NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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No. 1067

THE MINIMUM ENERGY LOSS PROPELLER

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SUMMARY

Various cases are presented of the solution of the problem of the most efficient propeller, more general cases being considered than the one by Betz in 1919: namely, that of a propeller under a limiting light load. The problem is solved directly and also with the aid of the Ritz method which became readily applicable after the author proposed a method for the solution of the propeller problem, in general, with the aid of trigonometric series. The design of a propeller with the aid of this method is given and an analysis is made of the effect of the fuselage and of the viscosity coefficient μ on the character of the solution of the variational problem.

SYMBOLS

$\overline{P}$ total power of propeller (nondimensional)

P_p useful power of propeller (nondimensional)

$\overline{\Gamma}$ nondimensional circulation $(k\Gamma/4\pi\omega R^2)$

k number of blades

ω angular velocity of propeller

R radius of propeller

$\overline{v}_{t1}$, $\overline{v}_{a1}$ nondimensional tangential and axial velocities induced in the plane of the propeller disk by the free helical vortices on the line of the bound vortex

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[1]Report No. 455, of the Central Aero-Hydrodynamical Institute, Moscow, 1939.

$\bar{r}$ nondimensional radius of propeller; symbol is also used for $\omega\bar{r}$ the nondimensional rotational velocity

$\bar{V}$ axial velocity (nondimensional)

$\bar{W}$ $\sqrt{\bar{V}^2+(\omega\bar{r})^2}$, resultant of $\bar{V}$ and $\omega\bar{r}$.

$\bar{w}_{n1}$ resultant induced (nondimensional) velocity $= \sqrt{\bar{v}_{t1}^{\,2} + \bar{v}_{a1}^{\,2}}$

p, W pressure and resultant velocity at infinity ahead of the propeller

p_2, $W_2 = \sqrt{(\omega r_2 - v_{t2})^2 + (V + v_{a2})^2}$, pressure and velocity at infinity behind the propeller

$\Delta p = p - p_2$

v_{a2}, v_{t2} axial and tangential induced velocities at infinity behind propeller

$v'_{a1} = v_{a2}/2$, $v'_{t1} = v_{t2}/2$

T propeller thrust

$\bar{w}_{n2}$ induced velocity at infinity behind propeller

φ blade setting

h propeller-fuselage interference coefficient

$\bar{v}'_i$ interference velocity of propeller-fuselage system

ON THE BETZ SOLUTION OF THE VARIATIONAL PROBLEM OF THE AIRPLANE PROPELLER

As is known, the credit belongs to A. Betz for giving an approximate solution of the problem of the propeller of minimum energy loss for the case of a finite number of blades. (See reference 1.) In general, the problem of the variable propeller in an ideal fluid may be stated thus: It is required to find the condition under which the integral expressing the total power of the propeller (in nondimensional units):

$$\overline{P} = \int_{\xi}^{1} \overline{\Gamma} \left(\overline{V} + \overline{v}_{a_1}\right) \overline{r} \, d\overline{r} = \text{minimum} \qquad (1)$$

while the integral

$$\overline{P}_p = \overline{V}\,\overline{P} = \overline{V} \int_{\xi}^{1} \overline{\Gamma} \left(\overline{r} - \overline{v}_{t_1}\right) d\overline{r} = \text{constant} \qquad (2)$$

where $\overline{P}_p$ is the useful nondimensional power of the propeller. In the integrals (1) and (2) $\overline{\Gamma}$ is the nondimensional circulation equal to $k\overline{\Gamma}/4\pi\omega R^2$.

where

k number of propeller blades

ω angular velocity of the propeller

R propeller radius

The magnitudes $\overline{v}_{t_1}$ and $\overline{v}_{a_1}$ are the nondimensional tangential and axial velocities induced in the plane of the propeller disk by the free helical vortices on the line of the bound vortex. These velocities are unknown functions of $\overline{\Gamma}$ the character the change of which with the nondimensional radius $\overline{r}$ is likewise unknown. An added condition imposed on the function $\overline{\Gamma}(\overline{r})$ for a finite number of blades with free tips is: $\overline{\Gamma}(0) = \overline{\Gamma}(1) = 0$.

It follows immediately from what was said in the foregoing, that the losses of the screw propeller are expressed as follows:

$$\overline{E} = \overline{P} - \overline{P}_p = \int_{0}^{1} \overline{\Gamma} \left(\overline{r}\,\overline{v}_{a_1} + \overline{V}\,\overline{v}_{t_1}\right) d\overline{r} \qquad (3)$$

It is not difficult to see that the expression in parenthesis is no other than the modulus of the vector product:

$$\left[\vec{\overline{W}}, \vec{\overline{w}}_{n1}\right] = \begin{vmatrix} \vec{\tau}^{\,0} & \vec{a}^{\,0} & -\vec{r}^{\,0} \\ \omega\overline{r}, & \overline{V}, & 0 \\ -\overline{v}_{t1}, & \overline{v}_{a1}, & 0 \end{vmatrix}$$

in which $\vec{\tau}^{\,0}$, $\vec{a}^{\,0}$ and $\vec{r}^{\,0}$ are unit vectors in the tangential, axial, and radial directions, respectively, and therefore

$$\omega\overline{r}\,\overline{v}_{a1} + \overline{V}\,\overline{v}_{t1} = \overline{W}\,\overline{w}_{n1}\,\sin(\widehat{\vec{W}\,\vec{w}_{n1}})$$

where

$$\overline{W} = \sqrt{\overline{V}^2 + \overline{r}^2}$$

$$\overline{w}_{n1} = \sqrt{\overline{v}^2_{t1} + \overline{v}^2_{a1}}$$

If it is assumed that $\vec{w}_{n1} \perp \vec{W}$, then

$$\omega\,\overline{r}\,\overline{v}_{a1} + \overline{V}\,\overline{v}_{t1} = \overline{W}\,\overline{w}_{n1}$$

and

$$\overline{E} = \int_0^1 \overline{\Gamma}\,\overline{W}\,\overline{w}_{n1}\,d\overline{r}$$

