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GAS MOTION IN A LOCAL SUPERSONIC REGION AND CONDITIONS OF POTENTIAL-FLOW BREAKDOWN

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Translation

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GAS MOTION IN A LOCAL SUPERSONIC REGION AND CONDITIONS OF POTENTIAL-FLOW BREAKDOWN*

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For a certain Mach number of the oncoming flow, the local velocity first reaches the value of the local velocity of sound $(M = 1)$ at some point on the surface of the body located within the flow. This Mach number is designated the critical Mach number M_{cr}. By increasing the flow velocity, a supersonic local region is formed bounded by the body contour and the line of transition from subsonic to supersonic velocity. As is shown by observations with the Toepler apparatus, at a certain flow Mach number $M > M_{cr}$, a shock wave is formed near the body that closes the local supersonic region from behind. The formation of the shock wave is associated with the appearance of an additional resistance defined as the wave drag.

In this paper, certain features are described of the flow in the local supersonic region, which is bounded by the contour of the body and the transition line, and conditions are sought for which the potential flow with the local supersonic region becomes impossible and a shock wave occurs.

In the first part of the paper, the general properties of the potential flow in the local supersonic region, bounded by the contour of the profile and the transition line, are established. It is found that at the transition line, if it is not a line of discontinuity, the law of monotonic variation of the angle of inclination of the velocity vector holds (monotonic law). An approximation is given for the change in velocity at the contour of the body. The flow about a contour having a straight part is studied.

In the second part of the paper, an approximation is given of the magnitudes of the accelerations at the interior points of the supersonic region. With the aid of these approximations, it is

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shown that for profiles convex to the flow the breakdown of the potential flow, associated with an increase of the Mach number of the oncoming flow, cannot be due to the formation of an envelope of the characteristics within the supersonic region.

On the basis of the monotonic law, the transitional Mach number M is found, beyond which the potential flow with local supersonic region becomes impossible.

I - PROPERTIES OF POTENTIAL FLOW WITH LOCAL SUPERSONIC REGION

1. Properties of Supersonic Flow Bounded by Solid Wall and Transition Line

The flow in the supersonic region about a wing profile of usual form with a flow about it of a Mach number higher than critical is considered. In accordance with these conditions, two types of such flow may occur:

(a) Transition from supersonic flow to subsonic flow. - In this case, the supersonic region is bounded by part of the solid contour AB and the transition line $\lambda = 1$, which is indicated by a dotted line, as shown in figure 1. The magnitude $\lambda = w/a_*$, where w is the absolute magnitude of the velocity and a_* is the critical velocity (the velocity at which the flow velocity is equal to that of sound);

(b) Transition from supersonic flow to subsonic flow with aid of shock wave. - In this case, the supersonic region is bounded by the solid contour AB, the transition line $\lambda = 1$, and the shock wave CB (fig. 2).

Through each point of the supersonic region, two characteristics of different families pass. The minimum angle between the direction of the velocity at a given point and each of the characteristics is equal to the Mach angle α.

If the velocity vector is rotated by the angle α in a counterclockwise direction, the direction of the vector will coincide with one of the characteristics that shall be denoted as the characteristic of the first family in contradistinction to the other characteristic, which shall be denoted as the characteristic of the second family.

In case (a), each of the characteristics drawn from an arbitrary point P of the contour has a point in common with the transition line $\lambda = 1$ (fig. 1).

In case (b), all the characteristics of the second family end on the transition line, but not all characteristics of the first family possess this property.

By displacing the point P on the contour in the direction toward the base of the shock wave B, it is found that originating from a certain point D of the contour, the characteristics of the first family no longer end on the transition line but intersect the shock wave (fig. 2). The essential difference of the flow with local supersonic region from other mixed flows (for example, flow from a Laval nozzle) lies in the fact that from each point of the contour there originates at least one characteristic that ends on the transition line.

The region ADCA, in which both characteristics drawn from the points of the contour end on the transition line, is denoted as region I; the region DCBD, in which only the characteristics of the second family fall on the transition line, is denoted as region II.

In case (a), the entire supersonic region coincides with region I, hence, all theorems derived in this paper and the approximations for the flow in region I, obtained from the single assumption of the ending of the characteristics of both families on the transition line, will also hold for case (a).

Hereinafter, there will often be considered, together with a certain point P of the contour, simultaneous points on the line $\lambda = 1$, which are the ends of the characteristics originating from point P. The end of the characteristic of the first family will always be designated by the same letter as the point of the contour but with subscript 1 and by the sign *. Similarly, the end of the second characteristic will be designated by subscript 2 with sign *, as shown in figures 1 and 2.

In the following derivations, the fundamental magnitudes are the inclinations of the tangents at an arbitrary point P of the contour to the axis of abscissas, denoted always by θ_k; the inclinations of the velocity vector on the transition line at points P_{1*} and P_{2*} are denoted by θ_{1*} and θ_{2*}, respectively. All three magnitudes θ_k, θ_{1*}, and θ_{2*} are functions of the arc length of the contour.

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An arbitrary point P on the contour in the supersonic region is considered. The characteristic of the first family PP_{2*} is represented in the plane of the hodograph of the velocity by an arc of an epicycloid of the second family $P'P_{2*}'$ (fig. 3). Let $\theta = \theta_{2*} - \theta_k$ be the polar angle between the points P_{2*}' and P_{1*}'. For the magnitude λ at the point P of the profile, the following equation is obtained:

$$\lambda = f(\theta_{2*} - \theta_k) \tag{1.1}$$

where $\lambda = f(\theta)$ is the equation of the epicycloid so set up that the equation $f(0) = 1$ holds. This equation, as is known, has the form (reference 2)

$$\theta = -\frac{1}{2}\left\{\sqrt{\frac{\chi+1}{\chi-1}} \text{ arc sin } \left[\chi - (\chi-1)\lambda^2\right] - \text{arc sin } \left[(\chi+1)\frac{1}{\lambda^2} - \chi\right]\right\} + \left(\sqrt{\frac{\chi+1}{\chi-1}} - 1\right)\frac{\pi}{2} \tag{1.2}$$

In determining θ_{2*} from the given λ and θ_k

$$\theta_{2*} = \theta_k + \varphi(\lambda) \tag{1.3}$$

where φ is the function reciprocal to f. The graphs of the functions $f(\theta)$ and $\varphi(\lambda)$ are given in figure 4.

By considering the characteristic of the first family starting from point P (fig. 3),

$$\lambda = f(\theta_k - \theta_{1*}) \qquad \theta_{1*} = \theta_k - \varphi(\lambda) \tag{1.4}$$

where f and φ are the same functions as in equations (1.1) and (1.3).

For the points of region I where the characteristics of both families end on the transition line, the following equation is obtained from equations (1.3) and (1.4):

$$\theta_k = \frac{\theta_{1*} + \theta_{2*}}{2} \tag{1.5}$$

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Thus

THEOREM 1. - The inclination of the contour at any point P of zone I is the arithmetic mean of the inclinations of the velocity vectors at the points of the transition line, which are the ends of the characteristics starting out from point P.

Equation (1.3) and the second equation (1.4) permit, for a certain distribution of the velocities on the given contour, determination of the inclinations of the velocity vectors at the ends of the characteristics lying on the transition line.

2. Monotonic Law of Change in Angle of Inclination of Velocity Vector on Transition Line

If in the equations of an adiabatic gas, the nondimensional velocity λ and the angle of inclination θ of the velocity vector to the x axis are taken as the unknown functions, the following equations are obtained:

$$(1 - M^2)\left(\cos\theta\frac{\partial\lambda}{\partial x} + \sin\theta\frac{\partial\lambda}{\partial y}\right) - \lambda\left(\sin\theta\frac{\partial\theta}{\partial x} - \cos\theta\frac{\partial\theta}{\partial y}\right) = 0$$

$$\sin\theta\frac{\partial\lambda}{\partial x} - \cos\theta\frac{\partial\lambda}{\partial y} + \lambda\cos\theta\frac{\partial\theta}{\partial x} + \lambda\sin\theta\frac{\partial\theta}{\partial y} = 0 \qquad (1.6)$$

The character of the change of the magnitude θ along the line $\lambda = \lambda_1$ = constant. On one side of this line, let $\lambda < \lambda_1$ and on the other side let $\lambda > \lambda_1$ (fig. 5). The normal to the line $\lambda = \lambda_1$ is drawn in the direction of decreasing velocity. By considering a certain point of this line and by taking the direction of the y axis to agree with the direction of the normal to this point, from equation (1.6)

$$(1 - M_1{}^2)\sin\theta_1\frac{\partial\lambda}{\partial n} - \lambda_1\sin\theta_1\frac{\partial\theta}{\partial s} + \lambda_1\cos\theta_1\frac{\partial\theta}{\partial n} = 0$$

$$\cos\theta_1\frac{\partial\lambda}{\partial n} - \lambda_1\cos\theta_1\frac{\partial\theta}{\partial s} - \lambda_1\sin\theta_1\frac{\partial\theta}{\partial n} = 0 \qquad (1.7)$$

where M_1 is the Mach number for $\lambda = \lambda_1$, and θ_1 is the angle between the velocity vector and the positive direction of the tangent to the line $\lambda = \lambda_1$ at a fixed point of this line. By eliminating from equations (1.7) the magnitude $\partial\theta/\partial n$ and by determining from the obtained equation $\partial\theta/\partial s$

$$\frac{\partial\theta}{\partial s} = \frac{1 - M_1{}^2 \sin^2 \theta_1}{\lambda_1} \frac{\partial\lambda}{\partial n} \tag{1.8}$$

In the subsonic flow, $1 - M_1{}^2 \sin^2 \theta_1 > 0$ for any θ and because $\partial\lambda/\partial n \leq 0$, then $\partial\theta/\partial s \leq 0$. Thus θ, in this case, changes monotonically along the line λ = constant.

