)|^2}{2\pi} d\omega + \int_{-\infty}^{\infty} |Y(\omega)|^2 \frac{P_n|\varphi_n(\omega)|^2}{2\pi} d\omega \tag{14}$$

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$$R^2 = \int_{-\infty}^{\infty} \left|\frac{Y(\omega)}{H_f(\omega)}\right|^2 \frac{P_s|\varphi_s(\omega)|^2}{2\pi} d\omega + \int_{-\infty}^{\infty} \left|\frac{Y(\omega)}{H_f(\omega)}\right|^2 \frac{P_n|\varphi_n(\omega)|^2}{2\pi} d\omega \tag{15}$$

The first term of each of these expressions is due entirely to signal, the second to noise. Although these equations are written in terms of the fixed filter $H_f(\omega)$ and the over-all system transfer function $Y(\omega)$ (instead of the compensating network as described in sketch (a)), they can be expressed in terms of the fixed filter and the compensating filter by the relation

$$Y(\omega) = H_c(\omega) H_f(\omega) \tag{16}$$

An expression for the optimum compensating network which minimizes (13) can be readily found. If we make the following definitions

$$\psi_s(\omega) = \frac{P_s|\varphi_s(\omega)|^2}{2\pi} \tag{17}$$

$$\psi_n(\omega) = \frac{P_n|\varphi_n(\omega)|^2}{2\pi} \tag{18}$$

then it is clear that the ψ's are analogous to spectral densities and that the solution for the optimum compensating network function, $H_{co}(\omega)$, is given by (see ref. 5 or 1)

$$H_{co}(i\omega) = \frac{1}{2\pi\Lambda^+(\omega)} \int_0^{\infty} e^{-i\omega t} \int_{-\infty}^{\infty} \frac{\overline{H_f(i\alpha)}\psi_s(\alpha) e^{i\alpha t}}{\Lambda^-(\alpha)} d\alpha\, dt \tag{19}$$

where

$$\Lambda(\omega) = [\overline{H_f(i\omega)} H_f(i\omega) + \rho][\psi_s(\omega) + \psi_n(\omega)]$$

As discussed in reference 1, the optimum compensating network H_{co} can only be determined by the simultaneous solution of equations (15) and (19) so as to result in the desired restriction. Evaluation of equation (14) then follows. Of the various factors involved in these equations the quantities which must normally be known are the input quantities $\psi_s(\omega)$ and $\psi_n(\omega)$, and the fixed network $H_f(\omega)$. Unfortunately even for very simple forms for these functions, the complexity of operations involved in the solution of these equations does not permit the general solution to be obtained explicitly. We will be interested in certain forms for these functions which will now be discussed.

Simplified forms for inputs and fixed network.- A very common and important form for the inputs which occurs in many physical problems and particularly in the interception of targets is the following

$$\psi_{\ddot{s}}(\omega) = \frac{\sigma}{\omega^2 + \xi^2} \tag{20}$$

or

$$\psi_s(\omega) = \frac{\sigma}{\omega^4(\omega^2+\xi^2)} \tag{21}$$

and

$$\psi_n(\omega) = N \tag{22}$$

where $\psi_{\ddot{s}}(\omega)$ corresponds to the second derivative of the signal. The form in (20) or (21) is valid for many stationary and nonstationary processes as will be seen. Furthermore this form is general enough to approximate a variety of experimentally determined input data. The other function $\psi_n(\omega)$, the noise frequency factor, is approximated by a constant. This is a good approximation, since in most physical situations the bandwidth of the actual noise is much broader than that of the optimum system transfer function $Y_o(\omega)$. Obviously, from equation (17) the actual filters which appear in sketch (c) would be

$$\varphi_s(s) = \frac{\sqrt{2\pi\sigma/P_s}}{s^2(s+\xi)} \equiv \alpha f(s) \tag{23}$$

$$\varphi_n(s) = \sqrt{N} \tag{24}$$

The functions φ and ψ are merely two different ways of expressing the signal input function. Thus the gain α of the actual filter in sketch (c) is related to the magnitude σ of the input frequency function by

$$\sigma = \frac{P_s\alpha^2}{2\pi} \tag{25}$$

The other function, H_f, is the fixed element representing the dynamics of the vehicle. This function is generally very complicated; when the output is taken to be a displacement quantity as it is here, the fixed network will be of the form

$$H_f(s) = k_f \frac{a_n s^n + a_{n-1} s^{n-1} + \ldots + 1}{s^2(b_m s^m + b_{m-1} s^{m-1} + \ldots + 1)} \tag{26}$$

Even for cases in which m and n are only 1 or 2, the solution of equation (19) cannot be achieved in general terms. However, there are several sound reasons to believe that such a complicated H_f is both unnecessary and undesirable, and that it can be approximated by

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$$H_f \approx \frac{k_f}{s^2} \tag{27}$$

The reasoning is as follows. First, it can be shown that the fixed network given by equation (27) is optimum. In this sense it should be noted that we are considering optimization within a class of optimum systems. That is, for each fixed network H_f there is an optimum over-all guidance system and corresponding minimum error. However, the relationship between the fixed network and the minimum error is so complicated there is no a priori way to tell which fixed network will be best. Nevertheless, the results of reference 2 show that the H_f given by equation (27) results in an over-all system which is the best of all these systems. Thus (27) is a desirable form for the fixed network and should be striven for. Second, even if the dynamics are not the ideal ones given in equation (27), their effect on increased error may be small. We certainly cannot tell from the equations just <u>how</u> sensitive the minimum error will be to changes in the fixed network from the optimum form in (27) since, as we have seen, we could not even tell whether the effect would be beneficial or detrimental. However, it has been shown in reference 2 that the effect on minimum error of the dynamic factors in equation (26) is small as long as the natural frequencies are not too low and the damping ratios are reasonably small. A good many vehicles fall in this category.

<u>Solution</u>.- With these forms for $\psi_s(\omega)$ and $H_f(\omega)$, it is possible to solve equation (19) exactly. This exact solution is given in appendix A, and it can be seen to be quite unwieldy. For practical purposes we would like to know if suitable simplifications can be made without sacrificing appreciable accuracy.

Such simplifications can be made as discussed in appendix A. The nature of this solution will now be outlined. First of all for the forms of $\psi_s(\omega)$, $\psi_n(\omega)$, and $H_f(\omega)$ given in equations (21), (22), and (27), the error equation (14), the restricted quantity equation (15), and optimum compensating network equation (19) can be made dimensionless by means of the following substitutions:

$$\left.\begin{aligned} \omega &= \beta x \\ \beta &= \sqrt[6]{\sigma/N} \\ \nu &= \xi/\beta \end{aligned}\right\} \tag{28}$$

Obviously, the parameters β and ν are associated entirely with the input characteristics (i.e., signal and noise). The dimensionless analog of equations (14), (15), and (19) become

$$\frac{E^2}{N\beta} = \int_{-\infty}^{\infty} \frac{|1-Y(x)|^2}{x^4(x^2+\nu^2)}\,dx + \int_{-\infty}^{\infty} |Y(x)|^2 dx \tag{29}$$

$$\frac{R^2 k_f^2}{N\beta^5} = \frac{A^2}{N\beta^5} = \int_{-\infty}^{\infty} \frac{|Y(x)|^2}{x^2+\nu^2}\,dx + \int_{-\infty}^{\infty} |Y(x)|^2 x^4 dx \tag{30}$$

$$H_{co}(x) = \frac{1}{2\pi\Lambda^+(x)} \int_0^{\infty} e^{-i\beta xt} \int_{-\infty}^{\infty} \frac{\overline{H_f(z)}\psi_s(z)e^{i\beta zt}}{\Lambda^-(z)}\,\beta\,dz\,dt \tag{31}$$

where

$$\Lambda(x) = \left[\overline{H_f(x)}H_f(x) + \rho\right][\psi_s(x) + \psi_n(x)]$$

Note that the left side of (30) becomes $R^2k_f^2$, where k_f is the vehicle gain. Since the fixed network has only two poles, both at zero, the quantity Rk_f is merely the acceleration. Thus

$$A^2 = R^2k_f^2 \tag{32}$$

It should also be noted that the functional form of the quantities in the above equations is changed by the transformation (28).

The solution of equation (31) for the optimum compensating network $H_{co}(\omega)$ is somewhat involved. For this reason the details of the derivation are discussed in appendix A. It is shown that if

$$\frac{4\nu^6}{27} \ll 1 \tag{33}$$

which is satisfied for nearly all cases of interest, the approximate optimum closed-loop transfer function, Y_o, can be expressed in dimensionless form as follows:

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$$Y_o(x) = \frac{-Px^2 + i\sqrt{2}\,(\sqrt{2}+\eta)x + 1}{(ix+1)(-x^2+ix+1)(-\eta^2x^2+i\sqrt{2}\eta x+1)} = Y_o(x;\nu_1,\eta) \tag{34}$$

or

$$Y_o(s) = \frac{\dfrac{P}{\beta_1^2}s^2 + \dfrac{\sqrt{2}(\sqrt{2}+\eta)}{\beta_1}s + 1}{\left(\dfrac{s}{\beta_1}+1\right)\left(\dfrac{s^2}{\beta_1^2}+\dfrac{s}{\beta_1}+1\right)\left(\dfrac{\eta^2}{\beta_1^2}s^2+\dfrac{\sqrt{2}\eta}{\beta_1}s+1\right)} \tag{35}$$

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$$= \frac{\dfrac{P}{\beta_1^2}s^2 + \dfrac{\sqrt{2}(\sqrt{2}+\eta)}{\beta_1}s + 1}{\dfrac{\eta^2}{\beta_1^5}s^5 + \dfrac{\sqrt{2}\eta(1+\sqrt{2}\eta)}{\beta_1^4}s^4 + \dfrac{(1+\sqrt{2}\eta)^2}{\beta_1^3}s^3 + \dfrac{(\sqrt{2}+\eta)^2}{\beta_1^2}s^2 + \dfrac{\sqrt{2}(\sqrt{2}+\eta)}{\beta_1}s + 1} \tag{36}$$

where

$$\left.\begin{aligned} P &= P(\nu_1,\eta) \qquad \text{(see eq. (A84))} \\ \gamma &= \rho^{1/4}k_f^{-1/2} \\ \eta &= \gamma\beta_1 \end{aligned}\right\} \tag{37}$$

The corresponding open-loop transfer function μ_o is

$$\mu_o(x) = -\frac{1}{(\sqrt{2}+\eta)^2 - P}\,\frac{-Px^2 + i\sqrt{2}(\sqrt{2}+\eta)x + 1}{x^2\left[-\dfrac{i\eta^2}{(\sqrt{2}+\eta)^2 - P}x^3 - \dfrac{\sqrt{2}\eta(1+\sqrt{2}\eta)}{(\sqrt{2}+\eta)^2 - P}x^2 + \dfrac{i(1+\sqrt{2}\eta)^2}{(\sqrt{2}+\eta)^2 - P}x + 1\right]} = \mu_o(x;\nu_1,\eta) \tag{38}$$

$$\mu_o(s) = \frac{\beta_1^2}{(\sqrt{2}+\eta)^2 - P}\,\frac{\dfrac{P_1}{\beta_1^2}s^2 + \dfrac{\sqrt{2}(\sqrt{2}+\eta)}{\beta_1}s + 1}{s^2\left\{\dfrac{\eta^2}{\beta_1^3[(\sqrt{2}+\eta)^2 - P]}s^3 + \dfrac{\sqrt{2}\eta(1+\sqrt{2}\eta)}{\beta_1^2[(\sqrt{2}+\eta)^2 - P]}s^2 + \dfrac{(1+\sqrt{2}\eta)^2}{\beta_1[(\sqrt{2}+\eta)^2 - P]}s + 1\right\}} \tag{39}$$

It will be seen that this solution for $Y_o(x)$ and $u_o(x)$ is dependent on only two dimensionless constants ν_1 and η. (The subscript 1 has been attached to ν and β in order to associate these quantities with the

inputs for which the system was optimized. Later a subscript 2 will also be used to associate these quantities with the actual input which may or may not be the same as that for which the system was optimized.) It is clear from the definition in (28) that ν_1 is dependent solely on the target characteristics. The other parameter, η, is dependent on the product of the input parameter β_1 and the vehicle parameter γ. The parameter γ is clearly associated with the vehicle from the above definitions; that is, γ depends on the vehicle gain k_f and on the Lagrangian multiplier ρ which is used to place the desired restriction on vehicle acceleration. The determination of ρ is discussed in later paragraphs. Obviously $\rho = \gamma = \eta = 0$ corresponds to the Wiener case with no restrictions, and in this case the transfer function simplifies considerably.

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Performance Equations of Optimum Systems

Having found the optimum system one can now determine the error and acceleration. Because we are interested in the performance of optimum systems for a variety of inputs, we must return to the more general expressions given in equations (11) and (12).

The error equation (11) is seen to consist of four terms which are defined to be, respectively,

$$E^2 = E_1^2 + E_2^2 + E_3^2 + E_n^2 \tag{40}$$

The first three terms are due to the signal while the last is due to noise. It is shown in appendix B that each of these components can be expressed in dimensionless form as follows

$$\frac{E_1^2}{\left(\frac{N_2\beta_2^6}{\beta_1^5}\right)} = \int_{-\infty}^{\infty} \frac{|1-Y_o(x) + x^2Q|^2}{x^4(x^2+\nu_2^2)}\,dx = -\frac{\pi i}{e_o}\frac{\lambda_1}{\lambda_2} = f_1(\nu_1,\nu_2,\eta) \tag{41}$$

$$\frac{E_2^2}{\left(\frac{N_2\beta_2^6}{\beta_1^5}\right)} = 4\pi i Q(a_2+Qa_4) = f_2(\nu_1,\nu_2,\eta) \tag{42}$$

$$\frac{E_3^2}{\left(\frac{N_2\beta_2^6}{\beta_1^5}\right)(\beta_1 T)} = 2\pi Q^2 = f_3(\nu_1,\nu_2,\eta) \tag{43}$$

$$\frac{E_n^2}{N_2\beta_1} = \int_{-\infty}^{\infty} |Y_o(x)|^2 dx = -\pi i \frac{\lambda_3}{\lambda_4} = f_4(\nu_1,\eta) \tag{44}$$

where λ_1, λ_2, λ_3, and λ_4 are determinants given in appendix B and Q is a parameter yet to be explained.

There are several general comments to be made concerning these equations. First, it will be seen that all the components of error can be readily evaluated by means of simple algebraic expressions or determinants. Second, the subscript 1 has been attached to certain quantities to identify them with the input for which the system is optimized, while the subscript 2 has been attached in order to identify these quantities with the actual input. Third, it can be seen that all the dimensionless components of error are functions of three parameters, ν_1, ν_2, and η. Actually for many cases, these components are functions of only two parameters. For example, for optimization problems (where the input is the same as that for which the system is designed), it is necessary that $\nu_1 = \nu_2$. In other problems, where the design input and the actual input are not the same, we will see that either ν_1 or ν_2 is zero. In these cases, the components will be functions of only two parameters. Fourth, it should be noted that the total error cannot be obtained in dimensionless form by adding equations (41) through (44) since the nondimensionalizing factors are not all the same for all components. And last is the factor Q which is somewhat involved and needs some explanation. From the definition in equation (B24),

$$Q = \beta_1^2 \lim_{s\to 0} \frac{1-Y_o(s)}{s(s+\xi_2)} \tag{45}$$

and (36) for the optimum Y_o, it can be shown that Q will always be zero unless $\nu_1 \neq 0$ and $\nu_2 = 0$, and in such a case its value will be dependent solely on ν_1 and η. That is,

$$\left.\begin{aligned} Q &= -a_3 - P = Q(\nu_1,\eta) \qquad && \text{if } \nu_1 \neq 0,\ \nu_2 = 0 \\ &= 0 && \text{otherwise} \end{aligned}\right\} \tag{46}$$

The physical meaning involved here is simple. For an accelerating target at least two integrations are always required in the forward part of the open-loop system. As can be shown from equation (36) for the optimum Y_o, the least number, namely 2, occurs when $\nu_1 \neq 0$, or when the system is optimized for a signal of the form $\psi_{\ddot{s}}(\omega) = \sigma/(\omega^2+\nu^2)$. In such a case Q will not be zero only if the signal input to this system is of the form σ/ω^2, that is, $\nu_2 = 0$. A nonstationary input of this form will be discussed in the next section.

Consider now evaluation of the restricted quantity given in equation (12). It is indicated in appendix B that when H_f and c_r are substituted in equation (12), the vehicle gain k_f will combine with R

so that the left side of (12) will become $R^2k_f^2$. Since there are no poles of the fixed network other than at zero the quantity Rk_f is merely acceleration. Equation (12) can then be expressed as

$$\left.\begin{aligned} A^2 &= A_1^2 + A_2^2 + A_3^2 + A_n^2 \\ &= A_s^2 + A_n^2 \end{aligned}\right\} \tag{47}$$

where the signal component is

$$A_s^2 = A_1^2 + A_2^2 + A_3^2 \tag{48}$$

Each of these components can be written in dimensionless form as

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$$\frac{A_1^2}{\left(\frac{N_2\beta_2^6}{\beta_1}\right)} = \int_{-\infty}^{\infty} \frac{|Y(x) - M_A|^2}{x^2 + \nu_2^2}\,dx = -\frac{\pi i}{e_o}\frac{\lambda_5}{\lambda_2} = g_1(\nu_1,\nu_2,\eta) \tag{49}$$

$$A_2^2 \equiv 0 \tag{50}$$

$$A_3^2 = P_s T\alpha_2^2 M_A^2 \tag{51}$$

$$\frac{A_n^2}{N_2\beta_1^5} = \int_{-\infty}^{\infty} x^4 |Y(x)|^2 dx = \frac{\pi i}{a_o}\frac{\lambda_6}{\lambda_4} = g_2(\nu_1,\eta) \tag{52}$$

where again the λ's are determinants and are given by the complete expressions in appendix B. As before, one can show that the components are functions of three parameters ν_1, ν_2, and η, although for most cases, the components will be functions of only two parameters. Also, the total acceleration cannot be obtained in dimensionless form by adding equations (49) through (52). Later, however, when we have reason to put $\beta_1 = \beta_2$ (or simply β), the components can be added. The parameter M_A is quite similar to the previous parameter Q. From its definition

$$M_A = \lim_{s\to 0} \frac{sY(s)}{(s+\xi_2)} = \lim_{s\to 0} \frac{s}{s + \xi_2} \tag{53}$$

it is clear that

$$\left.\begin{aligned} M_A &= 0 \qquad \text{if} \quad \nu_2 \neq 0 \\ &= 1 \qquad \text{if} \quad \nu_2 = 0 \end{aligned}\right\} \tag{54}$$

Physically, this condition is obviously dependent on the nature of the input signal.

Resumé

Because of the length of the previous sections, a resumé will now be given in order to summarize the problem, assumptions, and solution. The class of guidance systems considered above are those described in sketch (a) and the accompanying discussion. Our primary objective in this report has been discussed in the Introduction. Briefly, it is to relate performance, both optimum and off-design performance, explicitly to the guidance and control task, that is, to those parameters which are normally included in the statement of the problem.

Since a system cannot be optimized for all inputs and all restrictions, it is necessary to specify certain quantities. It is necessary to know something of the signal and noise inputs, the number of saturating elements, and the fixed network (see eq. (19) and succeeding discussion). Although a solution can be obtained for any choices of these quantities, we would like to choose them so that we can get an explicit solution and yet have these choices be as physically meaningful as possible. The choices which were made are:

(1) The signal is of the form given in equation (20). This form fits several inputs, as will be seen in following sections.

(2) The noise is stationary, white, and uncorrelated with the signal (see discussion following eq. (22)).

(3) Only the most critical saturating quantity (acceleration) and one fixed network need be considered (see p. 9 of ref. 1 for justification). Furthermore, even though the fixed network is generally very complicated, it can be adequately represented by $H_f = k_f/s^2$ (see the discussion in connection with eq. (27), and ref. 2).

There are, in addition, several assumptions made in obtaining the explicit solution:

(4) The system response time is less than the interval of flight time 0 to T (to be justified in a later section).

(5) The weighting factor P is independent of τ.

(6) The following inequality, which is not really essential, greatly simplifies the expressions (see eq. (33) and discussion).

$$\frac{4\nu^6}{27} \ll 1$$

The solution consists of two parts: the optimum transfer functions, and the performance equations for the optimum systems. The dimensionless optimum transfer functions are given by equations (34) and (38). It is

seen that they have been expressed in terms of two dimensionless parameters, ν_1 and η (defined in eqs. (28) and (37), and described below eq. (39)). The transfer function (34) is optimum for several inputs and forms the basis for optimum time-varying homing systems (as will be shown in later sections).

