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SUMMARY

The "general Prandtl number" $Pr' = \frac{A_q}{A} Pr$, aside from the Reynolds number determines the ratio of turbulent to molecular heat transfer, and the temperature distribution in turbulent friction layers. A_q = exchange coefficient for heat; A = exchange coefficient for momentum transfer.

A formula is derived from the equation defining the general Prandtl number which describes the temperature as a function of the velocity. For fully developed thermal boundary layers all questions relating to heat transfer to and from incompressible fluids can be treated in a simple manner if the ratio of the turbulent shear stress to the total stress τ_t/τ in the layers near the wall is known, and if the A_q/A can be regarded as independent of the distance from the wall.

The velocity distribution across a flat smooth channel and deep into the laminar sublayer was measured for isothermal flow to establish the shear stress ratio τ_t/τ and to extend the universal wall friction law. The values of τ_t/τ which resulted from these measurements can be approximately represented by a linear function of the velocity in the laminar-turbulent transition zone.

The effect of the temperature relationship of the material values on the flow near the wall is briefly analyzed. It was found that the velocity at the laminar boundary (in contrast to the thickness of the laminar layer) is approximately independent of the temperature distribution.

The temperature gradient at the wall and the distribution of temperature and heat flow in the turbulent friction layers were calculated on the basis of the data under equations (2) to (4). The derived formulas and the figures reveal the effects of the Prandtl number, the Reynolds number, the exchange quantities and the temperature relationship of the material values.

*"Die Wärmeübertragung in turbulenten Reibungsschichten." Z.f.a.M.M., vol. 20, no. 6, Dec. 1940, pp. 297-328.

That the form of the wall and the pressure drop affect the results is illustrated by the variation of the thermal behavior of the friction layers in the pipe, channel, and flat plate.

After a discussion of the different definitions of the heat transfer coefficient a new formula for the rate of heat transfer is given based on the maximum temperature difference. The new equation differs from that offered by Prandtl by an additional term that allows for the conditions in the laminar-turbulent transition zone.

INTRODUCTION

A survey of the literature on heat transfer in turbulent boundary layers discloses that the problem has been treated in numerous studies (reference 1). Because of its technological importance, the number of experimental projects in which empirical or semi-empirical formulas established for various conditions and for various applications preponderate.

The theoretical principles are but rarely treated. The literature therefore contains only a few general formulae. In Germany the formulas by Nusselt and Prandtl are most generally utilized. In the English literature it is customary to introduce the Reynolds analogy which upon generalization by G. I. Taylor leads to approximately the same results as the Prandtl theory.

The theories to date are based on simplifying assumptions, such as do not usually obtain in reality. The derived expressions therefore required extrapolation based on experimental results, the extension extending beyond the original range of validity. The practical point of view was maintained in arranging the semi-empirical equations and questions of the physical significance became secondary.

The research programs in heat transfer involving many technically important special cases in the turbulent region fail to allow the deduction of a general theory without limitations. The solution of this problem is very closely related to the research of the flow processes in direct proximity of the wall.

Before proceeding to an analysis of these questions a brief survey of the available theoretical contributions should be of interest.

REVIEW OF PREVIOUS CONTRIBUTIONS

The school of Nusselt has made great strides in the study of heat transfer problems by the use of the theory of similarity, particularly in arranging the various subdivisions in reasonable order. The great technical importance of the model studies is that it does not require the exact knowledge of the individual processes, and that simple formulae are obtainable for practical use, even in complicated cases. But since no details of the physical mechanism are secured the results can be of a preliminary nature only.

Reynolds (reference 2) attempted to define the rules of heat transfer from the point-to-point variation of the flow pattern. He proceeds from the assumption that the turbulent mechanism of heat transfer is the same as the mechanism of the momentum transfer. But his considerations are still incomplete for practical application and only through supplementary considerations by Taylor (reference 3) and Stanton (reference 4) were the results of Prandtl accomplished.

Prandtl (reference 5) also starts from the assumption that heat and momentum are transferred by the same mechanism. A complete analogy between these phenomena does not exist, however, unless similar boundary conditions obtain, when the nondimensional $\mu c_p/\lambda$ (termed the Prandtl number, Pr) is equal to unity, and when the pressure drop is negligible (as, for instance, in flow past a flat plate). In contrast the momentum transfer with pressure drop (pipes and channels) is described by equations which differ from those of heat transfer and momentum.

In order to treat the technically important case of flow through a pipe Prandtl postulated fictitious heat sources in the stream, by means of which a sufficient similarity of the equations of heat and momentum transfer is obtained. The Reynolds concept was taken, that is, that in a very thin layer near the wall practically all of the transfer is by molecular action and that outside of this layer only the turbulent exchange mechanism is effective, while the molecular conductivity may be neglected.

The heat source postulate then leads to a simple equation between the heat transfer and the resistance to flow, which may be written in the form:

$$\alpha_m = \frac{q_0}{T_m} = \frac{c_p}{1 + \frac{u_a}{u_m}(Pr - 1)} \times \frac{\tau_0}{u_m} \qquad (1)$$

where

α_m heat transfer coefficient referred to the mean temperature

T_m, c_p specific heat

q_0 density of heat flow at the wall

τ_0 shear stress at the wall

u_m mean flow velocity

u_a velocity at the "boundary" of transition from laminar to turbulent flow

To use equation (1) the ratio u_a/u_m must be known. In the absence of experimental data of the extremely thin wall layer, Prandtl (reference 6) used the following reasoning to evaluate u_a. In the laminar layer a linear velocity increase exists, the slope of which is fixed by the shear at the boundary. In the turbulent core the 1/7th power law holds for Reynolds numbers below 10^6. The plane in which the two velocities coincide is called the boundary between the laminar layer and the turbulent zone. The exact determination of the boundary velocity u_a is to follow from the heat measurements.

The heat transfer data available to date indicate that Prandtl's formula does not hold for large Prandtl numbers. In consequence there have been proposed various corrections to this formula in order to meet the requirements of practice.

The basis of the discrepancies lies in the idealization of the transition from laminar to turbulent flow. This transition is naturally continuous, hence an intermediate layer exists in which the viscous and turbulent shear stresses are of the same order of magnitude. Since the transition to turbulence occurs close to the wall, it has not been possible so far to measure the velocity distribution in the intermediate layer with sufficient accuracy. Von Kármán (reference 7) has estimated the exchange conditions in the transitional region based on an extrapolation of Nikuradse's velocity measurements in the direction of the wall. Based on his postulates, von Kármán gives the formula for the heat transfer coefficient as:

$$\frac{\alpha_1}{\alpha} = 1 + \alpha\sqrt{\alpha_1}\ [(Pr-1) + b \ln (1+c\ (Pr-1))] \qquad (2)$$

where

α_1 heat transfer coefficient for $Pr - 1$

a, b, c, constants (reference 8)

An improvement of the theory has been carried out by Taylor (reference 9). Starting from the postulates of Reynolds-Taylor, the latter discusses the error of the analogy between heat transfer and momentum transfer for flow accompanied by pressure drop. Taylor calculates the temperature profile which corresponds to a velocity profile measured by Stanton at $Pr = 1$. The temperature profile differs somewhat from the velocity profile, that is, the temperature gradient at the wall (and hence also the heat transfer coefficient) is lower by several percent than the wall velocity gradient.

Of great practical interest is the variation of the heat transfer coefficient for non-isothermal flow in which the material values vary with temperature. Apparently this problem has not yet been solved analytically. The theories to date imply isothermal flow (material properties not a function of temperature or space). Since large temperature differences do occur in practice, the proper mean magnitudes of the material properties are introduced into the isothermal expressions (reference 10).

While the present report was in the press, two further articles dealing with turbulent heat transfer have appeared, one by Mattioli (reference 10a) and the other by Hofmann (reference 10b).

Starting from special theoretical concepts with respect to the turbulent mechanism, Mattioli extrapolates the turbulent velocity distribution into the semi-laminar zone in order to deduce from this velocity concept the presumption of equal exchange quantities for heat transfer and momentum transfer the magnitude of the turbulent heat transfer. A careful analysis of the difficult derivation shows that the important phenomena near the boundary are not adequately defined. In addition to the semi-laminar layer there is presumed to exist a wall layer (which is established from the heat transfer measurements of Bühne and from other fluid flow measurements mentioned above) which is much greater in thickness than the laminar layer. Mattioli is therefore forced to assume a substantial exchange in his wall layer. Since the Mattioli theory connot describe accurately this exchange near the wall, the temperature change in the wall layer is put proportional to $(Pr)^m$, where m is established from heat transfer measurements.

It is worth noting, however, that Mattioli quantitatively allows for effect of temperature on the viscosity. For this purpose a generalized distance parameter is introduced in a manner similar to that employed in the present report (see equation (30)).

Hofmann calculates the temperature distribution and the heat transfer coefficient with special consideration of the laminar layer whereby the usual simplifying postulates are retained. The concept of a thermal boundary between the turbulent cord and a boundary layer is also adopted and the thickness of this layer is discussed. In contrast with von Kármán, progress is made in that the laminar layer thickness for high Prandtl numbers is introduced. The arbitrarily chosen velocity distribution near the wall lies above the test points of the present report.

The position taken by Hofmann that the heat transfer depends solely on the velocity distribution and on the Prandtl mixing length requires a correction. Basic to every theory is a hypothesis of the turbulent diffusion of heat. If the ratio of the exchange quantities for heat and momentum transfer is chosen (Hofmann tacitly presumes the identity of these quantities), then the laws of heat transfer follow at once direct from the velocity profile without the aid of any turbulence theory, hence without the help of the Prandtl mixing length, which in consequence drops out again in the course of the Hofmann calculation.

THE PROBLEMS

In order to avoid subsequent corrections and to present the hydrodynamic theory of the turbulent heat transfer coherently the following assignments are to be solved:

1. To derive a general equation for heat transfer into which the technologically important boundary conditions and the flow phenomena, particularly in the transitional layer, can be introduced.

2. To measure the flow processes near the wall for technologically important cases, particularly smooth surfaces, rough surfaces, almost isothermal flow, non-isothermal flow, and so forth.

3. To introduce the obtained data on wall adjacent flow into the general expression to build special formulae which can be checked by heat transfer measurements.

The following statements are made relative to these problems:

The presentation of the general theory should be clear from a physical point of view and it should be simple in order that it may be utilized in practice.

The derivation of a generally applicable equation for heat transfer is carried through in a simple manner. In contrast the measurement of the flow distribution near the wall presents considerable difficulties. In order to obtain practical test data especially thick boundary layers are essential. This requirement implies large flow sections and low flow velocities, that is, low dynamic pressures and low pressure drops must be measured.

The conditions become complicated if the flow is not isothermal. Through the influence of the temperature field, not only the material properties but also the flow phenomena are changed.

The presence of roughness introduces further complications. It is true that flow on rough walls has been extensively studied and the laws of the "nuclear flow" in pipes are well known but there is no dependable knowledge of the flow processes near the wall between the protuberances.

The experimental exploration of the flow distribution near the wall is a broad field of research which can only be accomplished piecemeal. The author first explored the data available near the wall. While these studies are not complete, they have progressed far enough to enable a theoretical treatment of the heat transfer at a smooth wall.

A particularly important sub-task consists in checking the applicability of the theoretical formulae by means of heat transfer measurements as the theory contains postulates relative to the mechanism of heat transfer which require confirmation by experience. If necessary the theoretical assumptions must be modified to fit the experimental facts. The heat transfer measurements can be employed with great benefit to clarify the questions of turbulence structure.

THE PRANDTL NUMBER

The hydrodynamic equation for the continuity of heat flow (equation (47)) is not sufficient for predicting the temperature distribution in the friction layers. It requires another equation for the temperature which takes into account the requirements of the system under consideration. (This temperature equation, looked for, places the continuity of heat flow equation (equation (47)) in the position of a special condition that must always be satisfied.).

The Prandtl number $Pr = \frac{\mu c_p}{\lambda}$ governs the form of the temperature profile. It is logical, therefore, to begin with the Prandtl number concept. To secure a differential equation necessitates a determinating equation for Pr that holds for each point in the fluid. Since the individual factors in Pr have "point" significance, the derivation of such an equation is possible.

Let q equal the density of the heat flow, and τ the shearing stress of the density of momentum transfer. Assume that the heat flow and momentum flow act in the same direction ($\pm y$) at a given point, which is perpendicular to the mean velocity u (time average) at this point. In the system under consideration y is measured perpendicular to the wall and u parallel to it.

The total momentum τ consists of a portion τ_m by the molecular transfer and a portion τ_t by the turbulent exchange motion. The same holds true for the heat flow. Hence

$$\tau = \tau_m + \tau_t \tag{3}$$

$$q = q_m + q_t \tag{4}$$

With

μ coefficient of viscosity

λ thermal conductivity

A exchange coefficient for momentum

A_q corresponding coefficient for heat

c_p specific heat

u, T time averages of velocity and temperature, respectively

u', v' velocity fluctuations in the x, y, directions, respectively

T' corresponding temperature fluctuation

$$\tau_m = \mu \frac{du}{dy} \tag{5}$$

$$\tau_t = A \frac{du}{dy} = - \rho \overline{u' v'} \tag{6}$$

$$q_m = \lambda \frac{dT}{dy} \tag{7}$$

$$q_t = c_p A_q \frac{dT}{dy} = - c_p \rho \overline{T' v'} \tag{8}$$

The coefficients μ, λ, A, A_q, are defined by these equations. There presentation of τ_t and q_t in terms of the fluctuating components is for the present irrelevant, but will be clarified in Chapter 9.

Equations (5) to (8) then yield the following proportion:

$$\frac{q_t}{q_m} = \frac{A_q}{A} Pr \frac{\tau_t}{\tau_m} = Pr' \frac{\tau_t}{\tau_m} \tag{9}$$

Accordingly the ratio of turbulent to molecular heat flow is proportional to the ratio of turbulent to molecular shear stress. The proportionality factor is $Pr' = (A_q/A)Pr$, a quotient which is called "general Prandtl number."

Equation (9) thus leads to an extension of the concept of the Prandtl number for turbulent flow with $P' = A_q/A \ Pr$ instead of Pr. Only in the case where the exchange coefficients are the same for momentum transfer and for heat transfer will Pr and Pr' be equal.

