Simple Two-Dimensional Corrections for One-Dimensional Pulse Tube Models

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Abstract

One-dimensional oscillating flow models are very useful for designing pulse tubes. They are simple to use, not computationally intensive, and the physical relationship between temperature, pressure and mass flow are easy to understand when used in conjunction with phasor diagrams. They do not possess, however, the ability to directly calculate thermal and momentum diffusion in the direction transverse to the oscillating flow. To account for transverse effects, lumped parameter corrections, which are obtained though experiment, must be used. Or two-dimensional solutions of the differential fluid equations must be obtained.

A linear two-dimensional solution to the fluid equations has been obtained. The solution provides lumped parameter corrections for one-dimensional models. The model accounts for heat transfer and shear flow between the gas and the tube. The complex Nusselt number and complex shear wall are useful in describing these corrections, with phase relations and amplitudes scaled with the Prandtl and Valensi numbers. The calculated ratio, α, between a two-dimensional solution of the oscillating temperature and velocity and a one-dimensional solution for the same shows α scales linearly with Va for $Va < 30$. In this region $\alpha < 0.5$, that is, the enthalpy flow calculated with a two-dimensional model is 50% of a calculation using a one-dimensional model. For $Va > 250$, $\alpha = 0.8$, showing that diffusion is still important even when it is confined to a thing layer near the tube wall.

Introduction

A previous paper[1] that examined the scaling parameters for pulse tubes is based on a two-dimensional analysis of anelastic oscillating and compressible low Mach number flow of a gas contained in a tube of thin but finite wall thickness.[2] Anelasitic flows are characterized by low Mach numbers and oscillating frequencies much less than system resonance frequency. This approximation is appropriate when acoustic and shock wave energies are negligible relative to the energy needed to compress and expand the bulk gas.[3] For a tube radius and a tube wall thickness of $O(10^{-1})$ and $O(10^{-2})$ smaller than the tube length, respectively, and for a tube with z axial coordinate scaled from 0 to 1, where the cold end is at $z = 0$, the scaling reduces the problem to 8 dimensionless groups as shown in Table 1. The velocity phase angle at $z = 0$ and $z = 1$ is ϕ_U which is scaled from 0 to 1 (corresponding to 0° to 360°). Three of the dimensionless groups are relevant to transverse (radial) diffusion: the Valensi number, Va; the Prandtl number, Pr; and the Fourier number of the tube wall, Fo. This paper explores the use of Va, Pr and Fo in providing lumped-parameter corrections of transverse diffusion for one-dimension models.

Table 1. Dimensionless groups for oscillating compressible flow in the open tube of a pulse tube.

	Range	Name	Definition	Comments
ε	10^{-3} to 10^{-1}	expansion parameter	$\tilde{U}_0^*/(\omega^* L^*) = d_0^*/L^*$	ratio of displacement length, d_0^*, to tube length L^*, $\omega^* = 2\pi f^*$
λ	10^{-8} to 10^{-5}	elasticity parameter	$\gamma \frac{\mathrm{M}^2}{\varepsilon} = \gamma \frac{\tilde{U}_0^*}{a^*} \frac{\omega^* L^*}{a^*}$	$\gamma = \frac{C_p^*}{C_v^*}$ and $a^* = \sqrt{\gamma R^* T_0^*}$
M	10^{-5} to 10^{-3}	Mach number	$\tilde{U}_0^*/a^*$	ratio of velocity at $z = 0$ to speed of sound
Va	1 to 10^2	Valensi number	$r_w^{*2}\omega^*/\nu^*$	squared ratio of tube inner radius to viscous diffusion length
Pr	0.7	Prandtl number	ν^*/α^*	squared ratio of viscous to thermal diffusion length of gas
Fo	0 to 10^2	Fourier number	$\alpha_w^*/(l^{*2}\omega^*)$	squared ratio of tube wall thermal diffusion length to tube wall thickness
$\tilde{U}_L$	0 to 1	velocity ratio	$\tilde{U}_L^*/\tilde{U}_0^*$	ratio of velocity amplitude at $z = 1$ to amplitude at $z = 0$
ϕU	-0.5 to 0	velocity phase angle		velocity phase angle where U_0 at $z = 0$ leads U_L at $z = 1$