The performance equations for the optimum system consist of the error equations (41) through (44) and the acceleration equations (49) through (52). All of them are given in dimensionless form as functions of the three parameters ν_1, ν_2, and η. These equations can be used for constant-coefficient systems and time-varying homing systems, and for certain stationary and nonstationary inputs. They can also be used for evaluating off-design performance. Such uses of these equations are illustrated in following sections.

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OPTIMUM PERFORMANCE

It will be the purpose of this section to show how the solution just derived applies to several distinct types of optimization problems, and to present the corresponding theoretical optimum performance curves. The problems considered involve different types of systems and inputs, both time-invariant systems and time-varying homing systems, and several types of signal characteristics both stationary and nonstationary.

Time-Invariant Systems

There are many guidance and control problems that can be described by constant-coefficient differential equations with stationary signal inputs which possess frequency characteristics of the form (20). It is of interest to enumerate some random processess which fall in this category. All have been described previously elsewhere, although in different terms. When they are expressed in terms of the same definitions, these processes and their descriptions can be summarized as in figure 1. It will be seen that in signals A and B, both the amplitude and the interval length are random variables, while in C only the interval is random.

The optimum performance curves for systems subjected to signal inputs of the form (20) can be readily obtained in dimensionless form from the performance equations presented in the last section. For this case let us see what some of the parameters should be. First, since we are concerned here only with optimization problems, the input to the system is to be identical with that for which the system is optimized. In this case β_1 equals β_2 and ν_1 equals ν_2, and we may therefore use simply β and ν (also N with no subscript). Furthermore, since the system is to be optimized for inputs of the form (20), it is necessary that ν not be zero. Second, it is clear that the form for the signal input given in equation (20) is applicable to all the processes in figure 1 provided the

symbols σ and ξ are properly interpreted in terms of the particular process of interest. Third, for stationary processes the input weighting function P_k in equation (2) is 2π; that is, $P_s = P_N = 2\pi$. Fourth, the factors Q and M_A which are involved in the performance equations can be seen from equations (46) and (54) to be both zero. To summarize the parameters involved, then, we must have in the performance equations

$$\left.\begin{aligned} \nu_1 &= \nu_2 = \nu \neq 0 \\ \beta_1 &= \beta_2 = \beta \\ M_A &= Q = 0 \end{aligned}\right\} \tag{55}$$

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As a result of these values, several components can be dropped from the performance equations since

$$E_2 = E_3 = A_3 = 0$$

Also, since $\beta_1 = \beta_2 = \beta$ the components of error and acceleration can be combined to give

$$\frac{E^2}{N\beta} = \frac{E_1{}^2}{N\beta} + \frac{E_n{}^2}{N\beta} \tag{56}$$

$$\frac{A^2}{N\beta^5} = \frac{A_1{}^2}{N\beta^5} + \frac{A_n{}^2}{N\beta^5} \tag{57}$$

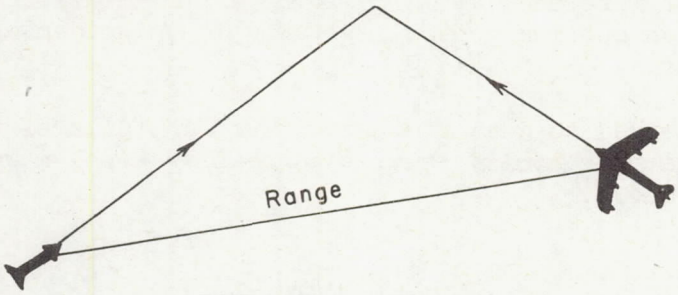
Each of these components is still given by the equations (41), (44), (49), and (52). It is only the labeling of the left side of these equations which has been changed since β does not occur on the right. Plots of each of the components of error and acceleration as well as their total are presented as functions of ν and η in figures 2(a) through 2(d). The dependent quantities may be taken to represent either ensemble or time averages.

The curves presented in figures 2(a) through 2(d) may be used to evaluate optimum performance for any set of conditions for which $4\nu^6/27 \ll 1$. Given any signal or target parameter and noise as defined by ν, β, and N, and a vehicle with any rms acceleration capability A, the factor η can be found from figure 2(d). From this η, the minimum error can be determined from figure 2(b), and the optimum system transfer functions from equations (35) and (39). If, however, one is not interested in knowing the optimum transfer function, the intermediate parameter η can be eliminated as shown in figure 2(e), and the minimum error can be obtained directly as a function of available vehicle acceleration.

The impulse responses of the optimum systems are of fundamental importance. They are useful in establishing minimum launching ranges; they determine the error or miss due to an impulse of signal or noise; and they determine the minimum duration of certain nonstationary inputs for which the solution is valid, as will be seen. The impulse responses are shown in dimensionless form in figure 3. Note that ν has a much smaller effect on these responses than does η.

Time-Varying Systems

In this section is considered a class of important control system problems which is characterized by time-varying differential equations. The nature of this class of problems is illustrated in the following sketch by the example of a homing missile intercepting a bomber target. It is seen that this example is characterized by the fact that the range between



Sketch (d)

the missile and target changes continuously with time. This range variation is due to physical facts and hence cannot be avoided. Since range enters into the coefficients of the differential equation describing this problem, the problem is necessarily time varying. The time-varying situation illustrated here may be recognized as belonging to a large and important class of problems such as mid-course or terminal guidance in interplanetary flight, fire control, and aircraft landing.

This time-varying problem has the same ingredients as does the time-invariant problem just discussed, that is, the target maneuver, the noise, and the missile maneuverability. In addition, however, the system is constrained to operate with a forced time variation representing the varying range. Furthermore, in this type of problem we are concerned with minimizing the error only at a particular time T (the time of arrival at the destination), but restricting the vehicle capabilities at all times t_2 previous to T. Thus ensemble averages are particularly meaningful. From these remarks then one can see that this time-varying problem is similar in concept but basically more complicated than the time-invariant case discussed in the previous section.

It is not at all obvious that the solution for the time-invariant case given in the previous section has any connection with the present

time-varying problem; however, a relationship can be shown. The optimization of the time-varying problem has been studied in references 4 and 6. There an equivalence was shown between the time-invariant and the time-varying homing problems, so that the solution given in the present report forms the basis for the optimization of the time-varying system. More specifically, it is clear from the results of references 4 and 6 that:

(1) The optimum performance curves (for the rms ensemble average of the error at time T, or miss, and the rms ensemble average of the acceleration) given in this report in figures 2(a) through 2(e) are valid for the time-varying problem.

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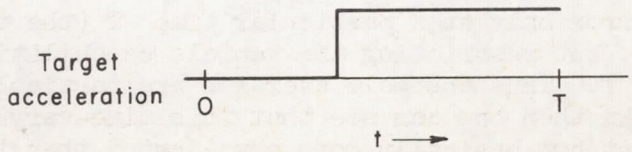
(2) The synthesis of the optimum homing system can be accomplished by combining the results of this report and those of reference 4 for the homing study. That is, the transfer function $Y_0(s)$ given in equation (35) of this report can be substituted for $H(s)$ in equation (22) of reference 4. From this latter equation one may then synthesize an optimum control system which, incidentally, is also time-varying.

It is not intended to deal further with the details of the optimization of the time-varying homing systems since this problem was the subject of references 4 and 6.

Nonstationary Inputs

One of the assumptions generally made in the Wiener theory and in Newton's modification of this theory is that the input process must be stationary. However, in many physical problems, especially in the interception of targets, the input process must be considered as essentially nonstationary. The reason is that all real target maneuvers will have a finite beginning and a finite end. Consequently, it will be desirable to examine the applicability of the solution presented earlier to certain nonstationary cases of interest in the interception problem.

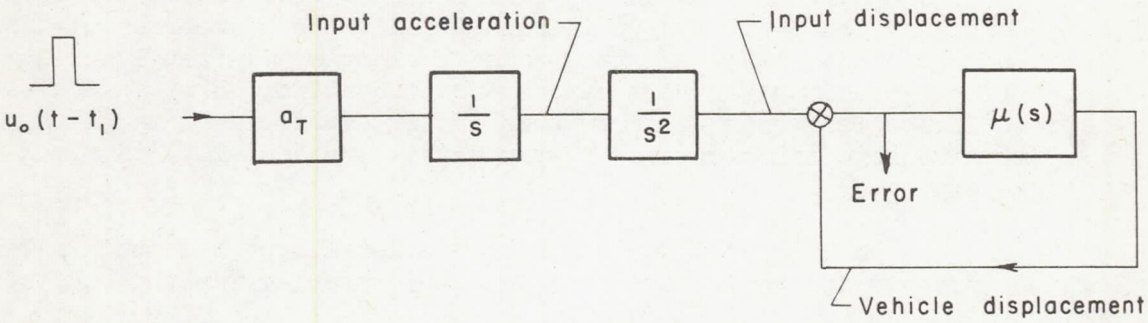
Nonstationary step signal.- One type input of interest in the target interception problem is a step of target acceleration as is indicated in the following sketch, where the beginning of the step is equally likely



to occur anywhere within the interval of interest. This interval starts at some finite time, $t = 0$, when the vehicle is launched and extends to

some time T when the vehicle reaches the target. Such an input is a random process which is distinctly nonstationary. It should be noted that this type of maneuver has been used previously in reference 7. Optimization for this input will be shown to be a special case of the solution presented earlier. As will be seen, the solution given here is one which enables the results for this and other inputs to be unified.

Let us consider first the problem of merely evaluating the mean-square ensemble average of the error at time T due to a signal only. In the real case of interest, of course, a noise signal would be added to this input. The real time block diagram with a pure signal input is shown in the following sketch where the time t_1 at which the input



impulse occurs is uniformly distributed in 0 to T. This diagram is obviously of the same general form that was used previously in sketch (b). We have

$$\varphi_s(s) \equiv \alpha f(s) = \frac{a_T}{s^3} \tag{58}$$

$$W_s(s) = 1 - Y(s) \tag{59}$$

where Y(s) is the over-all closed-loop transfer function. Now let us see what the values of the parameters involved in the solution will be. First, using equations (17), (58), and the fact that $P_s = 1/T$ (since t_1 is uniformly distributed), we have

$$\psi_s(\omega) = \frac{a_T^2}{2\pi T\omega^6} \tag{60}$$

This latter function is certainly not a spectral density and can only be called a frequency function which is associated with the signal. However, one can see that this factor for the nonstationary process enters the equations exactly as would a spectral density for a stationary process. Hence one can derive minimum errors and optimum system transfer functions

even though the process is distinctly nonstationary. Second, by comparing equations (60) and (21) we see we need only make the following definitions.

$$\left.\begin{aligned} \sigma &= \frac{a_T^2}{2\pi T} \\ \xi &= 0 \end{aligned}\right\} \tag{61}$$

Third, since the system is to be optimized for this input, it is necessary that $\xi_1 = 0$, and since the system is to be subjected to this input, $\xi_2 = 0$. In dimensionless terms (see eq. (28)), this amounts to

$$\nu_1 = \nu_2 = \nu = 0 \tag{62}$$

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Fourth, from equations (46) and (54) we see that

$$\left.\begin{aligned} Q &= 0 \\ M_A &= 1 \end{aligned}\right\} \tag{63}$$

Fifth, since the actual input is the same as that for which the system is optimized,

$$\beta_1 = \beta_2 = \beta$$

The performance curves can now be readily obtained from the solution given in equations (41) through (44) and (49) through (52). Note that again the E_2 and E_3 components of error are zero; also $A_3^2 = \alpha_2^2 = a_T^2$ and is therefore not a function of ν_1, ν_2, or η. Thus the performance equations simplify to

$$\frac{E^2}{N\beta} = \frac{E_1^2}{N\beta} + \frac{E_n^2}{N\beta} \tag{64}$$

$$\frac{A^2 - a_T^2}{N\beta^5} = \frac{A_1^2}{N\beta^5} + \frac{A_n^2}{N\beta^5} \tag{65}$$

The components on the right are still given by equations (41), (44), (49), and (52), and they are now functions of only one parameter, η, since the ν's are zero.[3] At this point it is both illuminating and useful to examine the transition that occurs from the stationary case just presented

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[3]Note that the E_1 and A_1 components given by equations (B37) and (B72) appear to be indeterminant. However, that this is not the case is clear by expanding λ_1, λ_2, and λ_5 along the last row. Hence one can: (1) simply eliminate the last row and column of λ_1, λ_2, and λ_5 or (2) merely use a very small value of ν_2, say 10^{-4}.

to this nonstationary case. From the details of the equations which are presented in appendix B one can deduce that E_1, E_n, (and therefore E, the total error) and A_n components are part of a continuous transition from the stationary case. That is, all of these components will be identical with that for the stationary case for $\nu_1 = \nu_2 = 0$ (see figs. 2(a), (b), and (c)). For clarity, plots of these components are repeated in figures 4(a) and (b). The remaining two components, A_1 and A_3, comprise the total signal component of the acceleration and their values are given by the appropriate equations. The transition for these two components is not as simple but can be explained as follows. In the stationary case when $\nu_1 = \nu_2 \neq 0$, the A_3 component was zero. As ν_1 and ν_2 approach zero, the other component A_1 grows without bound because the first integral in equation (30) is improper in the limit. Hence it cannot be plotted on figure 2(c). Physically, this is associated with a final steady value of the acceleration. In taking this into account as was done in the derivation one gets a different equation for A_1 (in which $M_A = 1$ instead of zero) and in addition another component, A_3. In other words as we progress to the nonstationary case, the A_1 for the stationary case becomes, in the limit, the two terms A_1 and A_3. As shown in equation (65) the new term A_3 (which is simply a_T) can be moved to the left side of the equation since it is not a function of ν or η. The A_1 component is plotted in figure 4(b). The total error, equation (64), and the total acceleration, equation (65), are plotted in figures 4(a) and (b). If η is eliminated, the data in these two curves can be cross-plotted to give figure 4(c), that is, the minimum dimensionless error directly as a function of dimensionless acceleration. These curves may be used to evaluate optimum performance in precisely the same manner as described on page 19 except that they are now much simpler because only the parameter η is involved.

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The optimum system transfer functions for this case are still given by equations (35) and (39) if we put $\nu_1 = 0$. We have

$$Y_o(s) = \frac{\dfrac{(\sqrt{2}+\eta)^2}{\beta_1^2} s^2 + \dfrac{\sqrt{2}(\sqrt{2}+\eta)}{\beta_1} s + 1}{\left(\dfrac{s}{\beta_1} + 1\right)\left(\dfrac{s^2}{\beta_1^2} + \dfrac{s}{\beta_1} + 1\right)\left(\dfrac{\eta^2}{\beta_1^2} s + \dfrac{\sqrt{2}\eta}{\beta_1} s + 1\right)} \tag{66}$$

$$\mu_o(s) = \frac{\beta_1^3}{(1+\sqrt{2}\eta)^2} \frac{\dfrac{(\sqrt{2}+\eta)^2}{\beta_1^2} s^2 + \dfrac{\sqrt{2}(\sqrt{2}+\eta)}{\beta_1} s + 1}{s^3\left[\dfrac{\eta^2}{\beta_1^2(1+\sqrt{2}\eta)^2} s^2 + \dfrac{\sqrt{2}\eta}{\beta_1(1+\sqrt{2}\eta)} s + 1\right]} \tag{67}$$

The impulse responses of this system described by equation (66) are given in figure 3 by the $\nu_1 = \nu_2 = 0$ curves; their significance has been

discussed previously. One can note that the form of the open-loop transfer function is now different; that is, there are now three integrations required in the forward loop rather than only two. This difference is readily explained. The P which was previously very complicated and given by equation (A84) approaches a very simple value, $P = (\sqrt{2}+\eta)^2$, as ν_1 approaches zero. This has the effect of increasing one of the time lags in the forward part of the loop, and in the limit it becomes an integration--a first-order lag term with infinite time lag. The physical reason for this extra integration is quite simple. It is due to the assumption about the input, that is, that there will be one and only one switch of the acceleration. Actually, this can never be stated with certainty. Saying that the target definitely maneuvers only once during the flight interval is certainly not the same as saying it is not likely to maneuver more than once, and this difference leads ultimately to the difference in the number of integrations required of the optimum guidance system.

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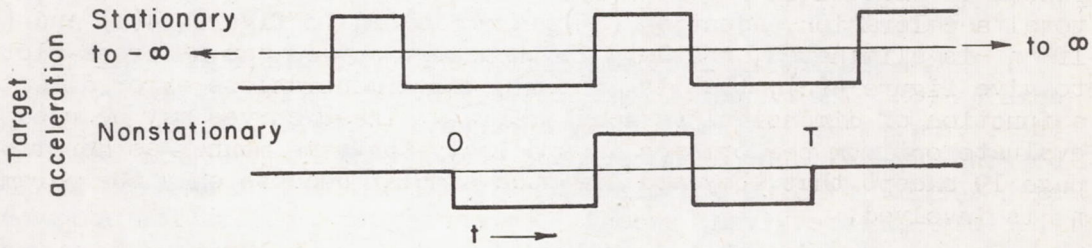
Nonstationary signals derived from stationary ones.- Other types of nonstationary signals of interest are those derived from stationary signals. The situation is illustrated below where the top sketch shows the



stationary maneuver which extends in both directions to infinity. The real maneuver, however, will necessarily start at some finite time called zero and will end at some finite time T when the vehicle reaches the target, as indicated above. This process might be termed stationary in the interval 0 to T, since it is part of a stationary process.

Since the upper of the two inputs does not occur in nature, it is often stated that this input is unrealistic and cannot be used. It is true that the stationary theory is applicable, strictly speaking, only to the upper of the two inputs above. However, the nonstationary character of the lower input is due to the mathematical definition of stationarity. In the practical case it is clear that it makes little difference to the vehicle, so far as error is concerned, whether the process persists over an infinite or finite period so long as the process begins at a time before the interception point by an amount equal to or greater than the system response time. (Of course, the process may terminate any time after time T without affecting the results.) In other words, an infinite period is, for practical purposes, simply one which is longer than the system response time. Thus when response times are short, results presented previously

in figure 2 for time-varying and time-invariant systems apply to the nonstationary case cited. Fortunately most interception situations fall in this category. The impulse responses which were presented in generalized form in figure 3 can be readily used to verify this condition.

OFF-DESIGN PERFORMANCE

In previous sections we have considered only optimization problems, that is, problems in which the system is subjected to precisely the same input for which it was specifically designed. Here we will consider off-design performance, that is, the deterioration in the error when the system is optimized for one input but subjected to a different input. It is important to note that the actual input might be different from the design input for two reasons. First, the two inputs, the actual and design inputs, might be describable by the same type of process, but the numerical values of the parameters for each input might be different. Such situations would occur, for example, if a noise level were different from the design value, or if there were a change in signal magnitude. Second, the actual and design inputs might be different because the type of process is different.

In the following sections different off-design cases will be considered. The manner in which the previous solution can be applied to these cases will be immediately apparent. However, to avoid confusion between the various parameters and components of error and acceleration, each section will be organized thusly: first the parameters involved in the solution are given; second, the error and acceleration components are arranged in tabular form together with the figure number for components which are plotted. It should be noted that the components are given individually since they cannot be combined because of different nondimensionalizing factors. Where possible, the components are combined. It should also be noted that certain curves will be identical with previous curves. Rather than repeat these curves it will only be necessary to alter the ordinate to the dimensionless form indicated. In so doing, the ordinates of the curves must be expressed as rms values.

Actual and Design Inputs of Same Type, Values Different

Stationary inputs.- Let us assume that a system has been optimized for any of the stationary signals (in addition to noise) which have been discussed previously. Then, let us assume that any (or all) of the specific values of the input process for which the system was optimized are changed. The solution already obtained is applicable to these problems. Since the actual input and design input are both the same type of stationary process, but the numerical values describing the processes are not necessarily the same, we must have

$$\nu_1 \neq \nu_2 \neq 0$$

$$\beta_1 \neq \beta_2$$

Also from equations (46) and (54) we see that

$$Q = 0$$

$$M_A = 0$$

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The error and acceleration components for this case can now be summarized as follows:

$$\frac{E_1^2}{\left(\frac{N_2\beta_2^6}{\beta_1^5}\right)} = f_1(\nu_1,\nu_2,\eta)$$ not plotted; see equation (B37)

$$E_2 = 0$$

$$E_3 = 0$$

$$\frac{E_n^2}{N_2\beta_1} = f_4(\nu_1,\eta)$$ rms values plotted in figure 2(a)

$$\frac{A_1^2}{\left(\frac{N_2\beta_2^6}{\beta_1}\right)} = g_1(\nu_1,\nu_2,\eta)$$ not plotted; see equation (B72)

$$A_3 = 0$$

$$\frac{A_n^2}{N_2\beta_1^5} = g_2(\nu_1,\eta)$$ rms values plotted in figure 2(c)

It can be seen that the E_1 and A_1 components are functions of the three variables ν_1, ν_2, and η. To display these components with adequate accuracy, a good many curves would be required, and they are therefore not plotted. (This is the only case in which three variables are required.) In a specific application the desired curves could be readily obtained from the equations indicated. The other components are functions of only the two variables, ν_1 and η.