Since equation (9) refers to flow in which turbulent and molecular shearing stresses act, it is particularly suitable for the representation of the physical phenomena in the transitional layer. The treatment of the heat transfer in the present report therefore starts from the transitional flow, the "fully turbulent" core and the laminar motion at the wall being treated as special cases. (In the proximity of the wall the exchange mechanism perpendicular to the wall is not possible; therefore the turbulent friction disappears and the momentum transfer is accomplished by internal friction only. Because of the turbulent pressure fluctuations, the stream velocity near the wall also experiences fluctuations. The continuity of this fluid flow is largely maintained by the lateral transverse fluctuations, so that the wall flow glides practically parallel to the surface. In this sense the viscous wall flow is "laminar.")

A picture of the physical significance of the Prandtl number is best obtained by observation of the transitional layer for extremely high values of Pr' (very viscous fluids). In this case practically only turbulent heat transfer exists ($q_t >> q_m$) at those places in the transition region where only small turbulence exists ($\tau_t << \tau_m$). In this extreme case the molecular heat transfer is so small, that even a slight convection signifies a form of "short circuit" for the heat flow. Therefore the temperature profiles for high Prandtl Numbers are "smoothed."

Even for the special case of $Pr' = 1$ and q/τ = constant, the temperature profile can be fixed readily. It is

$\frac{q_t}{q_m} = \frac{\tau_t}{\tau_m}$ or $\frac{q}{\tau} = \frac{q_m}{\tau_m} \approx \frac{dT}{du}$. In this case the profiles of the temperature and velocity agree with each other. (The condition that $q \approx \tau$ is well satisfied in the friction layer of a flat plate.)

THE GENERAL TEMPERATURE EQUATION

The temperature distribution follows from the equation of the molecular heat stream $q_m = \lambda \frac{dT}{dy}$; q_t must, therefore, be replaced by $(q - q_m)$ in equation (9). Introducing the ratio q/τ equations (5) and (7) then gives:

$$\frac{q_m}{\tau_m} = \frac{\lambda}{\mu}\frac{dT}{du} = \frac{q/\tau}{1 + (Pr' - 1)\tau_t/\tau} \qquad (10)$$

The boundary condition at the wall is to be introduced in this general equation. That is,

$$\left(\frac{dT}{du}\right)_o = \frac{\mu_o\, q_o}{\lambda_o\, \tau_o} \qquad (11)$$

and when augmented by (11), equation (10) is integrated to:

$$T - T_o = \left(\frac{dT}{du}\right)_o \int_o^u \frac{\dfrac{\mu\lambda_o}{\mu_o\lambda}\,\dfrac{q\,\tau_o}{q_o\tau}}{1 + \left(\dfrac{A_q}{A}\,Pr - 1\right)\dfrac{\tau_t}{\tau}}\, d\,u \qquad (12)$$

The factor $\left(\frac{dT}{du}\right)_o$ is determined by extension of the integral over the total velocity field of the friction layer.

The temperature-velocity quotient $\left(\frac{dT}{du}\right)_o = \left(\frac{dT}{dy}\Big/\frac{du}{dy}\right)_o$ is a measure for the amount of heat transferred to the wall. The heat transfer at the wall is obtained from the temperature distribution.

Equation (12), although designed to calculate the temperature distribution, has general application. The considerations so far are based solely on known definitions properly rearranged and combined and no special assumptions relative to the flow have entered the computations except the boundary condition of a laminar wall layer.

The above derivation indicates that a general result can be secured without employing the hydrodynamic equations (46) and (47)). This is due to the fact that the basis of each theory, entirely independent of the method of calculation, is a postulate related to the exchange mechanism. (For instance T' is set proportional to u', or more generally $A_q = A$, or as in the case in point, A_q/A is to be determined later.) This simple hypothetical content of the theory is seen also from the presentation of the simple Prandtl analogy for the present subject is treated in such a manner as to make this step possible.

To complete the temperature equation (12) the magnitudes A_q/A, $q\tau_0/q_0\tau$ and τ_t/τ must be known. (These quotients are introduced later in order that the effect of each postulate may be observed independently. Also the various deviations between the theory and experiment reveal at a glance the direction in which the assumptions must be modified.)

In order to carry through a calculation A_q/A is assummed to be constant. The value of A_q/A is to be determined from experimental data.

The quotient $q\tau_0/q_0\tau$ cannot be fixed arbitrarily. The heat stream q is related to the temperature T through the differential continuity equation (see equation (47) generalized Fourier-Poisson) of heat flow. But a first approximation of the temperature distribution can be obtained by assuming that the layer for heat transfer is of about the same thickness as the friction layer.

In this case the heat flow disappears where the shear stress is zero, while on the wall $q/q_0 = 1$ and $\tau/\tau_0 = 1$. Thus the total range of the friction layer can be expressed with

$$\frac{q\tau_0}{q_0\tau} = 1 + k \tag{13}$$

where k is small compared to 1 at least in proximity of the wall.

In the turbulent friction layers the velocity gradient is steep near the wall. The largest part of the velocity region u lies in a zone where k is small. So for the integration T over u of equation (12), $(q\tau_0/q_0\tau) \sim 1$

may be put in first approximation. (In the entrance zones where the wall temperature changes suddenly this approximation is not possible. For such cases the heat boundary layer is much thinner than the friction layer and it therefore plays an important role in the variation in heat flow. Thermal entrance lengths in existing friction layers are quite short however (see Latzko, Z.a.M.M., Bd. 1 (1921) p. 268), so that when assuming $(\mu \lambda_0/\mu_0 \lambda) \sim 1$ and, τ_t/τ is known, the integration can be completed.

This procedure yields a first approximation of the temperature distribution by means of which the heat flow can be evaluated. The heat flow distribution then affords a second approximation for the temperature distribution which is practically adequate for the case of constant material properties.

Several quantitative conclusions can be drawn from equation (12) relative to the temperature profile of various friction layers which coincide approximately with the stress quotient (τ_t/τ) (such as, for example, the flow through a pipe, channel or flat plate) at equal Reynolds numbers, where the velocity distribution obeys the "universal law").

At the flat plate $\left(\frac{dq}{dy}\right)_0 = 0$ and likewise $\left(\frac{d\tau}{dy}\right)_0 = 0$.

The assumption $k \sim 0$ is therefore well satisfied over the greater part of the velocity field of the flat plate. No appreciable differences obtained here between the first and second approximation and the final solution of T. (Even though τ and q are very similar at the plate, they are not coincident, for q depends on Pr' while τ does not. Therefore there will exist for the plate, a small difference between the actual temperature profile and the first approximation of T.)

For flow with pressure drop a far from negligible difference exists between the second and first approximations (that is, between the actual profile and that of the "plate profile" of the temperature.) By pressure drop $(d\tau/dy)_0 < 0$, but at the flat channel wall $(dq/dy)_0 = 0$ and in the pipe $(dq/dy)_0 > 0$. Hence it follows for the temperature distributions that the pipe profile differs more from the plate profile than the channel profile, that is, according to equation (12), the temperature rise at the pipe wall is flatter than at the channel wall and even more so than at the flat plate (see fig. 4). For a quantitative treatment of model problems it is advisable to integrate the temperature equations by sections, that is, the laminar section, the transitional region and the real turbulent layer. The boundary at the end of the laminar zone is designated by the subscript a, and the beginning of the turbulent layer by b_o, whence, after introducing the substitution equation (13), equation (12) gives

$$\tau\left(\frac{du}{dT}\right)_o = \int_o \frac{\mu\lambda_o}{\mu_o\lambda}\, du \qquad (0 < T < T_\alpha) \quad \text{(12a)}$$

$$\tau\left(\frac{du}{dT}\right)_o = \int_o^a + \int_a \frac{\frac{\mu\lambda_o}{\mu_o\lambda}(1 + k)}{1 + (Pr' - 1)\,\frac{\tau_t}{\tau}}\, d\,u \qquad (T_a < T < T_{bo}) \quad \text{(12b)}$$

$$\tau\left(\frac{du}{dT}\right)_o = \int_o^a + \int_a^{b_o} + \frac{\left(\frac{A\ c_{p_o}}{A_q\ c_p}\right)_t \left(u - u_{b_o} + \int_{b_o} k\, du\right)}{Pr_o\left(1 - \left(1 - \frac{1}{Pr'_t}\right)\left(\frac{\tau_m}{\tau}\right)_t\right)} \qquad (T_{bo} < T < \Theta) \quad \text{(12c)}$$

(Θ = max. temp. difference between the wall and the flowing fluid.) In addition it should be observed that k may be disregarded for the laminar region. A general disregard of k in the main fluid stream is not tenable. The subscript t indicates a mean value for the turbulent region (formed over u).

In the actual turbulent region it is to be noted that for small values of Reynolds numbers τ_t/τ is considerably smaller than unit (see fig. 2), correspondingly τ_m/τ is not negligible. However to an approximation $(\tau_t/\tau) = (\tau_t/\tau)_t$ = constant. The point where $\tau_t/\tau = (\tau_t/\tau)_t$ is the turbulent boundary designated with b_o.

To utilize the temperature equations the variation of (τ_t/τ) and the boundary velocities u_a, u_b (and correspondingly u_{b_o}) must be known. This involves the flow distribution near the walls, with limitation to the processes at the smooth wall and to flows obeying the universal velocity distribution equation.

VELOCITY DISTRIBUTION AT A SMOOTH WALL

The measurements by Nikuradse (reference 11) have shown that the turbulent velocity distribution can be approximately represented by the following equation:

$$\frac{u}{u^*} = 5.75 \ln y^* + B \tag{14}$$

where the dimensionless shearing stress velocity is defined by

$$u^* = \sqrt{\frac{\tau_o}{\rho}} \tag{15}$$

and the dimensionless wall distance by

$$y^* = \frac{u^* y}{\nu} \tag{16}$$

The constant B depends on conditions at the wall. For smooth walls B is approximately 5.5. Equation (14) is a straight line on semi-logarithmic paper as shown in figure 1.

The velocity distribution for the laminar wall layer can also be represented by means of u/u^* and y^*. Rearrangement of the Poiseuille equation results in

$$\frac{u}{u^*} = y^*\left(1 - \frac{\eta}{2}\right) \tag{17}$$

where $\eta = y/r$ and r is the radius of the pipe or channel for the equation in the sub-layer. In general the laminar layer is so thin that $\eta/2$ may be neglected compared to unity and therefore practically

$$\frac{u}{u^*} = y^* \tag{17a}$$

Equation (17a) is therefore the universal equation for the velocity distribution in the laminar zone. It is shown in figure 1 as the curve which passes through the point $\log y^* = 1$, $u/u^* = 10$.

The flow conditions in the transitional layer are not very well established experimentally. This sublayer adjacent to the wall is usually so thin that accurate measurements of the velocity can hardly be made. The closest wall proximity was probably reached by Stanton with his surface tube (reference 12). But even these test data are insufficient for the present arguments.

Since the application of our theory is predicted on the knowledge of the shear stress ratio (τ_t/τ) in wall proximity, a wall layer of such thickness was required as to render a measurement of the wall flow possible.

The thickness of the laminar layer y_a and the boundary velocity u_a are fixed by definite values of u/u^* and y^*; y_a increases with decreasing u^* according to equation (16). The reduction of u^* is limited by the fact that at too low shear forces the critical Reynolds number is undercut and so the entire flow becomes laminar. It is therefore appropriate to introduce the Reynolds number $Re = u_m d/\nu$ in the place of u^*. Then the thickness of the wall layer is

$$y_a = \frac{y_a^* d}{Re \sqrt{\zeta/8}} \tag{18}$$

where the so-called resistance coefficient ζ is defined in the usual manner as: $\dfrac{\Delta p}{\Delta l} = \zeta \dfrac{\rho u^2_m}{2\ d}$

$$\frac{\tau_o}{\rho \cdot u_m^2} = \frac{\zeta}{8} \tag{19}$$

(ζ decreases slightly with Re).

The thickness of the wall layer grows with the diameter d of the pipe or channel and decreases with the Reynolds number. For a given Reynolds number y_a/d is independent of the choice of flowing medium.

The important number y_a^*, the exact value of which is not yet known, lies below 10 according to available measurements. The critical Reynolds number is 3000, and the corresponding $\zeta \sim 0.04$. Herewith

$$y_a < \frac{d}{20} \qquad (20)$$

This equation reveals that even for the lowest possible Reynolds number the stream diameter must be fairly great in order that a probe can be introduced into the laminar wall layer. (With considerations to the influence of the wall in the probe, the wall layer should be as thick as possible.)

But the achievement of a sufficient boundary layer thickness by increasing the stream diameter introduces fundamental difficulties. If the increase in diameter is to achieve the purpose desired, the Reynolds number may not be increased (see equation 18). This means that the velocity must be decreased in the same proportion as the diameter is increased. As a result the dynamic pressure and the pressure drop are reduced quadratically with the stream diameter, that is, with the boundary layer thickness.

This is exemplified at the dynamic pressure of the mean velocity u_m, for which introducing $Re = du_m/\nu$, we get

$$\frac{\rho}{2} u_m^2 = \frac{\mu^2 \ Re^2}{2 \ \rho \ d^2} \qquad (21)$$

To fix the order of magnitude of this dynamic pressure several numerical values are inserted. Let $Re \sim 3000$, $d = 25$ centimeter (utilized in measurements reported later in this paper). For air as the flowing medium ($\mu = 1.8 \times 10^{-4}$, $\rho = 1.2 \times 10^{-3}$) it is

$$\frac{\rho}{2} u_m^2 \sim 2 \times 10^{-3} \text{ mm } H_2O$$

The pressure drop to be measured is of the same order of magnitudes and is equal to $\rho\ u_m^2$ at $Re = 3000$ for $\Delta l = 100\ d$. For flowing water these pressures are about four times larger and only when utilizing a viscous oil does the magnitude become equal to 1 millimeter of water. If these pressures are to be measured to within 1 percent then the sensitivity of the manometer must be in the range 10^{-2} to 10^{-5} millimeters of water.

The problem of precise measurement of the flow phenomena close to the wall for non-isothermal flow involves the technical difficulty of measuring extremely small pressure differences. (The velocities can be determined without the use of pressure measurements. In the boundary layer itself a hot wire anemometer or a thread anemometer can be used in place of a pitot tube. These devices must be calibrated and the calibration at best depends on pressure measuring devices. In addition pressure drop measurements are desired to check the effective shear stress.) Such measurements can be made with the micromanometer designed by the author which has an upper limit of sensitivity of 10^{-6} millimeters of water (reference 13).

The turbulent flow measurements reported here were made in a rectangular channel 25 centimeters high, 1 millimeter wide and 16 millimeters long and with a maximum velocity of 80 centimeters per second. Fine pitot tubes and hot wires were utilized. The hot wire anemometer was calibrated in the parabolic distribution of a 3 centimeter high × 30 centimeter wide laminar channel in which at similar distances from the wall, the same τ_0 obtained as in the turbulent channel. τ_0 was evaluated from the maximum velocity as well as from the pressure drop.