Results of model

How thermal diffusion affects bulk pressure and temperature phasors

Figure 1 shows how thermal diffusion in the gas and tube wall affect the bulk pressure and temperature phasors, p_1 and T_b, respectively, for $U_L = 1$, $\phi_U = -0.1$; Fo = 0 and 100; and Pr = 0.7 and Va= 1, 30 and 100. The bulk temperature phasor is the integrated oscillating temperature profile over the tube radius; U_L is the normalized velocity amplitude at $z = 1$ and p_1 is the bulk oscillating pressure. The condition for an isothermal wall boundary is Fo $\rightarrow$ 1 (thick wall relative to thermal penetration). The condition for an adiabatic wall is Fo >> 1 (thin wall relative to thermal penetration). For Va >> 1, the tube inner radius is much larger than the thermal penetration distance in the gas. The shaded areas of the graphics in column 1 indicate the velocity phasors at locations in the tube between $z = 0$ and $z = 1$.

Figure 1, column 1, shows that for an adiabatic wall condition (Fo = 100) temperature and pressure phasors are in-phase as would be expected. The thermal penetration distance within the tube wall is much larger than the tube wall itself, thus there is no thermal time lag in the tube wall. The tube wall temperature closely follows the oscillating temperature; likewise, the tube wall does little to constrain the oscillating gas temperature amplitude. This can clearly be seen from the oscillating gas radial temperature profiles given in column 2, where the curves designated with a "*" are for Fo = 100. As seen in column 2, the gas temperature is not "pinned" at $T = 0$ at $r = 1$ (the gas/tube wall interface), but instead "floats", thereby having less thermal diffusion effect.

Often in one-dimensional models, it is assumed that adiabatic conditions on the gas are present (Fo >> 1). This condition is not likely achieved for real for pulse tubes, however. For example, a stainless steel tube with a wall thickness of 0.01 cm and thermal diffusivity of 0.045 cm^2/sec fillled with gas oscillating at 30 Hz will have Fo $\approx$ 2.4.

A non-adiabatic condition in the gas can have profound effects on the gas temperature phasor. For the isothermal condition (Fo $\rightarrow$ 0) the oscillating bulk temperature, T_b lags the oscillating pressure p_1 by about 15°. This is illustrated in Fig. 1, column 1. More importantly, the temperature phasors are shifted out-of-phase relative to the velocity phasors (shaded area). This reduces enthalpy flow, since enthalpy flow depends on the cosine of the phase angle between velocity and temperature. Even at large Va = 100, where it might be expected that T_b (Fo $\rightarrow$ 0) will approach T_b (Fo=100), there is still a significant detrimental phase shift of T_b (Fo $\rightarrow$ 0) out of the shaded velocity phasor area, that is, T_b is never really in-phase with the gas velocities. The phasor diagrams given in column 1 would indicate that the open tube of most pulse tubes probably do not operate as ideal adiabatic systems.

Column 2 of Fig. 1 shows the corresponding oscillating temperature for isothermal conditions (Fo $\rightarrow$ 0). Diffusion has a large effect on gas temperature for isothermal wall conditions. For small Va, this tends to dampen the temperature amplitude of the gas. As Va increases, the dampening of the gas temperature lessens. However, diffusion still constrains the the temperature oscillations near the wall. And since the area averaged bulk temperature scales with the square of the radius, this still constitues a large effect, even at large Va.