Nonstationary inputs.- Let us now see how the solution applies to the situation when the system is optimized for specific values of the step maneuver and noise, but the values of the actual signal and noise are different. In this case we must have

$$\nu_1 = \nu_2 = 0 \qquad Q = 0$$

$$\beta_1 \neq \beta_2 \qquad M_A = 1$$

From these values, the performance curves can be obtained from the solution given in equations (41) through (44) and (49) through (52). The components are

$$\frac{E_1^2}{\left(\frac{N_2\beta_2^6}{\beta_1^5}\right)} = f_1(\eta)$$ rms values plotted in figure 4(a)

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$$E_2 = 0$$

$$E_3 = 0$$

$$\frac{E_n^2}{N_2\beta_1} = f_4(\eta)$$ rms values plotted in figure 4(a)

$$\frac{A_s^2 - a_T^2}{\left(\frac{N_2\beta_2^6}{\beta_1}\right)} = \frac{A_1^2}{\left(\frac{N_2\beta_2^6}{\beta_1}\right)} = g_1(\eta)$$ rms values plotted in figure 4(b)

$$A_3^2 = a_T^2$$

$$\frac{A_n^2}{N_2\beta_1^5} = g_2(\eta)$$ rms values plotted in figure 4(b)

We see that since the A_3 component is a constant, it has been combined with the total signal component A_s as follows

$$A_s^2 - a_T^2 = A_1^2$$

In this way, total acceleration can be obtained directly from figure 4(b). Also note that all the components (other than A_3) are functions of only the one variable η.

Actual and Design Inputs of Different Types

System optimized for stationary signal, actual signal nonstationary.- Let us assume that a system has been optimized for any of the stationary signals (in addition to noise) which we have discussed previously, but that this system is to be evaluated against a different type of maneuver such as the nonstationary single step maneuver also discussed before. The previous solution can be shown to be immediately applicable. From the discussion of earlier sections it is clear that we must have

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$$\nu_1 \neq 0 \qquad Q \neq 0$$

$$\nu_2 = 0 \qquad M_A = 1$$

$$\beta_1 \neq \beta_2$$

For these values the error and acceleration components can be summarized as follows

$$\frac{E_1^2+E_2^2}{\left(\frac{N_2\beta_2^6}{\beta_1^5}\right)} = f_1(\nu_1,\eta) + f_2(\nu_1,\eta) \qquad \text{rms values plotted in figure 5(a)}$$

$$\frac{E_3^2}{\left(\frac{N_2\beta_2^6}{\beta_1^5}\right)(\beta_1 T)} = f_3(\nu_1,\eta) \qquad \text{rms values plotted in figure 5(b)}$$

$$\frac{E_n^2}{N_2\beta_1} = f_4(\nu_1,\eta) \qquad \text{rms values plotted in figure 2(a)}$$

$$\frac{A_s^2-a_T^2}{\left(\frac{N_2\beta_2^6}{\beta_1}\right)} = \frac{A_1^2}{\left(\frac{N_2\beta_2^6}{\beta_1}\right)} = g_1(\nu_1,\eta) \qquad \text{rms values plotted in figure 5(c)}$$

$$A_3^2 = a_T^2$$

$$\frac{A_n^2}{N_2\beta_1^5} = g_2(\nu_1,\eta) \qquad \text{rms values plotted in figure 2(c)}$$

It is worth noting that now none of the components are zero and that they are all functions of the two parameters ν_1 and η. Also, as in the previous section, the A_3 component can be combined with the total signal component A_s. In a later section a specific example will be considered. It is also worth noting that the E_1 and A_1 components appear to be

indeterminant. This is not the case, however. Possibly the simplest procedure (which is readily shown from the equations) is merely to eliminate the last row and column of λ_1, λ_2, and λ_5.

System optimized for nonstationary signal, actual signal stationary.- For this situation let us assume that the system has been optimized for the nonstationary signal (and noise) discussed previously, that is, a single step of acceleration occurring any time during the time of flight. The actual signal, however, is to be any of the possible stationary signals described before. In this case we see that we must have

$$\nu_1 = 0 \qquad Q = 0$$

$$\nu_2 \neq 0 \qquad M_A = 0$$

$$\beta_1 \neq \beta_2$$

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The error and acceleration components now are

$$\frac{E_1^2}{\left(\frac{N_2\beta_2^6}{\beta_1^5}\right)} = f_1(\nu_2,\eta) \qquad \text{rms values plotted in figure 6(a)}$$

$$E_2 = 0$$

$$E_3 = 0$$

$$\frac{E_n^2}{N_2\beta_1} = f_4(\eta) \qquad \text{rms values plotted in figure 4(a)}$$

$$\frac{A_1^2}{\left(\frac{N_2\beta_2^6}{\beta_1}\right)} = g_1(\nu_2,\eta) \qquad \text{rms values plotted in figure 6(b)}$$

$$A_3 = 0$$

$$\frac{A_n^2}{N_2\beta_1^5} = g_2(\eta) \qquad \text{rms values plotted in figure 4(b)}$$

Note only that the E_1 and A_1 components are functions of ν_2 and η while the noise components are functions of only η. In the next section a specific example of this case will be considered.

EXAMPLE OF EFFECT OF TYPE OF SIGNAL INPUT

The effect on performance of the type of signal input is of particular interest in guidance system design. The reason is that one hardly ever knows with certainty what type of process the input will be. Consequently, two aspects of the problem are important:

1. The effect on performance of the type of signal input for which the system is optimized, and

2. The effect on performance of subjecting a system optimized for one type of input to an input of different type.

The two problems posed above really amount to a comparison of inputs. The question is: What is a good way to compare two inputs which are different types of processes? Unfortunately, the answer is not clear cut. We can see that since the processes are different and therefore the parameters describing the processes are different, the parameters could be chosen arbitrarily and independently. In this case the two effects on performance we are examining could be arbitrarily large. The previous sections could be used to obtain the exact amount of this difference. However, a more reasonable comparison might be based on finding conditions under which the inputs are equivalent in terms of performance. It would appear that β might be a good parameter on which to base the equivalence. For one thing, from the definition $\beta = \sqrt[6]{\sigma/N}$ it can be seen that β contains all the information about the input signal and noise. Furthermore the dominant modes in the optimum transfer function are determined by β. Thus one might expect that different types of inputs which have the same β would be approximately equivalent.

Let us illustrate the latter approach by a specific example. We will compare the nonstationary input described on page 21 with the stationary random input, Case C. For a realistic comparison, it is reasonable that the maximum acceleration should be limited to the same value for both inputs. Since the values of β for these two inputs are

$$\left.\begin{aligned} &\text{Stationary input C:} \quad \beta = \sqrt[6]{\frac{2a_T^2}{\pi N \bar{l}}} \\ &\text{Nonstationary input:} \quad \beta = \sqrt[6]{\frac{a_T^2}{2\pi N T}} \end{aligned}\right\} \qquad (68)$$

we see that these will be the same only if

$$\bar{l} = 4T$$

Now let us examine the effect on performance when the above criterion is used. It is clear that the results of the previous sections contain the desired answers. However, since the nondimensionalizing factors were not always the same, it will be necessary to remove these factors by using specific numerical values. Since the equations and curves are dimensionless, comparisons could be made for other cases of interest. Let us take arbitrarily, the flight time, the maximum acceleration, and the noise magnitude to be the following values:

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$$T = 10 \text{ sec}$$

$$a_T = 0.95 \text{ g}$$

$$N = 15 \text{ ft}^2/\text{radian/sec}$$

then for the stationary maneuver we would have $\bar{l} = 40$ seconds. For these conditions $\beta = 1.0$ and $\nu_1 = \nu_2 = \nu = 0.05$.

Now we can answer the first question, the effect of choice of target maneuver on which system design is based. This effect is shown by comparing the performances of two systems each optimized for the two inputs just discussed. The performance curves can be readily obtained from figures 2(e) and 4(c), and the result is shown in figure 7 where minimum theoretical error for each input is plotted against the vehicle rms acceleration capability. It can be seen the differences between these two curves is quite small over the entire range of vehicle acceleration and amounts to only a few feet. Thus the difference in optimizing for these two apparently different inputs is small provided the β's are the same. It is of interest to note that this difference in performance is small even though the actual change in acceleration for the stationary maneuver is twice as severe as for the nonstationary maneuver.

The second question, the effect of using an input for which the system was not designed, can also be readily answered. By utilizing the data in figures 5 and 6, one can readily evaluate the effect of using the nonstationary input with a system optimized for the stationary input and vice versa. The result is shown in figure 7 by the two sets of points, rather than curves, in order to avoid confusion. It might be a little surprising that the deterioration in error from the optimized curves is so slight.

CONCLUDING REMARKS

It seems desirable to emphasize the viewpoint which has been taken in this report. In general, optimization theory expresses certain types of operations to be performed in order to determine an optimum system. One would like to be able to carry out these operations in order to draw general conclusions about the best theoretical performance which can be achieved and the design of optimum guidance systems from a knowledge of the guidance and control task. It cannot be done without some loss, or narrowing down of this task. That is, it is impossible to choose a system which is optimum for all inputs and all restrictions. Here the desired solution has been obtained: first, by restricting the class of input signals but yet restricting it to a very useful and important class which includes certain stationary and nonstationary inputs; second, by simplifying the form of the fixed network or output element as was indicated by the results of a previous study; and third, by making approximations in the analysis so that a simple explicit solution could be obtained without sacrificing significant accuracy. The latter two are actually not very restrictive.

There are many ways in which the solutions presented here might be used. First, they might be used to determine the best theoretical performance which could be achieved for any specific case where the vehicle and target characteristics are quantitatively known. The result might then be compared to the performance of any other system to indicate possibilities for improvement. Second, the results might be used in preliminary design to evaluate the relative importance of each of the factors which affect minimum error. Such evaluations are useful in determining those design changes which would be worthwhile in attaining smaller errors. Or last, the results might be used to reach conclusions about the effect of different input signals, or target motions, as was discussed in the last section.

It is believed that the solution given might also be applied to other signal inputs not considered here. An example of such a signal which is of practical importance in the interception problem is a target maneuver consisting of a single switch of acceleration from a negative to a positive value at some random time.

An important extension is needed to the nonlinear problem. It will be recalled that the results presented here are based on rms values of the restricted quantity, the acceleration. However, most systems have hard limits. To insure that the system remains linear, the rms values must be chosen small enough compared to the saturation limits. In most cases the answer so obtained is near optimum, that is, the rms values required to keep the system essentially linear are still large enough that there is not much deterioration in error even from the infinite rms value. However, there is an increasing number of guidance systems which must

operate under more difficult and adverse circumstances in which the vehicle maneuverability becomes small. In these cases the error increases rapidly and it is important to utilize the maneuverability of the vehicle in a more efficient way, that is, by nonlinear control.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Oct. 24, 1960

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APPENDIX A

DERIVATION OF OPTIMUM SYSTEM TRANSFER FUNCTION

It will be the purpose of this appendix to derive an approximate solution for the optimum system transfer function Y_o. As described in the text we must first find the optimum compensating network H_{co}. An expression for H_{co} was given in the text in dimensionless form by equation (31) and it is repeated here.

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$$H_{co}(x) = \frac{1}{2\pi\Lambda^+(x)} \int_0^\infty e^{-i\beta xt} \int_{-\infty}^\infty \frac{\overline{H_f(z)}\psi_s(z)e^{i\beta zt}}{\Lambda^-(z)} \beta \, dz \, dt$$

where

$$\Lambda(x) = \left[\overline{H_f(x)}H_f(x)+\rho\right]\psi_{ii}(x) \qquad \text{(A1)}$$

In these expressions x is a dimensionless angular frequency related to ω by

$$\omega = \beta x \qquad \text{(A2)}$$

$$\beta = \epsilon\sqrt{\frac{\sigma}{N}} \qquad \text{(A3)}$$

where the parameters σ and N are related to the input quantities as given in equations (21) and (22). The quantity $\psi_{ii}(x)$ in (A1) is defined as the sum of signal and noise frequency functions, respectively.

$$\psi_{ii}(x) = \psi_s(x) + \psi_n(x) \qquad \text{(A4)}$$

The quantities Λ^+ and Λ^- are defined as the factors of Λ with poles and zeroes in the upper and lower half planes, respectively. Thus

$$\Lambda = \Lambda^+\Lambda^- \qquad \text{(A5)}$$

The input quantities $\psi_s(x)$ and $\psi_n(x)$ are, from equations (21) and (22)

$$\psi_s(x) = \frac{N}{x^4(x^2+\nu^2)} \tag{A6}$$

$$\psi_n(x) = N \tag{A7}$$

where

$$\nu = \frac{\xi}{\beta} \tag{A8}$$

It might appear a little odd that N would appear in equation (A6) for the signal frequency function, but this is due to the nondimensionalizing factor in (A3). Note also that some of the poles of $\psi_s(x)$ lie on the real axis which the theory in deriving equation (A1) does not permit. This problem was discussed in appendix A of reference 1, where the correct procedure was indicated. The procedure consisted in modifying equation (A6) so as to displace the poles at zero slightly off the real axis, and after the final answer was obtained letting the magnitude of this displacement go to zero. Such a procedure is quite unwieldy. In the interests of keeping the expressions as simple as possible, the form in (A6) can be used if one is careful to remember at the critical points that the poles should actually be slightly off the real axis. The remaining function in (A1) is H_f, and as has been discussed in the text, it can be well approximated by equation (27). Thus in terms of dimensionless frequency,

$$H_f(x) = -\frac{k_f}{\beta^2 x^2} \tag{A9}$$

Having dispensed with explanations and definitions, we will be concerned in the following paragraphs with the solution of (A1).

Let us find some of the functions needed in equation (A1). Starting with $\psi_{ii}(x)$, we can combine (A4), (A6), and (A7).

$$\psi_{ii}(x) = \psi_s(x) + \psi_n(x)$$

$$= N \frac{x^6+\nu^2x^4+1}{x^4(x^2+\nu^2)} \tag{A10}$$

$$= N \frac{p(x)[-p(-x)]}{x^4(x+i\nu)(x-i\nu)} \tag{A11}$$

If we let x_{pm} represent roots in the upper half plane, we have

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$$p(x) = \prod_{m=1}^{3} (x-x_{pm})$$

$$= x^3 + b_2x^2 + b_1x + b_o \tag{A12}$$

where

$$\left.\begin{aligned} b_2 &= -(x_{p1}+x_{p2}+x_{p3}) \\ b_1 &= x_{p1}x_{p2} + x_{p1}x_{p3} + x_{p2}x_{p3} \\ b_o &= -x_{p1}x_{p2}x_{p3} \end{aligned}\right\} \tag{A13}$$

Similarly,

$$-p(-x) = \prod_{m=1}^{3} (x+x_{pm})$$

$$= x^3 - b_2x^2 + b_1x - b_o \tag{A14}$$

Multiplying, we get

$$p(x)[-p(-x)] = x^6 + (2b_1-b_2^2)x^4 + (b_1^2-2b_ob_2)x^2 - b_o^2 \tag{A15}$$

Comparing equations (A10) and (A15), we see that the following relation between the b's must exist

$$b_1^2 - 2b_0b_2 = 0 \tag{A16}$$

Also it is clear that

$$b_0 = i \tag{A17}$$

These latter relations will be needed later.

Next we know that because of (A9),

$$\overline{H_f(x)}H_f(x) = \frac{k_f^2}{\beta^4 x^4} \tag{A18}$$

Therefore

$$\overline{H_f(x)}H_f(x) + \rho = \rho\,\frac{x^4+(k_f^2/\rho\beta^4)}{x^4} \tag{A19}$$

$$= \rho\,\frac{q(x)q(-x)}{x^4} \tag{A20}$$

where, if we let x_{qn} represent roots in the upper half plane, the polynomial $q(x)$ is

$$q(x) = \prod_{n=1}^{2}(x-x_{qn})$$

$$= x^2 + c_1x + c_0 \tag{A21}$$

and

$$q(-x) = \prod_{n=1}^{2} (x+x_{qn})$$

$$= x^2 - c_1 x + c_o \tag{A22}$$

with

$$\left.\begin{aligned} c_1 &= -(x_{q1}+x_{q2}) \\ c_o &= x_{q1}x_{q2} \end{aligned}\right\} \tag{A23}$$

Obviously,

$$c_o{}^2 = \frac{k_f{}^2}{\rho\beta^4} \tag{A24}$$

Now combining equations (A20) and (A11) we have for $\Lambda(x)$ in (A1),

$$\Lambda(x) = \left[\overline{H_f(x)}H_f(x)+\rho\right]\psi_{ii}(x)$$

$$= \rho N \frac{q(x)q(-x)p(x)[-p(-x)]}{x^8(x+i\nu)(x-i\nu)} \tag{A25}$$

Splitting this expression according to equation (A5) we have

$$\Lambda^+(x) = \rho N \frac{q(x)p(x)}{x^4(x-i\nu)} \tag{A26}$$

$$\Lambda^-(x) = \frac{q(-x)[-p(-x)]}{x^4(x+i\nu)} \tag{A27}$$

Here we have to recall the remarks made at the beginning of this appendix in order to see how to split the x^8 in (A25).

Now, the first integral in equation (A1) may be evaluated by utilizing equations (A6), (A9), and (A27), and the result is

$$I_1 = \int_{-\infty}^{\infty} \frac{\overline{H_f(z)}\psi_s(z)e^{i\beta zt}}{\Lambda^-(z)} \beta \, dz \tag{A28}$$

$$= -\frac{k_f N}{\beta} \int_{-\infty}^{\infty} \frac{e^{i\beta zt}}{z^2(z-i\nu)q(-z)[-p(-z)]} dz \tag{A29}$$

$$\equiv -\frac{k_f N}{\beta} \int_{-\infty}^{\infty} f(z)dz \tag{A30}$$

This integral is readily evaluated by considering z to be a complex variable and integrating in the upper half plane. We must recall again that the second order pole at the origin would actually have been displaced slightly above the real axis if we had taken the trouble to do so at the beginning of the analysis. Thus there are two poles within the contour, a second order pole at the origin and a simple pole at $z = i\nu$. It is easy to show after a little algebra that

$$\mathrm{Res}(0) = \lim_{z\to 0} \frac{d}{dz} [z^2 f(z)]$$

$$= \frac{\beta}{\nu b_o c_o} t - \frac{i\nu(b_o c_1 + b_1 c_o) + b_o c_o}{\nu^2 b_o{}^2 c_o{}^2}$$

$$\equiv \frac{\beta}{\nu b_o c_o} t + \gamma_1 \tag{A31}$$

$$\text{Res}\,(i\nu) = \lim_{z \to i\nu} [(z-i\nu)f(z)]$$

$$= \frac{e^{-i\beta\nu t}}{-\nu^2[-p(-i\nu)]q(-i\nu)}$$

$$\equiv \gamma_2 e^{-\beta\nu t} \tag{A32}$$

Thus I_1 in equation (A30) is

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$$I_1 = -\frac{k_f N}{\beta} 2\pi i[\text{Res}(0)+\text{Res}(i\nu)]$$

$$= -\frac{2\pi i N k_f}{\beta}\left(\gamma_1 + \frac{\beta}{\nu\beta_o c_o} t + \gamma_2 e^{-\beta\nu t}\right) \tag{A33}$$

The second integral in equation (A1) is denoted by I_2 and is merely a Fourier transform of I_1.

$$I_2 = \int_0^\infty I_1 e^{-i\beta x t} dt \tag{A34}$$

$$= -\frac{2\pi i N k_f}{b_o c_o} \frac{-(\gamma_1+\gamma_2)b_o c_o x^2 + \left(i\gamma_1 \nu b_o c_o + \frac{i}{\nu}\right) x + 1}{(i\beta x)^2(\nu+ix)} \tag{A35}$$

$$= \frac{2\pi i N k_f}{b_o c_o} \frac{T_{\alpha_1}{}^2 x^2 + 2\zeta_{\alpha_1} T_{\alpha_1} x + 1}{\beta^2 x^2(\nu+ix)} \tag{A36}$$

Note that in ivaluating equation (A34) it is necessary to carry along the pole displacements discussed earlier in order to have suitable convergence factors. In the limit as these displacement terms approach zero, one then obtains equation (A36).