The measurements were made very difficult because the low velocities were easily disturbed by external causes. For instance, small temperature differences between the air stream and the wall (induced by unavoidable fluctuations in room temperature) caused observable changes in the velocity distribution. Therefore the turbulent velocity profile was almost always slightly unsymmetrical and thus u^* was different on the upper and lower wall.

On the top of that the recorded pressure drops yielded an average u^* which was too great as compared with the results of other authors. The channel flow obviously was not completely developed (in a tube the length of 64 diameters would have been sufficient.) But since the pressure drop was held constant for all test points, it was possible to determine the mean u^* by comparison with indisputable measurements of other authors at higher values of y^*. For this purpose measurements of Nikuradse were utilized omitting those for which the wall correction was questionable. (Similarly the measurements of the Stockholm report which do not lie in the range of others of Nikuradse's measurements and are obviously too high, have been omitted.)

The results of these measurements near the boundary are shown in figure 1. The u/u^* points approach the laminar curve very gradually. It is reached at approximately $u/u^* = 1.5$, a value which is substantially lower than that usually assessed.

The value $u_a/u^* = 1.5$ is however still uncertain and it must finally be based on much more accurate measurements.

It is also true, that an accurate determination of the limit where $du/dy \sim (du/dy)_0$ is not possible from velocity measurements. For this purpose heat transfer measurements at high Prandtl numbers will serve better to determine the laminar boundary. From heat transfer measurements by Bühne it would appear that u_a/u^* is somewhat larger than 2.

The recorded velocity distribution in the transitional region can be approximated at:

$$\ln\left(\frac{b-u/u^*}{b-a}\right) = \frac{y_a^*\left(1-\frac{\eta_a}{2}\right)-y^*\left(1-\frac{\eta}{2}\right)}{b-a} \sim \frac{y_a^*-y^*}{b-a} \tag{22}$$

where

$$a = \frac{u_a}{u^*} \qquad b = \frac{u_b}{u^*} \quad \text{and}$$

u_a is the velocity at the laminar boundary and u_b a suitably selected velocity at the turbulence boundary.

As before $\eta/2$ can be neglected compared to 1 and equation (22) then becomes a universal law. In figure 1 equation (22) is presented for $a = 2$, $b = 15$. The curve is dashed above u/u^* 15, where it loses its physical significance. The measurements are satisfactorily represented by this equation.

It remains to be explained why the velocity distribution in the transitional zone was approximated by equation (22) although some other similar function had been possible.

The ratio τ_t/τ is required. To fix the ratio, differentiate equation (22):

$$\frac{d\left(\frac{u}{u^*}\right)}{(1 - \eta)\, dy^*} = \frac{b - u/u^*}{b - a} \tag{23}$$

where

$$\frac{d\,\frac{u}{u^*}}{dy^*} = \frac{\mu}{\rho u^{*2}}\,\frac{du}{dy} = \frac{\tau_m}{\tau_o} \tag{24}$$

The total shear stress for developed flow with pressure drop is

$$\tau = \tau_o\,(1 - \eta) \tag{25}$$

(near the wall one may set $\tau \sim \tau_o$). Then, solving, one obtains:

$$\frac{\tau_m}{\tau} = \frac{u_b - u}{u_b - u_a} \tag{26}$$

$$\frac{\tau_t}{\tau} = \frac{u - u_a}{u_b - u_a} \tag{27}$$

Since equation (22) is confirmed quite well by the measurements for $u_a < u < u_b$ the true variation of τ_t should not differ much from equation (27). An uncertaintly exists, of course, at the limits a and b.

There follows from equation (27) by introducing the often used ratio $\varphi = u/U$

$$\frac{\tau_t}{\tau} = \frac{\frac{U}{u^*} \varphi - a}{b - a} \tag{27a}$$

where U is the maximum velocity at the edge of the friction layer. U/u^* is, in accordance with equation (14) a function of r^* (that is, the value of y^* at the border of the friction layer based on the distance r from the walk). The relation between Re and r^* is given by the identity.

$$\frac{U}{2\, u_m} R_e = \frac{U}{u^*} r^* = \frac{U}{u^*} \frac{r u^*}{\nu} \tag{28}$$

where

u_m mean velocity

$\frac{u_m}{U} = \varphi_m$ a function of Re

Figure 2 shows τ_t/τ for different values of φ with Reynolds numbers as the parameter as calculated from equation (27a) for the transitional layer and by equation (14) for the turbulent region. The constants a and b were chosen at 2 and 15.5, respectively.

The actual τ_t/τ distribution no doubt differs from that shown in figure 2 for small values of φ. But the difference between the velocity distribution as expressed by equation (22) and the laminar curve is less than the scatter of the experimental points (see fig. 1) so that nothing certain may be said relative to the actual τ_t/τ variation near the laminar boundary nor of the laminar boundary itself.

Details of the variation of τ_t/τ play at first no part. In contrast with earlier work in which the friction layer was divided into two regions in which τ_t/τ varied from 0 to 1, it should for the first suffice to approximately describe the processes in the transitional zone.

The earlier division of the friction layer into a laminar and a turbulent region is indicated by two vertical lines in figure 2. The dotted line represents $Re = 4 \times 10^4$ and $a = 2$ and the dot dashed line represents $Re = 4 \times 10^4$ but $a = 8.8$, which is the value chosen by Prandtl in 1928. τ_t/τ was defined as zero up to $a = 2$ (or 8.8) and unity for greater values of y^*.) At high Prandtl numbers where the transitional layer can be regarded as part of the turbulent zone core with respect to heat transfer (see equation (9)) $a = 2$ is in good agreement while $a = 8.8$ results in a heat transfer rate which is too small.

THE EFFECT OF TEMPERATURE RELATIONSHIP OF THE MATERIAL VALUES ON THE FLOW PHENOMENA

If the material values are functions of the temperature then the flow distribution across a section will be changed as mentioned above and also reduction in temperatures in the direction of flow causes hydrodynamic changes for all fluids which are compressible. In this instance, in principle at least, there exist no velocity profiles which are similar, the same statement holds for the temperature profiles both considered as a function of length.

Since the magnitudes of the temperature differences and the differences in the temperature coefficients of each property enter into the evaluation of the profiles, a general solution of the problem is hardly possible and the study restricted to the simple case of similar temperature profiles which are practically achieved at relatively low differences.

If the viscosity of an isothermal friction layer is changed from ν_1 to ν_2 and if the remainder of the variables, particularly u^* do not change, then equation (14) reveals a parallel displacement of the turbulent velocity profile (see fig. 3a) with a velocity difference of

$$\Delta u = 5.75\ u^* \ln \frac{\nu_2}{\nu_1} \qquad (29)$$

From this it follows that the viscosity has practically no influence in the fully turbulent region, but affects solely the boundary velocity near the wall. (The turbulent flow slides at the wall at a higher or lower velocity equal in magnitude to Δu.)

If the viscosity in the turbulent core of the isothermal flow plays no part its influence for non-isothermal flow is limited to the effect due to its variation. Viscosity variations in the turbulent core are not great, for the temperature variations are not great. We may, therefore, generalize the laws established for isothermal flow by omitting the effect of viscosity in the turbulent core and by replacing the isothermal viscosity ν in equation (14) by a suitably defined laminar layer viscosity ν_l.

Recently the resistance measurements of Rohonczi (reference 14) for non-isothermal flow of hot water being cooled in a tube were published. The measurements could not be adequately correlated if the friction factors were plotted against Reynolds numbers in which the viscosity is evaluated at the mean fluid temperature. In contrast the correlation is satisfactory if the viscosity ν_0 is evaluated at the wall temperature.

The rest of the discrepancies can be eliminated if a viscosity slightly less than that corresponding to the wall is employed in the Reynolds numbers. As far as the author could determine the results of Rohonczi can be satisfactorily correlated and are in agreement with those of Blasius-Nikuradse if the Reynolds number is referred to the mean laminar layer viscosity ν_l and a is put equal to 2. (Rohonczi chose ν_0 as the correct viscosity due to an error in conclusion from similarity reasoning in which the differential equation for isothermal flow was applied to non-isothermal flow. In addition the ν_l values of Rohonczi do not achieve coincidence of the isothermal and non-isothermal results. Up to this time the thickness of the laminar sub-layer was chosen too thick, resulting in a sublayer temperature which was too high and a value of νl which is too low. Only for one set of data at high Reynolds number will the results yield to adequate correlation.)

Consider next the influence of a uniform viscosity variation on the flow conditions near the wall. If the friction velocity u^* is not changed, then the boundary velocities $u_a = a\, u^*$ $u_b = b\, u^*$ are maintained since a and b are universal constants. That is, only the layer thicknesses change to

$$y_a = \frac{\nu\, y_a{}^*}{u^*} \qquad y_b = \frac{\nu\, y_b{}^*}{u^*}$$

In figure 3a the velocity profiles for uniform changes of viscosity are shown in which, for the sake of simplicity, the transitional layer is included with the turbulent core. Curve 1 is the original profile. Reducing the viscosity yields profile 2 with one-half the laminar layer thickness. Increasing the viscosity by 50 percent yields profile 3 with a corresponding laminar-layer thickness of 1.5 of the original layer.

The case of a locally variable viscosity such as obtains in non-isothermal flow stipulates a generalization of the dimensionless distance: $y^* = yu^*/\nu$ For the viscous wall layer the following simple possibility presents itself:

$$y^* = u^* \int_0 \frac{dy}{\nu} \qquad u^* = \sqrt{\frac{\tau_o}{\rho_o}} \tag{30}$$

The applicability of this concept must be established by experiment. But it may be stated that this concept (equation (30)) is more satisfactory than the original and that one can predict well those cases in which the property-temperature quotient is not too great by employing equation (30).

The ratio u/u^* can be generalized by re-arrangement of equation (3) for the laminar wall layer.

$$\frac{\rho\, d\, y}{\mu} = \frac{\rho\, d\, u}{\tau_o}$$

The following indentity holds for the laminar layer:

$$\frac{1}{\rho_o u^*}\int_0 \rho \; d\,u = u^*\int_0 \frac{dy}{\nu} = y^* \qquad (31)$$

So yu^*/ν is replaced by $u^*\int_0 \frac{dy}{\nu} = y^*$, then u/u^* must be replaced by $\frac{1}{\rho_o u^*}\int_0 \rho\, du$ in order to preserve the universal representation of the Poiseuille law.

Under the postulate that there exists a certain critical number in this representation, the laminar boundary of the non-isothermal flow must be at the same value as for the isothermal flow $y_o^* = a$. For the ratio u_a/u^* equation (31) then gives

$$\frac{u_a}{u^*} = \frac{2a\rho_a}{\rho_o + \rho_a} \qquad (32)$$

if in the first approximation $\rho(T)$ and $T(u)$ are linear. At constant density $u_a/u^* = a$ as was the case for isothermal flow.

In incompressible fluids the laminar boundary velocity therefore always approximates to the same value a_u^* no matter what the viscosity variation in the laminar layer may be. The integration limit u_a in equations (12a) to (12c) can therefore be retained for non-isothermal flow also.

The effect of the viscosity expresses itself in the thickness of the laminar layer, according to equations (30) and (31):

$$y_a = \frac{1}{u^*}\int_0^a \nu \; dy^* = \frac{\nu_o a}{u^*}\int_0^1 \frac{\mu}{\mu_o}\, d\left(\frac{u}{u_a}\right) \qquad (33)$$

These ratios are expressed qualitatively in figure 3b, that is, for specified values of u^* and μ_o at the wall. The viscosity in the sub-layer is smaller in Profile 2, greater in Profile 3, than the viscosity μ_o for isothermal flow illustrated in velocity Profile 1.

It is seen that the parallel shift of the turbulent profile in the non-isothermal case is much less than by the uniform variation of μ. The velocity change may be expressed approximately from equation (14) as:

$$\Delta u = 5.75\ u^* \ln \left(\int_0^1 \frac{\mu}{\mu_o}\, d\left(\frac{u}{u_a}\right)\right) \tag{34}$$

Because of the neglect of the transitional layer this shift is less than the true Δu. An improvement is possibly obtainable with the integral $\int_0^1 \frac{\mu\ \tau_m}{\mu_o \tau}\, d\left(\frac{u}{u_o}\right)$

THE TEMPERATURE DISTRIBUTION AND THE TEMPERATURE GRADIENT AT A SMOOTH WALL

To further evaluate equations (12a) and (12c) $\mu\lambda_o/\mu_o\lambda$ is substituted for $c_{po}\ Pr/c_p\ Pr_o$ and the dimensionless ratios $\varphi = u/U$, $\vartheta = T/\Theta$ introduced.

To simplify the calculation a constant Prandtl number $Pr_{(\overline{u})}$ and a constant specific heat $c_{p_{\overline{u}}}$ is introduced for the transitional layer. For the turbulent region itself c_p is equal to c_{pt}, a constant. Further the ratio of the exchange quantities is assumed identical. (For completely laminar flow the ratio A_q/A loses of course its significance.)

Defining a mean Prandtl number for the laminar layer as:

$$Pr_l = \frac{1}{\varphi_a} \int_0^{\varphi_a} \frac{c_{p_o}}{c_p}\, Pr\, d\varphi \tag{35}$$

and putting

$$e = \int_0^{\varphi} \frac{\frac{c_{p_o}}{c_p}\, Pr' k}{1 + (Pr' - 1)\frac{\tau_t}{\tau}}\, d\varphi \tag{36}$$

it follows from equations (12a) to (12c) upon the introduction of equation (27a) and the application of $Pr'_o = A_q/A\ Pr_o$ and $Pr'_l = A_q/A\ Pr_l$ that:

$$Pr'_o \left(\frac{d\varphi}{d\vartheta}\right)_o \vartheta = Pr'_l \varphi_a + e \sim Pr'_l \varphi_a \qquad (0 < \vartheta < \vartheta_a) \quad (37a)$$

$$Pr'_o \left(\frac{d\varphi}{d\vartheta}\right)_o \vartheta = Pr'_l \varphi_a + e + \frac{c_{po}}{c_{p\overline{u}}} \frac{\varphi_b - \varphi_a}{1-1/Pr'_{\overline{u}}} \ln\left(1 + (Pr'_{\overline{u}}-1) \frac{\varphi-\varphi_a}{\varphi_b-\varphi_a}\right)$$

$$(\vartheta_a < \vartheta < \vartheta_{bo}) \quad (37b)$$

$$Pr'_o \left(\frac{d\varphi}{d\vartheta}\right)_o \vartheta = Pr'_l \varphi_a + e + \frac{c_{po}}{c_{p\overline{u}}} \frac{\varphi_b - \varphi_a}{1-1/Pr'_{\overline{u}}} \ln Pr'_{\overline{u}} + \frac{c_{po}}{c_{pt}} (\varphi-\varphi_b)$$

$$(\vartheta_{bo} < \vartheta < 1) \quad (37c)$$

From equation (37c) the temperature gradient at the wall is:

$$\frac{c_{pt}}{c_{po}} Pr'_o \left(\frac{d\varphi}{d\vartheta}\right)_o = 1 + e_1 + \varphi_a\left(\frac{c_{pt}}{c_{po}} Pr'_l - 1\right) + (\varphi_b - \varphi_a) \frac{\frac{c_{pt}}{c_{p\overline{u}}} \ln Pr'_{\overline{u}}}{1 - 1/Pr'_{\overline{u}}} - 1 \qquad (38)$$

---

where e_1 is the value of e for $\varphi = 1$ multiplied by c_{pt}/c_{po}. (See equation (36).)