By considering the transition line where $\lambda_1 = 1$ and by starting from the assumption that the transition line is not a line of discontinuity (reference 1), that is, that all the derivatives $\partial\theta/\partial s$, $\partial\theta/\partial n$, $\partial\lambda/\partial s$, and $\partial\lambda/\partial n$ are finite on the transition line, the following equation is obtained from equation (1.8)

$$\frac{\partial\theta}{\partial s} = \cos^2 \theta \frac{\partial\lambda}{\partial n} \tag{1.9}$$

and therefore the condition $\partial\theta/\partial s \leq 0$. Thus

THEOREM 2. - If a point moves along the transition line so that the region of subsonic velocity lies to the left, the velocity vector will monotonically turn in the clockwise direction.

The condition $\partial\theta/\partial s = 0$ along the transition line, in the case where a transition from the subsonic to the supersonic velocity occurs, was previously obtained by S. A. Christianovich.

The property of monotonicity is not, however, characteristic for supersonic flow. In this case, the inequality $1 - M^2 \sin^2 \theta > 0$ and the inequality $1 - M^2 \sin^2 \theta < 0$ could hold and therefore the value of $\partial\theta/\partial s$ can change sign.

From further discussion, it will be clear that the fact expressed by theorem 2 determines, to a considerable extent, the character of the flow in the local supersonic region and also the possibility or impossibility of the simultaneous existence of the subsonic and supersonic flows without change in the potential character of the flow. This fact will hereinafter be termed the "law of monotonicity on the transition line," or simply the "monotonic law."

3. Property of Monotonic Change of Velocity Magnitude and Its Inclination along Characteristics

By making use of the results of section 2, it will be proved that:

THEOREM 3. - If in the supersonic region there is given a section of the characteristic of one family, such that the characteristics of the other family originating from the points of this segment end on the transition line, the angle of inclination of the velocity vector and the magnitude of the velocity are monotonic functions along the given segment of the characteristic.

By considering a certain point C on the segment AB of the characteristic, draw from the points A, B, and C to the line of transition the characteristics of the family different from the family to which the characteristic AB belongs (fig. 6). The ends of these characteristics lying on the transition line are A_*, B_*, and C_*, respectively.

In the plane of the hodograph, points A, B, and C are represented by the points A', B', and C', respectively, lying on a certain epicycloid γ; and the points A_*, B_*, and C_* are represented by the points A_*', B_*', and C_*', respectively, lying on the circle $\lambda = 1$. Each pair of points, A' and A_*', B' and B_*', and C' and C_*' lie on one of the epicycloids of the family different from the one to which the epicycloid γ belongs (fig. 7).

As the point C' moves along the epicycloid γ and its corresponding point C_*' moves along the circle $\lambda = 1$, the polar angles θ at the points C' and C_*' either simultaneously decrease or simultaneously increase because each straight line, θ = constant, intersects the epicycloid only at one point. When point C moves in the same direction from point A to point B along the segment of the characteristic AB, the point C_* moves along the transition line from point A_* to point B_* likewise in one direction as the characteristics of one family do not intersect. In accordance with theorem 2 of the preceding section, it follows that the point C_*' likewise moves along the circle $\lambda = 1$ in the same direction. Hence, the polar angle θ for the point C' varies monotonically and, as follows from the properties of the epicycloid, the magnitude of the velocity λ likewise varies monotonically for point C'.

In the supersonic region I about the profile determined in section I, both families of characteristics satisfy the conditions of theorem 3 so that in moving along any characteristic, the inclination of the velocity vector θ and the magnitude of the velocity λ vary monotonically. For motion directed toward the transition line along the characteristics of the first family, both magnitudes θ and λ monotonically decrease; whereas the characteristics of the second family (the left one), θ monotonically increases and λ monotonically decreases. From the obtained property of the monotonic variation of the magnitude of the velocity and the angle of inclination of the velocity vector along each of the characteristics of region I, it follows that this region is represented as a single sheet on the corresponding region of the hodograph.

In the supersonic region II, only the characteristics of the first family satisfy the conditions of theorem 3. On moving along the characteristics away from the profile, both magnitudes θ and λ monotonically decrease. On moving along the characteristics of the second family in region II, the property of monotonicity of the change in the magnitudes θ and λ does not, in general, hold.

By making use of the known properties of the characteristics in the plane of the flow and in the plane of the velocity hodograph, the directions of concavity, as shown in figure 8, are obtained in the region of applicability of theorem 3 for small Mach numbers. This direction of concavity of the characteristics, as follows from the results of S. A. Christianovich (reference 1), takes place only in the case $M < M_0 = 2/\sqrt{3 - \chi} = 1.565$. By making use of the results mentioned, it is found that if $M = M_0$ for a certain point K in the region of applicability of theorem 3, the characteristics have the appearance represented in figure 9. Point K is always a point of inflection of the characteristics.

4. Variation of Velocity at Profile in Supersonic Region

It is assumed, as is usually the case, that in the supersonic region the profile is everywhere directed convex to the flow. In the supersonic region at the profile, an arbitrary point K and the characteristic of the second family KK_{2*} are assumed to originate at this point (fig. 10). For point K from equation (1.1), $\lambda = f(\theta_{2*} - \theta_k)$ where f is the function introduced according to equation (1.1).

$$\frac{d\lambda}{d(-\theta_k)} = f'(\theta_{2*} - \theta_k)\left[1 + \frac{d\theta_{2*}}{d(-\theta_k)}\right] \tag{1.10}$$

In accordance with theorem 2, $d\theta_{2*}/d(-\theta_k) \leq 0$; hence, by making use of equation (1.10)

$$\frac{d\lambda}{d(-\theta_k)} \leq f'(\theta_{2*} - \theta_k) \tag{1.11}$$

But $\theta_{2*} - \theta_k = \varphi(\lambda)$, where φ is the reciprocal function of f. Hence

$$f'(\theta_{2*} - \theta_k) = f'[\varphi(\lambda)] = \frac{d\lambda}{d\theta}$$

where the differentiation is effected along the epicycloid in the plane of the hodograph (fig. 11). Two points A and B are assumed to be on the epicycloid, the polar angle between which is equal to $d\theta$. The value of λ is also assumed to be greater at point B than at point A.

From point A, an arc of the circle λ = constant is drawn to the intersection with the straight line OB at point C, the length of the arc being taken equal to $d\sigma$. The angle between the tangents to the arc of the circle AC and the arc of the epicycloid AB at point A is equal to the Mach angle α. Hence, $d\lambda = \text{tg}\,\alpha\, d\sigma$ and because $d\sigma = \lambda d\theta$, then $d\lambda/d\theta = \lambda \text{tg}\,\alpha$.

The graph of the function $f'[\varphi(\lambda)] = \lambda \text{tg}\,\alpha$ is given in figure 5. The inequality (1.11) therefore assumes the form

$$\frac{d\lambda}{d(-\theta_k)} \leq \lambda \text{tg}\,\alpha \tag{1.12}$$

As is shown by the preceding inequality, this value depends only on the magnitude λ at the given point. By comparing equation (1.10) and inequality (1.11), it is found that inequality (1.12) may be expressed as an equation if and only if at a given point of the profile $d\theta_{2*}/d\theta_k = 0$. If this relation holds true on a certain segment AB of the profile (fig. 12), the condition θ_{2*} = constant is obtained. All the characteristics of the second family starting from AB are then represented in the plane of the hodograph by the same epicycloid of the second family γ (fig. 13) because at their ends, lying on the line $\lambda = 1$, the velocity is constant. Hence, the entire region $ABB_{2*}A_{2*}$ (fig. 12) is also represented in the plane of the hodograph by the segment of the epicycloid A'B', upon which the points of the segment AB will also lie.

The characteristics of the first family in the region $ABB_{2*}A_{2*}$ must, in this case, be straight lines with constant velocity on each, that is, in this region a certain Meyer flow (rarefaction) must occur. However, as will be shown in section 6, the realization of such flow under the conditions of the problem is impossible. Hence, in the relation (1.12), the equality can hold only at certain points of the contour.