Now from equation (A1) the optimum compensating network can be found.

$$H_{co}(x) = \frac{1}{2\pi\Lambda^{+}(x)} I_2 \tag{A37}$$

Substituting equations (A26) and (A36) into (A37) gives

$$H_{co}(x) = \frac{k_f}{b_o c_o \beta^2 \rho} \frac{x^2(T_{\alpha 1}{}^2 x^2 + 2\zeta_{\alpha 1} T_{\alpha 1} x + 1)}{p(x)q(x)} \tag{A38}$$

It can be seen from equation (A12) that

$$p(x) = \prod_{m=1}^{3} (x - x_{pm})$$

$$= b_o \prod_{m=1}^{3} \left(\frac{x}{-x_{pm}} + 1 \right) \tag{A39}$$

Similarly, from equation (A21)

$$q(x) = \prod_{n=1}^{2} (x - x_{qn})$$

$$= c_o \prod_{n=1}^{2} \left(\frac{x}{-x_{qn}} + 1 \right) \tag{A40}$$

By utilizing equations (A17) and (A24), (A38) becomes

$$H_{co}(x) = - \frac{\beta 2}{k_f} \frac{x^2(T_{\alpha 1}{}^2 x^2 + 2\zeta_{\alpha 1} T_{\alpha 1} x + 1)}{\prod_{m=1}^{3} \left(\frac{x}{-x_{pm}} + 1 \right) \prod_{n=1}^{2} \left(\frac{x}{-x_{qn}} + 1 \right)} \tag{A41}$$

When we later determine the values of the roots in the denominator, it will be seen that some of these factors combine to give

$$H_{co}(x) = -\frac{\beta^2}{k_f}\frac{x^2(T_{\alpha 1}{}^2x^2+2\zeta_{\alpha 1}T_{\alpha 1}x+1)}{(T_{\beta 1}x+1)(T_{\gamma 1}{}^2x^2+2\zeta_{\gamma 1}T_{\gamma 1}x+1)(T_{\mu 1}{}^2x^2+2\zeta_{\mu 1}T_{\mu 1}x+1)} \tag{A42}$$

where

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$$\left.\begin{aligned}
T_{\beta 1} &= \frac{1}{-x_{p1}} & 2\zeta_{\mu 1}T_{\mu 1} &= -\left(\frac{x_{q1}+x_{q2}}{x_{q1}x_{q2}}\right)\\
T_{\gamma 1}{}^2 &= \frac{1}{x_{p2}x_{p3}} & T_{\alpha 1}{}^2 &= -(\gamma_1+\gamma_2)b_oc_o\\
2\zeta_{\gamma 1}T_{\gamma 1} &= -\left(\frac{x_{p2}+x_{p3}}{x_{p2}x_{p3}}\right) & 2\zeta_{\alpha 1}T_{\alpha 1} &= \left(i\gamma_1\nu b_oc_o + \frac{i}{\nu}\right)\\
T_{\mu 1}{}^2 &= \frac{1}{x_{q1}x_{q2}}
\end{aligned}\right\} \tag{A43}$$

Since, as has been shown in the text, the compensating network H_{co} and the over-all closed-loop system transfer functions Y_o are related by

$$Y_o(x) = H_{co}(x)H_f(x) \tag{A44}$$

we can combine equations (A9), (A42), and (A44)

$$Y_o(x) = \frac{T_{\alpha 1}{}^2x^2 + 2\zeta_{\alpha 1}T_{\alpha 1}x + 1}{(T_{\beta 1}x+1)(T_{\gamma 1}{}^2x^2+2\zeta_{\gamma 1}T_{\gamma 1}x+1)(T_{\mu 1}{}^2x^2+2\zeta_{\mu 1}T_{\mu 1}x+1)} \tag{A45}$$

The expressions in (A42) for the compensating network and in (A45) for the system transfer function are the general forms expressed in terms of various roots. We must now relate these roots to the various input parameters and the restricted quantity in order to get explicit expressions for the transfer functions. Let us look first at x_{pm} which are the roots of

$$p(x)[-p(-x)] = x^6 + \nu^2 x^4 + 1 = 0 \tag{A46}$$

The substitution

$$x^2 = \frac{1}{y} \tag{A47}$$

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reduces equation (A46) to

$$y^3 + \nu^2 y + 1 = 0 \tag{A48}$$

The standard form for the transformed cubic is

$$y^3 + py + q = 0 \tag{A49}$$

where we see that

$$\left.\begin{aligned} p &= \nu^2 \\ q &= 1 \end{aligned}\right\} \tag{A50}$$

The general cubic in equation (A49) can be solved by Cardan's method. However, instead of assuming a solution of the form $Y = u + v$, it is neater to use

$$y = v - u$$

Then following the usual development we can show that

$$u^3 = A$$

$$v^3 = B$$

where

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$$A = \frac{q}{2} + \frac{1}{2}\sqrt{q^2 + \frac{4p^3}{27}} = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4\nu^6}{27}} \tag{A51}$$

$$B = -\frac{q}{2} + \frac{1}{2}\sqrt{q^2 + \frac{4p^3}{27}} = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4\nu^6}{27}} \tag{A52}$$

and we agree to take the positive square root. Then the first root y_1 is

$$y_1 = \sqrt[3]{B} - \sqrt[3]{A} \tag{A53}$$

where we take the real roots. Since A is larger than B, define C as

$$C^2 \equiv -y_1 = \sqrt[3]{A} - \sqrt[3]{B} \tag{A54}$$

Obviously

$$\left.\begin{aligned} -1 \leq y_1 \leq 0 \\ 0 \leq C \leq 1 \end{aligned}\right\} \tag{A55}$$

Then we see that

$$x_{p1} = \frac{1}{\sqrt{y_1}} = \frac{i}{C} \tag{A56}$$

From (A43),

$$T_{\beta 1} = -\frac{1}{x_{p1}} = iC \tag{A57}$$

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It is well known that the other two roots, y_2 and y_3, of the cubic can be expressed in terms of the first root y_1. However, it is clear from equation (A43) that we do not need these roots. It is only necessary to know $x_{p2}x_{p3}$ and $x_{p2} + x_{p3}$. We may find these quantities from (A13) and (A16); that is, from the four equations

$$\left.\begin{aligned} b_1^2 - 2b_0b_2 &= 0 \\ b_2 &= -(x_{p1}+x_{p2}+x_{p3}) \\ b_1 &= x_{p1}x_{p2} + x_{p1}x_{p3} + x_{p2}x_{p3} \\ b_0 &= -x_{p1}x_{p2}x_{p3} \end{aligned}\right\} \tag{A58}$$

the four unknowns, b_2, b_1, $x_{p2}x_{p3}$, and $x_{p2} + x_{p3}$, can be found. The quantities b_0 and x_{p1} are known and are given by (A17) and (A56), respectively. First, from the last equation in (A58) we get

$$x_{p2}x_{p3} = \frac{1}{ix_{p1}} = -C \tag{A59}$$

Hence

$$T_{\gamma 1}^2 = \frac{1}{x_{p2}x_{p3}} = -\frac{1}{C} \tag{A60}$$

Now we are left with the first three equations with the three unknowns b_2, b_1, and $x_{p2} + x_{p3}$. Solving, we get

$$b_2 = \frac{1}{iC} - iC\sqrt{\frac{2}{C} - C^2} \tag{A61}$$

$$b_1 = -C - \sqrt{\frac{2}{C} - C^2} \tag{A62}$$

$$x_{p2} + x_{p3} = iC\sqrt{\frac{2}{C} - C^2} \tag{A63}$$

where we must take the positive square root in order to get a positive damping ratio in the optimum transfer function. From equations (A59) and (A63) we can now obtain

$$2\zeta_{\gamma 1}T_{\gamma 1} = -\frac{x_{p2}+x_{p3}}{x_{p2}x_{p3}} = i\sqrt{\frac{2}{C} - C^2} \tag{A64}$$

Next let us find the roots x_{qn}. From (A19) and (A20),

$$q(x)q(-x) = x^4 + \frac{k_f}{\rho\beta^4} \tag{A65}$$

Let us define

$$\left.\begin{aligned} \gamma^4 &= \frac{\rho}{{k_f}^2} \\ \eta &= \gamma\beta \end{aligned}\right\} \tag{A66}$$

Then (A65) is

$$q(x)q(-x) = x^4 + \frac{1}{\eta^4} \tag{A67}$$

The roots are

$$\left.\begin{aligned} x_{q1} &= \frac{1}{\eta}\exp\left(i\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}\eta} + \frac{i}{\sqrt{2}\eta} \\ x_{q2} &= -\frac{1}{\eta}\exp\left(-i\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}\eta} + \frac{i}{\sqrt{2}\eta} \end{aligned}\right\} \quad \text{(A68)}$$

From these roots and equation (A23) we can obtain c_1 and c_o which will be needed later.

$$c_1 = -i\,\frac{\sqrt{2}}{\eta} \quad \text{(A69)}$$

$$c_o = -\frac{1}{\eta^2} \quad \text{(A70)}$$

We can also obtain, from (A43), the dynamic factors

$$T_{\mu 1}{}^2 = \eta^2 \quad \text{(A71)}$$

$$2\zeta_{\mu 1}T_{\mu 1} = i\sqrt{2}\eta \quad \text{(A72)}$$

The dynamic terms in the numerator of (A42) or (A45) are harder to find. Their values are given in equation (A43), and they depend on γ_1 and γ_2. From equations (A31) and (A32) these parameters are

$$\gamma_1 = -\frac{i\nu(b_o c_1 + b_1 c_o) + b_o c_o}{\nu^2 b_o c_o{}^2} \quad \text{(A73)}$$

$$\gamma_2 = -\frac{1}{\nu^2[-p(-i\nu)][q(-i\nu)]} \quad \text{(A74)}$$

Let us find $2\zeta_{\alpha 1}T_{\alpha 1}$ first. Utilizing (A73) we get

$$2\zeta_{\alpha 1}T_{\alpha 1} = \frac{b_1}{b_o} + \frac{c_1}{c_o} \tag{A75}$$

Putting in the values of the b's and c's from equations (A69), (A70), (A62), and (A17) we arrive at

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$$2\zeta_{\alpha 1}T_{\alpha 1} = i\left(\sqrt{2}\eta + C + \sqrt{\frac{2}{C} - C^2}\right) \tag{A76}$$

Note that C is a function of ν, so that $2\zeta_{\alpha 1}T_{\alpha 1}$ is a function of only the dimensionless parameters ν and η.

The other dynamic term in the numerator of (A42) or (A45) is $T_{\alpha 1}{}^2$ given by

$$T_{\alpha 1}{}^2 = -(\gamma_1+\gamma_2)b_o c_o \tag{A77}$$

Putting (A73) and (A74) in (A77), and utilizing equation (A76) just developed, we have

$$T_{\alpha 1}{}^2 = \frac{b_o c_o}{\nu^2[-p(-i\nu)][q(-i\nu)]} + \frac{i\nu(b_o c_1 + c_o b_1)}{\nu^2 b_o c_o} + \frac{1}{\nu^2} \tag{A78}$$

$$= \frac{1}{\nu^2}\left\{\frac{1}{\left[\frac{-p(-i\nu)}{b_o}\right]\left[\frac{q(-i\nu)}{c_o}\right]} + i\nu(2\zeta_{\alpha 1}T_{\alpha 1}) + 1\right\} \tag{A79}$$

Note that all the factors in this expression can be shown to be functions of only ν and η. This form could be used for computational purposes. However, there are two reasons for modifying (A78) further. First, the explicit function of ν and η is not displayed. Second, as ν becomes small, $T_{\alpha 1}{}^2$ apparently becomes large; actually, it does not, since by performing some of the operations in the brackets we can show that the ν^2 cancels out. From (A67),

$$\frac{1}{q(-x)} + \frac{\eta^4}{\eta^4 x^4 + 1} q(x)$$

Then by use of (A21)

$$\frac{1}{q(-i\nu)} = \frac{\eta^4}{\eta^4 x^4 + 1} (-\nu^2 + ic_1\nu + c_o) \tag{A80}$$

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Similarly, from (A46) it can be shown that

$$p(i\nu)[-p(-i\nu)] = 1$$

so that

$$\frac{1}{-p(-i\nu)} = p(i\nu) = -i\nu^3 - b_2\nu^2 + ib_1\nu + b_o \tag{A81}$$

Then equation (A78) becomes

$$T_{\alpha 1}{}^2 = \frac{\eta^4 c_o{}^2 b_o{}^2}{\nu^2(\eta^4\nu^4+1)} \left(-\frac{\nu^2}{c_o} + i\frac{c_1}{c_o}\nu + 1\right)\left(-\frac{i\nu^3}{b_o} - \frac{b_2}{b_o}\nu^2 + i\frac{b_1}{b_o}\nu + 1\right) + \frac{i}{\nu}\left(\frac{c_1}{c_o} + \frac{b_1}{b_o}\right) + \frac{1}{\nu^2} \tag{A82}$$

As shown by equations (A69) and (A70) the c's are functions of η while from (A61) and (A62), the b's are functions of ν. Putting in these relations and multiplying, we can eventually show that

$$T_{\alpha 1}{}^2 \equiv -P$$

$$= -\frac{1}{\eta^4\nu^4+1}\left\{\left[\eta^2 + (C+\sqrt{2}\eta)\sqrt{\frac{2}{C} - C^2} + \sqrt{2}\eta C + \frac{1}{C}\right] - \left[\eta^2 C + (\eta^2+\sqrt{2}\eta C)\sqrt{\frac{2}{C} - C^2} + \frac{\sqrt{2}\eta}{C} + 1\right]\nu\right.$$

$$\left. + \left[-\eta^4 + \sqrt{2}\eta + \frac{\eta^2}{C} + \eta^2 C\sqrt{\frac{2}{C} - C^2}\right]\nu^2 - \left[\eta^2 - \sqrt{2}\eta^5 - \eta^4\left(C + \sqrt{\frac{2}{C} - C^2}\right)\right]\nu^3\right\} \qquad \text{(A83)}$$

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Note that only ν and η are involved; C is a function of ν as given previously by equation (A54).

For convenience let us summarize all the exact values for the dynamic terms of the optimum transfer function which we have just found.

$$\left.\begin{aligned}
T_{\beta 1} &= iC \\
T_{\gamma 1}{}^2 &= -\frac{1}{C} \\
2\zeta_{\gamma 1}T_{\gamma 1} &= i\sqrt{\frac{2}{C} - C^2} \\
T_{\mu 1}{}^2 &= -\eta^2 \\
2\zeta_{\mu 1}T_{\mu 1} &= i\sqrt{2}\eta \\
T_{\alpha 1}{}^2 &\equiv -P \\
&= -\frac{1}{\eta^4\nu^4+1}\left\{\left[\eta^2 + (C+\sqrt{2}\eta)\sqrt{\frac{2}{C} - C^2} + \sqrt{2}\eta C + \frac{1}{C}\right] - \left[\eta^2 C + (\eta^2+\sqrt{2}\eta C)\sqrt{\frac{2}{C} - C^2} + \frac{\sqrt{2}\eta}{C} + 1\right]\nu\right. \\
&\quad \left. + \left[-\eta^4 + \sqrt{2}\eta + \frac{\eta^2}{C} + \eta^2 C\sqrt{\frac{2}{C} - C^2}\right]\nu^2 - \left[\eta^2 - \sqrt{2}\eta^5 - \eta^4\left(C + \sqrt{\frac{2}{C} - C^2}\right)\right]\nu^3\right\} \\
2\zeta_{\alpha 1}T_{\alpha 1} &= i\left[\sqrt{2}\eta + C + \sqrt{\frac{2}{C} - C^2}\right]
\end{aligned}\right\} \qquad \text{(A84)}$$

where

$$C^2 = \sqrt[3]{A} - \sqrt[3]{B} \qquad 0 \leq C \leq 1$$

$$A = \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4\nu^6}{27}}$$

$$B = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{4\nu^6}{27}}$$

It will be seen that even though all of the dynamic terms in the optimum transfer function are dependent on only two parameters, η and ν, the transfer function is quite unwieldy, and should be simplified if possible. For this purpose let us look at equations (A51) and (A52). For all cases of practical interest it can be shown that

$$\frac{4\nu^6}{27} \ll 1 \tag{A85}$$

so that

$$\left.\begin{aligned} A &\approx 1 \\ B &\approx 0 \\ C &\approx 1 \end{aligned}\right\} \tag{A86}$$

The dynamic terms can then be shown to reduce to

$$\left.\begin{aligned} T_{\beta 1} &= i \\ T_{\gamma 1}{}^2 &= -1 \\ 2\zeta_{\gamma 1}T_{\gamma 1} &= i \\ T_{\mu 1}{}^2 &= -\eta^2 \\ 2\zeta_{\mu 1}T_{\mu 1} &= i\sqrt{2}\eta \\ T_{\alpha 1}{}^2 &\equiv -P \\ &= -\frac{1}{\eta^4\nu^4+1}\left\{(\sqrt{2}+\eta)^2 - (1+\sqrt{2}\eta)^2\nu + [\sqrt{2}\eta(1+\sqrt{2}\eta) - \eta^4]\nu^2 + [\sqrt{2}\eta^4(\sqrt{2}+\eta) - \eta^2]\nu^3\right\} \\ 2\zeta_{\alpha 1}T_{\alpha 1} &= i\sqrt{2}(\sqrt{2}+\eta) \end{aligned}\right\} \tag{A87}$$

We may now write the approximate closed-loop dimensionless transfer function, $Y_o(x)$, as follows.

$$Y_o(x) = \frac{-Px^2+i\sqrt{2}(\sqrt{2}+\eta)x+1}{(ix+1)(-x^2+ix+1)(-\eta^2x^2+i\sqrt{2}\eta x+1)} \tag{A88}$$

where the P in the numerator is the only complicated term. From (A88), using $s = i\omega = i\beta x$, we have

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$$Y_o(s) = \frac{\frac{P}{\beta^2}s^2 + \frac{\sqrt{2}(\sqrt{2}+\eta)}{\beta}s + 1}{\left(\frac{s}{\beta}+1\right)\left(\frac{s^2}{\beta^2}+\frac{s}{\beta}+1\right)\left(\frac{\eta^2}{\beta^2}s^2+\frac{\sqrt{2}\eta}{\beta}s+1\right)} \tag{A89}$$

$$= -\frac{\frac{P}{\beta^2}s^2 + \frac{\sqrt{2}(\sqrt{2}+\eta)}{\beta}s + 1}{\frac{\eta^2}{\beta^5}s^5 + \frac{\sqrt{2}\eta(1+\sqrt{2}\eta)}{\beta^4}s^4 + \frac{(1+\sqrt{2}\eta)^2}{\beta^3}s^3 + \frac{(\sqrt{2}+\eta)^2}{\beta^2}s^2 + \frac{\sqrt{2}(\sqrt{2}+\eta)}{\beta}s + 1} \tag{A90}$$

$$\equiv \frac{N(s)}{D(s)}$$

One may readily obtain the corresponding open-loop transfer function as given in the text by equations (38) and (39). Note that the dimensionless form of the transfer function is a function of only two variables: ν which is dependent on the input, and $\eta(= \beta\gamma = \beta\rho^{1/4})$ which is dependent on the input and the restricted quantity.

APPENDIX B

PERFORMANCE EQUATIONS OF OPTIMUM SYSTEMS

The optimum transfer function has been derived in appendix A and we would like now to derive the performance equations for such an optimum system. As indicated in the text we must return to equations (11) and (12). Let us look at the error equation (11) first. The error consists of four components

$$E^2 = E_1^2 + E_2^2 + E_3^2 + E_n^2 \tag{B1}$$

$$E_1^2 = \frac{P_s}{2\pi}\int_{-\infty}^{\infty}\left|\varphi_s(\omega)[1-Y_o(\omega)] - \frac{c_\epsilon}{i\omega}\right|^2 d\omega \tag{B2}$$

$$E_2^2 = 2P_s c_\epsilon \lim_{s\to 0}\left\{\varphi_s(s)[1-Y_o(s)] - \frac{c_\epsilon}{s}\right\} \tag{B3}$$

$$E_3^2 = P_s c_\epsilon^2 T \tag{B4}$$

$$E_n^2 = \frac{P_N}{2\pi}\int_{-\infty}^{\infty}\left|\varphi_n(\omega)Y_o(\omega)\right|^2 d\omega \tag{B5}$$

Before proceeding to expand the above equations, we will indicate certain things done in succeeding paragraphs that are common to all of these equations. The optimum transfer function $Y_o(s)$ in the above equations has been shown to be a function of only three parameters: the missile parameter η, and the two input parameters β_1 and ν_1. The subscript 1 is used to associate them with the input for which the system is optimized. In contrast, all of the parameters other than those in Y_o in the above equations are clearly associated with the input to which the system is subjected, and this may or may not be the same as that for which the system was optimized. In order to distinguish between these two situations, a subscript 2 will be attached to certain parameters to associate them with the actual input. Thus equations (20) through (25) would have the subscript 2 attached to σ, ξ, N, and α. Also from (28)

$$\beta_2 = \sqrt[6]{\frac{\sigma_2}{N_2}} \tag{B6}$$

When the equations are made dimensionless by setting $\omega = \beta_1 x$, as in appendix A, it will be seen (e.g., see eq. (B23)) that in equation (28) we would have

$$\nu_2 = \frac{\xi_2}{\beta_1} \tag{B7}$$

In the case of β_1 and β_2 the distinction is not very important because it will be seen in the text that in many cases of interest we will need to put $\beta_1 = \beta_2$ anyway. The distinction between ν_1 and ν_2 is more important.