The quantities $\varphi_a = a\ u^*/U$ and $\varphi_b = b\ u^*/U$ are known functions of r^* and Re (see equations (14) and (28)) if a and b are fixed. On the other hand, the material values Pr_l and $Pr_{\overline{u}}$ must be defined more explicitly. (To be discussed later.)

In equations (37c) and (38) φ_{bo} was replaced by φ_b because a slight variation while defining the turbulence boundary has practically no effect on the calculation of the total temperature. (At the boundary position φ_{bo} the temp. difference over the turbulent region is greater by

$\frac{\varphi_b - \varphi_{bo}}{Pr'_o}\left(\frac{d\vartheta}{d\varphi}\right)_o$ than at φ_b, while the temp. difference in the transitional zone is reduced by approx. the same amount.)

In addition to the very small error terms, the expression

$$(\varphi - \varphi_{bo})\ (1-1/Pr'_o)\left(\frac{\tau_m}{\tau}\right)_t$$

is also emmitted, since it is smaller than 0.02 (see fig. 2) even for the largest $(\tau_m/\tau)_t$ at small Reynolds numbers.

The number e_1 accounts for the effect of the variation of q/τ on the temperature gradient at the wall. Since this term is less than unity (see fig. 10) it plays no important part except at low Prandtl numbers as is seen from equation (38). The integration of equation (36) between b and unity usually suffices to calculate e_1 and by this operation Pr' and C_p disappear:

$$e_1 \sim \int_b^1 k\ d\ \varphi \tag{39a}$$

While the errors due to the inaccuracy of the specification of τ_t/τ by means of equation (27) tend to disappear for high Pr, for $Pr' = 1$ the only term which is in error is e_1 (for τ_t/τ is eliminated) to the extent that the errors due to the material values can be discounted.

For $Pr' = 1$

$$e_1 \sim \int_0^1 \frac{c_{pt}}{c_p} k\ d\ \varphi \tag{39b}$$

Here the integration from 0 to b for pipe flow is appropriate, if the Reynolds number is small (in this event k in the transitional zone cannot be neglected.)

For constant material values and for $\varphi_b = \varphi_a$, $q/\tau \sim q_0/\tau_0$ ($e \sim 0$) equations (38a) to (37c) and (38), give the Prandtl approximations:

$$\vartheta \left(\frac{d\varphi}{d\vartheta}\right)_0 = \varphi \qquad (0 < \vartheta < \vartheta_a) \tag{40a}$$

$$\vartheta \left(\frac{d\varphi}{d\vartheta}\right)_0 = \varphi_a + \frac{1}{Pr'} (\varphi - \varphi_a) \qquad (\vartheta_a < \vartheta < 1) \tag{40b}$$

$$Pr' \left(\frac{d\varphi}{d\vartheta}\right) = 1 + \varphi_a (Pr' - 1) \tag{41}$$

(Prandtl employed Pr instead of Pr'. He also used u_m and T_m as reference values rather than U and Θ.)

The fourth term of equation (36) which accounts for the conditions in the transitional layer is particularly important for average values of the Prandtl number. But at high values of Pr' the fourth term is small compared with the third and for constant material values Prandtl's equation (41) is approximately obtained again.

At $Pr' \sim 1$, and constant material values equation (38) simplifies to:

$$Pr' \left(\frac{d\varphi}{d\vartheta}\right)_0 = 1 + e_1 + \frac{\varphi_a + \varphi_b}{2} (Pr'-1) \tag{42}$$

In flow with pressure drop, consideration of the heat flow distribution which enters into the e_1 term yields a smaller temperature increase at the wall than by the assumption $q/\tau = q_0/\tau_0$. For instance $(d\vartheta/d\varphi)_0$ is not equal to unity at $Pr' = 1$ and $Re = 4.10^5$ but in a channel is only about 0.94 and in the pipe approximately 0.91 (see fig. 10).

In figure 4 $(d\vartheta/d\varphi)_0$ is presented as a function of Pr' for constant material values with Re as the parameter, a = 2 and b = 15.5. The solid curves refer to the flow through a pipe. For $Re = 4 \times 10^5$ the dot-dashed curve is that of a channel while the dashed curves refer to a flat plate.

The asymptotic limit value of $(d\vartheta/d\varphi)_o$ for extremely high Prandtl numbers is $1/\varphi_a$. The temperature gradient at the wall $(d\vartheta/dy)_o$ is therefore at the most $1/\varphi_a$ times greater than the corresponding velocity gradient.

For $Pr = 0.72$ (air and other gases at room temp.), $(d\vartheta/d\varphi)_o \sim 0.8$ at the plate, if $A_q = A$ (see equation (42) and fig. 4).

Elias (reference 15) has established, for the flow along a heated plate, that the temperature and velocity profiles are similar, that is, that $(d\vartheta/d\varphi)_o \sim 1$.

This value for $(d\vartheta/d\varphi)_o$ holds, however, for $(A_q/A) Pr \sim 1$. From this it follows that $A_q/A \sim 1/0.72 \sim 1.4$.

A similar result was obtained by Lorenz and Friedrichs (reference 16) in their experiments with air flowing through heated pipe. $Re \sim 10^5$, $(d\vartheta/d\varphi)_o \sim 0.97$. This value lies at $Pr' \sim 1.08$ as may be seen from figure 4 (equation (12)). From this it follows that $A_q/A = 1.5$.

The question regarding the ratio of the exchange quantities, however, cannot be considered solved, hence no specified value A_q/A will be ascertained.

Figure 5 illustrates the temperature distribution $\vartheta(\varphi)$ for various Pr' at $Re = 4.10^4$. The solid curves indicate the second approximation for pipe flow, the dashed curves represent the first approximations which approximately correspond to the temperature distribution along the plate. The division into three flow regions is indicated by the lines φ_a = constant, φ_b = constant (that is, for $a = 2.0$ and $b = 15.5$).

In figure 6 the temperature distributions of figure 5 are plotted against the dimensionless wall distance η. For purposes of clarity only the case of $Pr' = 1$ for the first approximation (also a near approx. for the plate) is presented. This curve also represents the velocity profile for $Re = 4.10^4$ for the universal velocity distribution curve approximately holds for the pipe as well as for the plate.

As to the non-isothermal problem the least trouble is in the choice of the value of c_p since the specific heat varies but slightly with the temperature. In many cases one may write $c_{po} = c_{p\overline{u}} = c_{pt}$ in which event equation (38) is greatly simplified.

For a more accurate analysis the approximate range of the pertinent temperatures must be known. Employing the subscriptsutilized to describe the material values and arranging the temperatures in the order of increasing temperature results in

$$0 < T_l < T_a < T_{\overline{u}} < T_b < T_u < T_t < \Theta$$

Here, in addition to the material value temperatures, the boundary temperatures T_a and T_b as well as the mean temperature of the flowing fluid T_u are introduced. (The definitions of T_u and T_t depend on the variations of the material values and are very complicated, (see derivation of equation (38) But it is not necessary to consider this matter further here.)

Since the principal mass of the fluid is turbulent T_u and T_t are quite similar so that in general T_t can be replaced by the known T_u. At high Pr, temperature $T_{\overline{u}}$ agrees with T_t, hence with T_u (fig. 5); but at low Pr, $T_{\overline{u}}$ is substantially lower than T_u. Since $T_{\overline{u}}$ is applicable only to the term of the transitional layer, the approximate value from figure 5 will suffice.

Of particular influence on the heat transfer is the temperature relation of the Prandtl number in the laminar layer if a very viscous fluid is involved. (For viscous fluids the major resistance to heat transfer is offered by the laminar layer. Since the laminar layer thickness y_a varies with the temp. viscosity history of the fluid (see equation 33) and since the temp. variation of Pr is fixed primarily by the viscosity (c_p and λ vary but slightly with temp.) Y_a depends on Pr/Pr_o. But it may not be said that the thickness of the laminar layer is a function of Pr for this is a heat transfer factor and the laminar layer thickness depends on a purely hydrodynamic variable as seen from equation 33.) In this event a mere estimate of Pr_l by means of a cursory temperature T_l may introduce a serious error. The prediction of Pr by means of equation (35) is therefore indicated.

To simplify the calculation a proportionality between velocity and temperature for the non-isothermal laminar layer is assumed:

$$T = \varphi \left(\frac{d\vartheta}{d\varphi}\right)_0 \cdot \Theta \tag{43}$$

Further the temperature variation of $\frac{Pr}{c_p}$ is approximated by the linear equation:

$$\frac{Pr}{c_p} = \frac{Pr_0}{c_{p0}} (1 + m\ T) \tag{44}$$

where m is an empirical constant.

With the assumptions (43) and (44), equation (35) gives

$$\frac{Pr_l}{Pr_0} = 1 + \frac{m}{2} \varphi_a \left(\frac{d\vartheta}{d\varphi}\right)_0 \Theta = 1 + \frac{m}{2} T_a \tag{45}$$

A first approximation to the temperature gradient $\left(\frac{d\vartheta}{d\varphi}\right)_0$ at the wall is obtained from figure 4 where for the Prandtl number Pr' the value of the wall, Pr'_0 may be chosen.

For a more accurate solution of Pr_l (to be discussed elsewhere) the real non-linear functions $T(\varphi)$ and $Pr(T)$ must be used instead of the linear relations given by equations (43) and (44).

THE DISTRIBUTION OF THE HEAT FLOW DENSITY

IN A CHANNEL AND A PIPE

The differential equation for the equilibrium of forces in a fluid with allowance for the continuity equation and omission of density variations reads:

$$\rho \frac{\partial \overline{w}}{\partial t} + \rho\ \nabla (\overline{w};\overline{w}) = -\ \text{grad}\ \rho + u \nabla^2 \overline{w} \tag{36}$$

where $\overline{w}$ is the velocity vector.

The differential equation for thermal convection and conduction is written similarly:

$$\rho\, c_p \frac{\partial T}{\partial t} + \rho\, c_p\, \nabla\,(w;T) = \epsilon + \lambda\ \nabla^2 T \tag{47}$$

where

ϵ source of density per unit volume.

A formal analogy between the equations of momentum and heat exist therefore for flows with grad $p \neq 0$ only in the presence of spatial heat sources in such flows. Even though the internal friction of a fluid is small in technical applications, the variable ϵ is retained in the equation for future consideration of the analogy.

Equations (46) and (47) are next applied to the completely developed turbulent flow in a flat rectangular channel and in a pipe. For this type of flow the nonuniform terms cancel out by averaging and likewise the derivatives of the mean velocity along the principal flow x.

With $u = \overline{u} + u'$ and v' denoting the velocity and fluctuating components of the velocity in the x and y directions, respectively, where bars represent mean velocities with time and the primes represent instantaneous variations from the mean, the scalar equation for two-dimensional channel flow in terms of mean values is

$$\rho \frac{\partial}{\partial y}\left(\overline{u'\ v'}\right) = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} \tag{48}$$

In this equation the bars are omitted from the pressure and velocity terms as was done previously. The bars are used only to represent the mean products of fluctuating quantities.

The mean momentum interchange $-\rho\,\overline{u'\ v'}$, which may be regarded as a stress attitude is identical with the turbulent shear stress τ_t, while $\mu \frac{du}{dy}$ is the viscous shear stress τ_l. (τ_t is the turbulent momentum transport in the −y direction. For $\frac{du}{dy} > 0$ τ_t is likewise positive; since the positive u' are associated with the negative v' and vice versa on the average. For the exchange process the higher u velocities arrive from greater and the low velocities from closer wall distances.)

Hence τ is the total shearing stress

$$\frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x} = \text{constant} \qquad \text{(channel)} \quad (47)$$

The pressure drop is constant since the flow is fully developed.

The heat balance in two-dimensional channel flow follows from equation (47)

$$\rho\, c_p \left(u \frac{\partial T}{\partial x} + \frac{\partial}{\partial x} \overline{(T'\ u')} + \frac{\partial}{\partial y} \overline{(T'\ v')} \right) = \epsilon + \lambda \frac{\partial^2 T}{\partial y^2} \qquad (50)$$

Assuming that no great changes in $\overline{(T'\ u')}$ occur in the direction of flow x as certainly is the case for fully developed temperature distributions, the second term on the left side of equation (50) may be neglected. There remains, then, only the fluctuation product. $\rho\, c_p\, \overline{T'\ v'}$ which is equivalent to the turbulent heat transport $-q_t$ perpendicular to the wall. Introducing the total heat flow $q = q_t + \lambda \frac{\partial T}{\partial y}$, further affords:

$$\frac{\partial q}{\partial y} = \rho\, c_p\, u \frac{\partial T}{\partial x} - \epsilon \qquad \text{(channel)} \quad (51)$$

For fully developed flow in a pipe the following equation in cylindrical coordinates (instead of equation (49) for the channel) is obtained

$$2 \frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x} = \text{constant} \qquad \text{(pipe)} \quad (49a)$$

and (instead of equation (51) for the channel) the equation:

$$\frac{\partial}{\partial y}\left(q(1-\eta)\right) = (1-\eta)\left(\rho\ c_p\ u\frac{\partial T}{\partial x} - \epsilon\right) \quad \text{(pipe)}\ (51a)$$

where $\eta = \frac{y}{r}$ (r = radius of pipe or 1/2 width of channel).