In the supersonic region I, the characteristics of the first family likewise end on the line $\lambda = 1$. By applying considerations analogous to those previously mentioned, the second inequality along the profile in region I is obtained.

$$\frac{d\lambda}{d(-\theta_k)} \geq -\lambda \operatorname{tg} \alpha \tag{1.13}$$

In the case where the equality in relation (1.13) is attained on a certain segment of the profile AB in the region bounded by the characteristics of the first family originating at points A and B, the transition line, and the profile, a Meyer flow (compression) takes place, the straight lines being the characteristics of the second family (fig. 14). As will be shown in section 6, such flow likewise cannot be realized under the conditions of the problem and therefore, in relation (1.13), the equality can be attained only at certain points of the contour. By combining relations (1.12) and (1.13), the following inequality is obtained on the contour of region I:

$$\left|\frac{d\lambda}{d(-\theta_k)}\right| \leq \lambda \operatorname{tg} \alpha \tag{1.14}$$

Inequality (1.13) is of little interest in the case of a shock wave that closes the supersonic region because in this case in region I, the velocity generally does not decrease so that the relation refers mainly to the case of the flow with supersonic region without a shock wave. In this case, region I coincides with the entire supersonic region.

The equation for the change in the velocity can also be obtained for the case where a certain segment of the profile is concave to the flow. Thus, for any point C on the segment AB (fig. 15)

$$\lambda = f(\theta_{2*} - \theta_k) \qquad \frac{d\lambda}{d\theta_k} = f'(\theta_{2*} - \theta_k)\left(\frac{d\theta_{2*}}{d\theta_k} - \qquad\right)$$

But $f'(\theta_{2*} - \theta_k) = \lambda \operatorname{tg} \alpha$, hence

$$\frac{d\lambda}{d\theta_k} \leq - \lambda \operatorname{tg} \alpha \tag{1.15}$$

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By making use of the law of monotonicity, $d\theta_{2*}/d\theta_k \leq 0$ and therefore from inequality (1.15)

$$\frac{d\lambda}{d\theta_k} \leq - \lambda \operatorname{tg} \alpha \tag{1.16}$$

[NACA Comment: equation (1.16) is apparently in error.]

Thus, on the segment of the profile having a concavity facing the flow, the velocity drops and, as follows from relation (1.16), its drop cannot be too slow. Inequality (1.16) differs essentially from relations (1.12) and (1.13), which show that on the convex segment of the profile, the velocity cannot vary too rapidly.

Strictly speaking, these conditions are true up to those changes in the flow that are brought about by an oblique shock wave (fig. 15). By considering the characteristic of the second family $C_{2*}C_1$ and its prolongation C_1C, it is found that the transition through the shock wave in the flow plane corresponds to the displacement along the segment of the strophoid $C'_1\tilde{C}_1$ in the plane of the velocity hodograph (fig. 16). Hence, the transforms of the segments C_1C_{2*} and C_1C of the characteristic lie on two different epicycloids of the second family. However, by making use of the fact that at the point C'_1 the strophoid and epicycloid have a common tangent and the same radius of curvature, it is found that with an accuracy up to small magnitudes of the third order, relative to the changes in velocity in the shock wave, these two epicycloids may be considered as coinciding. Inequality (1.16) holds with the same degree of accuracy.

5. Flow in Supersonic Region Arising from Presence of Straight Segment on Profile

The straight segment AB is assumed to be on the profile in the supersonic region. The characteristic of the second family CC_{2*} originates from the arbitrary point C, which is assumed to be on the straight segment AB (fig. 17). At point C, the equation is $\lambda = f(\theta_{2*} - \theta_k)$.

As point C moves along segment AB in the flow direction, the magnitude θ_k remains constant because of the rectilinearity of the segment and the magnitude θ_{2*}, in accordance with theorem 2, does not increase. Hence, using the fact that f is a monotonically increasing function of its argument: 1107

THEOREM 4. - <u>On a straight segment of a profile in a supersonic region, the velocity in the flow direction does not increase.</u>

It is assumed that the straight segment is located in the supersonic region I (fig. 18).

From the ends of the straight segment to the transition line, the characteristics of the first family AA_{1*} and BB_{1*} and of the second family AA_{2*} and BB_{2*} are drawn. Let $\theta_k = \theta_0$ on AB. From point C near point A, the characteristic of the second family is drawn to the intersection with the characteristic AA_{1*} at point C'. From theorem 3, it is found that $\theta(A) \geq \theta(C') \geq \theta(C)$, but $\theta(A) = \theta(C) = \theta_0$, hence $\theta(C') = \theta_0$ and, as again follows from theorem 3, $\theta = \theta_0$ over the entire segment of the characteristic CC'.

By moving point C from point A to point B, it is found that the characteristics AA_{1*} and BB_{2*} necessarily intersect and in the triangle ABD, formed by the segment AB and these characteristics, $\theta = \theta_0 = \text{constant}$. Hence, it follows that $\lambda = \text{constant}$ and the characteristics of both families are rectilinear.

The entire region $ABB_{2*}A_{2*}$ is represented in the plane of the velocity hodograph by a single epicycloid of the second family, as all the characteristics of the second family are represented by a single epicycloid passing through the point $\theta = \theta_0$, $\lambda = \lambda(A)$. Hence, over the entire region considered, a certain Meyer flow (rarefaction) will take place with straight characteristics of the first family. Similarly, the entire region $ABB_{1*}A_{1*}$ is represented in the plane of the hodograph by a single epicycloid of the first family, over which a Meyer flow (compression) will take place with straight characteristics of the second family.

As was previously stated and as will be shown in section 6, the realization of Meyer flows under the conditions of the problem is impossible.

6. Impossibility of Meyer Flow in Local Supersonic Region

In the plane x,y, a Meyer flow is assumed to be between the characteristics C_1 and C_2 of the second family with straight characteristics of the first family (fig. 19).

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From a certain point A of characteristic C_1, the straight characteristic of the second family is drawn to its intersection at point B with the characteristic C_2. On the straight line AB, the Mach angle α = constant, hence, it intersects the characteristics C_1 and C_2 at the same angle $\gamma = \pi - 2\alpha$.

Set $AB = l$ and denote by β the angle formed by the straight line AB with the axis of abscissas. The straight characteristic A_1B_1, infinitely near AB, is considered. The segments of the straight lines AB and A_1B_1 are denoted, respectively, by ds_1 and ds_2 on the characteristics C_1 and C_2 and the angle between AB and A_1B_1 is denoted by $d\beta$. Thus, the following equations are obtained:

$$ds_1 = \frac{r d\beta}{\sin \gamma} \qquad ds_2 = \frac{(r + l)\, d\beta}{\sin \gamma} \tag{1.17}$$

where r is the distance between point A and the point of intersection of the straight lines AB and A_1B_1.

Set the length of the segment A_1B_1 equal to $l + dl$. From figure 18, the equation $dl = (ds_2 - ds_1) \cos \gamma$ is obtained, or by making use of equations (1.17)

$$dl = l \operatorname{ctg} \gamma \, d\beta \tag{1.18}$$

In the plane of the hodograph, the characteristic C_1 is represented by an epicycloid.

The directions AB and A_1B_1 coincide with the direction of the normals to this epicycloid drawn at the points A' and A'_1, corresponding to the points A and A_1 (fig. 20). The angle between the radius vector and the direction of this normal is equal to the Mach angle α. Hence, $\beta = \alpha + \theta$ and $d\beta = d\alpha + d\theta$.

Thus, by making use of equation (1.18)

$$\frac{dl}{l} = \operatorname{ctg} \gamma (d\alpha + d\theta) = - \operatorname{ctg} 2\alpha \, d\alpha + \operatorname{ctg} \gamma \, d\theta \tag{1.19}$$

or by integrating

$$\ln \frac{l}{l_0} = \ln \sqrt{\frac{\sin 2\alpha_0}{\sin 2\alpha}} + \int_{\theta_0}^{\theta} \operatorname{ctg} \gamma \, d\theta \tag{1.20}$$

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where l_0, α_0, and θ_0 characterize a certain initial point.

For $\alpha > \pi/4$ and $\theta > \theta_0$, both terms on the right of equation (1.20) are positive so that the following inequalities hold:

$$\ln \frac{l}{l_0} > \ln \sqrt{\frac{\sin 2\alpha_0}{\sin 2\alpha}} \qquad l > l_0 \sqrt{\frac{\sin 2\alpha_0}{\sin 2\alpha}} \tag{1.21}$$

If $\lambda \rightarrow 1$, then $\alpha \rightarrow \pi/2$ and $l \rightarrow \infty$.

An entirely analogous result is obtained for the Meyer flow between the characteristics of the first family with straight characteristics of the second family. Thus

THEOREM 5. - A Meyer flow, between two characteristics of one family with straight characteristics of the second family, cannot be entirely realized up to the line $\lambda = 1$ in a finite region.