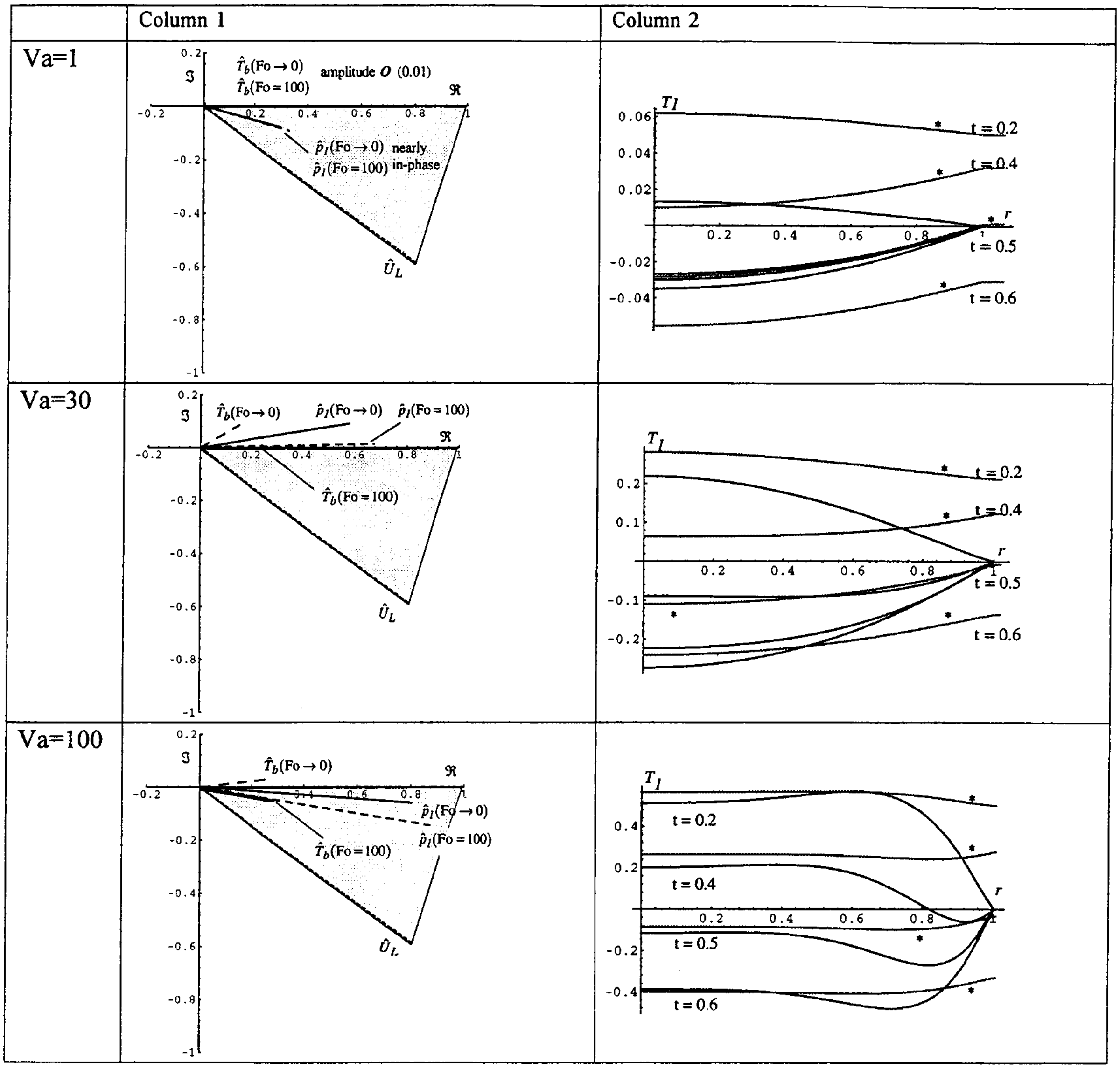

Figure 1. Column 1 shows the effect of Va and Fo on pressure, p_1, and bulk temperature, T_b, phasors for $\tilde{U}_L = 1$, $\phi_U = -0.1$. Reference velocity phasor $\hat{U}_0$ along real axis with unit amplitude. Column 2 shows the temperature, $T_1(r)$. Temperature profiles for Fo $\to 0$ are pinned at $r = 1$, and profiles for Fo = 100 (identified with '*') float at $r = 1$.

A useful relation quantifying the relation between bulk temperature and heat transfer to the tube wall is contained in the complex Nusselt number, $\hat{\mathrm{Nu}}$.

$$\hat{\mathrm{Nu}} = \frac{\hat{q}_w e^{it}}{\left(\hat{T}_w - \hat{T}_b\right) e^{it}} \tag{1}$$

Kornhauser and Smith[4] have examined this for basic pulse tube (BPT) systems (velocity at $z = 1$ is zero) and rectangular coordinates. For orifice pulse tube (OPT) systems (velocity at $z = 1$ is non-zero) the complex Nusselt number is the the same as for BPT-systems. This is seen in Fig. 2 where the $\hat{\mathrm{Nu}}$, is the same for both the BPT and OPT. The conclusion is that $\hat{\mathrm{Nu}}$ does not depend on $\hat{U}_L$ and ϕ_U, thus the relation for $\hat{\mathrm{Nu}}$ proposed by Kornhouaser for BPT-type systems with planar geometries can be used for similar OPT systems. At small PrVa, steady heat transfer correlations for low in a tube can be used since the dominance of diffusion constrains $\hat{\mathrm{Nu}}$ and $\hat{U}_0$ to being in phase (not shown in Fig. 2). At larger PrVa, deviations occur resulting in a phase shift of $\hat{\mathrm{Nu}}$ away from steady-state correlations.