By introducing the shear stress at the wall τ_o there follows for the pipe and the channel from equations (49a) and (19), respectively:

$$\tau = \tau_o\ (1-\eta) \quad \text{(channel and pipe)}\ (52)$$

For the case of fully developed temperature distribution to which the argument is confined the partial differential equations of heat flow become ordinary differential equations. (The entrance lengths for fluid flow and heat diffusion are independent of each other. For instance, a sudden change of wall temperature may introduce a thermal entrance in a hydrodynamically developed flow.) The similarity of the temperature profiles states that the decrease of temperature per unit temperature $-dT/T$ in distance dx at each distance y from the wall is a constant. Therefore $-dT/dx =$ constant T. The omission of ϵ leaves instead of equations (51) and (51a)

$$\frac{dq}{dy} = \text{constant}\ \ u\ T \quad \text{(channel)}\ (53)$$

$$\frac{d}{dy}\left(q(1-\eta)\right) = \text{constant}\ (1-\eta)\ u\ T \quad \text{(pipe)}\ (53a)$$

T is defined, as before, as the temperature excess over that at the wall.

It should be noted, however, that a fully developed temperature distribution is possible only at small excess temperatures T. In general $u(\partial/\partial x)(\rho\ c_p\ T)$ instead of $\rho\ c_p\ u\frac{\partial T}{\partial x}$ would have to be reckoned with. For the following it is assumed that change of profile remains within such limits that equations (53) and (53a) remain applicable with sufficient accuracy.

Integration of equations (53) and (53a) and introduction of the limit values ($q = q_0$ at $\eta = 0$ and $q = 0$ at $\eta = 1$) in conjunction with the nondimensional $\vartheta = T/\Theta$ and $\varphi = u/U$ gives:

$$\frac{q}{q_0} = 1 - \frac{\int_0^\eta \vartheta \varphi \, d\eta}{\int_0^1 \vartheta \varphi \, d\eta} \qquad \text{(channel)} \quad (54)$$

$$\frac{q}{q_0}(1-\eta) = 1 - \frac{\int_0^\eta \vartheta \varphi (1-\eta) d\eta}{\int_0^1 \vartheta \varphi (1-\eta) d\eta} \qquad \text{(pipe)} \quad (54a)$$

A good approximation for q/q_0 is obtained with the application of the temperature equations (37a) to (37c). Although the introduction of the simplified temperature equations (40a) to (41) is sufficient.

The velocity distribution of the turbulent region is represented by the well known power law:

$$\varphi = \eta^n \qquad (55)$$

where $0.18 > \eta > 0.10$ for $4.10^3 < Re < 4.10^6$.

The power law represents the velocity distribution even better for large η's than the logarithmic law and is especially suitable for the present calculation. In wall proximity the velocity is of course less than that calculated by the power law and the error of the derivation is small only at high Reynolds numbers where the power law must be used near the wall.

The use of equations (40a) to (41) and equation (55) then yields approximately:

$$\int_0^\eta \vartheta \varphi \, d\eta \sim \frac{(1 + n - n^2)(Pr' - 1)\varphi_a + \varphi}{(1 + 2n)((Pr' - 1)\varphi_a + 1 + e)} \varphi^{1+1/n} \qquad (56)$$

Here the lower limit φ_a is replaced by 0. Because the value of this integral is practically zero at the limit φ_a since the exponent $1 + 1/n$ is high and the integration from 0 to φ_a yields an integral which is very small.

Thus the density of heat flow for channel flow at high Reynolds numbers approximates to:

$$\frac{q}{q_o} = 1 - \frac{(Pr'-1)(1+n)\varphi_a+\varphi}{(Pr'-1)(1+n)\varphi_a+1}\varphi^{1+1/n} \quad \text{(channel)} \quad (57)$$

where $\varphi = \eta^n$, so that q/q_o may be represented as a function of η.

In a similar fashion the heat flow through the fluid in a pipe is approximately, according to equation (54a):

$$\frac{q}{q_o}(1-\eta) = 1$$

$$- \frac{(Pr'-1)(1+2n)(2+n-(1+n)\varphi^{1/n})\varphi_a+(1+n/2)(2+2n-(1+2n)\varphi^{1/n})\varphi}{(Pr'-1)(1+2n)\varphi_a+1+n/2}\varphi^{1+1/n}$$

$$\text{(pipe)} \quad (58)$$

A good view of the variation of q/q_o may be obtained for the special cases of $Pr' = 1$ and $Pr' \rightarrow \infty$ as substituted into equations (57) and (58):

$$\frac{q}{q_o} = 1 - \eta^{1+2n} \quad (\text{channel } Pr' = 1) \quad (57a)$$

$$\frac{q}{q_o} = 1 - \eta^{1+n} \quad (\text{channel } Pr' = \infty) \quad (57b)$$

$$\frac{q}{q_o}(1-\eta) = 1 - (2+2n-(1+2n)\eta)\eta^{1+2n} \quad (\text{pipe } Pr' = 1) \quad (58a)$$

$$\frac{q}{q_o}(1+\eta) = 1 - (2+n-(1+n)\eta)\eta^{1+n} \quad (\text{pipe } Pr' = \infty) \quad (58b)$$

The q/q_0 curves flatten out with increasing values of Pr and Re (decreasing n), that is, they approach $q/q_0 = 1 - \eta$ which is the limit of equations (57a) and (58a) for $n = 0$. It further follows that the quantities k and e decrease with increasing Pr' and Re (see equations (13) and (39) and fig. 10).

Figures 7 and 8 show $(q/q_0)(\varphi)$ and $(q/q_0)(\eta)$, respectively, for the pipe and channel at $Re = 4 \times 10^4$. Figure 9 reveals $(q/q_0)(\eta)$ for various values of Re at $Pr' = 0.72$ and $Pr' = 200$. These curves were computed for φ by the true velocity distribution as shown in figure 1 rather than power law.

The variation of the heat flow density in proximity of the wall is noteworthy, where $(dq/d\eta)_0 = 0$ for the channel, but $(dq/d\eta)_0 = q_0$ for the pipe (see equations (53) and (53a)). The rise of the heat flow density of the pipe beyond the value q_0 is due to the fact that the total heat flow $Q \sim q(1-\eta)$ near the wall is practically constant, while the section through which the heat flows, decreases with $(1-\eta)$. In channel flow no cross-sectional area changes occur; thus the heat flux density and the total heat flow are always directly proportional.

At mid-channel (and pipe, respectively) the variation of the heat flow density is characterized by,

$$\left(\frac{dq}{d\eta}\right)_1 = -\frac{q_0}{\vartheta_u \varphi_m}$$

(see equations (54) and (54a)), $\vartheta_u = \frac{T_u}{\Theta}$ and $\varphi_m = \frac{u_m}{U}$ denoting the dimensionless magnitudes of the mean stream temperatures T_u and the mean velocity (u_m), respectively. Since these magnitudes are smaller than unity, the negative slope of q/q_0 is greater than unity, and is greater for the pipe than for the channel.

Equations (57) and (58a) enable the calculation of k and e_1 (see equations (13) and (39)) through which the second approximation to the temperature is secured. In figure 10 the e_1 term for pipe flow is shown plotted against Pr' for different Re.

(As mentioned above (equation (9)) G. I. Taylor has computed a second approximation to the temp. distribution for the case of $Pr = 1$. His arguments rest on Reynolds' analogy. In the calculation $\partial T/\partial x$ is assumed independent of the distance from the wall which corresponds to $\partial q/\partial y$ proportional to u (instead of $u\,T$). Throughout a correction term, which is too small, results (for instance at $Re = 4.10^4$ it is 5 percent instead of 9 percent, as is the case for high Pr (that is, for $T \sim$ constant)). (See fig. 10.)

COMPARISON BETWEEN MOMENTUM INTERCHANGE AND HEAT TRANSFER

Supplemental to this theory an attempt is made to compare the differential equations of heat and momentum and to indicate that the historic heat source theorem also leads to a generalization of the Prandtl number.

The similarity of the differential equations (46) and (47) is so obvious that it need not be discussed further. However, it is necessary to analyze the existing differences.

One substantial departure lies in the fact that the heat equation contains no term corresponding to the pressure drop in the momentum equation. This difference can, however, be removed in some cases (as Prandtl has shown) in first approximation by substituting a suitably chosen heat source density ϵ.

Physically this artifice has the following significance. The momentum of a flowing fluid can be maintained by a pressure gradient. The heat content of the fluid is, in contrast, reduced by the transfer through the walls, unless heat is produced in the fluid itself (such as by a current of electricity flowing through the fluid). To complete the analogy between heat transfer and momentum exchange the volume heat sources must be so disposed that the temperature and velocity profiles are similar. In the particular case where the velocity distribution remains constant in the axial direction (fully developed flow in a pipe or channel) the temperature profile should be maintained likewise.

A further difference between equations (46) and (47) rests on the fact that equation (46) is a vector equation and equation (47) is a scalar equation, hence only one component of equation (46) can be compared with equation (47). The analogy is, therefore, carried out for fully developed plane channel flow. By concentrating on the special case of plane flow, the problem is much clearer, and affords more far reaching conclusions than from equations (46) and (47).

For this flow equations (46) and (47) give:

$$\frac{\partial \tau}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{d\overline{u}}{dy} - \rho\, \overline{u' \; v'}\right) = \frac{\partial p}{\partial x} \tag{49b}$$

$$\frac{\partial q}{\partial y} = \frac{\partial}{\partial y}\left(\lambda \frac{d\overline{T}}{dy} - \rho\, c_p\, \overline{T' \; v'}\right) = -\,\epsilon + \rho\, c_p\, \overline{u}\, \frac{\partial \overline{T}}{\partial x} \tag{51b}$$

In order to avoid misinterpretations, the mean values are again represented by bars.

In the most general form, the equations for τ and q are analogous. But because fully developed flow has been postulated, the terms with $\partial u^2/\partial x$, $\frac{\partial}{\partial y}\,(\overline{u}\;\overline{v})\;\frac{\partial}{\partial y}\,(\overline{T}\;\overline{v})$ drop out, while the term with $\frac{\partial}{\partial x}\,(\overline{u}\;\overline{T})$ in the heat equation remains. The problem, then, is to choose ϵ so that the term with $\partial\overline{T}/\partial x$ vanishes.

As shown above $\partial q/\partial y \sim -\overline{u}\;\partial T/\partial x \sim \overline{u}\;\overline{T}$ (see equation (53)). Thus for a given heat source distribution $\epsilon = k\,\overline{u}\;\overline{T}$, a suitable choice of k will cause $\partial\overline{T}/\partial x$ to diminish to zero for every y, without in any way modifying the temperature distribution $\overline{T}$ or the exchange $\overline{T' \; v'}$. With this choice of ϵ the actual temperature profile will be retained; it simply remedies the earlier decrease in temperature in the direction of flow.

However this heat density which varies with the distance from the wall cannot be compared with the pressure gradient which is constant over the section. So in order to carry through the consideration of the analogy it is necessary that ϵ = constant. In this case also it may be stipulated that $\partial\overline{T}/\partial x$ should vanish at each position of y. Then the heat flow is:

$$\frac{dq}{dy} = - \epsilon = \text{constant}$$

Under this condition $\overline{T}$ is of course no longer the actual temperature, but rather an approximation to the temperature, which is that the postulate ϵ = constant yields too great a temperature gradient at the wall because the wall layers are heated excessively by constant source density. But in view of the fairly well compensated temperature profiles the error must be small.

Now the identity of equations (49b) and (51b) can be adduced by putting conformably to Prandtl, $\mu c_p = \lambda$ ($Pr = 1$) and $T = \beta u$ (β = constant).

But the equation $T = \beta u$ is only one possible solution. The solution is, in fact, somewhat special, since it not only requires the time averages of the velocity and the temperature proportional, but the fluctuations u' and v' themselves to be proportional at every instant.

$$\overline{T} = \beta \overline{u} \tag{59}$$

$$T' = \beta u' \tag{59a}$$

These equations are obviously fulfilled if the mechanism of transfer of the u-component of the momentum and the mechanism of heat transfer are completely similar. This may occur in particular cases.

Consider next the general case where the correlation coefficient between u and T is less than 1. To this purpose the turbulent terms in equations (49b) and (51b) are expressed by $A\ d\overline{u}/dy$ and $c_p\ A_q (dT/dy)$. The identity of the equations is attained when $\frac{A}{\mu} = \frac{A_q\ c_p}{\lambda}$ (that is, $Pr' = 1$) for each distance y from the wall and when equation (59) is satisfied.

Equation (59) thus represents an approximate solution of the heat equation for the case defined by $Pr' = 1$. In this case assumptions relative to the fluctuations of u' and T' are no longer required.

The solution (equation (59)) stipulates that the fictitious source strength ϵ be fixed by:

$$\epsilon = - \beta \, c_p \frac{A_q}{A} \frac{dp}{dx} \tag{60}$$

On the basis of the postulate $dq/dy = -\epsilon$, the source strength ϵ may be defined in terms of heat flow at the wall q_o:

$$\epsilon = q_o/r \tag{61}$$

(r is the half-width of the channel). Introducing the shear stress at the wall for the pressure drop, it follows from equation (60) that:

$$q_o = \beta \, c_p \frac{A_q}{A} \tau_o \tag{62}$$

(This change of form of the equations has the advantage that the form of the fluid boundary (whether pipe, channel, plate, and so forth) which is unessential for these considerations does not affect the result.) This equation may be derived also, for the quasi-plane case of the pipe. The constant β can be expressed by appropriate standard values (for instance, by the maximum values) the mixed mean temp. of $\overline{T}$ or $\overline{u}$.

At $Pr' \neq 1$, the analogy is not complete for the total fluid, but only for the turbulent core in which the terms of molecular conduction can be neglected.

Then equation (59) is appropriately replaced by:

$$\overline{T} - T_a = \beta \, (\overline{u} - u_a) \tag{59b}$$

where T_a and u_a are time averages at "the point of transition to the laminar flow". For the rest the calculation is the same and equation (62) holds for $Pr \neq 1$. Solely β becomes another proportionality factor.

For the prediction of β the maximum values of Θ or U, the temperature and velocity, respectively, can be used. It is also possible, however, to introduce the sectional averages T_m and u_m (as was done in the Prandtl derivation) since the section of the laminar layer of the flow is negligible compared with the total flow section. With T_a expressed in terms of u_a and Pr':

$$T_a = \beta \, Pr' \, u_a \tag{63}$$

- an equation which results upon the application of equation (62) to the laminar layer, - equation (59b) gives:

$$\beta = \frac{\Theta/U}{1 + (Pr'-1)u_a/U} \tag{64}$$

or

$$\beta = \frac{T_m/u_m}{1 + (Pr'-1)u_a/u_m} \tag{64a}$$

The following useful conclusion can be drawn from equations (64) and (64a):

$$\vartheta_m = \frac{\varphi_m + \varphi_a(Pr'-1)}{1 + \varphi_a(Pr'-1)} \tag{65}$$

$\left(\vartheta_m = \frac{T_m}{\Theta};\ \varphi_m = \frac{u_m}{U}\right)$ The dimensionless mean temperature is identical at $Pr' = 1$ with the dimensionless mean velocity (within the framework of the present approx.). With increased Prandtl number ϑ_m approaches unity.