To compute the true angle between the velocities $\vec{W}$ and $\vec{w}_{n1}$ use is made of the Bernoulli equation which is written down for a streamline through the propeller blade. If the pressure and velocity at infinity ahead of the propeller are p and W while at infinity behind the propeller they are p_2 and $W_2 = \sqrt{(\omega r_2 - v_{t2})^2 + (V + v_{a2})^2}$ the Bernoulli equation gives immediately:

$$\left(V + \frac{v_{a2}}{2}\right) v_{a2} - \left(\omega r - \frac{v_{t2}}{2}\right) v_{t2} = \frac{\Delta p}{\rho}$$

where $\Delta p = p - p_2$. Denoting for briefness $v_{a2}/2$ and $v_{t2}/2$, respectively, by v'_{a1} and v'_{t1} thus

$$(V + v'_{a1})\, v'_{a1} - (\omega r - v'_{t1})\, v'_{t1} = \frac{1}{2}\,\frac{\Delta p}{\rho} \qquad (a)$$

The magnitude $\Delta p/\rho$ at infinity behind the propeller is constant along the same streamline but changes in passing from one streamline to another and is a periodic function of the polar angle σ on which the fluid particles of the streamline considered are displaced relative to a certain initial helical surface which may be taken as one of the vortex surfaces originating at the propeller blades. The same can be said with regard to the induced velocities v_{t2} and v_{a2}. The period of all these magnitudes is equal to $2\pi/k$ where k is the number of blades. For heavily loaded propellers the magnitude $\Delta p/\rho$ may reach large values For lightly loaded propellers this magnitude is small in comparison with the values $V v_{a2}$ and $\omega r\, v_{t2}$. For a propeller with an infinite number of blades $\Delta p/\rho$ is given by

$$\frac{\Delta p}{\rho} = \int_{r_2}^{R_2} \frac{v^2_{t2}}{r}\, dr$$

In most of the present day theories of the lightly loaded propeller it is assumed that $\vec{w}'_{n1} \perp \vec{W}$ (Betz, Prandtl, Kawada, etc.) to which the relations as follows correspond:

$$v'_{a1} = w'_{n1} \cos\beta,\ v'_{t1} = w'_{n1} \sin\beta$$

hence (fig. 1) it follows that

$$V v'_{a1} - \omega r\, v'_{t1} = 0$$

It is not difficult to see that the assumption of the perpendicularity of $\vec{w}'_{n1}$ to $\vec{W}$ is equivalent to neglecting in formula (a) the magnitudes v'^2_{a1} and $v'^2_{t1} - \Delta p/2\rho$, that is, to the linearization of the problem. Such linearization is possible only for the case of lightly loaded propellers when the magnitude v'_{a1} is actually small by comparison with V and $v'^2_{t1} - \Delta p/2\rho$ is small by comparison with ωr. Moreover, the foregoing theories assume also that:

$$v'_{t1} = \frac{v_{t2}}{2} = v_{t1}$$

$$v'_{a1} = \frac{v_{a2}}{2} = v_{a2}$$

where v_{t1} and v_{a1} are the induced velocities in the plane of the propeller. This second assumption is equivalent to the assumption that the helical vortices lie on the surfaces of circular cylinders and have a constant axial pitch.[1] In what follows it is assumed in correspondence with what has been said that

$$v_{a1} = w_{n1} \cos \beta$$

$$v_{t1} = w_{n1} \sin \beta$$

$$V v_{a1} = \omega r v_{t1} = 0$$

The assumption of the foregoing relations involves certain errors in the determination of the velocities v_{a1} and v_{t1}. The effect of these errors on the values of the velocities $\omega r - v_{t1}$ and $V + v_{a1}$ in the expressions for the thrust and power is very small, however, because the velocities v_{t1} and v_{a1} for light and moderate loads are small by comparison with ωr and V but nevertheless not so small that they can be neglected. Thus rejection of the component $\left(w'^2_{n1} - \frac{1}{2}\frac{\Delta p}{\rho}\right)$ by comparison with $Vv'_{a1} - \omega r v'_{t1}$ in formula (a) does not at all mean that v_{t1} should be neglected by comparison with ωr and v_{a1} by comparison with V in the expressions for the thrust and power. Thus, for example, an error even of 10 percent in the determination of v_{a1} when taken equal to 0.1V gives in the

[1]In the linearized theory this assumption is a simple consequence of neglecting v_a and v_t by comparison with V and ωr in the formulas of Biot-Savart for determining v_{a1} and v_{t1}.

expression for $V + v_{a_1}$ an error of less than 1 percent as can easily be verified by computation. Check computations of propellers show that the foregoing assumptions with regard to v_{t_1} and v_{a_1} permit the obtaining of values for P and T in very good agreement with the results of experiment.

THE PROBLEM OF BETZ AND ITS SOLUTION

It is quite evident that the problem of the propeller of maximum efficiency is equivalent to the problem of the propeller with minimum energy loss since finding the minimum $\overline{P}$ for given $\overline{P}_p$ is equivalent to finding the minimum of $\overline{E} = \overline{P} - \overline{P}_p$. An exact solution of this problem presents very great difficulties and for this reason it is necessary to solve it by making some simplifying assumptions. Depending on the character of these assumptions various solutions are obtained.

The preliminary problem solved by Betz is the following: To find the conditions for which

$$\overline{E} = \int_0^1 \overline{\Gamma}\, \overline{W}\, \overline{w}_{n_1}\, d\overline{r} = \text{minimum} \tag{4}$$

if

$$\overline{P}_p = \int_0^1 \overline{\Gamma}\, \overline{r}\, d\overline{r} = \text{constant} \tag{5}$$

Thus, in his initial solution Betz neglects the velocity $\overline{v}_{t_1}$ by comparison with $\omega\overline{r}$ and assumes that $\vec{w}_{n_1} \perp \vec{W}$. The detailed solution of the problem for the case assumed by Prandtl that $\vec{w}_{n_1} \perp \vec{w}_1$ was presented in CAHI Report No. 324. In view of the importance of the Betz problem for further discussion, the method of its solution will be briefly presented, particularly since the method proposed by Betz himself is not very clear and at times raises some doubt as to its rigor.