Let us look first at the noise component E_n. From the discussion following equation (2) we know that $P_n = 2\pi$. Next, we may substitute equation (24) into (B5) (using the subscript 2). And finally we can make the angular frequency ω dimensionless by using the same factor as was used in appendix A; that is, by letting $\omega = \beta_1 x$. Then equation (B5) becomes

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$$\frac{E_n^2}{N_2\beta_1} = \int_{-\infty}^{\infty} |Y_o(x)|^2 dx \tag{B8}$$

$$= \int_{-\infty}^{\infty} \frac{|N(x)|^2}{D(x)D(-x)} dx \tag{B9}$$

where

$$Y_o(x) = \frac{N(x)}{D(x)} \tag{B10}$$

Such an expression occurs very frequently and can be evaluated by the method of reference 8; that is, an integral of the form

$$I_n = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g(x)}{h(x)h(-x)} dx \tag{B11}$$

where

$$\left.\begin{aligned} g(x) &= b_o x^{2n-2} + b_1 x^{2n-4} + \ldots b_{n-1} \\ h(x) &= a_o x^n + a_1 x^{n-1} + \ldots a_n \end{aligned}\right\} \tag{B12}$$

can be evaluated by the following very simple but elegant expression

$$I_n = \frac{(-1)^{n+1}}{2a_o} \frac{N_n}{D_n} \tag{B13}$$

where D_n is the determinant given by

$$D_n = |d_{mr}|$$

$$d_{mr} = a_{2m-r} \qquad 0 \geq 2m-r \leq n$$

$$= 0 \qquad \text{otherwise}$$

and N_n is the same as D_n except that the first column of N_n is replaced by $b_0, b_1, \ldots, b_{n-1}$. The $N(x)$ and $D(x)$ in equation (B9) can be obtained from equation (34) and they are

$$N(x) = -Px^2 + i\sqrt{2}(\sqrt{2}+\eta)x + 1 \tag{B14}$$

$$D(x) = i\eta^2x^5 + \sqrt{2}\eta(1+\sqrt{2}\eta)x^4 - i(1+\sqrt{2}\eta)^2x^3$$

$$-[(\sqrt{2}+\eta)^2]x^2 + i\sqrt{2}(\sqrt{2}+\eta)x + 1 \tag{B15}$$

In evaluating (B9) we have, on comparing (B9) and (B11),

$$|N(x)|^2 = g(x)$$

$$D(x) = h(x)$$

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and we see that $n = 5$. Thus $h(x)$ and $g(x)$ are

$$\left.\begin{aligned} g(x) &= b_0x^8 + b_1x^6 + b_2x^4 + b_3x^2 + b_4 \\ h(x) &= a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 \end{aligned}\right\} \tag{B16}$$

Then it is clear that

$$\left.\begin{aligned} a_0 &= i\eta^2 \\ a_1 &= \sqrt{2}\eta(1+\sqrt{2}\eta) \\ a_2 &= -i(1+\sqrt{2}\eta)^2 \\ a_3 &= -[(\sqrt{2}+\eta)^2] \\ a_4 &= i\sqrt{2}(\sqrt{2}+\eta) \\ a_5 &= 1 \end{aligned}\right\} \tag{B17}$$

$$\left.\begin{aligned} b_0 &= 0 \\ b_1 &= 0 \\ b_2 &= P^2 \\ b_3 &= 2(\sqrt{2}+\eta)^2 - 2P = 2(-a_3-P) \\ b_4 &= 1 \end{aligned}\right\} \tag{B18}$$

The solution of (B9) according to equation (B13), then, is

$$\frac{E_n^2}{N_2\beta_1} = 2\pi i I_5$$

$$= -\pi i \frac{\lambda_3}{\lambda_4} = f_4(\nu_1,\eta) \tag{B19}$$

where λ_3 and λ_4 are the following determinants:

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$$\lambda_3 = \begin{vmatrix} 0 & a_1 & a_0 & 0 \\ b_2 & a_3 & a_2 & a_1 \\ b_3 & 1 & a_4 & a_3 \\ 1 & 0 & 0 & 1 \end{vmatrix} \qquad \lambda_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 1 & a_4 & a_3 & a_2 \\ 0 & 0 & 1 & a_4 \end{vmatrix} \tag{B20}$$

This is the result given in equation (44). Since λ_3 and λ_4 are simpler in determinant form, they will not be expanded.

Next we will consider the first component of error E_1. From equations (8), (10), and (23),

$$c_\epsilon = \lim_{s\to 0} s\varphi_s(s)W_s(s) = \lim_{s\to 0} \frac{s\alpha_2[1-Y(s)]}{s^2(s+\xi_2)}$$

$$= \alpha_2 \lim_{s\to 0} \frac{1-Y(s)}{s(s+\xi_2)} \equiv \alpha_2 M_\epsilon \tag{B21}$$

Putting (23) and (B21) in (B2) gives

$$E_1^2 = \frac{P_s\alpha_2^2}{2\pi} \int_{-\infty}^{\infty} \frac{|1-Y_0(\omega) + \omega^2 M_\epsilon - i\omega\xi_2 M_\epsilon|^2}{\omega^4(\omega^2+\xi_2^2)}\, d\omega \tag{B22}$$

This latter equation can be made dimensionless, as before, if we let $\omega = \beta_1 x$. Then

$$E_1^2 = \frac{P_s\alpha_2^2}{2\pi} \frac{1}{\beta_1^5} \int_{-\infty}^{\infty} \frac{|1-Y_0(x) + \beta_1^2 M_\epsilon x^2 - i\beta_1^2 M_\epsilon \nu_2 x|^2}{x^4(x^2+\nu_2^2)}\, dx \tag{B23}$$

Let us define Q:

$$Q \equiv \beta_1^2 M_\epsilon = \beta_1^2 \lim_{s\to 0} \frac{1-Y(s)}{s(s+\xi_2)} \tag{B24}$$

Now if, in the optimum transfer function given in (A45), we put in the exact values of the parameters given in (A84), it can be shown that if $\xi_1 \neq 0$ (or $\nu_1 \neq 0$), $1 - Y(s)$ will have two factorable s's in the numerator, while if $\xi_1 = 0$, it will have three. Thus it is clear that

$$\begin{aligned} Q &\neq 0 \quad \text{if } \xi_1 \neq 0\ ,\ \xi_2 = 0 \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{B25}$$

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The value of Q if not zero can be deduced from (B24) and (36) for the approximate optimum transfer function to be

$$\begin{aligned} Q &= (\sqrt{2}+\eta)^2 - P \\ &= -a_3 - P \end{aligned} \tag{B26}$$

The last term in the numerator of (B23) is obviously zero. Now from the definition of β in equations (28) and (25) an alternate expression for β_2 is

$$\beta_2 = \sqrt[6]{P_s \alpha_2^2 / 2\pi N_2} \tag{B27}$$

Then (B23) becomes

$$\frac{E_1^2}{\left(\frac{N_2\beta_2^6}{\beta_1^5}\right)} = \int_{-\infty}^{\infty} \frac{|1-Y_0(x) + Qx^2|^2}{x^4(x^2+\nu_2^2)}\, dx \tag{B28}$$

This latter expression is given in the text as equation (41). Putting (B10) in (B28),

$$\frac{E_1^2}{\left(\frac{N_2\beta_2^6}{\beta_1^5}\right)} = \int_{-\infty}^{\infty} \frac{|D(x) - N(x) + x^2 QD(x)|^2}{D(x)D(-x)x^4(x^2+\nu_2^2)}\, dx \tag{B29}$$

By obtaining the N(x) and D(x) from equations (B14) and (B15) we can show

$$\begin{aligned} D(x) - N(x) + x^2QD(x) = x^2 \Big\{ & iQ\eta^2x^5 + Q\sqrt{2}\eta(1+\sqrt{2}\eta)x^4 + [i\eta^2 - iQ(1+\sqrt{2}\eta)^2]x^3 \\ & + [\sqrt{2}\eta(1+\sqrt{2}\eta) - Q(\sqrt{2}+\eta)^2]x^2 + [-i(1+\sqrt{2}\eta)^2 \\ & + iQ\sqrt{2}(\sqrt{2}+\eta)]x - [(\sqrt{2}+\eta)^2 - P - Q] \Big\} \end{aligned} \tag{B30}$$

Thus in comparing (B29) to (B11) we must have

$$h(x) = D(x)(ix+\nu_2) \tag{B31}$$

$$g(x) = \left| \frac{D(x) - N(x) + x^2 Q D(x)}{x^2} \right|^2 \tag{B32}$$

and we see that since $D(x)$ is fifth order, $n = 6$. The $h(x)$ and $g(x)$ in equation (B12) will now be of the form

$$h(x) = e_0x^6 + e_1x^5 + e_2x^4 + e_3x^3 + e_4x^2 + e_5x + e_6 \tag{B33}$$

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$$g(x) = r_0x^{10} + r_1x^8 + r_2x^6 + r_3x^4 + r_4x^2 + r_5 \tag{B34}$$

After considerable algebra we find that the e's and r's can be expressed in terms of the a's just defined in equation (B17) as follows:

$$\left.\begin{aligned}
e_0 &= ia_0 \\
e_1 &= ia_1 + \nu_2 a_0 \\
e_2 &= ia_2 + \nu_2 a_1 \\
e_3 &= ia_3 + \nu_2 a_2 \\
e_4 &= ia_4 + \nu_2 a_3 \\
e_5 &= ia_5 + \nu_2 a_4 \\
e_6 &= \nu_2
\end{aligned}\right\} \tag{B35}$$

$$\left.\begin{aligned}
r_0 &= -Q^2 a_0^2 \\
r_1 &= -2Qa_0^2 \\
r_2 &= -a_0^2 + Q^2(-2a_0a_4 + 2a_1a_3 + a_3^2) \\
r_3 &= 2Q[-a_0a_4 - a_2^2 + a_1a_3] \\
&\quad + Q^2(-2a_2a_4 + a_3^2) \\
r_4 &= 2(a_1 + Qa_3)(a_3 + P + Q) - (a_2 + Qa_4)^2 \\
r_5 &= (a_3 + P + Q)^2
\end{aligned}\right\} \tag{B36}$$

Now we have

$$\frac{E_1^{\,2}}{\left(\frac{N_2\beta_2^{\,6}}{\beta_1^{\,5}}\right)} = 2\pi i I_6$$

$$= -\frac{\pi i}{e_0}\frac{\lambda_1}{\lambda_2} = f_1(\nu_1,\nu_2,\eta) \tag{B37}$$

where λ_1 and λ_2 are the following determinants

$$\lambda_1 = \begin{vmatrix} r_0 & e_0 & 0 & 0 & 0 & 0 \\ r_1 & e_2 & e_1 & e_0 & 0 & 0 \\ r_2 & e_4 & e_3 & e_2 & e_1 & e_0 \\ r_3 & e_6 & e_5 & e_4 & e_3 & e_2 \\ r_4 & 0 & 0 & e_6 & e_5 & e_4 \\ r_5 & 0 & 0 & 0 & 0 & e_6 \end{vmatrix} \qquad \lambda_2 = \begin{vmatrix} e_1 & e_0 & 0 & 0 & 0 & 0 \\ e_3 & e_2 & e_1 & e_0 & 0 & 0 \\ e_5 & e_4 & e_3 & e_2 & e_1 & e_0 \\ 0 & e_6 & e_5 & e_4 & e_3 & e_2 \\ 0 & 0 & 0 & e_6 & e_5 & e_4 \\ 0 & 0 & 0 & 0 & 0 & e_6 \end{vmatrix} \tag{B38}$$

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Now for the E_2 component of error given by equation (B3). Using equations (23) and (B21), and making them dimensionless as before, we can show that

$$\lim_{s\to 0}\left\{\varphi_s(s)[1-Y(s)] - \frac{c_\epsilon}{s}\right\} = -\frac{\alpha_2}{\beta_1^{\,3}}\lim_{x\to 0}\frac{1-Y(x) + Qx^2 - i\nu_2 Qx}{x^2(ix+\nu_2)} \tag{B39}$$

Since the product $\nu_2 Q$ will always be zero, the last term may be dropped and equation (B3) can be written

$$E_2^{\,2} = \frac{2P_s\alpha_2^{\,2}Q}{-\beta_1^{\,5}}\lim_{x\to 0}\frac{1-Y(x) + Qx^2}{x^2(ix+\nu_2)} \tag{B40}$$

Utilizing equations (B10), (B14), and (B15), we get the expansion

$$E_2^{\,2} = \frac{2P_s\alpha_2^{\,2}Q}{-\beta_1^{\,5}}\lim_{x\to 0}\frac{(\ )x^5 + \ldots + [-i(1+\sqrt{2}\eta)^2 + i\sqrt{2}(\sqrt{2}+\eta)Q]x - [(\sqrt{2}+\eta)^2 - P - Q]}{D(x)(ix+\nu_2)} \tag{B41}$$

Now we can see that if $Q \neq 0$ ($\xi_1 \neq 0$, $\xi_2 = 0$ from eq. (B25)), the value of (B41) would be

$$E_2^{\,2} = \frac{2P_s\alpha_2^{\,2}Q}{\beta_1^{\,5}}\left[(1+\sqrt{2}\eta)^2 - Q\sqrt{2}(\sqrt{2}+\eta)\right] \tag{B42}$$

On the other hand if $Q = 0$ we can show from equation (B41) that $E_2 = 0$. For if $Q = 0$ and $\nu_2 \neq 0$, the value of $\lim \{\}$ will be either 0 or a constant, depending on whether $\nu_1 = 0$ or not. If $\nu_2 = 0$, the denominator approaches zero with x; however, according to equation (B25) it must be that $\nu_1 = 0$ so that the numerator also approaches zero with x; the ratio is a constant. Thus we see that (B42) is valid whether Q is zero or not. Rewriting (B42) in terms of the a's used previously we have

$$E_2^2 = \frac{2P_s\alpha_2^2 Q}{\beta_1^5}(ia_2 + iQa_4) \tag{B43}$$

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or using (B27)

$$\frac{E_2^2}{\left(\frac{N_2\beta_2^6}{\beta_1^5}\right)} = 4\pi Q(ia_2 + iQa_4) = f_2(\nu_1,\nu_2,\eta) \tag{B44}$$

which is the same as given by equation (42) in the text.

The E_3 component is quite simple and can be written in terms of the same quantities as the other components. By use of (B21) and (B27), (B4) becomes

$$E_3^2 = \frac{P_s\alpha_2^2}{2\pi N_2} 2\pi N_2 M_\epsilon^2 T$$

$$= \beta_2^6 \frac{N_2 T 2\pi Q^2}{\beta_1^4}$$

Therefore,

$$\frac{E_3^2}{\left(\frac{N_2\beta_2^6}{\beta_1^5}\right)(\beta_1 T)} = 2\pi Q^2 = f_3(\nu_1,\nu_2,\eta) \tag{B45}$$

All of the error components are now in a dimensionless form and are functions of three parameters, ν_1, ν_2, and η. As shown in the text, for most cases these components reduce to functions of only one or two parameters.

Now we will consider the equation for the restricted quantity given earlier by equation (12). The restricted quantity consists of four parts

$$R^2 = R_1^2 + R_2^2 + R_3^2 + R_n^2 \tag{B46}$$

where

$$R_1^2 = \frac{P_s}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\varphi_s(\omega)Y(\omega)}{H_f(\omega)} - \frac{c_r}{i\omega}\right|^2 d\omega \tag{B47}$$

$$R_2^2 = 2P_s c_r \lim_{s\to 0}\left[\frac{\varphi_s(s)Y(s)}{H_f(s)} - \frac{c_r}{s}\right] \tag{B48}$$

$$R_3^2 = P_s c_r^2 T \tag{B49}$$

$$R_n^2 = \frac{P_n}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\varphi_n(\omega)Y(\omega)}{H_f(\omega)}\right|^2 d\omega \tag{B50}$$

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It is worthwhile to note at this point a common feature of all these equations. When H_f and c_r are substituted in the above equations, all the left sides will become of the form $R^2k_f^2$, where k_f is the vehicle gain. Since there are no poles of the fixed network other than at zero, the quantity Rk_f is merely acceleration.

$$A^2 = R^2k_f^2 \tag{B51}$$

Thus rather than (B46) we will be interested in the equation

$$A^2 = A_1^2 + A_2^2 + A_3^2 + A_n^2 \tag{B52}$$

Starting first with the noise component, R_n, we see that in a manner similar to the noise component of error we can express equation (B50) as

$$\frac{R_n^2k_f^2}{N_2\beta_1^5} = \frac{A_n^2}{N_2\beta_1^5} = \int_{-\infty}^{\infty} x^4|Y_o(x)|^2dx \tag{B53}$$

Here we have put $P_n = 2\pi$, used equations (24) and (27), and nondimensionalized by setting $\omega = \beta_1 x$. Then using (B10),

$$\frac{A_n^2}{N_2\beta_1^5} = \int_{-\infty}^{\infty}\frac{x^4|N(x)|^2}{D(x)D(-x)}\,dx \tag{B54}$$

By comparing (B54) with (B11) we see we must let

$$g(x) = x^4|N(x)|^2 \tag{B55}$$

$$h(x) = D(x) \tag{B56}$$

We have from (B14),

$$x^4|N(x)|^2 = -Px^8 + [2(\sqrt{2}+\eta)^2 - 2P]x^6 + x^4 \tag{B57}$$

and $D(x)$ is given by (B15). Thus we see $n = 5$. The $h(x)$ and $g(x)$ will be of the form

$$\left.\begin{aligned} h(x) &= a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 \\ g(x) &= n_0x^8 + n_1x^6 + n_2x^4 + n_3x^2 + n_4 \end{aligned}\right\} \tag{B58}$$

Thus the a's are given by (B17) and

$$\left.\begin{aligned} n_0 &= P^2 = b_2 \\ n_1 &= 2(\sqrt{2}+\eta)^2 - 2P = b_3 \\ n_2 &= 1 \\ n_3 &= 0 \\ n_4 &= 0 \end{aligned}\right\} \tag{B59}$$

Then we have

$$\frac{A_n^2}{N_2\beta_1^5} = 2\pi i I_5$$

$$= \frac{\pi i}{a_0}\frac{\lambda_6}{\lambda_4} = g_2(\nu_1,\eta) \tag{B60}$$

where

$$\lambda_6 = \begin{vmatrix} b_2 & a_0 & 0 & 0 \\ b_3 & a_2 & a_1 & a_0 \\ 1 & a_4 & a_3 & a_2 \\ 0 & 0 & 1 & a_4 \end{vmatrix} \qquad \lambda_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 1 & a_4 & a_3 & a_2 \\ 0 & 0 & 1 & a_4 \end{vmatrix} \tag{B61}$$

Consider next the first component A_1 given by equation (B47). In a manner similar to deriving equation (B21) we find that

$$c_r = \lim_{s\to 0} s\varphi_s(s)W_s(s)$$

$$\equiv \frac{\alpha_2 M_A}{k_f} \tag{B62}$$

where

$$M_A = \lim_{s\to 0} \frac{sY(s)}{s+\xi_2} = \lim_{s\to 0} \frac{s}{s+\xi_2} \tag{B63}$$

Obviously there are only two possible values for M_A:

$$\left.\begin{aligned} M_A &= 0 \quad \text{for } \xi_2 \neq 0 \\ M_A &= 1 \quad \text{for } \xi_2 = 0 \end{aligned}\right\} \tag{B64}$$

Putting equations (23), (27), and (B62) in (B47), and then setting $\omega = \beta_1 x$, we have

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$$R_1^2 k_f^2 = A_1^2 = \frac{P_s \alpha_2^2}{2\pi} \int_{-\infty}^{\infty} \left| \frac{Y(x)}{\beta_1(ix+\nu_2)} - \frac{M_A}{i\beta_1 x} \right|^2 \beta_1 dx$$

$$= \frac{P_s \alpha_2^2}{2\pi} \frac{1}{\beta_1} \int_{-\infty}^{\infty} \frac{|ixY(x) - ixM_A - M_A\nu_2|^2}{x^2(x^2+\nu_2^2)} dx$$

$$= \frac{P_s \alpha_2^2}{2\pi} \frac{1}{\beta_1} \int_{-\infty}^{\infty} \frac{|Y(x) - M_A|^2}{x^2+\nu_2^2} dx \tag{B65}$$