Since it has been established that the correct heat flow distribution over the section is not necessary for the determination or the temperature distribution in first approximation, $q/\tau = q_0/\tau_0$ and this assumption compared with the assumption of a constant source strength, which as shown above, is necessary to establish the analogy. From $dq/d\eta = - \epsilon =$ constant, it follows that $q = q\ (1-\eta)$. On the other hand, for completely developed flow $\tau = \tau_0\ (1-\eta)$.

For fully developed flow our approximations therefore agree with the postulate of constant heat source distribution. The two idealizations differ, however, in their consequences for further theoretical treatment as well as in the justification of their physical admissibility, although both methods of treatment agree with great completeness as a basis of the turbulent velocity profile.

(Since the heat source theorem is designed to describe analogous phenomena which in detail are not analogous, the theory is definitely bounded, which limits its extension. The heat source theory is not purposed to consider particular force fields in detail, merely intended to reproduce and clarify the essential characteristics of the heat flow by comparison with known phenomena of the momentum transfer.)

HEAT VOLUME TRANSMITTED TO THE WALL

(a) Determination of the heat transfer from the temperature gradients at the wall.

From the temperature rise at the wall the unit heat rate at the wall follows directly at:

$$q_o = \lambda_o \left(\frac{dT}{dy}\right)_o = \frac{c_{po}\tau_o}{Pr_o\left(\frac{du}{dT}\right)_o} \tag{66}$$

This equation, is naturally general, is not affected by the type of flow nor by the time or space variation of τ_o and $(du/dT)_o$.

The rate of heat transfer Q over the area F is obtained by integrating equation (66). If U is the velocity and Θ the temperature difference of the fluid with respect to the wall at the boundary of the friction layer, the heat flow is:

$$Q = \int \frac{c_{po}\,\Theta\tau_o}{Pr_o\left(\frac{d\varphi}{d\vartheta}\right)_o U}\,dF \tag{67}$$

For an area over which Θ, U, and $(d\varphi/d\vartheta)_o$ are sensibly constant it follows further that:

$$\Delta Q = \frac{c_{po}\ \Theta}{Pr_o\left(\frac{d\varphi}{d\vartheta}\right)_o}\ \frac{\Delta W}{U} \tag{68}$$

where

ΔW resistance to flow offered by the area under consideration.

If a turbulent friction layer is involved $Pro\ (d\varphi/d\vartheta)_o$ in equation (68) is expressed by equation (38).

As to the permissible size of the area to which equation (68) can be applied in friction layers free to extend unhindered over the surface (that is, the actual boundary layers), Θ is, in general, the constant temperature corresponding to potential flow. In this case the admissible size of the area is dependent on the adequate constancy of U and $(d\varphi/d\vartheta)_o$ φ_a and φ_b, respectively, of equation (38) - φ_a and φ_b are functions of the Re of the velocity profile and so vary with the arc-length x).

In flow through pipes or channels, the Reynolds number for fully developed velocity profile is constant; but the maximum temperature Θ decreases. For which reason equation (68) holds only for pipes and channels if the flow section is sufficiently short. In long pipes the temperature drop must be accounted for, as shown in the next section.

(b) Consideration of the heat loss in the friction layer. The heat stream Q_u flowing through a flow section f is

$$Q_u = \int^f \rho\ c_p\ T\ u\ df$$

Introducing the mean flow temperature T_u (that is, the mean temp. of the fluid mass flowing through the flow section) affords

$$T_u = \frac{\int^f T\, u\, d f}{u_m\, f}$$

or for constant values of c_p and ρ:

$$Q_u = \rho\, c_p\, T_u\, u_m\, f$$

The heat volume given off at the wall over the arc length $(x-x_1)$, is equal to the difference $Q = Q_u - Q_{u_1}$ in the flow sections f and f_1. With the postulate that the profiles of the velocity and temperature are similar: $T_{u_1}/\Theta_1 = T_u/\Theta$, and $u_{m_1}/U_1 = u_m/U$, hence

$$Q = \rho\, c_p\, \vartheta_u\, \varphi_m\, (U_1\, \Theta_1\, f_1 - U\, \Theta\, f) \tag{69}$$

where $\vartheta_a = T_u/\Theta$ and $\varphi_m = u_m/U$.

Equation (69) can be utilized to check experimentally the theory (similarly combined with equations (38) and (68)). All quantities in equation (69) are readily measureable. A minor complication is introduced in fixing ϑ_u, for which a mixing chamber is required.

Also ϑ can be evaluated from the theory. For the plane case:

$$\vartheta_u = \frac{\int_0^1 \vartheta\, \varphi\, d\eta}{\varphi_m} = \frac{\int_0^1 \vartheta\, \varphi\, d\eta}{\int_0^1 \varphi\, d\eta} \quad \text{(three dimensional friction layers)}$$

and for the flow in a pipe

$$\vartheta_u = 2\frac{\int_0^1 \vartheta\, \varphi\, (1-\eta)\, d\eta}{\varphi_m} = \frac{\int_0^1 \vartheta\, \varphi\, (1-\eta)\, d\eta}{\int_0^1 \varphi(1-\eta)\, d\eta} \quad \text{(pipe)}$$

In figure 11, ϑ_u is shown for the flat plate and for the pipe as a function of Pr for several Reynolds numbers. The effect of the Reynolds number on ϑ_u is less than the effect of Pr, since φ(Re) appears in the numerator and the denominator.

(1) Pipe or channel flow.

In flow through a pipe or channel, the development of the friction layer is limited and the strength of the final friction layer is equal to one half the distance to the opposite wall. The mass of fluid moved in the fully developed friction layer does not change, and the heat given off by the fluid can be calculated from the reduction of temperature.

So, when no change in cross-sectional area is considered equation (69) gives:

$$Q = \rho\, c_p\, \vartheta_u\, u_m\, (\Theta_1 - \Theta) f \tag{70}$$

On the other hand, the heat transferred is also defined by equation (67). Here, it must be noted that, because of the similarity of the temperature profiles the percentage temperature drop $- dT/T$ over the length dx is the same at all distances from the wall. Hence it also applies to the mean temperature:

$$\ln \frac{\Theta_1}{\Theta} = \text{constant } (x - x_1)$$

and equation (67) yields

$$Q = \frac{c_p (\Theta_1 - \Theta)}{\ln \left(\frac{\Theta_1}{\Theta}\right) Pr_0 \left(\frac{d\varphi}{d\vartheta}\right)_0} \frac{W}{U} \tag{71}$$

where

Θ_1 given maximum initial temperature

W frictional resistance of the pertinent pipe or channel length.

(At small temperature differences $\Theta_1 - \Theta$ equation (71) changes to equation (68)).

By equating equations (70) and (71), the reduction of the mean temperature is:

$$\ln\left(\frac{\Theta_1}{\Theta}\right) = \frac{\alpha^*}{\vartheta_u} \frac{s}{f} (x - x_1) \tag{72}$$

Here the dimensionless variable α^* defined in equation (82) is introduced and s is the perimeter of the section f.

For pipes $s/f = 4/d$ for flat rectangular channels $s/f = 2/h$ (d = pipe diameter, b = channel width, h = channel height, $b \gg h$).

(2) Boundary layer flow.

If the flow along a wall is not bounded, the friction layer can develop unhindered, and while the boundary layer increases in thickness in the direction of flow, the maximum temperature Θ on the boundary of the friction layer remains, in general, unchanged.

If the surface of the body has the temperature Θ of the fluid, then the heat flow density at a particular point in the friction layer is equal to $\rho\ c_p\ \Theta_u$. But the cooling action of the wall lowers the temperature by $\Theta - T$. The "cold stream" through the section f of the boundary layer at the point in question is therefore:

$$Q = \rho\ c_p \int_0^f (\Theta - T)\ u\ d\ f \tag{74}$$

where

Q heat volume absorbed by the body surface up to the particular point x in unit time.

By introducing the dimensionless value of the mean flow temperature, equation (74) gives:

$$Q = \rho c_p \Theta u_m (1-\vartheta_u) f \tag{75}$$

the value of ϑ_u for any given velocity profile is approximately defined by the theory.

At constant maximum velocity U the heat absorbed over a length $x-x_1$ is fixed by the increase in section of the boundary layer thickness in the direction of flow, since φ_m and ϑ_u vary but little (φ_m increases and $(1-\vartheta_u)$ decreases with the Re of the boundary layer).

The effect of Pr on the heat transfer is expressed by the factor $(1-\vartheta_u)$. With increasing Pr the temperature profile becomes more blunt-nosed and ϑ approaches unity.

From the momentum lost in the boundary layer relative to potential flow the flow resistance W of the body can be written in a similar manner to that used for the heat diffusion Q

$$W = \rho U u_m (1-\varphi_u) f \tag{76}$$

Herein φ_u is defined by $\varphi_u \varphi_m f = \int^f \varphi^2 \, df$. The difference between the equations lies in the velocity U in potential flow which, in general, is not constant like Θ but varies with the arc-length x.

But the analogy between equations (75) and (76) is to be carried out for a surface area over which Θ and U are constant (a pressure drop is to be avoided). If $f-f_1$ is eliminated, and if ϑ_u and φ_u are the mean values over the particular arc length:

$$\Delta Q = c_p \Theta \frac{1-\vartheta_u}{1-\varphi_u} \frac{\Delta W}{U} \tag{77}$$

Here the heat transfer is expressed by the heat loss in the friction layer, while in equation (68) it appears in terms of the temperature gradient at the wall. Comparing equations (68) and (77) yields:

$$\vartheta_u = 1 - \frac{1 - \varphi_u}{Pr_o \left(\frac{d\varphi}{d\vartheta}\right)_o} \qquad (78)$$

For $Pr'_o = Pr_o = 1$, $(d\varphi/d\vartheta)_o = 1$ and hence $\vartheta_u = \varphi_u$, which is due to the similarity of the velocity and temperature profiles for zero pressure drop. $Pr_o\ (d\varphi/d\vartheta)_o$ increases with increased Pr and $\vartheta_u > 1$.

For the rest, equation (78) is easily verified for the simple friction layer (equations (40a) to (41)).

(c) Heat transfer coefficients.

The heat transfer coefficient α is defined by Newton's law of cooling:

$$\Delta Q = \alpha\ (T - T_o)\ \Delta F \qquad (79)$$

in which the heat transfer per unit time through the boundary area ΔF is put proportional to the temperature difference $(T - T_o)$ of the fluid and of the wall.

Originally the proportionality factor α was thought of as a pure material value comparable with the thermal conductivity and in the older literature was designated as the "outer thermal conductivity". With the increase of experimental data, it became more and more apparent that the flow phenomena adjacent to the wall contributed greatly to the heat transfer and varied in a complicated manner therewith. Hence the cooling law is only apparently simple, that is, when the simple form of equation (29) is maintained. all of the problems of heat transfer by convection are condensed in the factor α.

This naturally does not help to clarify the physical phenomena and later research has produced other axioms which throw light on the mechanism of heat transfer. For the practical application of research data it is, however, advantageous if complicated relations can be expressed by a single coefficient, the value of which is obtainable from graphs or tables.

In equation (79) the fluid temperature T remains undefined. What temperature between T_o and T_{max} to use is purely a matter of expediency. Only one point is necessary, namely, that the fluid temperature employed is adequately defined. The reference temperature must be relatively constant over the area under consideration. Since this condition holds true in all cases only over a small area ΔF the heat transfer coefficient must be defined as local quantity.

In the earlier derivations the heat transfer coefficient was based on the mean temperature T_m.

The calculation of a heat transfer coefficient $\alpha_m = q_o/T_m$ by means of equation (66) requires an expression for $\frac{T_m}{u_m}\frac{(du)}{(dt)_o}$ from the theory, if T_o is referred to the mean velocity u_m. Such an equation can be obtained from equation (37c) after forming a mean value of the velocity and the temperature over the section of the turbulent region. Since this section is not much smaller than the total flow section it can be approximated to:

$$Pr' \frac{T_m}{u_m}\left(\frac{du}{dT}\right)_o \sim 1 + e_m + \frac{u_a}{u_m}(Pr' - 1) + \frac{u_b - u_a}{u_m}\left(\frac{\ln Pr'}{1 - \frac{1}{Pr'}} - 1\right) \quad (38a)$$

This equation applies so much more as the boundary layer is thinner, that is, as the Reynolds number is greater.

Neglecting the term e_m and putting $u_a = u_b$, and $A_q = A$, equations (66) and (38a) give an expression for $\alpha_m = q_o/T_m$ which is identical with the Prandtl formula equation (1).

The mixed mean temperature T_u is usually employed rather than the mean temperature T_m which is difficult to measure. This requires an equation for $\alpha_u = q_o/T_u$ from the theory. Here a difficulty arises. Proper treatment of equation (37c) affords a formula for $(du/dT)_o$ in which the mean temperature of the turbulent region is the reference temperature. But then the equation includes a mean square value of u over the turbulent section in place of the mean velocity across the section.

By approximation the mean of the squares of u can, of course, be replaced by u_m (particularly at high Re) just as T_u can be roughly approximated instead of T_m. But this also means returning to equation (38a), that is, the Prandtl formula.

The difference between T_u and T_m played no part, however, in the earlier considerations. In view of the experimental difficulties, this difference usually lies well within the experimental error. Further, the omission of the transitional layer and the postulate of invariable material values accounted for larger discrepancies than this temperature difference.

In developing a theoretical equation for q_o/T_u which is in accord with experience, ten Bosch (reference (17)) proposed a semi-empirical equation. The form of the Prandtl equation was followed, but the constants were replaced by variables whose magnitudes were determined as a function of Re and Pr from measurements available, the resulting equation being:

$$\frac{q_o}{\rho\, c_p\, u_m T_u} = \frac{0.125\, \zeta}{1 + B\, Re^{-0.1}\, Pr^{-0.185}\, (Pr_g - 1)} \tag{80}$$

(for heating $\beta \sim 1.4$ for cooling $\beta \sim 1.12$. Pr_g refers to the layers near the wall. In the remainder of the formula the properties are fixed at the mean flow temp.)