The vortex sheet formed by the helical vortices at infinity may be considered as a surface of discontinuity of the potential Φ of the flow which takes place outside this surface. The circulation corresponding to a propeller element at any radius $\bar{r}$ will be equal to the potential difference between the points on the radius at each side of the surface of discontinuity; that is, will be equal to

$$\bar{\Gamma} = \bar{\Phi}_t - \bar{\Phi}_b = \bar{\Phi}_\Delta$$

where the subscripts t and b denote that the potential Φ is taken, respectively, at the "top" and "bottom" sides of the helical surface, where by the "top" side is meant that side which is in the direction of motion of the propeller. Assume therefore that the helical vortices have a constant axial pitch over their entire extent in which case it may be verified that the induced velocity in the plane of the propeller w_{n1} is equal to half the velocity w_{n2} at a great distance behind the propeller. It is assumed, moreover, that $\overrightarrow{w_{n2}} \perp \overrightarrow{W}$ and may then be written

$$w_{n2} = \frac{d\Phi}{dn}$$

that is, the derivative along the normal to the surface of the potential Φ the normal being in the direction from top to bottom side of the surface. On the basis of what has been said the expression for the losses may be written as follows:

$$E = \frac{k\rho}{2}\int_0^R (\Phi_t - \Phi_b)\frac{d\Phi}{dn} W\,dr = \frac{k\rho}{2}\iint_{f\ t} \Phi_t \frac{d\Phi}{dn}\,df - \frac{k\rho}{2}\iint_{f\ b} \Phi_b \frac{d\Phi}{dn}\,df$$

where k is the number of blades and $df = W\,dr\,dt$ is the element of surface swept out in time dt by the bound vortex of the blade so that $W dr = \int_0^{t=1} W\,dr\,dt$. This surface is two-sided and therefore the difference between the integrals in the formula for the losses may be written in the form

$$E = \frac{k\rho}{2}\iint_f \Phi \frac{d\Phi}{dn}\,df \tag{6}$$

From the second formula of Green there is obtained

$$E = \frac{k\rho}{2} \iint_f \Phi \frac{d\Phi}{dn} df = \frac{k\rho}{2} \left\{ \iiint_\tau \left(\vec{\nabla}\Phi\right)^2 d\tau + \iiint_\tau \Phi \nabla^2 \Phi d\tau \right\}$$

where $\vec{\nabla}$ is the Hamiltonian operator. Since the fluid in the case considered is incompressible $\nabla^2\Phi = 0$ and therefore

$$E = \frac{k\rho}{2} \iiint_\tau \left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2 \right] d\tau > 0 \qquad (7)$$

where $d\tau$ is the element of volume of the fluid displaced in time dt by a surface element in the direction of the normal. The magnitude $\rho d\tau = dm$ is the mass of fluid included within this volume and therefore the loss E represents the kinetic energy of the fluid at infinity displaced in unit time by the propeller blades. To solve the problem of the most efficient propeller, again pass to nondimensional notation and then obtain

$$\overline{E} = \frac{1}{2} \iint_{f^*} \Phi^* \frac{d\Phi^*}{dn} df^*; \quad \overline{P} = \iint_{f^*} \Phi^* \cos\beta \, df^*$$

Let two flows be given with corresponding potentials $\Phi^{*\prime}$ and $\Phi^{*\prime\prime}$. Now consider a third flow with potential $\Phi^{*\prime\prime} - \Phi^{*\prime}$ and shall then have

$$\overline{E}^{\prime\prime\prime} = \frac{1}{2} \iint_{f^*} (\Phi^{*\prime\prime} - \Phi^{*\prime}) \left(\frac{d\Phi^{*\prime}}{dn} - \frac{d\Phi^{*\prime}}{dn}\right) df^*$$

$$= \overline{E}^{\prime\prime} + \overline{E}^{\prime} - \frac{1}{2} \iint_{f^*} \left(\Phi^{*\prime} \frac{d\Phi^{*\prime\prime}}{dn} + \Phi^{*\prime\prime} \frac{d\Phi^{*\prime}}{dn}\right) df^*$$

$$= \frac{1}{2} \iiint_\tau \left[\vec{\nabla}(\Phi^{*\prime\prime} - \Phi^{*\prime})\right]^2 d\tau > 0. \qquad (8)$$

The condition of equality of thrust is written in the form

$$\iint_{f^*} \Phi^{*\prime} \cos\beta df^* = \iint_{f^*} \Phi^{*\prime\prime} \cos\beta df^* \tag{9}$$

On the other hand for an incompressible fluid, on the basis of the third formula of Green there is obtained

$$\iint_{f^*} \left(\Phi^{*\prime} \frac{d\Phi^{*\prime\prime}}{dn} - \Phi^{*\prime\prime} \frac{d\Phi^{*\prime}}{dn} \right) df^* = 0$$

and therefore

$$\overline{E}''' = \overline{E}'' + \overline{E}' - \iint_{f^*} \Phi^{*\prime} \frac{d\Phi^{*\prime\prime}}{dn} df^* > 0 . \tag{10}$$

Assuming that the flow with potential $\Phi^{*\prime\prime}$ possesses the property that

$$\frac{d\Phi_{\Delta}^{*\prime\prime}}{dn} = \overline{w}_2 \cos\beta ,$$

where $\overline{w}_2$ is a certain constant, and remembering the constancy of $\overline{P}_p$ there is obtained immediately

$$\overline{E}''' = \overline{E}' - \overline{E}'' > 0$$

and therefore

$$\overline{E}'' < \overline{E}' . \tag{11}$$

It follows that the flow corresponding to the propeller with the minimum loss of energy can be pictured as a solid vortex sheet at infinity moving in the axial direction with velocity $\overline{w}_2$.

The induced velocities in the plane of the propeller disks are obtained by the formulas:

$$\left.\begin{aligned} \overline{v}_{t1} &= \overline{w}_{n1} \sin\beta = \frac{w_2}{2} \frac{\overline{V}\overline{r}}{\overline{r}^2 + \overline{V}^2} , \\ \overline{v}_{a1} &= \overline{w}_{n1} \cos\beta = \frac{w_2}{2} \frac{\overline{r}^2}{\overline{r}^2 + \overline{V}^2} . \end{aligned}\right\} \tag{12}$$

From the proof given in the foregoing it is evident that the theorem holds for a propeller with any number of blades and in particular for a propeller with an infinitely large number of blades. In order to show that the foregoing limiting transition does not affect the character of the solution the expression is written for the losses in the case of the propeller with infinite number of blades. Therefore

$$\overline{E} = \int_0^1 \overline{\Gamma}\,\overline{w}_{n1}\,\overline{W}\,d\overline{r} = \int_0^1 \overline{\Gamma}^2 \frac{\overline{W}^2}{\overline{r}\,\overline{V}}\,d\overline{r}$$

since for $\vec{w}_{n1} \perp \vec{W}$

$$\overline{w}_{n1} = \frac{\overline{v}_{a1}}{\cos\beta} = \frac{\overline{v}_{t1}}{\sin\beta} = \frac{\overline{\Gamma}\,\overline{W}}{\overline{r}\,\overline{V}}$$