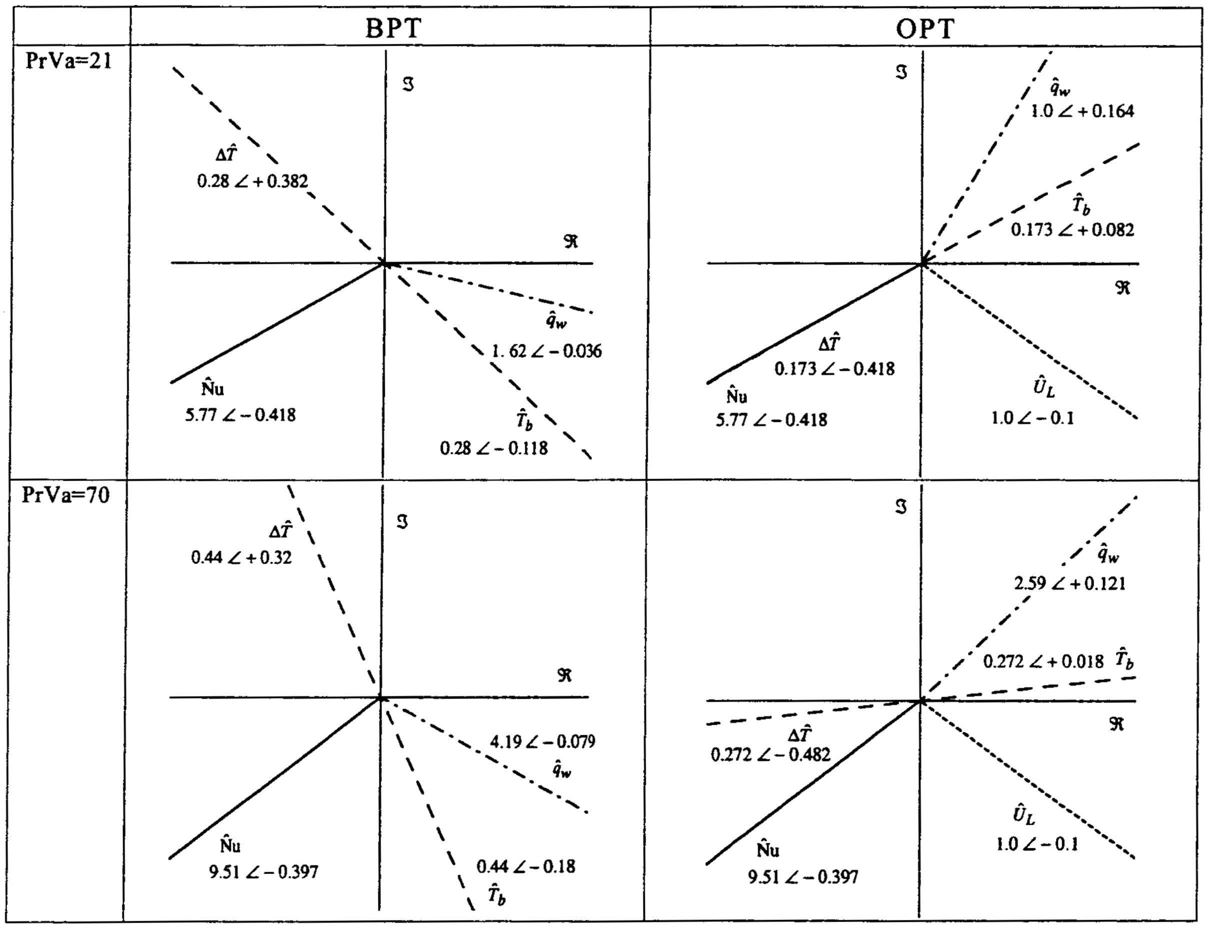

Figure 2. Effect of PrVa and Fo on heat transfer amplitude and phase. Reference velocity phasor $\hat{U}_0$ along real axis with unit amplitude.