If on the characteristic C_1, represented in figure 19, the equality $\lambda = 1$ is attained at a certain point D, then by making use of the preceding result, it is found that the tangent at point D to the characteristic C_1 is asymptotic for any characteristic C_2 placed relatively to C_1, as shown in figure 19[1].

From theorem 5, it follows that along the straight part of the profile in the local supersonic region, the magnitude of the velocity cannot be constant. If such were the case, then between the characteristics of the second family originating at the ends of this segment a Meyer flow would take place. This realization, because of the finiteness of the local supersonic region, is impossible as follows from theorem 5. Thus theorem 4 may be more definitely stated as follows:

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[1] According to S. A. Christianovich, if a Meyer flow originates from a straight characteristic on which $\lambda = 1$, the characteristics of the other family do not originate from this straight line.

THEOREM 6. - On the straight segment of a profile in the supersonic region, the velocity decreases.

II - CONDITIONS FOR BREAKDOWN OF POTENTIAL FLOW WITH LOCAL SUPERSONIC REGION

The occurrence of a wave resistance for a flow with large subsonic velocity about a body is assumed to be connected with the instant when at any point on the surface of the body a velocity is reached equal to the local velocity of sound.

The study of the results of a number of tests has shown, however, that the occurrence of a shock wave and therefore the arising of a wave resistance sometimes takes place beyond M_{cr}. Photographs obtained by the Toepler method show in certain cases a local supersonic region extended deep into the flow without any shock wave.

The necessity for the breakdown of the potential character of the flow with local supersonic region, that is, the necessity for the appearance of a shock wave, has not yet been theoretically demonstrated for a single case. It will therefore be of interest to describe certain conditions that are known to be accompanied by a breakdown of the potential flow.

1. Deformation of Contour

How the deformation of the contour can lead to the breakdown of a previously existing potential flow is considered.

A flow with local supersonic region about a curvilinear contour for a Mach number of the oncoming flow such that the flow is potential (fig. 21) is considered.

From points A and B of the contour, the characteristics of both families are drawn both in the flow plane and in the plane of the hodograph. Because of satisfying the condition of monotonicity of change of the angle on the transition line, the transformations of the points A_{2*}, B_{2*}, A_{1*}, and B_{1*} in the hodograph plane are located on the circle $\lambda = 1$ in the same order as on the transition line in the flow plane. The contour is deformed in such a way that the segment of the arc AB transforms into a straight segment (fig. 22).

By virtue of theorem 6, section I, the velocity along the straight segment AB can only decrease. By drawing the characteristics in the hodograph plane and by making use of theorem 1, it is found that the transforms of the points A_{2*}, B_{2*}, A_{1*}, and B_{1*} are not located on the circle $\lambda = 1$ in the order in which these points are arranged on the transition line in the flow plane. Consequently, the deformation of the contour breaks up the monotonicity on the transition line. Thus

THEOREM 7. - The characteristics of the first family that originate from the points of a straight segment of the contour can never end on the transition line but must fall on a shock wave.

If the contour deformation considered is effected continuously, it is evident that the breakdown of the potential flow occurs before the deformed segment becomes straight. This fact permits the conclusion that at a given Mach number of the oncoming flow, any profile convex to the flow can be so deformed that a new profile is again obtained convex to the flow, the flow about which for the same Mach number is no longer possible without a shock wave.

2. Impossibility of Formation of Line of Discontinuity Within Local Supersonic Region

Investigation will reveal whether the impossibility of realization of a continuous flow within the supersonic region for given boundary conditions, as expressed in the intersection of the characteristics of one family and the formation of a supersonic shock wave within the region, can be a cause for the breakdown of the potential character of the gas motion.

Kármán (reference 3) makes the assumption that the probable cause of the appearance of shock waves is the formation of an envelope of the Mach lines in the supersonic zone for a certain Mach number of the oncoming flow.

The envelope of the characteristics of one family, as shown by S. A. Christianovich (reference 1), coincides with the line of discontinuity that is determined as the geometric locus of the points at which at least one of the derivatives $\partial\theta/\partial x$, $\partial\theta/\partial y$, $\partial\lambda/\partial x$, or $\partial\lambda/\partial y$ becomes infinite. Hence, to investigate the problem of the possibility of the formation of supersonic shock waves within the supersonic region on increasing the Mach number of the oncoming flow, it is necessary to obtain the value of the acceleration at the points of the supersonic region. It will be shown that on the contour of the profile infinite accelerations cannot occur.

Given a certain flow in the supersonic region with continuous transition from supersonic to subsonic velocities (sec. I, pt. 1, case (a)); assume a certain point K on the contour in the supersonic region and take the x axis in the flow direction (fig. 23). Equations (1.6) then become

$$(M^2 - 1)\frac{\partial\lambda}{\partial x} - \lambda\frac{\partial\theta}{\partial y} = 0 \qquad \frac{\partial\lambda}{\partial y} - \lambda\frac{\partial\theta}{\partial x} = 0 \tag{2.1}$$

Thus

$$\frac{\partial\lambda}{\partial x} = \frac{d\lambda}{\partial\theta_k}\frac{\partial\theta}{\partial x} = -k\frac{d\lambda}{\partial\theta_k} = -\frac{1}{R}\frac{d\lambda}{d\theta_k} \tag{2.2}$$

$$\frac{\partial\theta}{\partial x} = -k = -\frac{1}{R} \tag{2.3}$$

where k is the curvature of the contour at point K and R is the radius of the curvature at this point. By making use of inequality (1.14) and equations (2.2) and (2.3)

$$\left|\frac{\partial\lambda}{\partial x}\right| \leq \frac{1}{R}\lambda \operatorname{tg}\alpha \qquad \left|\frac{\partial\theta}{\partial x}\right| = \frac{1}{R} \tag{2.4}$$

From equations (1.20)

$$\left|\frac{\partial\theta}{\partial y}\right| \leq \frac{1}{R}\operatorname{ctg}\alpha \qquad \left|\frac{\partial\lambda}{\partial y}\right| = \frac{\lambda}{R} \tag{2.5}$$

Relations (2.4) and (2.5) show that the magnitudes $\partial\theta/\partial x$, $\partial\theta/\partial y$, $\partial\lambda/\partial x$, and $\partial\lambda/\partial y$ are always finite on the contour within the supersonic region provided that everywhere $R \neq 0$. The case $R = 0$ (infinite curvature) at some point is a special case and never occurs on real profiles. Therefore, assume that on the contours considered, at all points $R > R_0$ where R_0 is a certain constant.

The maximum possible value of the acceleration at the interior points of a local supersonic region is computed. The differential equations of the characteristics of the first and second families in the plane x,y are, respectively,

$$dy = \operatorname{tg}(\theta + \alpha)\, dx \qquad dy = \operatorname{tg}(\theta - \alpha)\, dx \tag{2.6}$$

Moreover, along the characteristics of the first and second families, respectively, the relations hold (see for example reference 1)

$$\sigma + \theta = 2\xi = \text{constant} \qquad \sigma - \theta = 2\eta = \text{constant} \tag{2.7}$$

where

$$\sigma = 1 - \int_1^{\lambda} \sqrt{\frac{\lambda^2 - 1}{1 - [(\chi - 1)/(\chi + 1)]\lambda^2}}\, \frac{d\lambda}{\lambda} \tag{2.8}$$

From relations (2.6) and (2.7)

$$\frac{\partial y}{\partial \eta} = \operatorname{tg}(\theta + \alpha) \frac{\partial x}{\partial \eta} \qquad \frac{\partial y}{\partial \xi} = \operatorname{tg}(\theta - \alpha) \frac{\partial x}{\partial \xi} \tag{2.9}$$

By differentiating the first relation with respect to ξ, the second with respect to η, and by subtracting one from the other, the following equation is obtained for the value of x:

$$\frac{1}{\cos^2(\theta-\alpha)} \frac{\partial(\theta-\alpha)}{\partial \eta} \frac{\partial x}{\partial \xi} + \operatorname{tg}(\theta-\alpha) \frac{\partial^2 x}{\partial \xi \partial \eta} = \frac{1}{\cos^2(\theta+\alpha)} \frac{\partial(\theta+\alpha)}{\partial \xi} \frac{\partial x}{\partial \eta} + \operatorname{tg}(\theta+\alpha) \frac{\partial^2 x}{\partial \xi \partial \eta} \tag{2.10}$$

From relations (2.7)

$$\sigma = \xi + \eta \qquad \theta = \xi - \eta \tag{2.11}$$

By making use of these equations and choosing the coordinate system so that the direction of the x axis coincides with the direction of the velocity at some point A (fig. 24), the relation between the partial derivatives of x at the point considered is obtained from equation (2.10)

$$\frac{\partial^2 x}{\partial \xi \partial \zeta} + \frac{1 + \alpha'(\sigma)}{\sin 2\alpha} \frac{\partial x}{\partial \xi} + \frac{1 + \alpha'(\sigma)}{\sin 2\alpha} \frac{\partial x}{\partial \eta} = 0 \qquad \left(\alpha'(\sigma) = \frac{d\alpha}{d\sigma}\right) \tag{2.12}$$