Setting up the equation of Euler for the function $F^* = \overline{E} - \Lambda_1\overline{P}_p$ (where Λ_1 is a constant), write, according to the rules of the calculus of variation,

$$\frac{\partial F^*}{\partial\overline{\Gamma}} = 2\overline{\Gamma}\frac{\overline{W}^2}{\overline{r}\,\overline{V}} = -\Lambda_1\overline{V}\,\overline{r} = 0$$

hence

$$\overline{\Gamma} = \frac{1}{2}\Lambda\frac{\overline{r}^2\,\overline{V}^2}{\overline{W}^2} = \frac{\overline{w}_2}{2}\frac{\overline{V}\,\overline{r}^2}{\overline{r}^2 + \overline{V}^2} \qquad (13)$$

where

$$\Lambda_1\overline{V} = \overline{w}_2$$

and therefore

$$\overline{v}_{t1} = \frac{\overline{\Gamma}}{\overline{r}} = \frac{\overline{w}_2}{2}\frac{\overline{V}\,\overline{r}}{\overline{r}^2 + \overline{V}^2}$$

$$\overline{v}_{a1} = \frac{\overline{\Gamma}}{\overline{V}} = \frac{\overline{w}_2}{2}\frac{\overline{r}^2}{\overline{r}^2 + \overline{V}^2}$$

which accurately agrees with formulas (12).

The circulation distribution giving the required velocity distribution changes of course with the number of blades since the form of the functions $\bar{v}_{t_1}(\bar{\Gamma})$ and $\bar{v}_{a1}(\bar{\Gamma})$ varies with this number.

If the useful power of the blade is expressed in the form[1]

$$\bar{P}_p = \bar{V}\int_{\xi}^{1} \bar{\Gamma}\,\bar{r}\,d\bar{r} \tag{14}$$

as is done by Betz and the total power in the form

$$\bar{P} = \int_{\xi}^{1} \bar{\Gamma}\,(\bar{V} + \bar{v}_{a1})\,\bar{r}\,d\bar{r} \tag{15}$$

the expression for the losses must then be written as

$$\bar{E} = \int_{\xi}^{1} \bar{\Gamma}\,\bar{r}\,\bar{v}_{a1}\,d\bar{r} \tag{16}$$

To obtain the minimum of the integral (16) under the condition that the integral (14) remain constant is equivalent to the problem of finding the minimum of the integral (15) under the conditions:

$$\bar{P}_p = \bar{V}\int_{\xi}^{1} \bar{\Gamma}'(\bar{r}-\bar{v}'_{t1})\,d\bar{r} = \bar{V}\int_{\xi}^{1} \bar{\Gamma}''(\bar{r}-\bar{v}_{t1}'')\,d\bar{r}$$

and

$$\int_{\xi}^{1} \bar{\Gamma}'\,\bar{v}'_{t1}\,d\bar{r} = \int_{\xi}^{1} \bar{\Gamma}''\,\bar{v}_{t1}''\,d\bar{r} \tag{17}$$

that is, under the conditions of equal thrusts and equal rotational losses for the propellers compared.

The foregoing problem is thus a variational problem with stronger conditions imposed than the problem of Betz and refers to the propeller with maximum axial efficiency leading to the answer $\bar{v}_{a1}$ = constant as was shown in CAHI Report No. 324. Unfortunately in the latter report the

[1]That is, assuming the flow is irrotational.

restrictions imposed on the problem were not brought out with sufficient clearness and therefore on reading the second section of the third part the impression may be gathered that the solution $\overline{v}_{a1}$ = constant was contrasted with the solution $\overline{w}_1$ = constant (Betz solution); that is, a distinction was made between a propeller with minimum loss of energy and a propeller with maximum efficiency. Actually, it is not a question of such a contrast but simply of two different problems: namely, the propeller with minimum loss of energy and the propeller with maximum axial efficiency. The solution of Betz approaches more nearly the true solution of the problem of the propeller with maximum efficiency since there is approximately taken into account the change in the rotational losses in passing from one propeller to another with the same useful power. It must be said, however, that for propellers with the same diameters, angular speeds, and useful power, the rotational losses constitute almost a constant percent of the power $\overline{P}$.

CASE OF THE MODERATELY LOADED PROPELLER

The problem of Betz was solved actually for the case of a limiting light load on the propeller in which case only may be written

$$\overline{P}_p = \int_0^1 \overline{\Gamma}\, \overline{r}\, d\,\overline{r}$$

In the present section consider the case of a lightly and moderately loaded propeller for which is written

$$\left.\begin{aligned} \overline{P}_p &= \overline{V} \int_0^1 \overline{\Gamma}\,(\overline{r} - \overline{w}_{n1} \sin\beta)\, d\overline{r} \\ \overline{P} &= \int_0^1 \overline{\Gamma}\,(\overline{V} + \overline{w}_{n1} \cos\beta)\, \overline{r} d\overline{r} \end{aligned}\right\} \qquad (18)[1]$$

[1]Thus, in accordance with what was said in the foregoing, the equation is not linearized for the total and useful powers but use is made of the linearized theory only in determining the velocities v_{t1} and v_{a1}.

The foregoing expressions are satisfied, as check computations show, with sufficient accuracy for the previously mentioned class of propellers. The expressions (18) on the basis of the considerations of the previous section may be written in the form

$$\overline{P}_p = \overline{V}\int_0^1 \overline{\Gamma}\left(\overline{r} - \frac{1}{2}\frac{d\Phi^*}{dn}\sin\beta\right)d\overline{r} = \overline{V}\int_0^1 \overline{\Gamma}\,\overline{r}\,d\overline{r} - $$

$$-\frac{1}{2}\iint_{f_1^*} \Phi^* \frac{d\Phi^*}{dn} df_1^*,$$

$$\overline{P} = \int_0^1 \overline{\Gamma}\left(\overline{V} + \frac{1}{2}\frac{d\Phi}{dn}\cos\beta\right)\overline{r}\,d\overline{r} = \overline{V}\int_0^1 \overline{\Gamma}\,\overline{r}\,d\overline{r} +$$

$$+\frac{1}{2}\iint_{f_2^*} \Phi^* \frac{d\Phi^*}{dn} df_2^*,$$

where

$$df_1^* = \overline{V}\sin\beta\,d\overline{r}\,dt^*,$$

$$df_2^* = \overline{r}\cos\beta\,d\overline{r}\,dt^*$$

Now obtain the variations $\delta\overline{P}_p$ and $\delta\overline{P}$. For the first of these there is obtained

$$\delta\overline{P}_p = \overline{V}\int_0^1 \delta\overline{\Gamma}\,\overline{r}\,d\overline{r} - \frac{1}{2}\iint_{f_1^*} \delta\Phi^* \frac{d\Phi^*}{dn} df_1^* - \frac{1}{2}\iint_{f_1^*} \Phi^* \frac{d\delta\Phi^*}{dn} df_1^*,$$