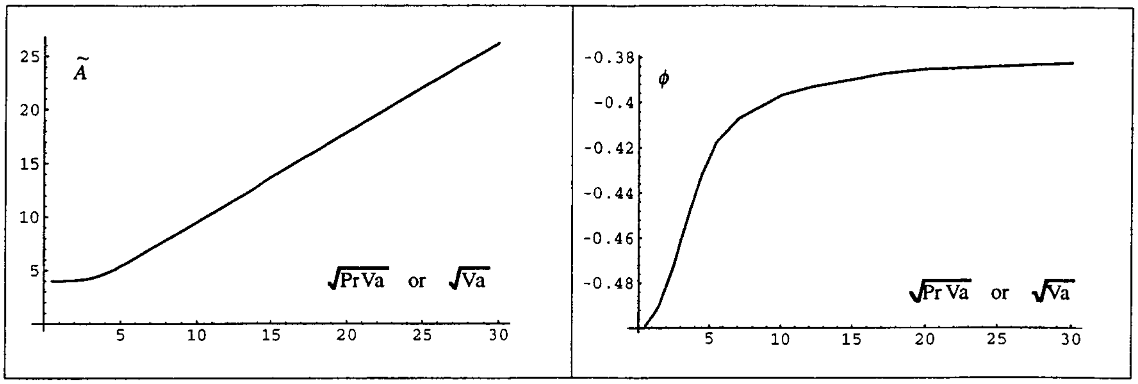

Figure 3. Amplitude and phase of the complex Nusselt number, $\hat{\text{Nu}}(\text{PrVa}) = \tilde{A}\, e^{i\phi}$, or complex wall shear factor, $\hat{\text{F}}(\text{Va}) = \tilde{A}\, e^{i\phi}$.

Figure 3 plots the amplitude and phase for $\hat{\text{Nu}}$ for increasing PrVa. The plot can be used for corrections to one-dimensional models in the form $\hat{\text{Nu}}(\text{PrVa}) = \tilde{A}\, e^{i\phi}$ for thermal modeling, and $\hat{\text{F}}(\text{Va}) = \tilde{A}\, e^{i\phi}$ to account for shear (friction). The usefulness of these relations can be seen from the one-dimensional equations for momentum transport, $i\hat{u}_{osc} = -\hat{p}_{2,z} + \hat{\text{F}}(\text{Va})\hat{u}_{osc}$, and heat transport, $i\hat{T}_{osc} = i\hat{p}_1 + \hat{\text{Nu}}(\text{PrVa})\hat{T}_{osc}$.

Enthlapy flow comparisons

Enthalpy flux is given by the time averaged product of oscillating temperature and oscillating velocity. Enthalpy flux integrated over the tube cross-sectional area gives enthalpy flow. Since the solutions obtained from ref. 2 are two-dimensional, enthalpy transport reflecting both temperature and velocity diffusion can be easily calculated

Figure 4, shows enthalpy transport for with $\tilde{U}_L = 1$ $\phi_U = -0.1$ and Fo $\to 0$. The plots of $\overline{h}_1(r)$ in column 1 are shown for z = 0, 0.5 and 1. For the case of PrVa = 0.7 and Fo $\to 0$, enthalpy flows in the reverse direction from $z = 0.6$ to $z = 1$. This is also shown in the corresponding plot of enthalpy flow vs. z in column 2. These plots show that operating with isothermal walls and small PrVa is not desirable.

The pulse tube should be operated where the thermal diffusion region is confined to a thin layer near the tube wall. This can be seen in for PrVa=70 in column 1. Here, enthalpy flux is positive throughout the tube, and is not dampened by diffusion at the tube wall. Column 2 shows how an adiabatic wall condition significantly increases enthalpy flow. The dashed-line is the near adiabatic condition of Fo = 100. Large PrVa and large Fo ideally is how a pulse OPT should be designed.