Let dl and ds be the lengths of the elementary segments of the characteristics of the first and second families, respectively. In passing from point A to neighboring points in the directions indicated in figure 24, dl and ds will be taken as positive

$$\frac{\partial x}{\partial \xi} = \frac{\partial x}{\partial l}\frac{dl}{d\xi} = -\cos\alpha\,\frac{dl}{d\xi} \qquad \frac{\partial x}{\partial \eta} = \frac{\partial x}{\partial s}\frac{ds}{d\eta} = \cos\alpha\,\frac{ds}{d\eta} \tag{2.13}$$

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where the derivative $dl/d\xi$ is taken along a characteristic of the second family and the derivative $ds/d\eta$ is taken along a characteristic of the first family. By making use of equations (2.13), relation (2.12) can be reduced to one of the two following forms:

$$\frac{\partial}{\partial \eta}\left(\cos\alpha\,\frac{dl}{d\xi}\right) + \frac{1+\alpha'(\sigma)}{\sin 2\alpha}\left(\cos\alpha\,\frac{dl}{d\xi}\right) - \frac{1+\alpha'(\sigma)}{2\sin\alpha}\,\frac{ds}{d\eta} = 0 \tag{2.14}$$

$$\frac{\partial}{\partial \xi}\left(\cos\alpha\,\frac{ds}{d\eta}\right) + \frac{1+\alpha'(\sigma)}{\sin 2\alpha}\left(\cos\alpha\,\frac{ds}{d\eta}\right) - \frac{1+\alpha'(\sigma)}{2\sin\alpha}\,\frac{dl}{d\xi} = 0 \tag{2.15}$$

Along the characteristic ξ = constant, relation (2.14) may be considered as an ordinary differential equation relative to the magnitude $\cos\alpha\,dl/d\xi$ and similarly along the characteristic η = constant, the relation (2.15) may be considered as an ordinary differential equation relative to $\cos\alpha\,ds/d\eta$. By setting $z = \cos\alpha\,dl/d\xi$, the following linear equation of the first order, based on equation (2.14), is obtained for determining z along the characteristic ξ = constant:

$$\frac{dz}{d\eta} + \frac{1+\alpha'(\sigma)}{\sin 2\alpha}\,z - \frac{1+\alpha'(\sigma)}{2\sin\alpha}\,\frac{dz}{d\eta} = 0 \tag{2.16}$$

The solution of this differential equation has the form

$$z = \exp\left[-\int_{\eta_0}^{\eta}\frac{1+\alpha'(\sigma)}{\sin 2\alpha}\,d\eta\right]\left\{z_0 + \int_{\eta_0}^{\eta}\exp\left[\int_{\eta_0}^{\eta}\frac{1+\alpha'(\sigma)}{\sin 2\alpha}\,d\eta\right]\frac{1+\alpha'(\sigma)}{2\sin\alpha}\,\frac{ds}{d\xi}\,d\eta\right\} \tag{2.17}$$

where η_0 is the value of η at a certain initial point on the characteristic considered, ξ = constant, and $z_0 = \cos\alpha_0(dl/d\xi)_0$ is the value of z at this point. The integral under the sign of the exponential function in equation (2.17) can be readily computed, if the magnitudes entering the expression under the integral are expressed in terms of the Mach angle α, by making use of the relation

$$\lambda = \sqrt{\frac{\chi+1}{2}} \, \frac{1}{\sqrt{\sin^2 \alpha + (\chi-1)/2}}$$

and the relations

$$d\eta = d\sigma = - d\theta \qquad \frac{d\lambda}{d\theta} = \lambda \operatorname{tg} \alpha$$

which hold along the characteristic of the first family ξ = constant, the first of which follows from relations (2.7) and (2.11) and the second of which was already obtained in section I, part 4. The following expression is thus obtained:

$$\int_{\eta_0}^{\eta} \frac{1 + \alpha'(\sigma)}{\sin 2\alpha} \, d\eta = \ln \left\{ \sqrt{\frac{\operatorname{tg} \alpha}{\operatorname{tg} \alpha_0}} \left[\frac{\sin^2 \alpha \left[\sin^2 \alpha_0 + 1/2(\chi-1)\right]}{\sin^2 \alpha_0 \left[\sin^2 \alpha + 1/2(\chi-1)\right]} \right]^{\frac{1}{2(\chi-1)}} \right\} \tag{2.18}$$

Hence, by replacing in equation (2.16) the magnitude z by its expression, and denoting by $G(\alpha,\alpha_0)$ the expression under the logarithm sign in (2.18)

$$\cos \alpha \frac{dl}{d\xi} = \frac{1}{G(\alpha,\alpha_0)} \left[\cos \alpha_0 \left(\frac{dl}{d\xi}\right)_0 + \int_{\eta_0}^{\eta} G(\alpha,\alpha_0) \frac{1+\alpha'(\sigma)}{2 \sin \alpha} \frac{ds}{d\eta} \, d\eta \right]. \tag{2.19}$$

An entirely similar solution of equation (2.15) gives

$$\cos \alpha \frac{ds}{d\eta} = \frac{1}{G(\alpha,\alpha_0)} \left[\cos \alpha_0 \left(\frac{ds}{d\eta}\right)_0 + \int_{\xi_0}^{\xi} G(\alpha,\alpha_0) \frac{1+\alpha'(\sigma)}{2 \sin \alpha} \frac{dl}{d\xi} \, d\xi \right] \tag{2.20}$$

By making use of the property of monotonicity of change of the magnitude of the velocity and the angle of inclination of the velocity vector along the characteristics in the local supersonic region (sec. I, pt. 3), it is noted that $dl/d\xi \geq 0$ and $ds/d\eta \geq 0$.

Moreover, $\alpha'(\sigma) > 0$. Hence, from equations (2.19) and (2.20), the following inequalities are obtained:

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$$\cos\alpha \frac{dl}{d\xi} \geq \cos\alpha_0 \left(\frac{dl}{d\xi}\right)_0 \frac{1}{G(\alpha,\alpha_0)} \tag{2.21}$$

$$\cos\alpha \frac{ds}{d\eta} \geq \cos\alpha_0 \left(\frac{ds}{d\eta}\right)_0 \frac{1}{G(\alpha,\alpha_0)} \tag{2.22}$$

In order to obtain the accelerations at a certain point K of the local supersonic region, the characteristics of the first and second families are drawn from this point up to the intersection with the contour at points A and B, respectively (fig. 25). Let $\alpha = \alpha_{01}$ and $\alpha = \alpha_{02}$ at points A and B, respectively. Also, let the radii of curvature at A and B be R_1 and R_2, respectively.

The inequalities (2.21) and (2.22) at the point K are considered where point B is taken for the initial point in inequality (2.21) and in inequality (2.22) point A is used. By making use of relations (1.14), (2.1), and (2.7), approximations are easily obtained for the magnitudes $d\xi/dl$ at point B and $d\eta/ds$ at point A

$$\left|\frac{d\xi}{dl}\right|_B \leq \frac{2}{R_2}\cos\alpha_{02} \qquad \left|\frac{d\eta}{ds}\right| \leq \frac{2}{R_1}\cos\alpha_{01} \tag{2.23}$$

By taking the direction of the x axis at point K along a streamline, the following equations are obtained:

$$\frac{\partial\theta}{\partial x} = -\frac{1}{2\cos\alpha}\left(\frac{d\xi}{dl} + \frac{d\eta}{ds}\right) \qquad \frac{\partial\theta}{\partial y} = \frac{1}{2\sin\alpha}\left(\frac{d\xi}{dl} - \frac{d\eta}{ds}\right)$$

$$\frac{\partial\lambda}{\partial x} = -\frac{\lambda\,\mathrm{tg}\,\alpha}{2\cos\alpha}\left(\frac{d\eta}{ds} - \frac{d\xi}{dl}\right) \qquad \frac{\partial\lambda}{\partial y} = -\frac{\lambda\,\mathrm{tg}\,\alpha}{2\sin\alpha}\left(\frac{d\xi}{dl} + \frac{d\eta}{ds}\right) \tag{2.24}$$

where the relation $\sigma'(\lambda) = -\mathrm{ctg}\,\alpha/\lambda$ is used. By making use of inequalities (2.21), (2.22), and (2.23) and equations (2.24) at point K, the following inequalities holding at this point are finally obtained:

$$\left|\frac{\partial\theta}{\partial x}\right| \leq \left[L(\alpha_{01},R_1) + L(\alpha_{02},R_2)\right] N(\alpha)\sqrt{\operatorname{tg}\alpha}$$

$$\left|\frac{\partial\theta}{\partial y}\right| \leq \left[L(\alpha_{01},R_1) + L(\alpha_{02},R_2)\right] N(\alpha)\sqrt{\operatorname{ctg}\alpha}$$

$$\left|\frac{\partial\lambda}{\partial x}\right| \leq \left[L(\alpha_{01},R_1) + L(\alpha_{02},R_2)\right] N(\alpha)\ \lambda\ \operatorname{tg}\alpha\sqrt{\operatorname{tg}\alpha}$$

$$\left|\frac{\partial\lambda}{\partial y}\right| \leq \left[L(\alpha_{01},R_1) + L(\alpha_{02},R_2)\right] N(\alpha)\lambda\sqrt{\operatorname{tg}\alpha} \qquad (2.25)$$

where

$$N(\alpha) = \left[\frac{\sin^2\alpha}{\sin^2\alpha + 1/2(\chi-1)}\right]^{\frac{1}{2(\chi-1)}} \qquad L(\alpha_0,R) = \frac{N(\alpha_0)}{R}\sqrt{\operatorname{ctg}\alpha_0}$$

The inequalities (2.25) show that the absolute values of the derivatives cannot exceed certain finite values, depending on the Mach number at the given point of the supersonic region and the minimum fixed radius of curvature on the part of the contour that is a boundary of the supersonic region.