Note that $M_A\nu_2$ is zero in the equation above (B65). Using (B27) now, (B65) becomes

$$\frac{R_1^2 k_f^2}{\left(\frac{N_2\beta_2^6}{\beta_1}\right)} = \frac{A_1^2}{\left(\frac{N_2\beta_2^6}{\beta_1}\right)} = \int_{-\infty}^{\infty} \frac{|Y(x) - M_A|^2}{x^2+\nu_2^2} dx \tag{B66}$$

which is the same as given in the text as equation (49). Now using equation (B10) we have

$$\frac{A_1^2}{\left(\frac{N_2\beta_2^6}{\beta_1}\right)} = \int_{-\infty}^{\infty} \frac{|N(x) - M_A D(x)|^2}{[D(x)(ix+\nu_2)][D(-x)(-ix+\nu_2)]} dx \tag{B67}$$

Using (B14) and (B15) it can be shown after a little algebra that

$$|N(x) - M_A D(x)|^2 = M_A^2\eta^4 x^{10} + \left\{M_A^2(1+\sqrt{2}\eta)^4 - M_A 2\sqrt{2}\eta(1+\sqrt{2}\eta)[M_A(\sqrt{2}+\eta)^2 - P]\right\} x^6$$
$$+ [M_A(\sqrt{2}+\eta)^2 - P]^2 x^4 + \left[2P(M_A-1) + 2(M_A-1)^2(\sqrt{2}+\eta)^2\right] x^2 + (M_A-1)^2$$

(B68)

The $D(x)(ix+\nu_2)$ is the same as was used in equation (B31), and $n = 6$. Thus in (B12),

$$\left.\begin{aligned} h(x) &= e_0x^6 + e_1x^5 + e_2x^4 + e_3x^3 + e_4x^2 + e_5x + e_6 \\ g(x) &= q_0x^{10} + q_1x^8 + q_2x^6 + q_3x^4 + q_4x^2 + q_5 \end{aligned}\right\} \quad (B69)$$

Since

$$\left.\begin{aligned} h(x) &= D(x)(ix+\nu_2) \\ g(x) &= |N(x) - M_A D(x)|^2 \end{aligned}\right\} \quad (B70)$$

we see that the e's are the same as given in equation (B35). The q's which are obtained from (B68) can be expressed in terms of the previously determined a's and the result is

$$\left.\begin{aligned} q_0 &= -M_A{}^2a_0{}^2 \\ q_1 &= 0 \\ q_2 &= -M_A{}^2a_2{}^2 + 2M_Aa_1(M_Aa_3+P) \\ q_3 &= (M_Aa_3+P)^2 \\ q_4 &= 2P(M_A-1) - 2a_3(M_A-1)^2 \\ q_5 &= (M_A-1)^2 \end{aligned}\right\} \quad (B71)$$

Finally we have, utilizing the method of equation (B13),

$$\frac{A_1{}^2}{\left(\frac{N_2\beta_2{}^6}{\beta_1}\right)} = 2\pi i I_6$$

$$= -\frac{\pi i}{e_0}\frac{\lambda_5}{\lambda_2} = g_1(\nu_1,\nu_2,\eta) \quad (B72)$$

where

$$\lambda_5 = \begin{vmatrix} q_0 & e_0 & 0 & 0 & 0 & 0 \\ 0 & e_2 & e_1 & e_0 & 0 & 0 \\ q_2 & e_4 & e_3 & e_2 & e_1 & e_0 \\ q_3 & e_6 & e_5 & e_4 & e_3 & e_2 \\ q_4 & 0 & 0 & e_6 & e_5 & e_4 \\ q_5 & 0 & 0 & 0 & 0 & e_6 \end{vmatrix} \qquad \lambda_2 = \begin{vmatrix} e_1 & e_0 & 0 & 0 & 0 & 0 \\ e_3 & e_2 & e_1 & e_0 & 0 & 0 \\ e_5 & e_4 & e_3 & e_2 & e_1 & e_0 \\ 0 & e_6 & e_5 & e_4 & e_3 & e_2 \\ 0 & 0 & 0 & e_6 & e_5 & e_4 \\ 0 & 0 & 0 & 0 & 0 & e_6 \end{vmatrix} \tag{B73}$$

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The second component, A_2, can be shown to be identically zero. Putting (23), (27), and (B62) in (B48) we can show

$$R_2^2 k_f^2 = A_2^2 = 2P_s\alpha_2^2 M_A \lim_{s\to 0}\left[\frac{sY(s) - M_A s - M_A\xi_2}{s(s+\xi_2)}\right]$$

$$= \frac{2P_s\alpha_2^2 M_A}{\beta_1} \lim_{x\to 0}\left[\frac{Y(x) - M_A}{ix+\nu_2}\right]$$

$$= \frac{-2P_s\alpha_2^2 M_A}{\beta_1} \lim_{x\to 0}\left\{\frac{(M_A-1) + [i\sqrt{2}(\sqrt{2}+\eta)(M_A-1)]x - [M_A(\sqrt{2}+\eta)^2 - P]x^2 + \ldots(\ )x^5}{ix+\nu_2}\right\} \tag{B74}$$

From the relation in (B64) we can see that in equation (B74)

$$\left.\begin{aligned} \lim_{x\to 0}\{\ \} &= -\frac{1}{\nu_2} \quad \text{for } M_A = 0 \\ &= 0 \quad \text{for } M_A \neq 0 \end{aligned}\right\} \tag{B75}$$

Combining (B75) and (B74) it is clear that $A_2^2 \equiv 0$.

It hardly matters what is done with the last term, A_3, since it is not a function of the parameters ν_1, ν_2, or η. Consequently it can be moved to the left side of equation (B52). For uniformity let us substitute (B62) in (B49) to give

$$R_3^2 k_f^2 = A_3^2 = P_s\alpha_2^2 M_A^2 T \tag{B76}$$

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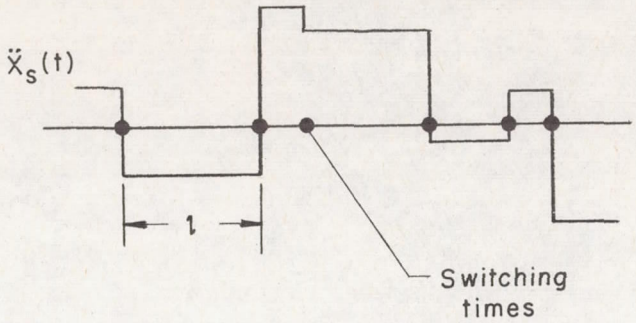
A. Amplitude random
Interval τ random

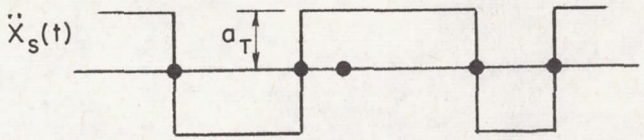
$P(\tau) = (1/\bar{\tau}) \exp(-\tau/\bar{\tau})$



$$\psi_{\ddot{s}}(\omega) = \frac{\lambda \widetilde{(\ddot{X}_s)^2}}{\pi(\omega^2 + \lambda^2)}$$

λ = Average switching rate

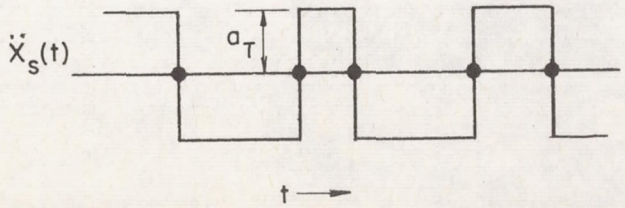
B. Same as A except
$\widetilde{\ddot{X}_s^2} = \ddot{X}_s^2 = a_T^2$



$$\psi_{\ddot{s}}(\omega) = \frac{\lambda a_T^2}{\pi(\omega^2 + \lambda^2)}$$

λ = Average switching rate

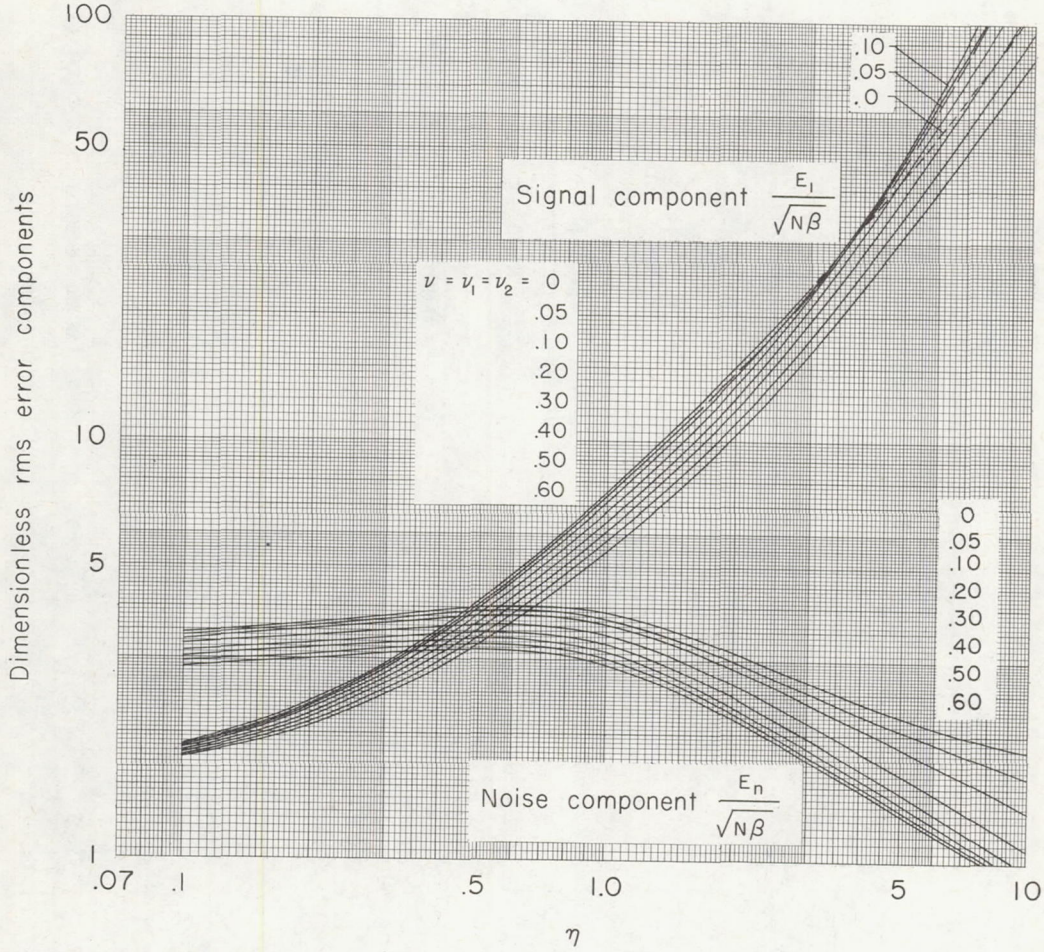
C. Amplitude changes sign at end of each interval.
Interval τ same as in A.



$$\psi_{\ddot{s}}(\omega) = \frac{k a_T^2}{\pi(\omega^2 + k^2)}$$

k = Twice average switching rate

Figure 1.- Example signal inputs.



(a) Error components.

Figure 2.- Optimum performance of time-invariant systems and time-varying homing systems with stationary inputs.

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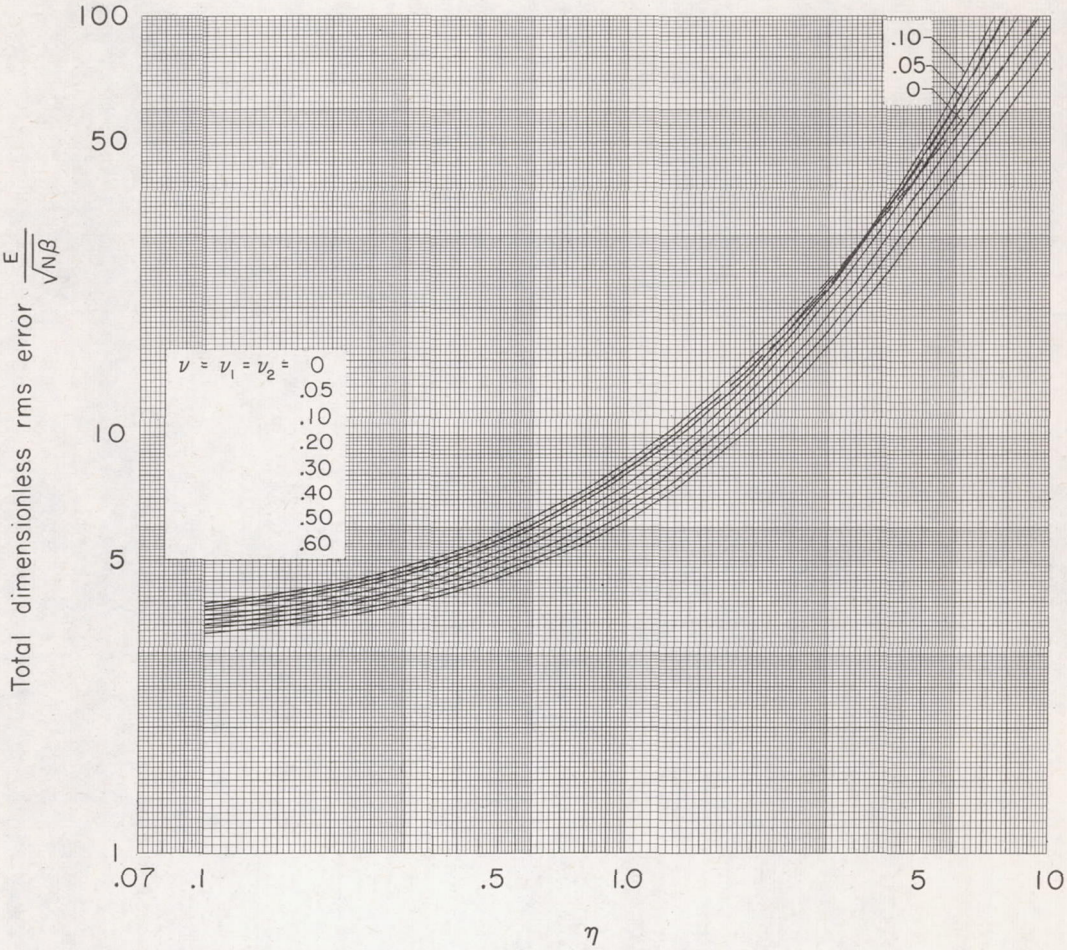
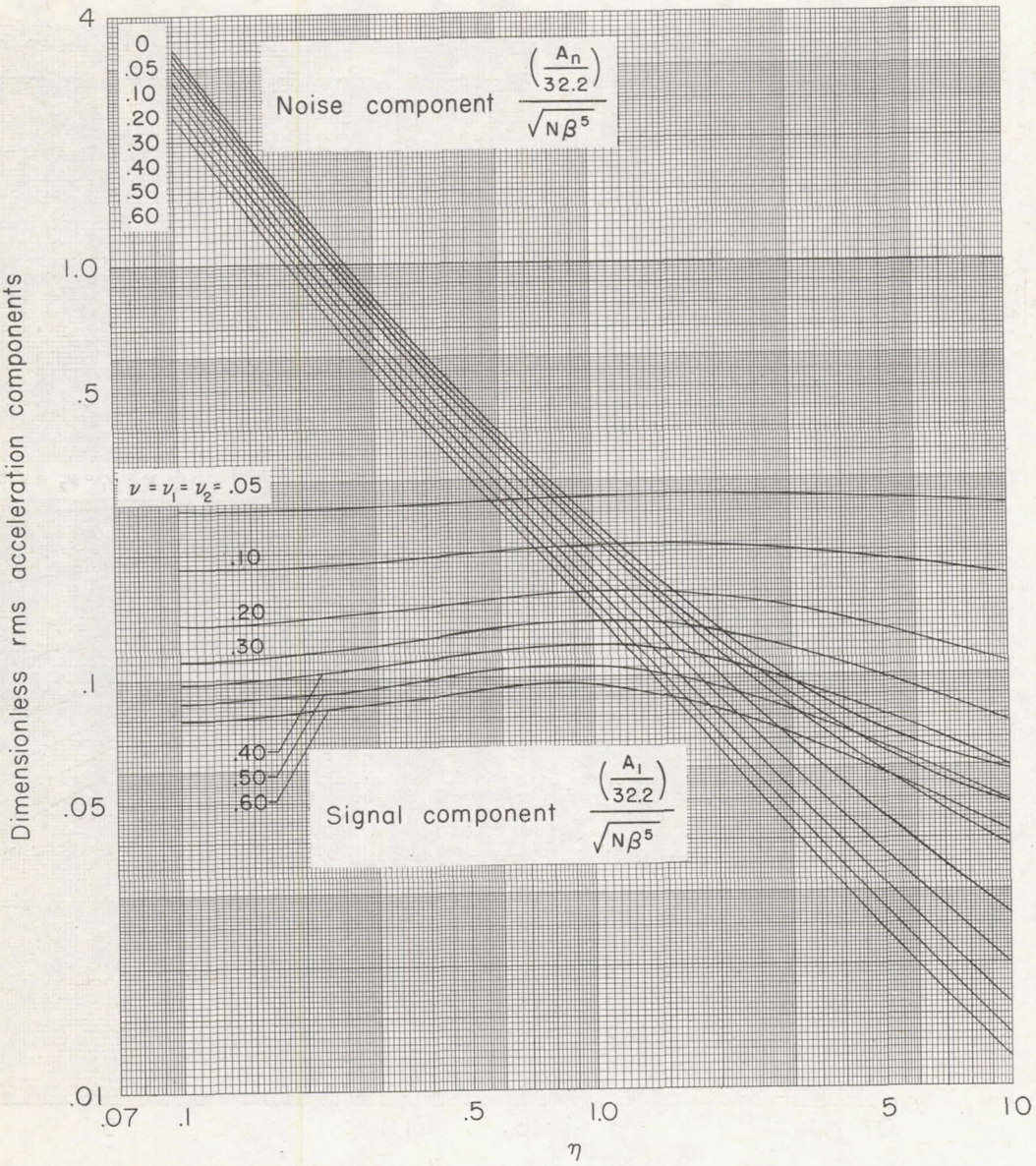


(b) Total error.

Figure 2.- Continued.



(c) Acceleration components.

Figure 2.- Continued.