But from the third formula of Green

$$\iint_{f^*}\left(\delta\Phi^*\frac{d\Phi^*}{dn} - \Phi^*\frac{d\delta\Phi^*}{dn}\right)df_1^* = \iiint_{\tau^*}(\delta\Phi^*\nabla^2\dot{\Phi}^* - \Phi^*\nabla^2\delta\Phi^*)\,d\tau^*,$$

which for an incompressible fluid gives

$$\iint_{f_1^*} \delta\Phi^* \frac{d\Phi^*}{dn} df_1^* = \iint \Phi^* \frac{d\delta\Phi}{dn} df_1^*$$

and therefore

$$\delta\overline{P}_p = \overline{V}\int_0^1 \delta\overline{\Gamma}\,\overline{r}\,d\overline{r} - \iint_{f_1^*} \delta\Phi^* \frac{d\Phi^*}{dn} df_1^* = \overline{V}\int_0^1 \delta\overline{\Gamma}\,\overline{r}\,d\overline{r} - \overline{V}\int_0^1 \delta\overline{\Gamma}\frac{d\Phi^*}{dn}\sin\beta\,d\overline{r}$$

In the same way there is obtained

$$\delta\overline{P} = \int_0^1 \delta\overline{\Gamma}\left(\overline{V} + \frac{d\Phi^*}{dn}\cos\beta\right)\overline{r}\,d\overline{r}$$

The condition of the minimum $\overline{P}$ for given $\overline{P}_p$ is expressed as

$$\delta\overline{P} - \Lambda\delta\overline{P}_p = \int_0^1 \delta\overline{\Gamma}\left[\overline{V}\,\overline{r}(1-\Lambda) + \frac{d\Phi^*}{dn}(\overline{r}\cos\beta + \Lambda\overline{V}\sin\beta)\right]d\overline{r} = 0.$$

Since the foregoing equation is true for any $\delta\overline{\Gamma}$, the condition must be satisfied that

$$\bar{V}\bar{r}(1-\Lambda)+\frac{d\Phi^*}{dn}(\bar{r}\cos\beta+\Lambda\bar{V}\sin\beta)=0,$$

hence, noting that

$$\cos\beta=\frac{r}{\bar{W}}, \quad \sin\beta=\frac{\bar{V}}{\bar{W}},$$

now obtain

$$\frac{1}{\bar{W}}\frac{d\Phi^*}{dn}=\frac{(\Lambda-1)\bar{V}\cdot\bar{r}}{\bar{r}^2+\Lambda\bar{V}^2},$$

hence

$$\bar{v}_1=\frac{1}{2}\frac{d\Phi^*}{dn}\sin\beta=\frac{\bar{w}'_2}{2}\frac{\bar{V}'\bar{r}}{r^2+\bar{V}'^2},$$

where

$$\bar{v}_{a1}=\frac{1}{2}\frac{d\Phi^*}{dn}\cos\beta=\frac{\bar{w}'_2}{2}\frac{\bar{r}^2}{\bar{r}^2+\bar{V}'^2}\frac{V'}{V},$$

$$\bar{w}_2=\frac{(\Lambda-1)\bar{V}}{\sqrt{\Lambda}}\equiv\frac{\Lambda_1\bar{V}}{\sqrt{\Lambda_1+1}}; \quad \bar{V}'=\bar{V}\sqrt{\Lambda}\equiv\bar{V}\sqrt{\Lambda_1+1}.$$

From the foregoing two formulas there is obtained

$$\bar{V}'^2=\bar{V}^2(\Lambda_1+1)$$

and therefore

$$\Lambda_1=\frac{\bar{V}'^2-\bar{V}^2}{\bar{V}^2},$$

hence

$$\bar{w}_2=\frac{(\bar{V}'^2-\bar{V}^2)}{\bar{V}'},$$

$$V'^2-\bar{w}_2\bar{V}'-\bar{V}^2=0$$

and

$$\bar{V}'=\frac{\bar{w}_2}{2}\pm\sqrt{\left(\frac{\bar{w}_2}{2}\right)^2+\bar{V}^2}=\frac{\bar{w}_2}{2}\pm\bar{V}\sqrt{1+\left(\frac{\bar{w}_2}{2\bar{V}}\right)^2},$$

where the positive sign corresponds to the physical meaning of the problem. For small values of the ratio $w_2/2V$ the equation may be written approximately

$$\bar{V}'\approx\bar{V}+\frac{\bar{w}_2}{2}$$

and therefore

$$\bar{v}_{t1}\approx\bar{w}_1\frac{\bar{r}(\bar{V}+\bar{w}_1)}{\bar{r}^2+(\bar{V}+\bar{w}_1)^2}, \qquad (19)$$

$$\bar{v}_{a1}\approx\bar{w}_1\frac{\bar{r}^2}{\bar{r}^2+(\bar{V}+\bar{w}_1)^2}$$

where $\bar{w}_1$ denotes $\bar{w}_2/2$. These formulas were proposed by Prandtl in 1919 without any proof for the case of moderately loaded propellers. In 1927, in volume VII of the Handbuch der Physik and also in 1932 in Ingenieur-Archiv, Heft 1, A. Betz gave a proof that formulas (19) are a solution of the variational problem of the screw propeller. His proof, however, was based on the so-called method of displacements and cannot be called entirely convincing.[1]

In concluding this section the equation is derived which must be satisfied by the potential Φ of the flow outside the vortex sheet. For this purpose now write the equation of continuity of the flow $\nabla^2\Phi = 0$ in cylindrical coordinates. Therefore

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \qquad (20)$$

In order to reduce this equation to a simpler form in the case of flow about a helical surface the new variables are introduced

$$\zeta = \theta - \frac{\omega}{V} z; \quad \frac{\omega r}{V} = \rho$$

then

$$\frac{\partial}{\partial z} = -\frac{\omega}{V}\frac{\partial}{\partial\zeta}; \quad \frac{\partial}{\partial\theta} = \frac{\partial}{\partial\zeta}; \quad \frac{\partial}{\partial r} = \frac{V}{\omega}\frac{\partial}{\partial\rho}$$

and the equation of continuity assumes the form

$$\frac{1}{\rho}\frac{\partial}{\partial r}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{\partial^2\Phi}{\partial\zeta^2}(1 + \rho^{-2}) = 0 \qquad (21)$$

This is the required equation for the potential Φ in terms of the independent variables ζ and ρ.