In connection with the theory of the present report, the heat transfer coefficient is most appropriately expressed in terms of the maximum temperature Θ, since only this temperature can be used as a reference quantity without reservation (see the derivation of equation (38)). The maximum temperature has in addition the advantage that it can be measured without the use of a mixing cup. (In cases where T_u can be measured more reliably than Θ, Θ maybe calculated as T_u/ϑ_u.) Hence the definition

$$\alpha = \frac{\Delta \dot{Q}}{\Theta\, \Delta F} = \frac{q_o}{\Theta}$$

or in nondimensional representation by dividing by $\rho_o\ c_{po}\ u_m$:

$$\alpha = \frac{\Delta Q}{\rho_o\ c_{po}\ u_m\ \Theta\ \Delta F} = \frac{q_o}{\rho_o\ c_{po}\ u_m\ \Theta} \tag{81}$$

In the technical literature the heat flow q_o is usually referred to $\rho\ c_p\ u_m\ T_u$ (see equation (80)). Further, it is customary to introduce the Nusselt number

$$Nu = q_o\ d/\lambda\ T_u$$

in which q_o is referred to as the heat flow $\lambda\ T_u/d$. These two dimensionless heat factors are related as follows:

$$\frac{q_o}{\rho\ c_p\ u_m\ T_u} = \frac{Nu}{Re\ Pr} = \frac{Nu}{Pe} = \frac{\alpha^*}{\vartheta u}$$

where

$$Pe = \frac{\rho\ c_p\ u_m\ d}{\lambda} = \text{Peclet number}$$

The older Nusselt number had proved itself in the representation of cases where the heat transfer phenomena were not to be separated. But, for those cases where statements can be made relative to the local heat transfer, and where q_o can be written directly in terms of $\lambda\ (dT/dy)_o$ it is proper to refer the known quantity q_o to the product $\rho\ c_p\ u\ T$ as will be seen in subsequent derivations.

In certain cases it may be desirable to compare the two dimensionless groups. But, in general, both Nu and Pe are superfluous if a well founded formula for the dimensionless group $q_o/\rho\ c_p\ u\ T$ is available.

In order that $q_o/\rho_o\, c_{po}\, u_m\, \Theta$ need not be repeated unnecessarily the symbol α^* has been introduced. The dimensionless factor α^* refers the heat volume q_o transferred to the wall to a heat volume $\rho_o\, c_p\, u_m\, \Theta$ flowing past the wall. The njmber α^* is, therefore, a kind of "efficiency" of the heat transfer. Ordinarily α^* is very small $(10^{-2} > \alpha^* > 10^{-5}$, see fig. 13) so that only a small fraction of the heat becomes useful for transfer.

Since α^* represents a locally defined quantity as well, equation (68) must be applied for the subsequent treatment of equation (81). Then equation (19) in conjunction with $\varphi_m = u_m/U$ give

$$\alpha^* = \frac{\zeta\, \varphi_m}{8\, Pr_o \left(\frac{d\varphi}{d\vartheta}\right)_o} \tag{82}$$

This formula is as general as equation (68) and the definition equation (19). It is therefore applicable, independent of the character of the flow.

Formula (82) had already been utilized to introduce α^* in equation (72) in order to establish the temperature drop which accompanies pressure drop. On the other hand, equation (72) can equally be used to define α^*.

$$a^* = \vartheta_u \ln \left(\frac{\Theta_1}{\Theta}\right) \frac{f}{F} \tag{72a}$$

which for a sufficiently short section affords

$$\alpha^* = \vartheta_u \frac{\Theta_1 - \Theta}{\Theta} \frac{f}{F} \tag{72b}$$

This equation is, with regard to equation (70) identical with the definition (equation (81)) for α^*.

For turbulent friction layers the $Pr_o\ (d\varphi/d\vartheta)_o$ from equation (38) must be introduced in equation (82). With

$\varphi_a = a \frac{u^*}{U}$, and $\varphi_b = b \frac{u^*}{U}$ and assuming $c_{po} \sim c_{p\overline{u}} \sim c_{pt}$ (which is permissible)

$$\alpha^* = \frac{0.125\ \zeta\ \varphi_m\ A_q/A}{1 + e_1 + a\ \frac{u^*}{U}\left(\frac{A_q}{A}\ Pr_l - 1\right) + (b-a)\ \frac{u^*}{U}\left(\frac{\ln(Pr_{\overline{u}} A_q/A)}{1 - A/A_q\ Pr_{\overline{u}}} - 1\right)} \qquad (83)$$

where

$\zeta, \varphi_m, U/u^*$ are known functions of Re

In equation (83) all variables and coefficients are known except a, b, and A_q/A. From flow measurements $a = 1.5$, $b = 15.5$. The value of 1.55 may be used since b occurs only in the term referring to the transitional layer. But the assigned value of a is very uncertain. Hence a and A_q/A must be determined from measured values of α^* according to equation (83).

Admittedly there exists a certain difficulty, involving two unknowns but equation (83) indicates that A_q/A scaresly affects α^* at high Pr, and a has but little influence on α^* at low Pr. Hence a can be from heat transfer measurements at high Prandtl numbers and A_q/A from similar measurements at small Prandtl numbers.

Figure 12 illustrates the measurements by Bühne (reference (18)) and Morris and Whitman (reference (19)) at high Prandtl numbers and the measurements by Rohonczi (l.c.) at low Prandtl numbers compared with predicted α^*. For the calculation $a = 2.2$ and $A_q/A = 1$ were chosen to achieve the best correlation of predicted with experimental results. The spread of the test points at high Pr is understandable because of the difficulty of measurement at high Pr and the fact that non-isothermal flow theory is not yet complete.

For constant or slightly variable material values only one value of Pr enters in equation (83). For this case (fig. 13) shows α^* as a function of Pr' at several Reynolds numbers. With the help of α^* established by equation (83), the heat diffusion ΔQ of a given area ΔF can be calculated.

$$\Delta Q = \rho_o \, c_{po} \, u_m \, \Theta \, \Delta F \, \alpha^* \tag{68a}$$

This is a form derived from equation (81) which is another form of equation (68).

The variable α^* is a point function and holds only over a small area. The area must be chosen just large enough so that Θ is sensibly constant over it.

Equation (68a) cannot be used for long pipes in which the temperature changes materially. Here the heat transfer may be computed by means of equation (70) if the temperature reduction $(\Theta_1 - \Theta)$ in the pipe length under consideration is known. The end temperature Θ can be predicted from equation (72) when the initial temperature Θ_1 is given.

As is seen the variable ϑ_u likewise plays a role. But this is only true for long pipe lengths for here the heat loss in the friction layer is decisive. For short pipe sections ϑ_u does not enter (this is also the region in which the conductance is applicable) (see equation(68a)). Then the temperatures are practically constant along the short surface length, and the heat diffusion ΔQ is given by equation (68a) as a function of the temperature gradients at the wall, which fact is included in equation (83) for α^*

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*The Kármán derivation appeared recently. For the correction the customary simplifying assumptions were made (uniformity of heat flow, shearing stress, and fluid properties). The constants of equation (2) are obtained by extrapolations of velocity measurements by Nikuradse to lower values of the distance from the wall. This extrapolation, however, is not confirmed by the measurements of the flow near the wall in the present work.

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*Stanton, with his "surface tube," probably attained the greatest closeness to the wall. But even these test data are not sufficient for the present considerations.

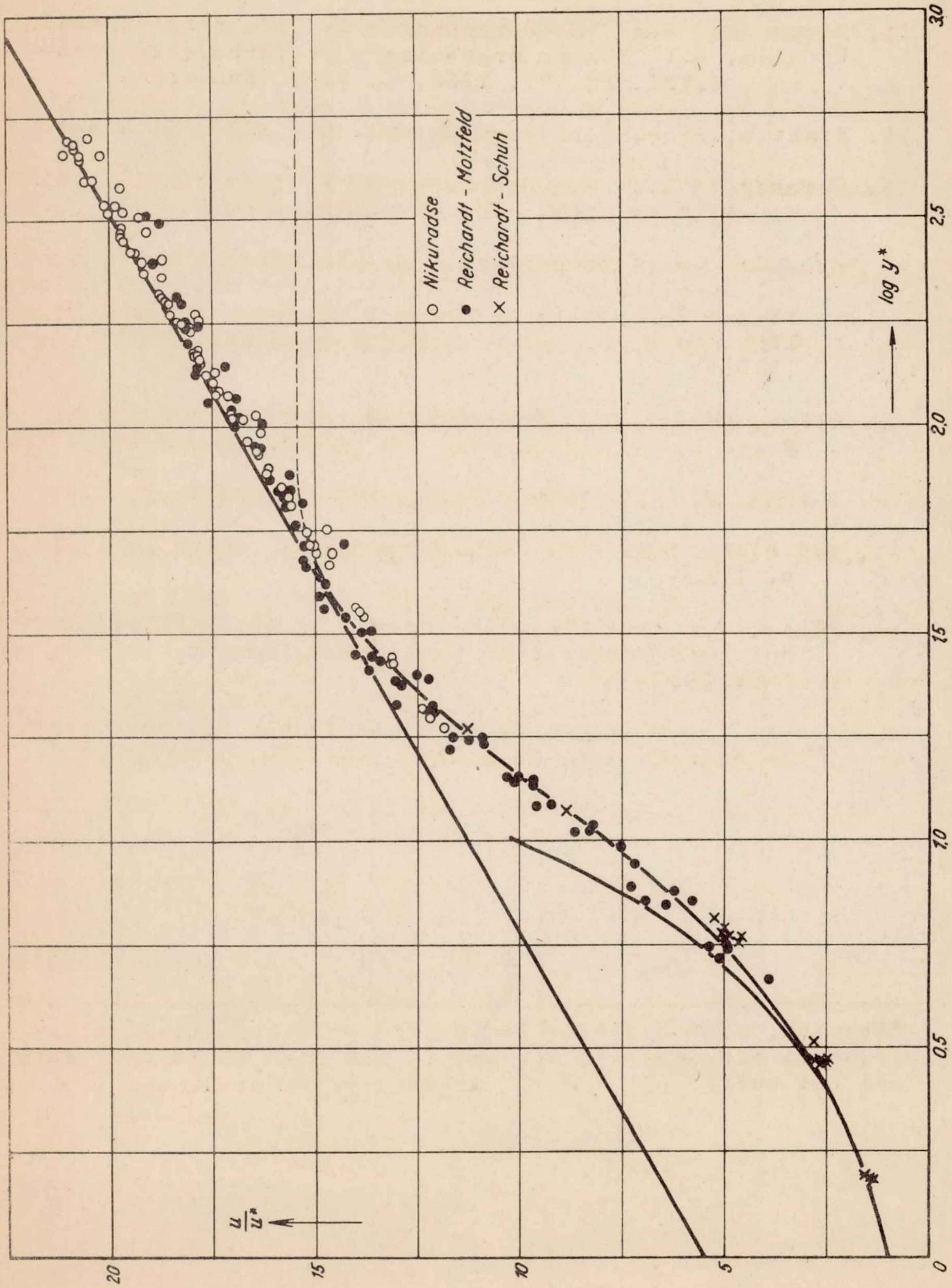

Figure 1.- Universal velocity distribution near the wall.

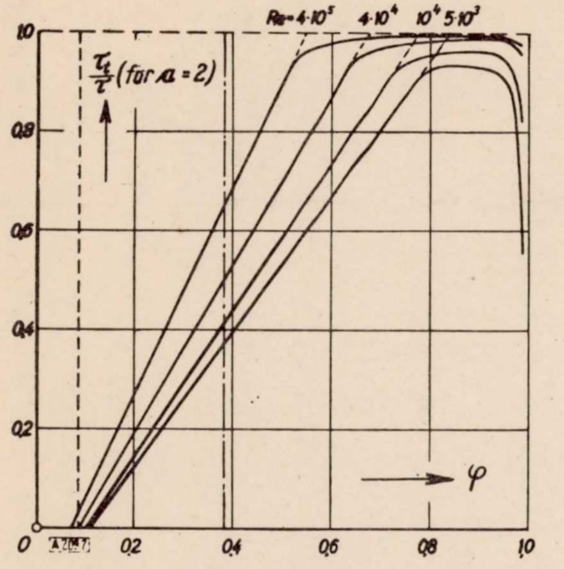

Figure 2.- The ratio τ_t/τ of the turbulent shear stress to the total shear stress as a function of the velocity ratio $\varphi = u/u_{max}$. Reynolds' number is the parameter.

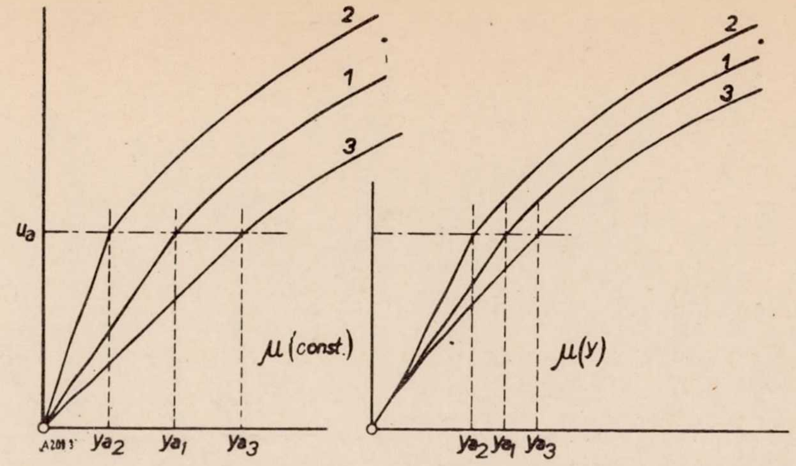

Figure 3.- Influence of viscosity changes on the velocity distribution near the wall for a given wall shear stress.