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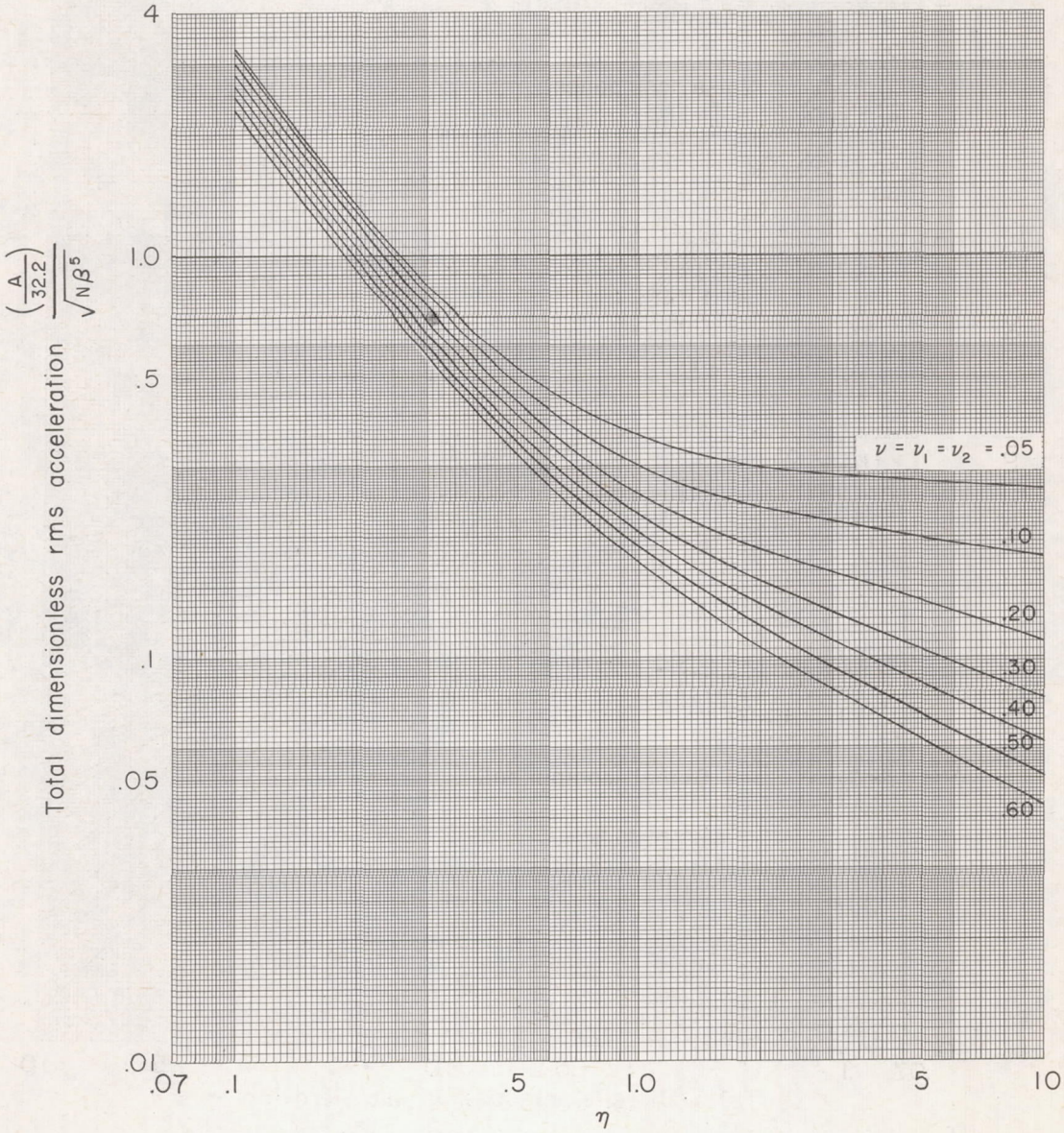
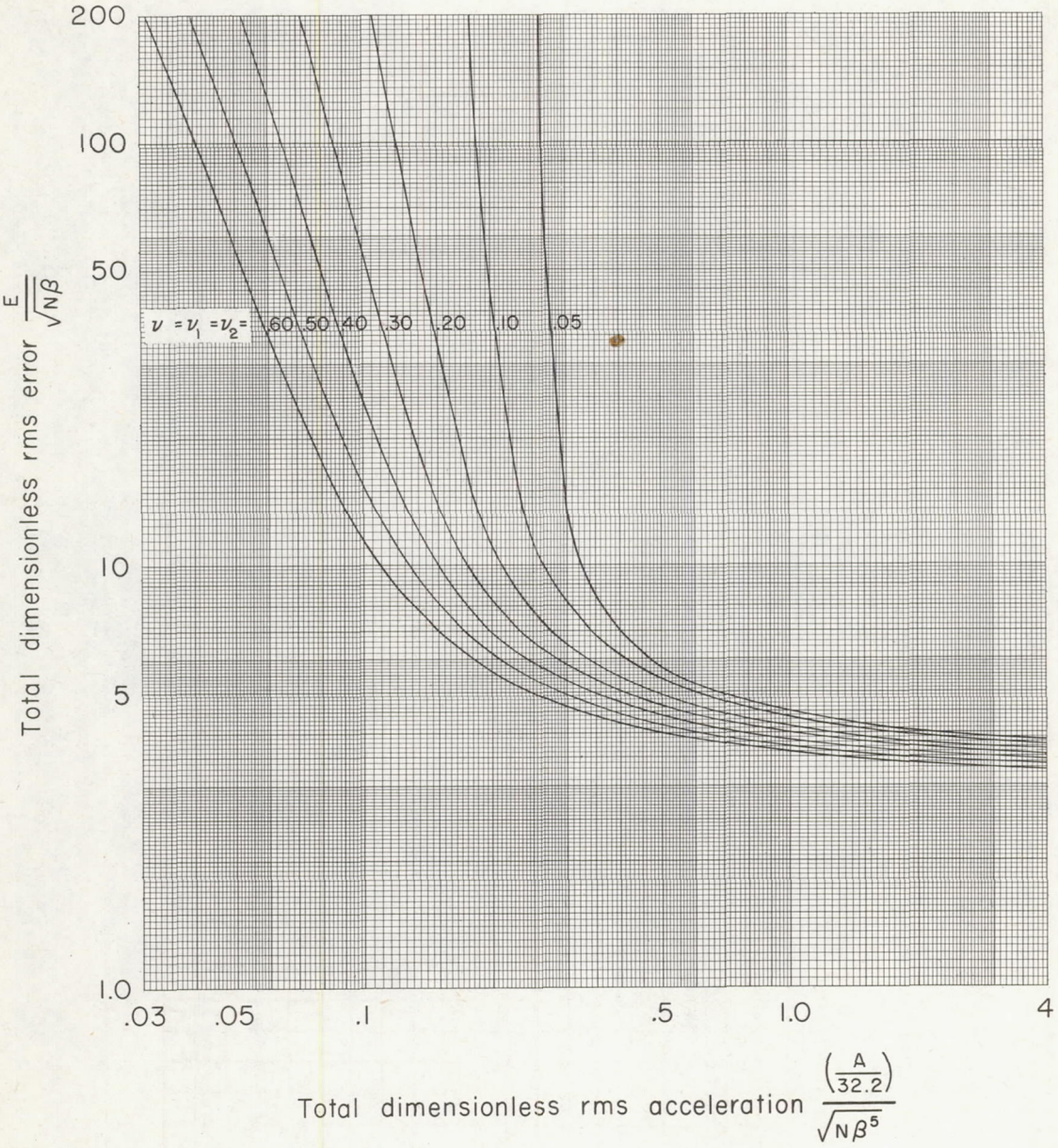


(d) Total acceleration.

Figure 2.- Continued.



(e) Total error as a function of total acceleration.

Figure 2.- Concluded.

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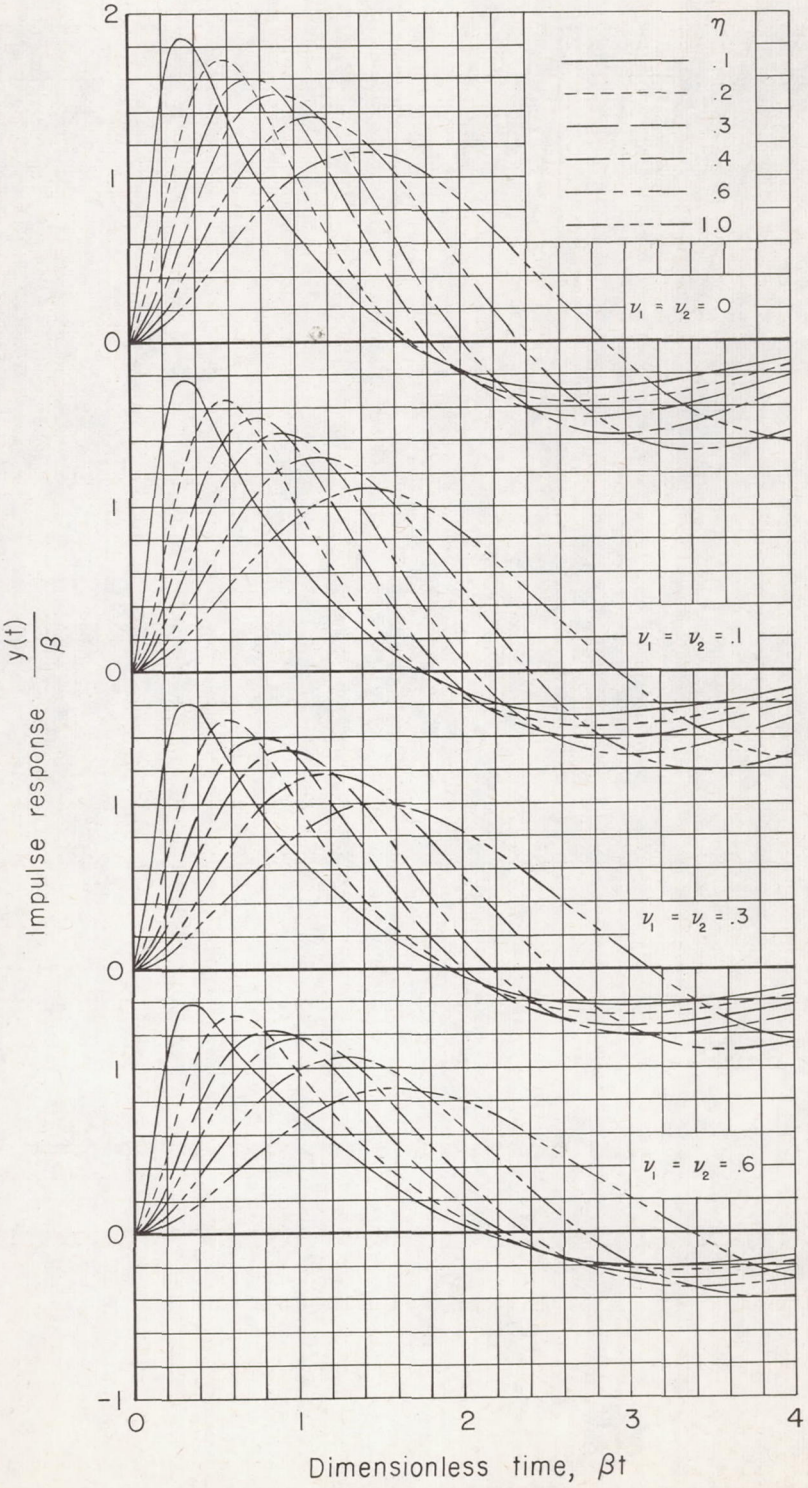
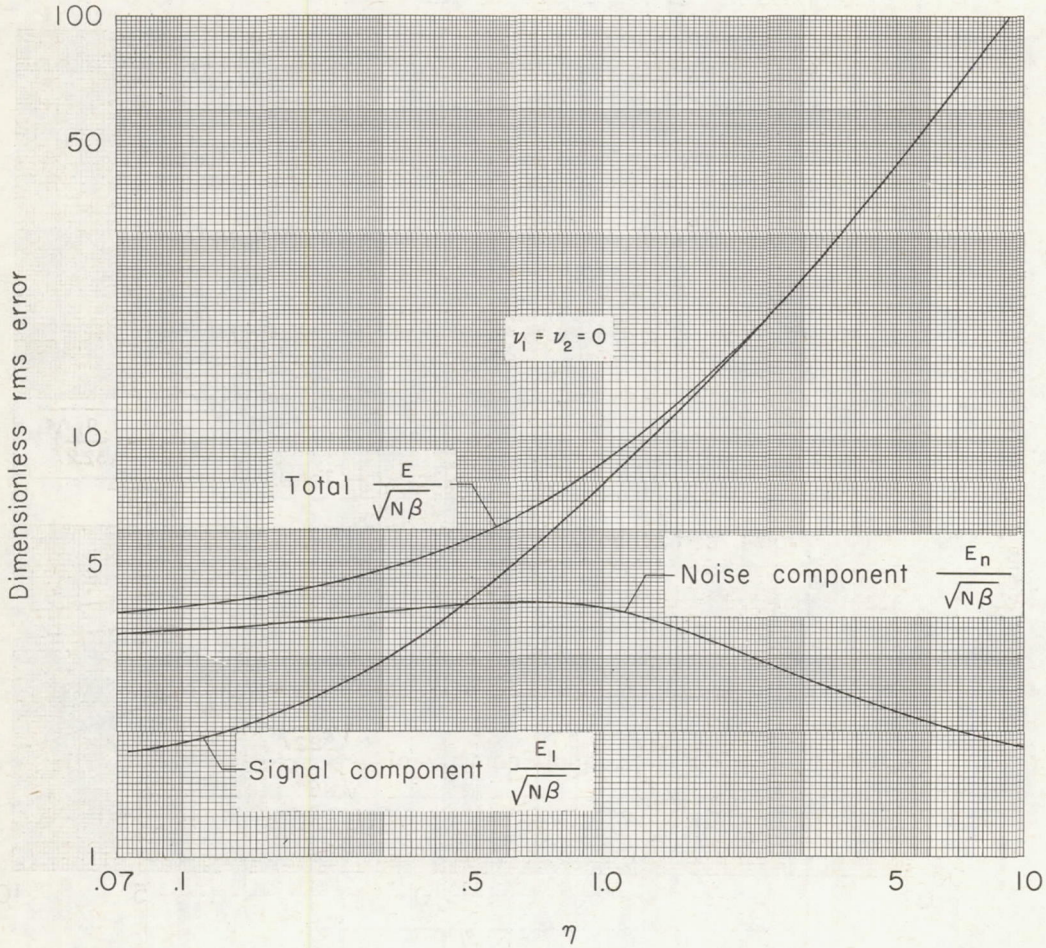


Figure 3.- Optimum impulse responses.



(a) Error.

Figure 4.- Optimum performance for nonstationary step acceleration signal.

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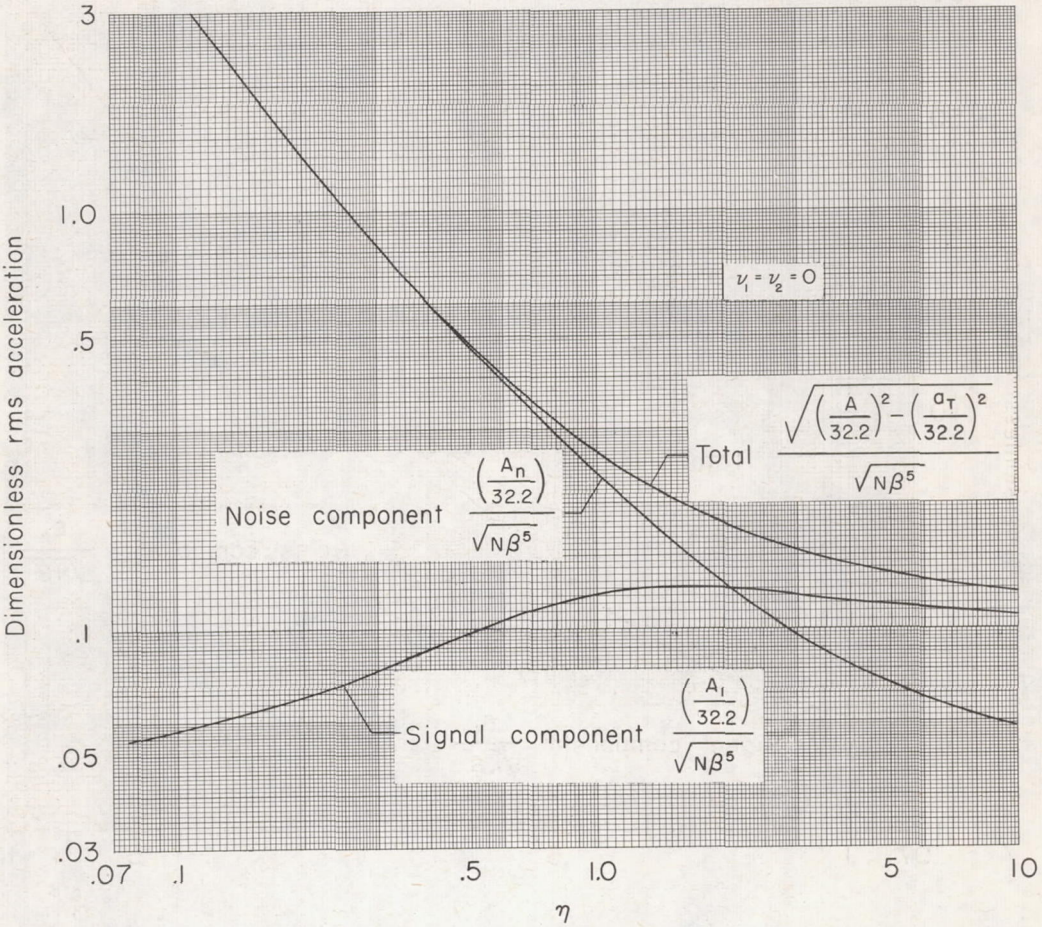
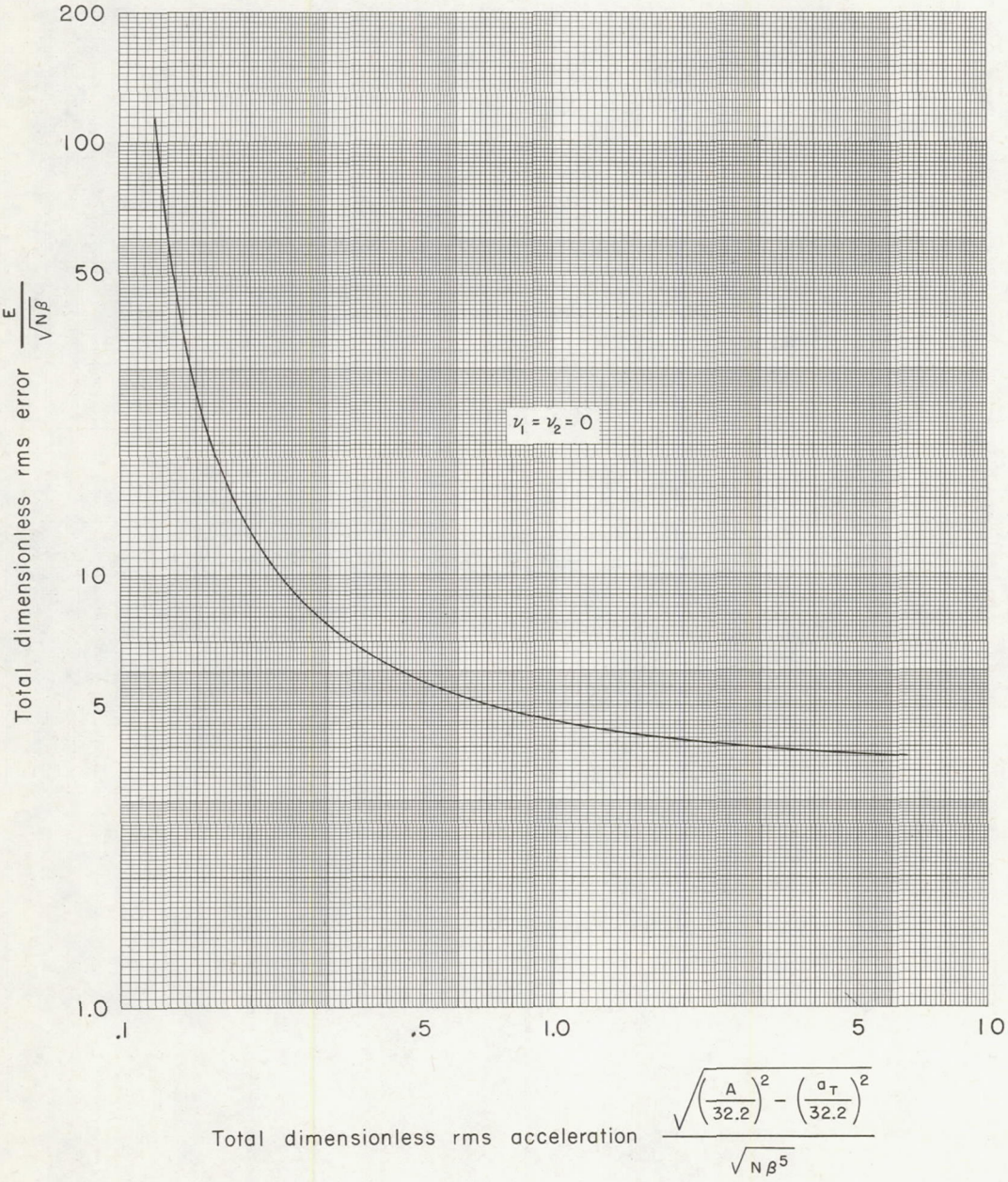


(b) Acceleration.

Figure 4.- Continued.



(c) Total error as a function of total acceleration.

Figure 4.- Concluded.