The additional conditions which must be satisfied by the function are the following: for $\rho \geqslant \rho_0$, where

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[1]At present a rigorous solution has been obtained of the variational problem for the case of the nonlinearized theory.

ρ_0 is the value of ρ at the tip of the blade Φ must be a single value and continuous function of ρ and ζ; moreover, Φ must be an odd periodic function of ζ and become zero for $\rho = \infty$. For $\rho < \rho_0$ Φ must undergo a discontinuity equal to Γ on the vortex surfaces.

On the surface of the vortex sheet for which $\zeta = 0$ the condition holds:

$$\frac{d\Phi}{dn} = v_{a2} \cos \beta - v_{t2} \sin \beta = \frac{\partial\Phi}{\partial\zeta} \frac{\omega^2r^2+V^2}{\sqrt{\omega^2r^2+V^2}} \frac{1}{Vr} = f(r)$$

where $f(r)$ is a given continuous function of r. For the case of a rigid helical surface this function is of the form

$$f(r) = w_2 \cos \beta = w_2 \frac{\overline{r}}{\sqrt{\overline{r}^2 + \overline{\overline{V}}^2}}$$

where w_2 is a constant magnitude.

Equation (21) was first obtained by Goldstein in 1929 in investigating the flow of a solidified vortex sheet of constant axial pitch.

SOLUTION OF THE PROBLEM OF THE MOST EFFICIENT PROPELLER WITH THE AID OF THE RITZ METHOD

As was shown earlier the problems of the propeller with finite number of blades are well solved with the aid of trigonometric series. Then assume that the circulation $\overline{\Gamma}_1$ is such that it can be expressed by the series

$$\overline{\Gamma}_1 = \sum_{n=1}^{\infty} A_n \sin n\theta, \tag{1}$$

the total induced velocity $\overline{w}_{n1}$ is expressed by the formula (see reference 2)

$$\overline{w}_{n1} = \frac{\pi}{\overline{L}} \frac{1}{\sin\theta} \sum A_n [n \cdot \sin n\theta + C_n(\theta)], \tag{2}$$

where C_n are known values depending on θ and on $\overline{V}$ and k as parameters, and $\overline{L}$ and θ ar(connected with the radii of the elements of the propelle by the equations

$$\overline{r} = \xi + \overline{L} - \overline{L}\cos\theta; \quad d\overline{r} = \overline{L}\sin\theta d\theta,$$

where $\overline{L}$ is half the effective part of the blade and ξ corresponds to its noneffective part. Since it is assumed that $\vec{w}_{n1} \perp \vec{W}$ it is clear that

$$\overline{v}_{a1} = \overline{w}_{n1}\cos\beta = \overline{w}_{n1}\frac{\overline{r}}{\sqrt{\overline{r}^2 + \overline{V}^2}}, \tag{3}$$

$$\overline{v}_{t1} = \overline{w}_{n1}\sin\beta = \overline{w}_{n1}\frac{\overline{V}}{\sqrt{\overline{r}^2 + \overline{V}^2}}. \tag{4}$$

Bearing in mind formulas (1), (2), (3), and (4) then write equations (18), section 3, in the form

$$\overline{P}_1 = \int_0^{\pi} \sum A_n \sin n\theta \left(\overline{V} + \frac{\pi}{\overline{L}}\frac{\cos\beta}{\sin\theta}\sum A_n a_n\right)\overline{r}\,\overline{L}\sin\theta d\theta, \tag{5}$$

$$\frac{1}{k}\overline{P}_p = \overline{P}_{1p} = \overline{V}\int_0^{\pi}\sum A_n \sin n\theta\left(\overline{r} - \frac{\pi}{\overline{L}}\frac{\sin\beta}{\sin\theta}\sum A_n a_n\right)\overline{L}\sin\theta d\theta, \tag{6}$$

where

$$\overline{r} = \xi + \overline{L} - \overline{L}\cos\theta, \quad a_n = n\sin n\theta + C_n. \tag{7}$$

Removing the parentheses formula (5) is readily reduced

$$\overline{P}_1 = \overline{V}\sum A_n \int_0^{\pi} \sin n\theta(\xi + \overline{L} - \overline{L}\cos\theta)\overline{L}\sin\theta d\theta +$$

$$+ \int_0^{\pi}\sum A_n \sin n\theta\ \pi\ \cos\beta(\xi + \overline{L} - \overline{L}\cos\theta)\sum A_n a_n d\theta \tag{8}$$

Then denote the first integral of this formula by I'_n and moreover, set

$$\pi \cos\beta \sin n\theta (\xi + \overline{L} - \overline{L}\cos\theta) = b_n, \tag{9}$$

Formula (8) may then be written as

$$\overline{P}_1 = \sum A_n I_n' + \int_0^\pi \sum A_n b_n \sum A_n a_n d\theta, \tag{10}$$

where

$$\sum A_n I_n' = \overline{V} \sum A_n \int_0^\pi \sin n\theta (\xi + \overline{L} - \overline{L}\cos\theta) \overline{L} \sin\theta d\theta =$$

$$= \overline{V} \frac{\pi}{2} L \left[(\xi + \overline{L}) A_1 - \overline{L} \frac{A_2}{2} \right].$$

The expressions under the integral signs may be written in the form

$$\begin{aligned} &\sum A_n b_n \sum A_n a_n = \\ &= A_1 b_1 (A_1 a_1 + A_2 a_2 + \ldots + A_n a_n) + \\ &+ A_2 b_2 (A_1 a_1 + A_2 a_2 + \ldots + A_n a_n) + \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots + \\ &+ A_n b_n (A_1 a_1 + A_2 a_2 + \ldots + A_n a_n) = \\ &= \sum_\nu \sum_\varkappa A_\nu A_\varkappa b_\nu a_\varkappa = \sum_\nu \sum_\varkappa A_\nu A_\varkappa G''_{\nu\varkappa}. \end{aligned} \tag{11}$$