The assumption that the cause of the formation of a shock wave is the envelope of the characteristics within the local supersonic region is therefore invalid. Thus, it is possible to formulate

THEOREM 8. - If for certain conditions determining the motion of the gas[2] a potential flow existed with local supersonic region, the formation of a shock wave within or on the boundary of the supersonic region, arising from any change in these conditions, cannot be preceded by the occurrence of infinite accelerations at the interior points of the supersonic region. That is, if the curvature of the part of the contour lying in the supersonic region does not become infinite.

[2]When the flow about the body is infinite, the conditions determining the motion of the gas are the Mach number of the oncoming flow and the shape of contour of the body.

The theorem obtained refers only to the internal points of the supersonic region. From inequalities (2.25), no conclusion can be drawn as to the finiteness of the accelerations on the transition line because the right-hand sides of three of them approach infinity as the transition line is approached. Hence, the question whether the shock waves are preceded by the occurrence of infinite accelerations at the points of the transition line remains open.

3. Criterion of Collapse of Potential Flow

In section II, part 1, it was shown that a deformation of the contour can lead to the collapse of the potential flow. Before the deformation of the curvilinear segment of the contour at its points, the inequality $d\lambda/d(-\theta_k) > -\lambda \operatorname{tg} \alpha$ holds. At the end of the deformation, however, at the points of the obtained straight segment, $d\lambda/d(-\theta_k) = -\infty$, as follows from theorem 6.

Hence, for any intermediate state of the deformed contour there is first attained, at some point of the segment, the equation

$$\frac{d\lambda}{d(-\theta_k)} = -\lambda \operatorname{tg} \alpha \tag{2.26}$$

This state of the deformed contour is, in a certain sense, critical because for further deformation of the contour, inequality (1.13) breaks down and thus the flow becomes impossible without a shock wave.

In the given case condition, equation (2.26) is the criterion of breakdown of the potential flow. Equation (2.26) may also be a criterion for the breakdown of the potential flow in the case where the flow is about a fixed contour but where the Mach number of the oncoming flow increases.

If for a certain Mach number, a potential flow existed with local supersonic region and at the points of the contour, the inequality

$$\frac{d\lambda}{d(-\theta_k)} > -\lambda \operatorname{tg} \alpha$$

were satisfied, then if with increase in the Mach number of the oncoming flow inequality (1.13) starting from a certain Mach number breaks down, this limiting Mach number would correspond to the attaining of equation (2.26) at some point of the contour.

The Mach number of the oncoming flow, for which at some point of the contour condition (2.26) is satisfied, will be denoted as the breakdown Mach number M^*.

The number M^* is thus the limiting Mach number beyond which the body is subject to wave resistance.

The breakdown criterion (2.26) can be conveniently represented in another form, by considering the magnitude

$$\theta_{1*} = \theta_k - \varphi(\lambda_k) \tag{2.27}$$

where $\varphi(\lambda)$ is a known function shown in figure 4. By differentiating this equation along the contour

$$\frac{d\theta_{1*}}{ds} = \left[1 - \varphi'(\lambda_k)\frac{\partial\lambda_k}{\partial\theta_k}\right]\frac{d\theta_k}{ds}$$

But $\varphi'(\lambda_k) = 1/(\lambda_k \operatorname{tg} \alpha_k)$ so that the following equation for the breakdown criterion is obtained equivalent to (2.26):

$$\frac{d\theta_{1*}}{ds} = 0 \tag{2.28}$$

It should be noted that the breakdown of inequality (1.13) at any point of the contour does not lead to the breakdown of the potential character of the flow near this point but makes impossible the ending of the characteristics of the first family from the neighborhood of this point on the transition line as the law of monotonicity would otherwise fail to hold at the transition line. The characteristics of the first family must, in this case, merge in the shock wave.

In order to determine the magnitude M^* of a given contour, it is necessary to know how for an increase in the Mach number of the oncoming flow, the value of the velocity at the points of the contour changes for a potential flow with local supersonic region.

At the present time, certain methods are known for the approximate solution of the problem of the flow about a fixed contour for Mach numbers exceeding M_{cr}. (See, for example, references 4 and 5.)

In all these methods, the convergence of the process of the successive approximations has not been proven; hence, the results of the computations made with the aid of these methods must be regarded with reserve. It seems probable, however, that up to the time when the potential flow is actually possible, these methods at least qualitatively represent the true character of the change in the distribution of the velocity on a fixed contour with increase in the Mach number of the oncoming flow.

It is thus of interest to see whether from these solutions there is a tendency toward the attainment on the contour of the condition (2.28) with increase in Mach number.

From the geometric data of a wall and distribution of the velocity along the wall, obtained in the work of Görtler (reference 5) for a Mach number of the oncoming flow $M = 0.9$, the angles of inclination θ_{1*} were computed using equation (2.27). The pattern of the lines of flow and the lines of equal velocities for this case is shown in figure 26. The dependence of θ_{1*} on the coordinate along the wall ξ is represented in figure 27.

As may be seen, the flow is near the breakdown. The monotonic decrease of the angle proceeds up to $\xi = 5.5$, after which the angle remains almost constant or slightly decreasing.

In figure 28 is shown the dependence of θ_{1*} on x for the flow about the profile, which is considered in an American investigation as $M = 0.75$ and $M = 0.83$. The velocity distribution over the contour and the boundaries of the local supersonic region are shown in figure 29. In figure 28, it is very clearly seen that the tendency toward a breakdown of the monotonicity on the transition line increases with increase in the velocity. At $M = 0.75$, somewhat exceeding M_{cr}, the angles decrease almost as fast as on the contour. For $M = 0.83$, the monotonicity of change of the angle θ_{1*} has already broken down. Hence, at $M < 0.83$ there is already a breakdown of the potential flow near this profile. Strictly speaking, the breakdown for $M = 0.83$ no longer has significance. Thus, in the given solutions, a tendency is revealed to attain on the contour condition (2.28) on increasing the Mach number of the oncoming flow.

4. Infinite Accelerations in Accurate Solutions

In section II, part 2, it is shown that an infinite acceleration on a contour can occur only if the curvature of the contour at any point is infinite.

An exact solution of a certain flow with local supersonic region with the occurrence of infinite acceleration at two symmetric points of the contour was given by Ringleb (reference 6). According to the results of this section, this flow indicates an infinite curvature of the contour at these points. Figure 30 shows the flow pattern and the velocity distribution. At points P and Q of the contour, Ringleb obtained infinite values of the derivative of the velocity, that is, infinite accelerations. It is found that in the plane of the hodograph, at the points corresponding to P and Q in the flow plane, the transform of the streamline and the epicycloid have a common tangent.

In the work of Kármán during 1941 (reference 3), the Ringleb solution is analyzed in detail with the object of showing for this type of flow the possible reasons for the formation of shock waves in the local supersonic regions. Kármán showed that if at the contour in the supersonic region there is a point with infinite acceleration and a value of $\partial\psi/\partial\theta \neq \infty$ (where ψ is the stream function and θ is the angle of inclination of the velocity vector) then, as in the case of Ringleb, the transform of the streamline in the plane of the hodograph touches the epicycloid at a point corresponding to the point with infinite acceleration in the flow plane. Kármán identifies this contact in the hodograph plane with the presence of an infinite acceleration at the corresponding point of the flow plane. He does not, however, observe the fact previously pointed out that the presence of an infinite acceleration at any point of the supersonic flow indicates the presence in it of an infinite curvature of the streamline, and associates the infinite acceleration, in this particular case, with the impossibility of a continuous flow in the general case. This lack of observation is indicated by the fact that with the aid of the previously mentioned condition of the tangency in the plane of the hodograph, Kármán seeks an infinite acceleration on the contour of the NACA-4412 profile, which has no infinite curvatures, making use of the experimental distribution of the velocity on the contour. In the plane of the hodograph, Kármán obtains the point of tangency of the transform of the contour with the epicycloid but corresponding to this point of the contour of the profile, infinite acceleration does not occur as should be the case. No satisfactory explanation of this fact contradicting his results is given.