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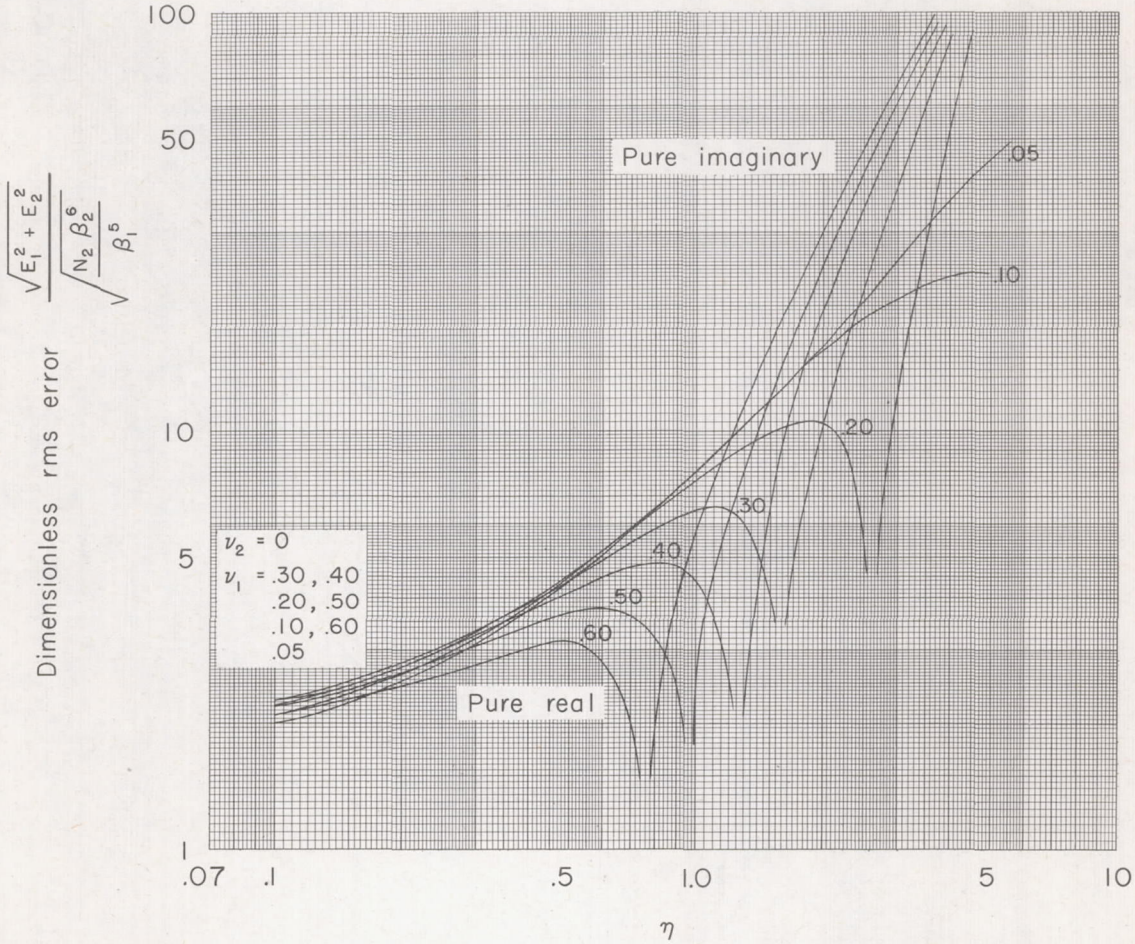
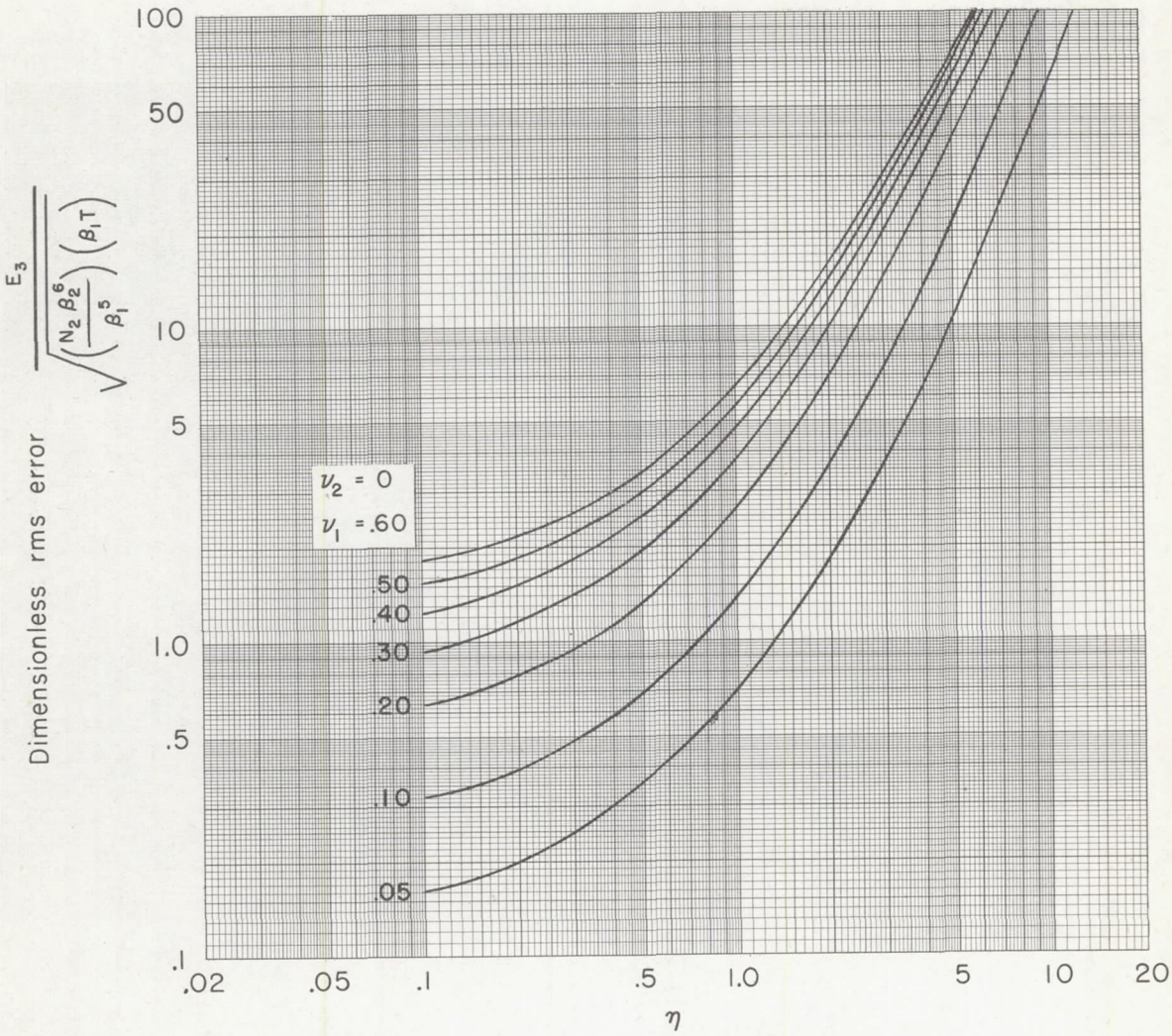


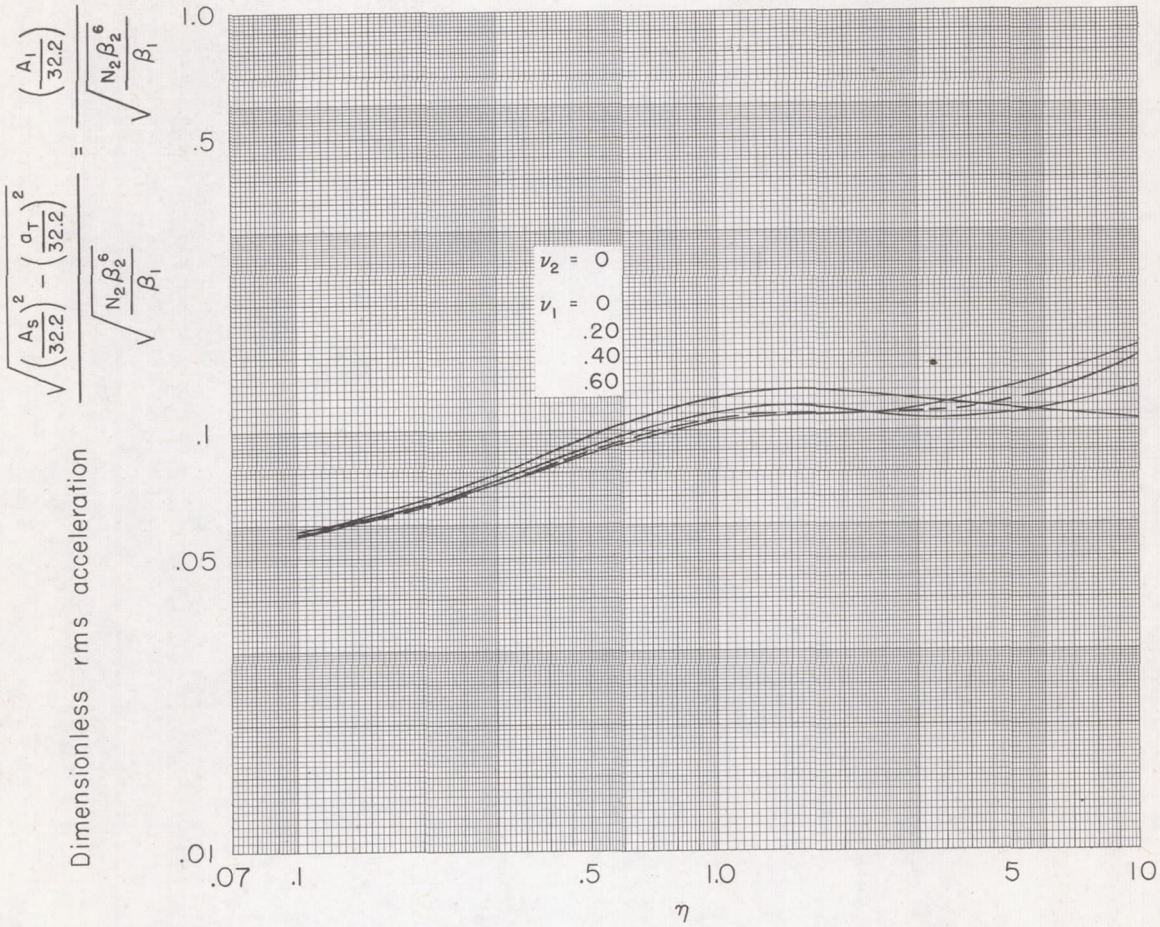
(a) Error components E_1 and E_2.

Figure 5.- Off-design performance of system optimized for stationary signal, actual signal nonstationary.



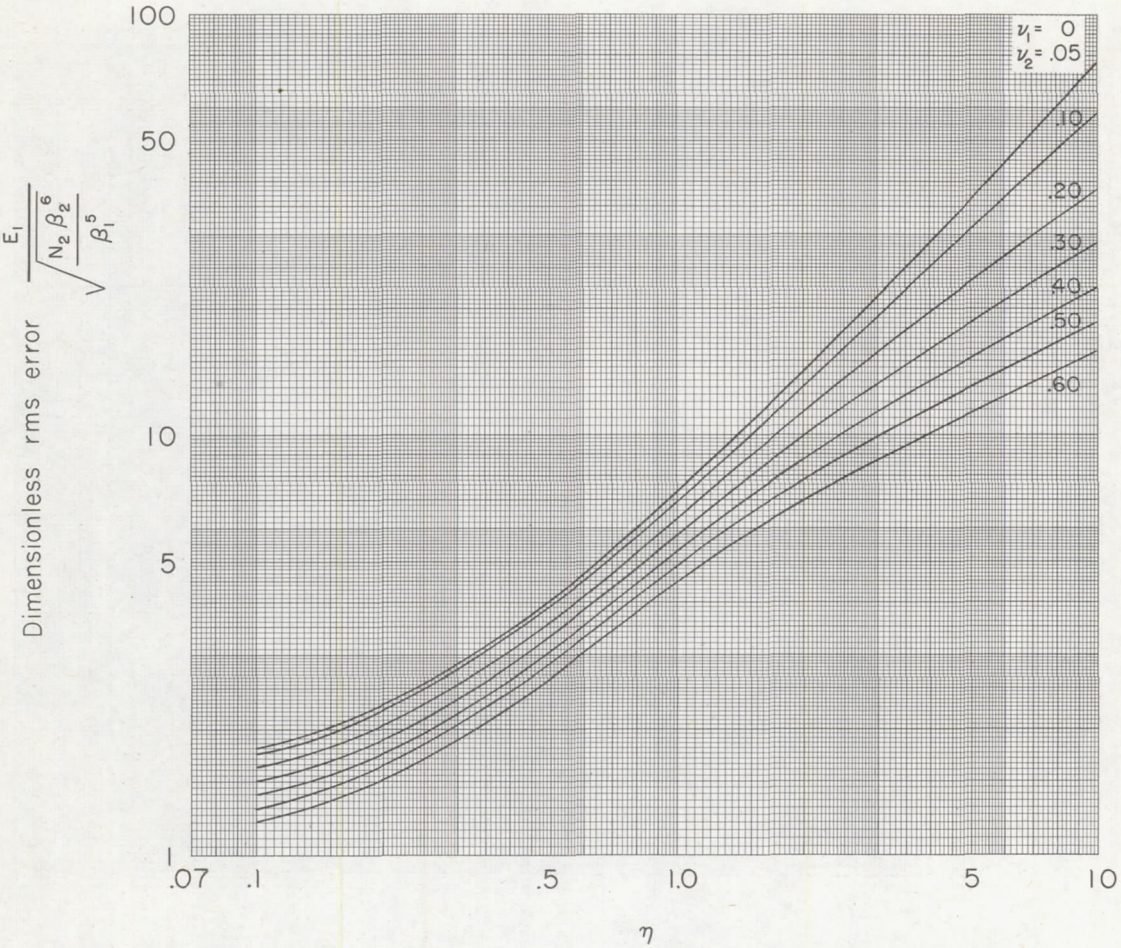
(b) Error component E_3.

Figure 5.- Continued.



(c) Acceleration components A_1 and A_s.

Figure 5.- Concluded.



(a) Error component E_1.

Figure 6.- Off-design performance of system optimized for nonstationary signal, acutal signal stationary.

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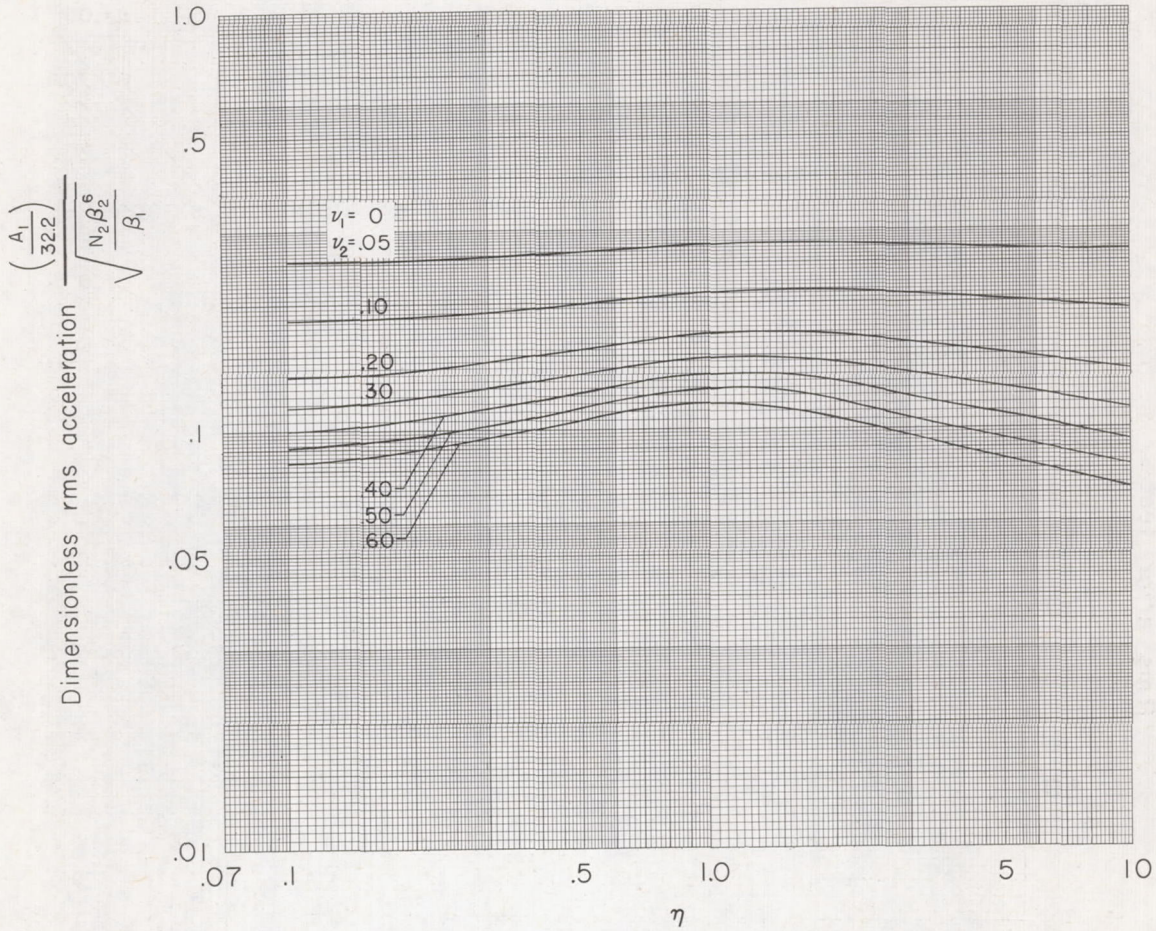

(b) Acceleration component A_1.

Figure 6.- Concluded.

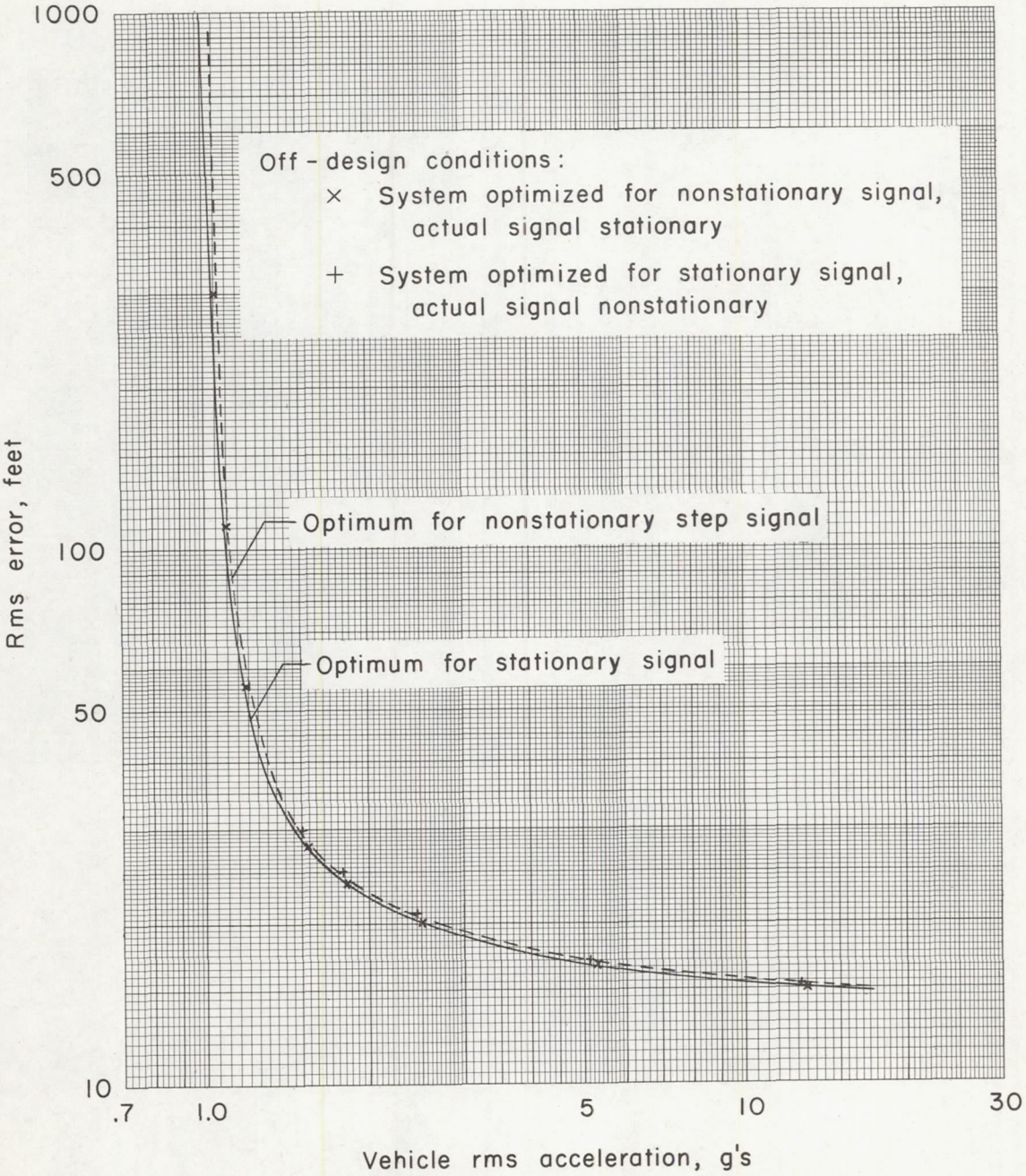

Figure 7.- Effect of type of signal input for example case.

A
2
7
4