Substituting this result in formula (10) there is obtained

$$\overline{P}_1 = \sum_n A_n I_n' + \sum_\nu \sum_\varkappa A_\nu A_\varkappa I''_{\nu\varkappa}, \tag{12}$$

where

$$n, \varkappa, \nu = 1, 2, 3 \ldots n$$

$$I''_{\nu\varkappa} = \int_0^\pi G_{\nu\varkappa} d\theta. \tag{13}$$

Passing to the expression for the useful power then find

$$\overline{P}_{1p} = \overline{V} \int_0^\pi \sum A_n \sin n\theta (\xi + \overline{L} - \overline{L}\cos\theta) \overline{L} \sin\theta d\theta -$$

$$- \overline{V} \int_0^\pi \sum A_n \pi \sin n\theta \sin\beta \sum A_n a_n d\theta. \tag{14}$$

The first integral is computed and gives

$$\sum_n A_n I_n' = \overline{V} \left[(\xi + \overline{L}) A_1 - \overline{L} \frac{A_2}{2} \right] \frac{\pi}{2} \overline{L}; \tag{15}$$

Setting

$$\pi \overline{V} \sin n\theta \sin\beta = l_n, \tag{16}$$

there is obtained

$$\overline{P}_{1p} = \sum_n A_n I_n' - \sum_\nu \sum_\varkappa A_\nu A_\varkappa I''_{\nu\varkappa}, \tag{17}$$

where

$$I''_{\nu\varkappa} = \int_0^\pi a_\varkappa l_\nu d\theta. \tag{18}$$

Thus there is finally obtained

$$\overline{P}_1 = \sum_n A_n I_n' + \sum_\nu \sum_\varkappa A_\nu A_\varkappa I''_{\nu\varkappa}, \tag{19}$$

$$\overline{P}_{1p} = \sum_n A_n I_n' - \sum_\nu \sum_\varkappa A_\nu A_\varkappa I''_{\nu\varkappa}. \tag{20}$$

Now proceed to the solution of the variational problem of interest: namely, to find the circulation distribution which renders the power $\overline{P}_1$ a minimum while maintaining the useful power of the propeller constant. It is assumed that the propellers having their circulation about the blades expressed by the trigonometric polynomial are being dealt with.

$$\overline{\Gamma}_1 = \sum_{n=1}^{m} A_n \sin n\theta.$$

Thus the problem reduces to finding the coefficients A_n. The LaGrange function will be of the form

$$F^* = \overline{P}_1 - \Lambda \overline{P}_{1p} = \sum_{n=1}^{m} A_n (I_n' - \Lambda I_n') + \sum_\nu^m \sum_\varkappa^m A_\nu A_\varkappa (I''_{\nu\varkappa} + \Lambda I''_{\nu\varkappa}) =$$

$$= (1 - \Lambda) \sum_{n=1}^{m} A_n I_n' + \sum_\nu^m \sum_\varkappa^m A_\nu A_\varkappa (I''_{\nu\varkappa} + \Lambda I''_{\nu\varkappa}).$$

Now obtain the derivative $\frac{\partial F^*}{\partial A_\nu}$ and equate it to zero;

$$\frac{\partial F^*}{\partial A_\nu} = (1 - \Lambda) I_n' + \sum_{\varkappa=1}^{m} A_\varkappa [I''_{\nu\varkappa} + I''_{\varkappa\nu} - \Lambda (I''_{\nu\varkappa} + I''_{\varkappa\nu})] = 0$$

$$\varkappa,\ n = \nu = 1, 2, 3 \ldots m.$$

There should be such derivatives which will give m equations with m + 1 unknowns where the (m + 1) the unknown will be Λ. Adding to these m equations the expression for the thrust there is obtained m + 1 equations. The solution of this system of equations for a practical computation of the propeller is very laborious and for this reason before practically applying the obtained results it is first necessary to do some preliminary work: namely, with given $\overline{V}$ and Λ to find A and P_1. Taking a series of values of Λ there is obtained a series of values for each A and for $\overline{P}_1$. Plotting the graphs of A_i and $\overline{P}_1$ against Λ for the parameter $\overline{V}$ the propellers may then be readily designed with the aid of interpolation. Now note that the above integrals are readily computed graphically once and for all for given n.

As computations have shown, a restriction to three coefficients A_i is sometimes sufficient. The velocity distribution corresponding to the problem just solved will be the same as that given by the formulas (19), section 3. The circulation distribution corresponding to the foregoing formulas can more conveniently be obtained by the method proposed by the author in CAHI Report No. 324. This method requires fewer computations than are required by the Ritz method, which is more of purely theoretical interest.

EFFECT OF THE FUSELAGE AND OF THE COEFFICIENT μ

Let $\overline{v}'_{a_1}$ and $\overline{v}'_{t_1}$ represent the axial and tangential (rotational) velocity components at a certain point of the propeller rotating in front of the fuselage. If the airplane moves with velocity $\overline{V}$ relative to the ground then the relative axial velocity of approach of the flow at the points of the propeller blade will be

$$\overline{V} - \overline{v}_i' + \overline{v}_{a_1}'$$

where v_i' is the velocity arising from the effect of the fuselage (retardation of the flow). This velocity for a fuselage of arbitrary shape is a function of the relative radius $\overline{r}$ of the elements of the propeller and also of the polar angle σ referred to a certain fixed direction in the plane of the propeller disk. In the case where the fuselage is a body of revolution the velocity $\overline{v}_i'$ is a function only of $\overline{r}$.

Now assume that the propeller blades are replaced by rectilinear lifting vortices on the basis of the theorem of Joukowski; then for $\mu = 0$ the following expression for the elementary power of the propeller:

$$d\overline{P} = \overline{\Gamma}'\overline{r}\left(\overline{V} - \overline{v}_i' + \overline{v}_{a_1}'\right) d\overline{r} \equiv (1 - h')\,\overline{\Gamma}'\overline{r}\left(\overline{V} + \overline{v}_{a_1}'\right) d\overline{r}$$

where

$$h' = \frac{v_i'}{V + v_{a_1}}$$

and the primes denote that all magnitudes are referred to the system propeller—fuselage.