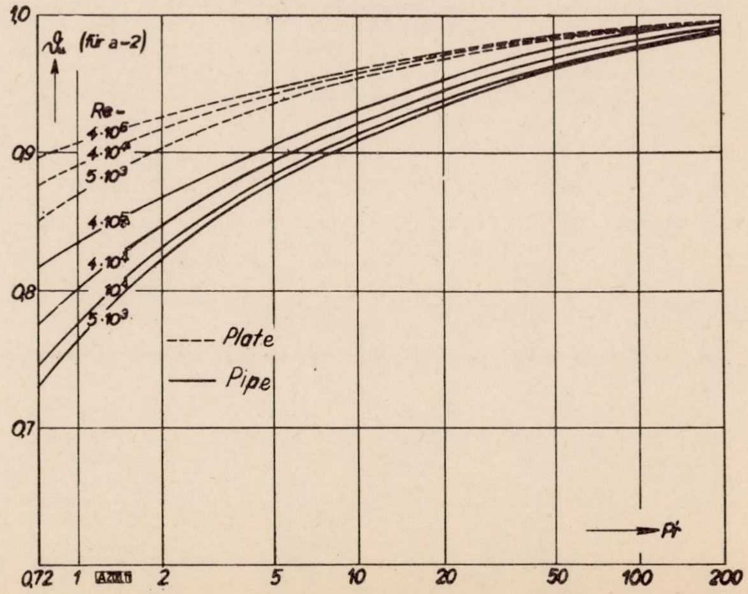

Figure 11.- The dimensionless mixed mean temperature $\vartheta_u = T_u/T_{max}$ as a function of Pr' with Re as the parameter for both pipe and flat plate.

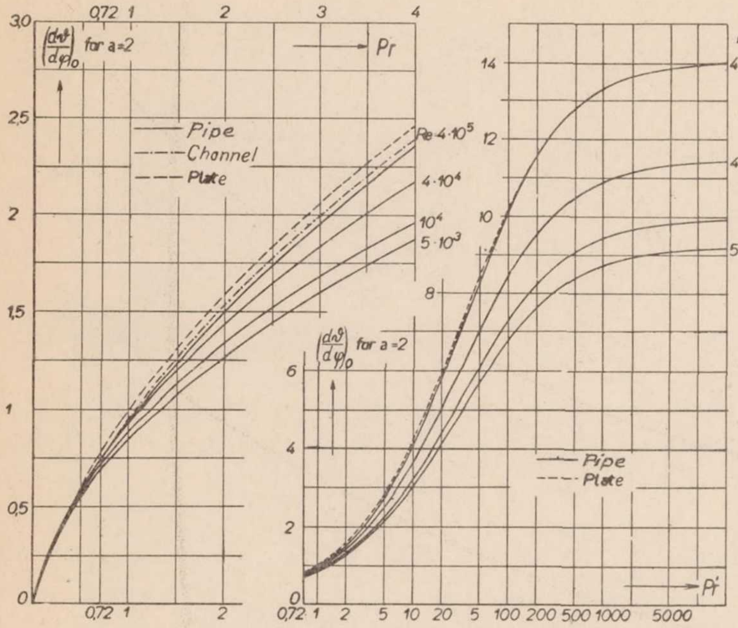

Figure 4.- $(d\vartheta/d\varphi)_o$, the ratio of the temperature gradient $(d\vartheta/dy)_o$ to the velocity gradient $(d\varphi/dy)_o$ both measured at the wall as a function of the generalized Prandtl number.

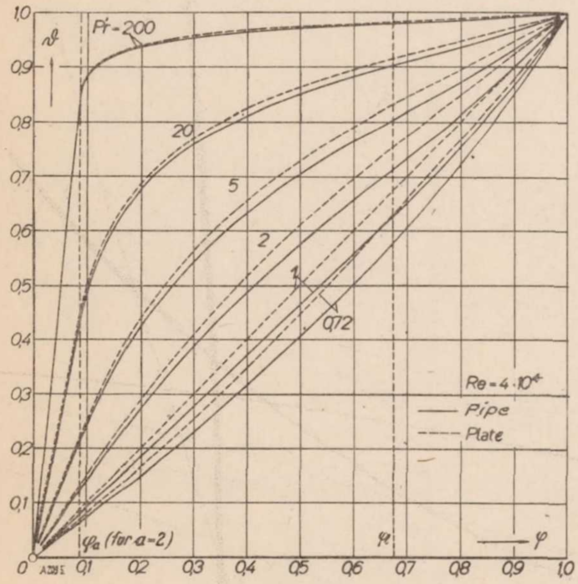

Figure 5.- Temperature (ϑ) as a function of velocity (φ) for flow through pipes and along plates with Pr' as the parameter. Re = 4 x 10^4.

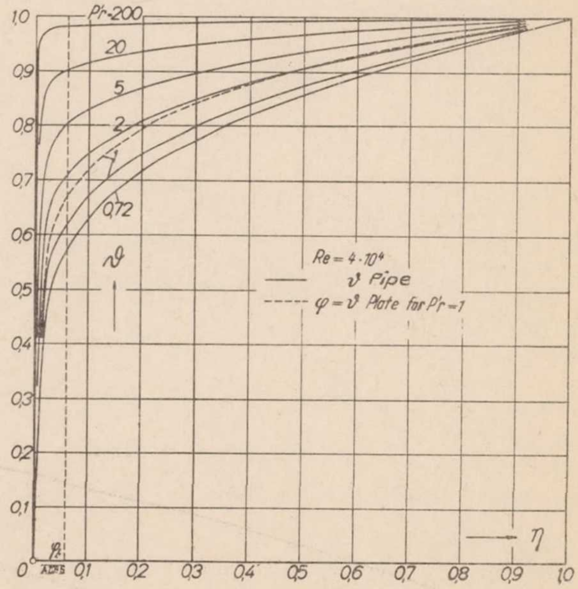

Figure 6.- Temperature (ϑ) as a function of η = (y/r) where y = the distance from the wall for the pipe with Pr' as the parameter and the velocity distribution $\varphi \cong \vartheta$ at a flat plate for Pr = 1. Re = 4 x 10^4

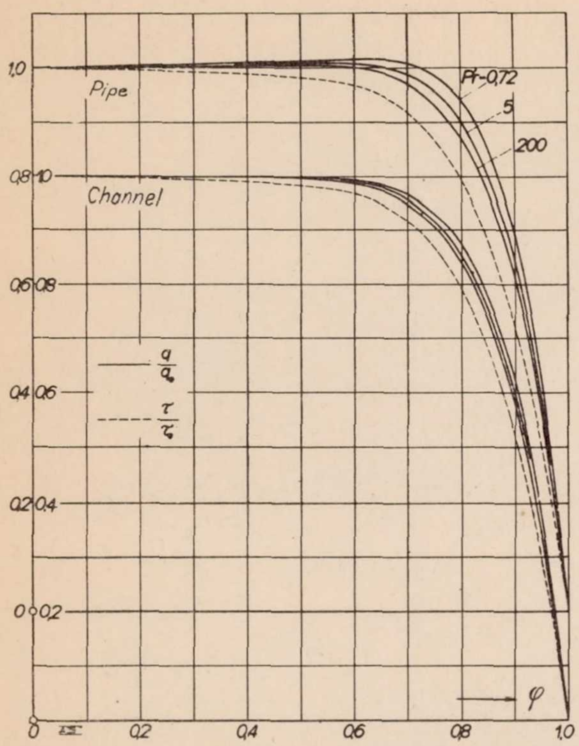

Figure 7.- Heat rate q/q_0 and shear stress (τ/τ_0) distribution in a pipe and in a channel as functions of the velocity φ, with Pr' as the parameter. Re = 4×10^4.

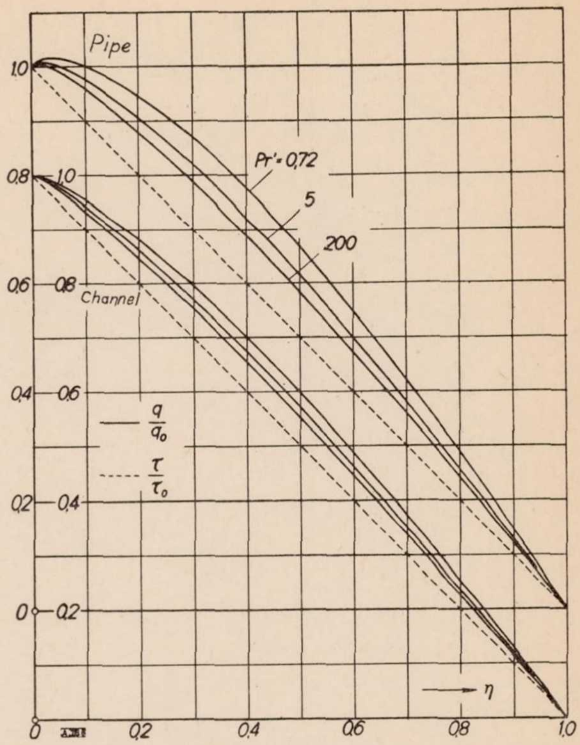

Figure 8.- Distribution of the heat rate (q/q_0) and shear stress (τ/τ_0) as a function of the wall distance η for Re = 4×10^4.

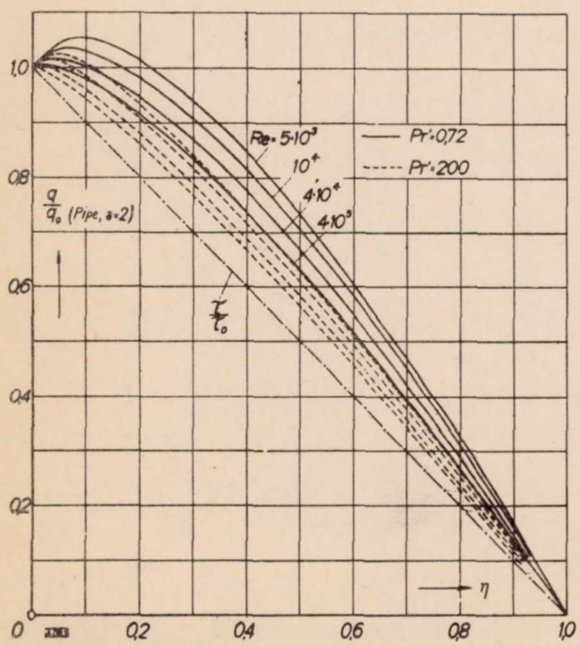

Figure 9.- Distribution of the heat rate q/q_0 as a function of η in a pipe for Pr' = 0.72 and Pr' = 200 with Re as the parameter.

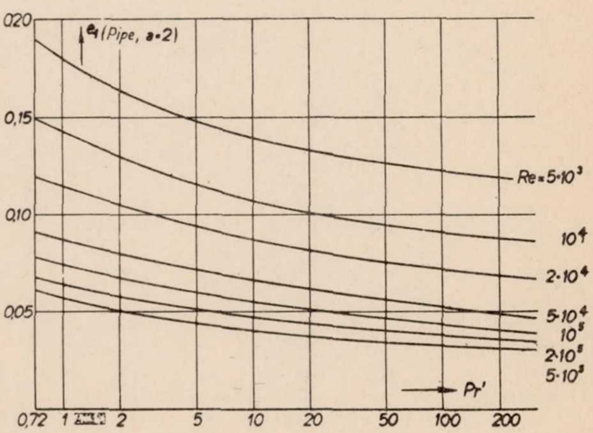

Figure 10.- The variable ρ, as a function of Pr with Re as the parameter. Flow in a pipe with a = 2.0.

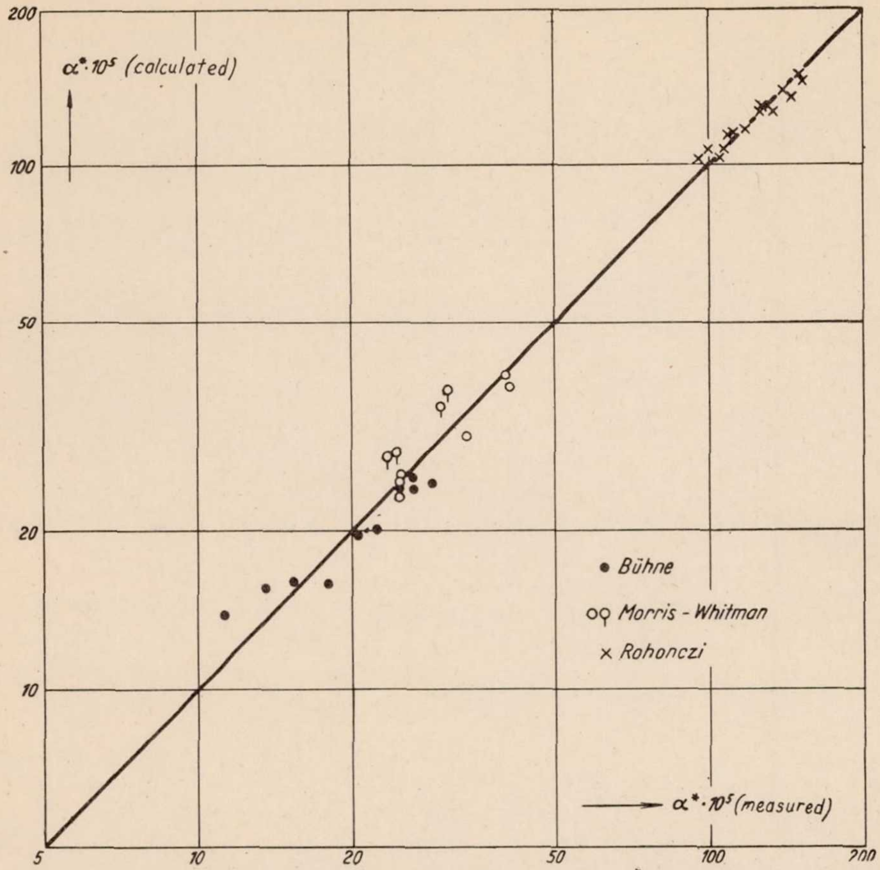

Figure 12.- Heat transfer coefficient α^* in tubes from the measurements of Bühne, Morris and Whitman and Rohonczi compared with the predicted values from equation 83. (a = 2.2, A_q/A = 1.1).

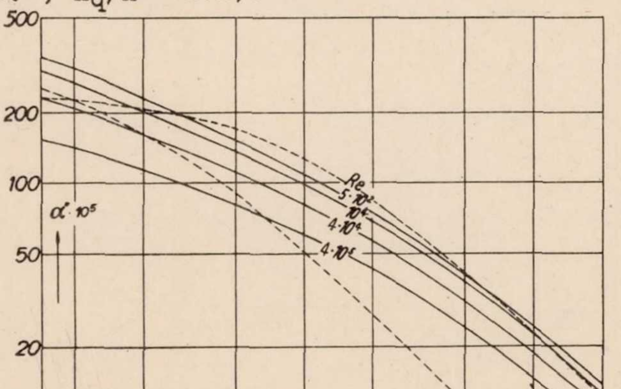

Figure 13.- Heat transfer coefficient α^* as a function of Pr from the theory with Reynolds' number as the parameter (properties invariable, a = 2.0, A_q/A = 1). Prandtl's predicted results are included for a = 2.0 (dashed) and a = 8.8 (dotted).