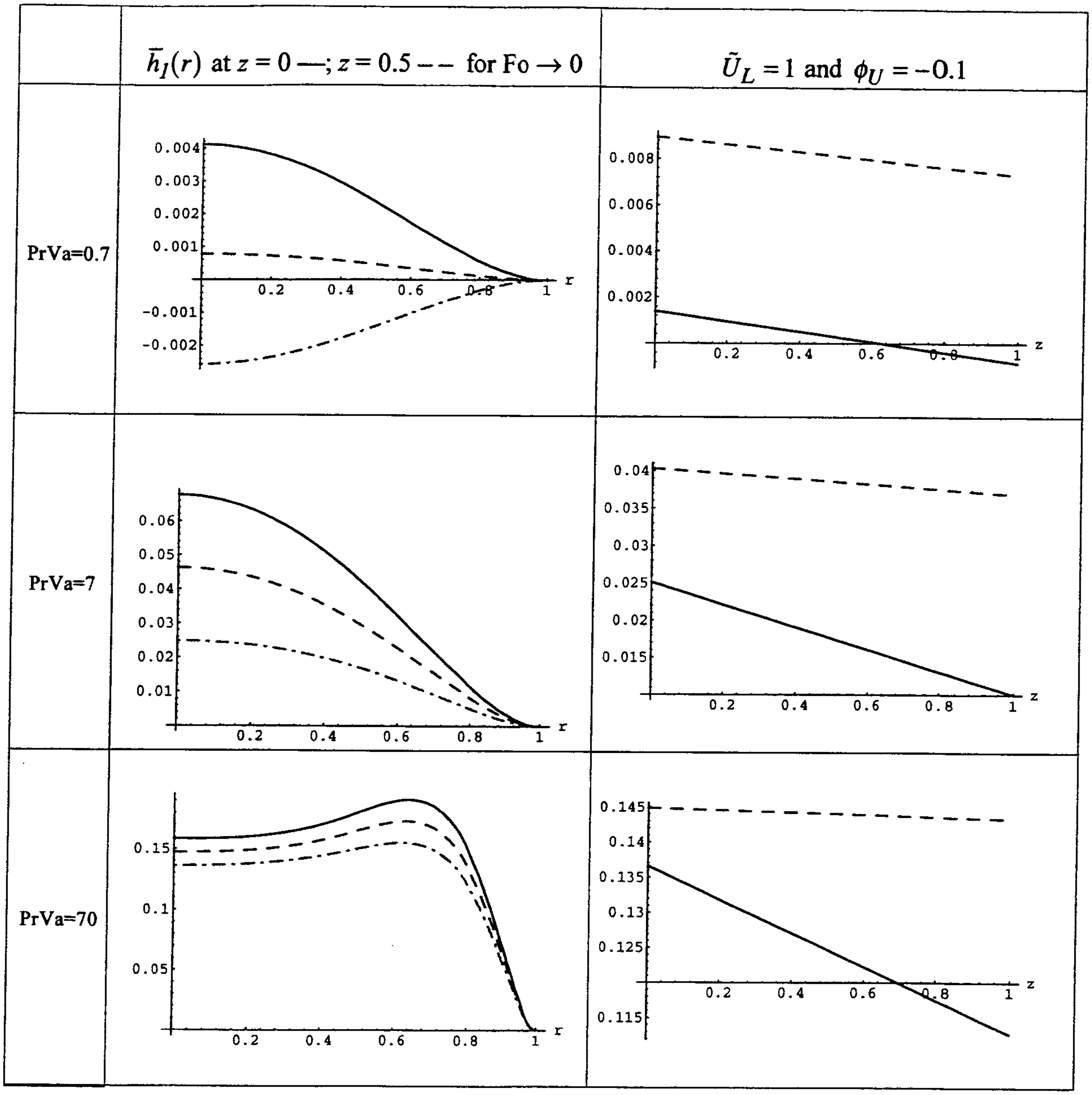

Figure 4. The effect of heat transfer on enthalpy flow, column 1 is the enthalpy flux; column 2 is the enthalpy flow as a function of z-position; Fo $\to 0$ —; Fo = 100 --.