This noncorrespondence is explained by the fact that the tangency in the plane of the hodograph can occur simultaneously with the condition $\partial\psi/\partial\theta = \infty$ at the point considered on the flow plane. An example of this is the Meyer flow, that is, the flow about a

certain contour for which the entire contour coincides in the plane of the hodograph with the epicycloid but nevertheless, at the points of this contour, the acceleration is finite if the curvature is finite.

Moreover, under the conditions of the problem of the flow with local supersonic region, the tangency in the plane of the hodograph of the transform of the contour with the epicycloid is equivalent to condition (2.26) at the point of the contour corresponding to the point of tangency, because at this point of the contour the acceleration is finite if the curvature of the contour is finite.

In recent times, investigations have appeared on exact solutions of the problem of flow with local supersonic regions where the equations of S. A. Chaplygin, in the plane of the hodograph, are used. Notwithstanding the fact that in these papers examples are given of the existence of potential flows with local supersonic region, the value of these investigations is limited by the fact that with change in the Mach number of the oncoming flow there occurs simultaneously a deformation of the contour of the body. Due to this fact, points with infinite acceleration appear within the supersonic region, which on the basis of the theorem proved in section II, part 2 indicates an infinite curvature of the contour at these points.

From the foregoing considerations, it is evident that it is a mistake to associate the occurrence of such a limiting line of the contour with the impossibility of a potential flow with local supersonic region about a fixed profile.

The investigation of the principal properties of a flow with local supersonic region must, in the opinion of the author, be carried out directly in the flow plane.

In the work described herein, all results were obtained directly from investigations in the flow plane.

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National Advisory Committee
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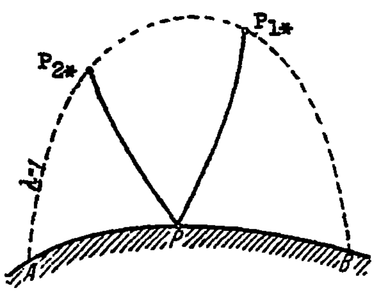


Figure 1.

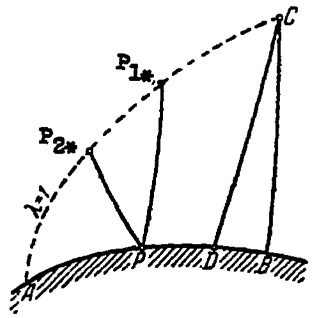


Figure 2.

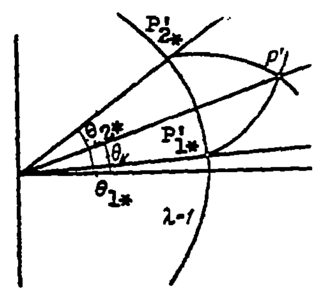


Figure 3.

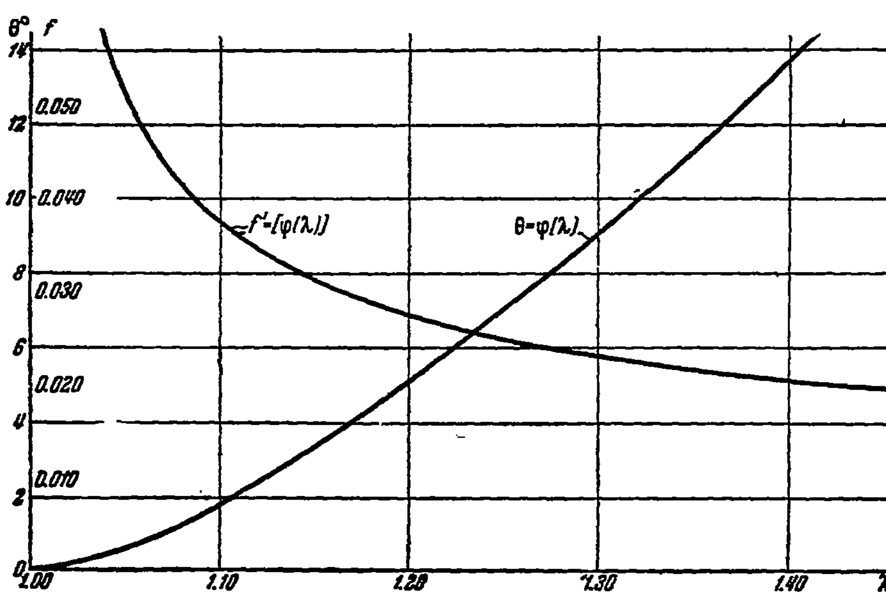


Figure 4.

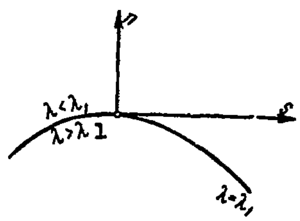
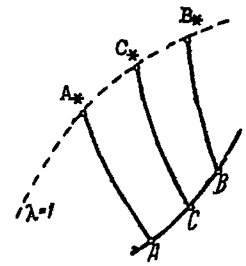


Figure 5.



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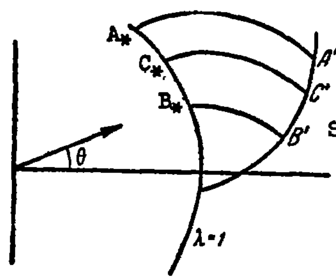

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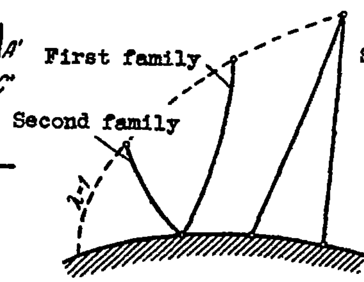

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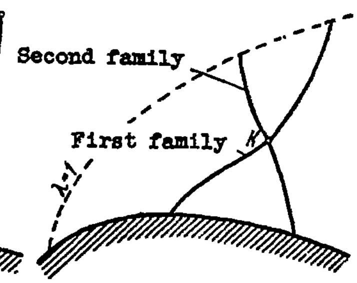

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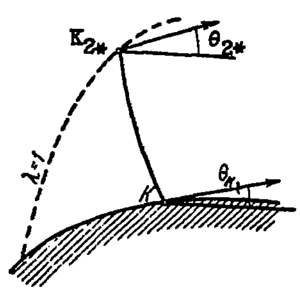

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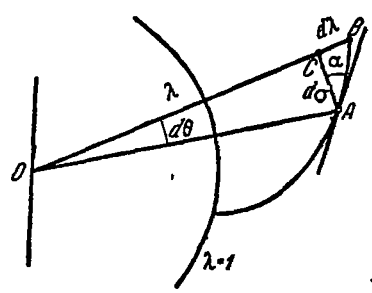

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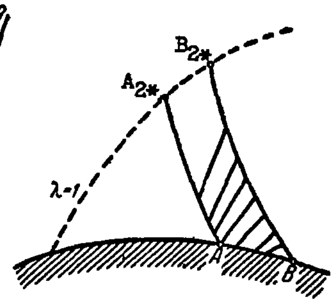

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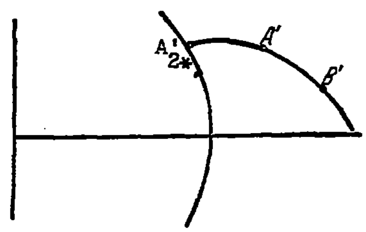

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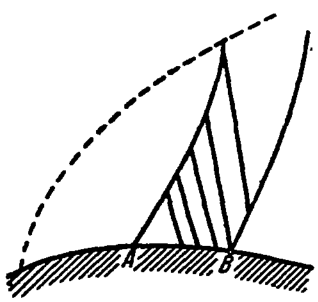

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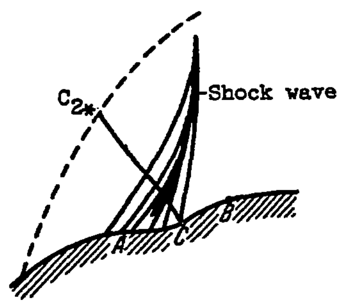

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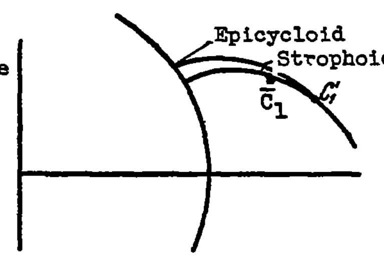

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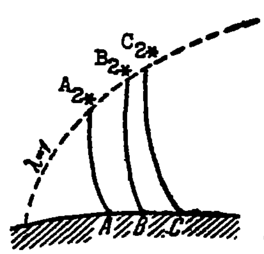

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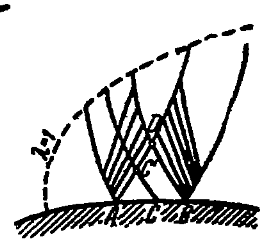

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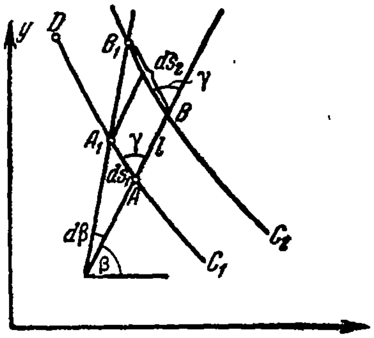

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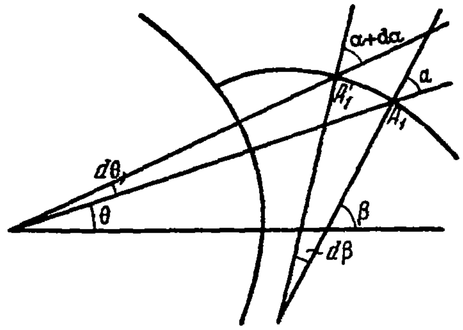

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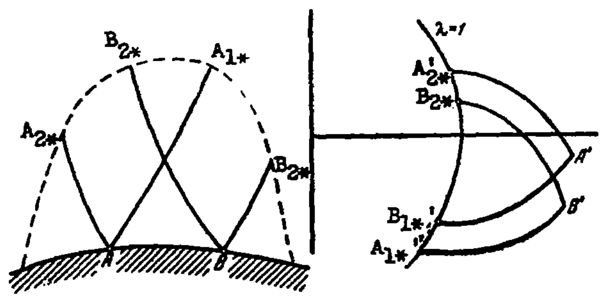
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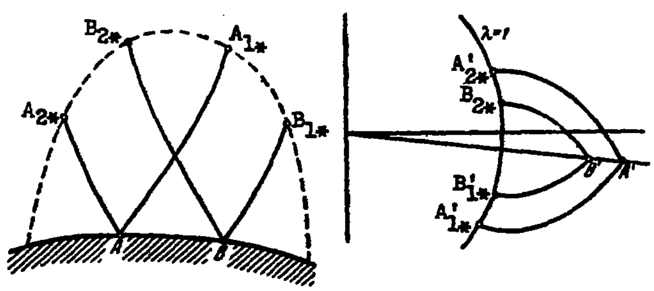
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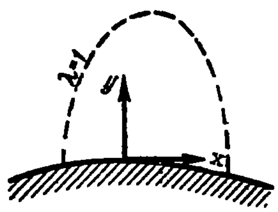

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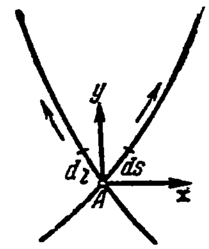

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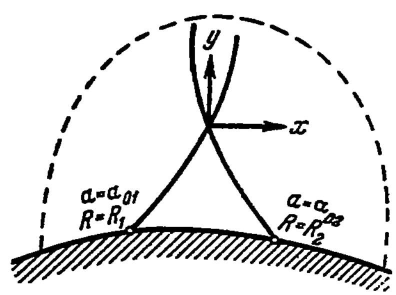

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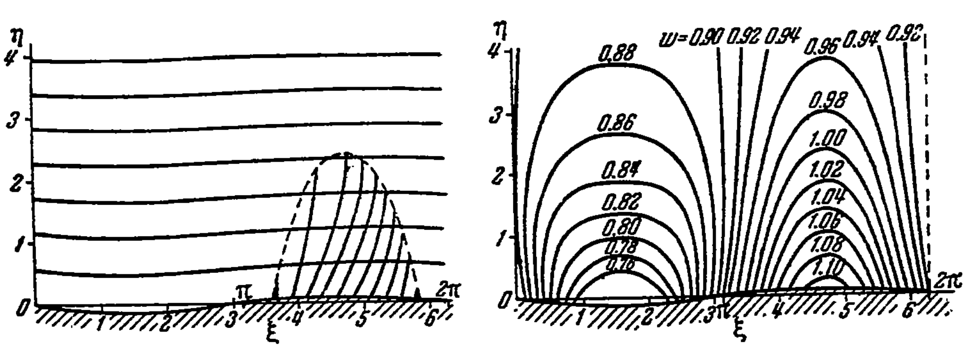

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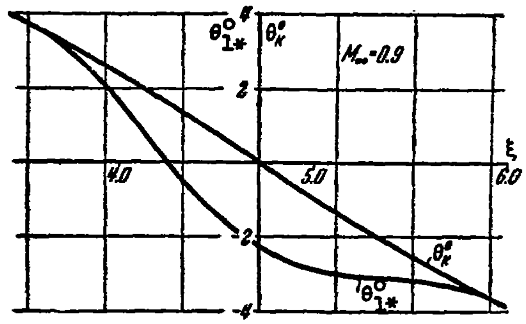

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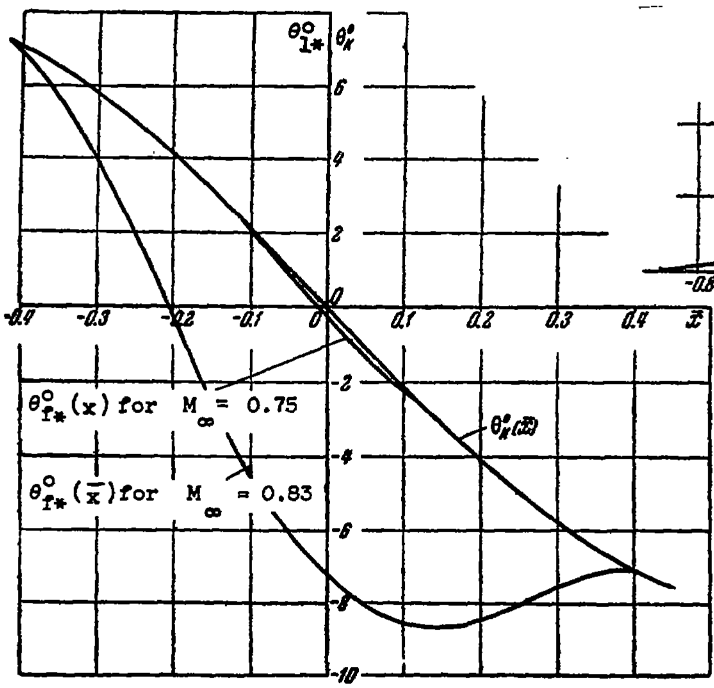

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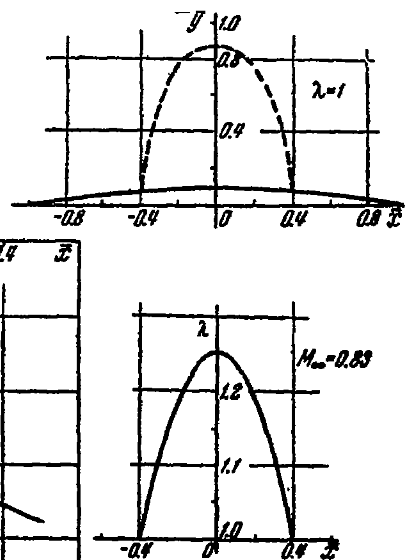

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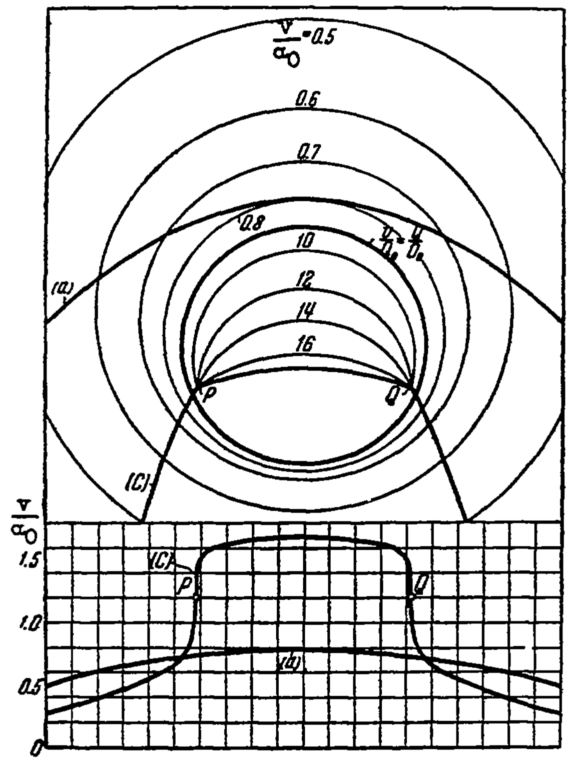

Figure 30.