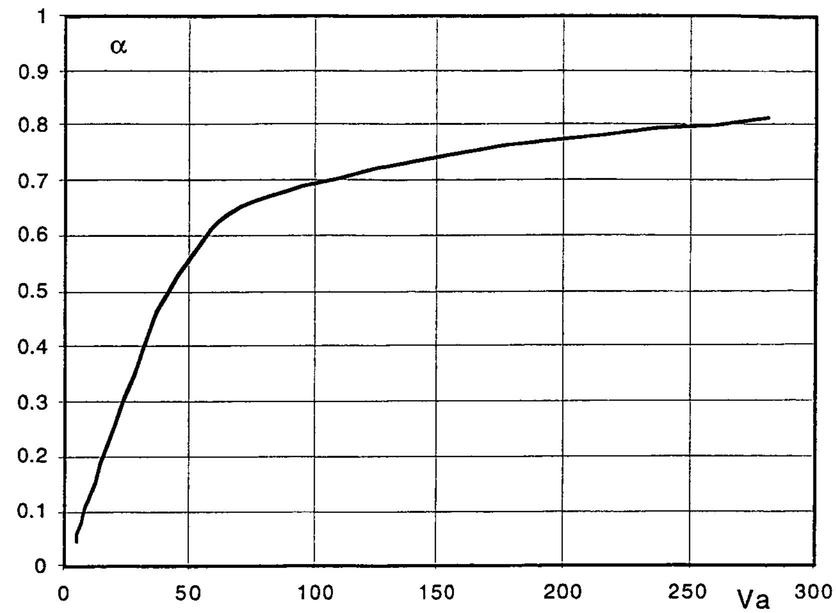

Figure 5. Correction factor, α, as a function of Va, Pr = 0.7. The correction factor is the ratio between the two dimensional enthalpy flow calculations of ref. 2 and that calculated for one-dimensional flow of ref. 4.

Figure 5 plots the correction factor, α, vs. Va with Pr=0.7 for use in correcting one-dimensional calculations for transverse thermal and viscous diffusion. The correction factor is the ratio between the calculated enthalpy flow given by ref[1] and the one-dimensional relation of ref 4. The plot of Fig. 5 is applicable for Pr = 0.5 to Pr = 0.9, and $\phi_U = 36°$ to 50°. It is readily seen that pulse tubes should be designed towards large Va so as to reduce the effects of diffusion to a thin boundary layer near the tube wall.

Discussion

The calculated leading order quantities for pressure, temperature, velocity and heat transfer, the mean-steady velocity and enthalpy flux fields, and the mean-steady temperature give an insightful understanding of the transport mechanisms for pulse tubes.

The orifice pulse tube (OPT) is a cooling device and so it does not rely on diffusion to obtain the appropriate phase angles between velocity and temperature. Phase angles are obtained through the velocity boundary conditions. The OPT should be operated where the thermal diffusion region is confined to a thin layer near the tube wall. This condition requires PrVa and Fo to both be large. The calculated plots of mean-steady velocity show that large Fo reduces mass streaming relative to Fo $\rightarrow$ 0. However, large Va tends to increase mass streaming.

Operating at small PrVa and small Fo is detrimental to an OPT because heat transfer between the gas and the tube wall: i) reduces the oscillating temperature amplitude near the tube wall, and ii) creates unwanted phase angles between velocity and temperature. Both of these effects will tend to reduce enthalpy flow. There is a practical limitation to having both PrVa and Fo very large, as these requirements lead to a system that must contain high pressures with a large diameter, thin-walled tube for a given frequency.

Heat transfer between the gas and the tube wall has an important effect on the pressure and temperature phasors.. When there is significant heat transfer between the gas and tube wall, $\mathrm{Fo} = O(1)$, the pressure and temperature phasors move out-of-phase relative to each other for both the BPT and OPT. Calculations indicate this to be as much as 20°. This is important because 1D models often assume adiabatic conditions on the gas and so there is a presumption that the temperature is always in-phase with pressure. Most pulse tubes operate at $\mathrm{Fo} = O(1)$ which is closer to isothermal wall conditions.

The complex Nusselt number is found to be independent of Fo, velocity amplitude ratio $\tilde{U}_L$, and velocity phase angle ϕ_U at the tube ends. When written in the form $\hat{\mathrm{Nu}}(\mathrm{PrVa}) = \tilde{A}\, e^{i\phi}$, $\tilde{A}$ is about 4 for PrVa < 3 and is linear with PrVa for PrVa > 25. The phase angle for PrVa < 0.5 is $\phi \rightarrow -0.5$ and for PrVa > 500, $\phi \rightarrow -0.38$. A similar relation for the complex shear wall factor exists using only Va as the independent parameter. The complex Nusselt number and shear wall factor can be used for one-dimensional linear oscillating flow in a tube to account for radial heat transfer or shear at the tube wall.

References

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