The total power absorbed by the propeller will be equal to

$$\overline{P} = \int_0^1 \overline{\Gamma}' \,(1 - h')\,(\overline{V} + \overline{v}_{a_1})\,\overline{r}\,d\overline{r}$$

The propeller does not impart the entire power $\overline{P}$ to the airplane but only a part since losses occur at the propeller. The magnitude of these losses will be given. The induced velocity v'_{a_1} according to the theorem of Joukowski gives rise to a force $\overline{\Gamma}\,\overline{v}_{a_1}d\overline{r}$ directed at right angles to the propeller radius opposite to the direction of rotation of the propeller. Since an element of the propeller rotates with velocity ωr the elementary losses corresponding to the force $\overline{\Gamma}'\overline{v}'_{a_1}d\overline{r}$ will equal

$$d\overline{E}_1 = \overline{\Gamma}'\overline{v}_{a_1}'\,\overline{r}\,d\,\overline{r}$$

In the same way the induced velocity $\overline{v}'_{t_1}$ gives rise to the elementary force $\overline{\Gamma}'\overline{v}_{t_1}'\,d\overline{r}$ parallel to the propeller axis opposite to the direction of forward motion of the propeller. The losses corresponding to this force will be

$$d\overline{E}_2 = \overline{V}\,\overline{\Gamma}'\overline{v}_{t_1}'\,d\overline{r}$$

Thus the induced losses of the propeller will be equal to

$$\overline{E} = \int_0^1 \overline{\Gamma}'\,(\overline{V}\,\overline{v}_{t_1}' + \overline{r}\,\overline{v}_{a_1}')\,d\overline{r}$$

It follows that the useful power which may be taken from the propeller blade is equal to

$$\overline{P}_p = \overline{P} - \overline{E} = \overline{V}\int_0^1 \overline{\Gamma}'\,(\overline{r} - \overline{v}_{t_1}')\,d\overline{r} - \int_0^1 \overline{\Gamma}'\overline{v}_i'\,\overline{r}\,d\overline{r}$$

The magnitude $\overline{P}_p$ is the power which is disposable by the propeller—fuselage system since at the propeller blades only the losses $\overline{E}$ are developed for $\mu = 0$. At light loads when the values of $\overline{v}'_{t_1}$ are not large and for not very thick fuselages when $\overline{v}'_i$ is small by comparison with $\overline{V}$ the expression for $\overline{P}_p$ may be approximately written as:

$$P_p \approx \overline{V} \int_0^1 \overline{\Gamma}'(1 - h'')(\overline{r} - \overline{v}'_{t_1})d\overline{r}$$

where

$$h'' = \frac{\overline{v}_i'}{\overline{V}}$$

The magnitudes h' and h'' taken in the system propeller-fuselage depend on $\overline{\Gamma}$, a condition which renders difficult the solution of the variational problem, especially since the finding of h' and h'' is very complicated. If, however, in remaining within the limits of the linearized theory and assuming that

$$h' \approx h'' \approx h = \frac{\overline{v}_i}{\overline{V}} = f(r,\sigma)$$

where $\overline{v}_i$ is the velocity produced in the fluid by the isolated fuselage the fundamental equation of the variational problem (the Euler equation) will be

$$(1-h)\frac{\partial}{\partial \overline{\Gamma}'}\left\{\overline{\Gamma}'\overline{V}\,\overline{r}(1-\Lambda) + \overline{\Gamma}'(\overline{V}\Lambda\overline{v}_{t_1}' + \overline{r}\,\overline{v}_{a_1}')\right\} = 0$$

hence it follows that the solution of this problem will be the same as for the case of the isolated propeller since, as before,

$$\overline{\Gamma}' = \Phi^*_t - \Phi^*_b; \quad \overline{w}_{n1} = \frac{\partial\Phi^*}{\partial n}$$

It is noted also that within the limits of the linearized theory the effect of μ is likewise excluded since the effect of μ on the total and useful powers of the propeller is expressed through the terms:

$$\overline{P}_p^{(\mu)} \approx \overline{V}^2 \int_0^1 \mu\overline{\Gamma}\,d\overline{r} = \overline{V}^2 \int_0^1 C_D\,\overline{W}\,\overline{b}\,d\overline{r}$$

$$\overline{P}_\mu \approx \int_0^1 \mu\overline{\Gamma}\,\overline{r}^2\,d\overline{r} = \int_0^1 C_D\,\overline{W}\,\overline{b}\,\overline{r}^2 d\overline{r}$$

The magnitude C_D at the below critical angles of attack is almost constant with respect to C_L (or what amounts to the same thing with respect to Γ). This magnitude may be expressed, for example, by the empirical formula of Toussaint

$$\frac{1}{2}C_D = 0.00612(Wc)^{-0.15}(1+1.11\delta)\left(1+0.1\frac{C_L}{2}\right) + 0.076\delta^2 + 0.018f \quad \text{(a)}$$

where

W numerical value of the velocity in m/sec;

c numerical value of the chord

δ relative thickness of the profile

f relative maximum camber

From the foregoing formula it may be seen that the effect of C_L on C_D is very small and for this reason the value of μ[1] cannot change the character of the solution of the variational problem. In making use of formula (a) this effect may, moreover, easily be taken into account, the functions $\delta(\bar{r})$ and $f(\bar{r})$ being given:

DESIGN OF PROPELLER FOR THE CASE $\bar{v}_{a1}$ = const

In the general case the expression for the power may be written in the form:

$$\bar{P}=\int_{\xi}^{1}\bar{\Gamma}\{(\bar{V}+\bar{v}_{a1})+\mu(\bar{r}-\bar{v}_{t1})\}\bar{r}\,d\bar{r}$$

or assuming that the induced velocity $\vec{w}_{n1}\perp\vec{W}$ and replacing μ by μ_m there is obtained for $\bar{V}+\bar{v}_{a1}$ = constant[2]:

$$\bar{P}=k(\bar{V}+\bar{v}_{a1})\int_{\xi}^{1}\bar{\Gamma}_1\bar{r}d\bar{r}+k\mu_m\int_{\xi}^{1}\bar{\Gamma}_1\left(\bar{r}-\frac{\bar{V}\bar{v}_{a1}}{\bar{r}}\right)\bar{r}d\bar{r}.$$

Assuming that the circulation $\bar{\Gamma}_1$ is expressed in the form of a trigonometric series:

$$\bar{\Gamma}_1=\sum A_n\sin n\theta,$$

with

$$\bar{r}=\xi+\bar{L}-\bar{L}\cos\theta,$$

---

[1]If the effect of C_L and C_D is neglected.

[2]It is noted that the design of the propeller for the case v_{a1} = const necessarily requires $\xi\neq 0$. Otherwise for $\bar{r}\to 0$ it should be $v_{t1}\to\infty$ which does not correspond to reality.

there is obtained

$$\int_{\xi}^{1}\bar{\Gamma}\bar{r}\,d\bar{r}=\sum\int_{\xi}^{\pi}A_n\sin n\theta\ \ (\xi+\bar{L}-\bar{L}\cos\theta)\bar{L}\sin\ \theta d\theta=\frac{\pi}{2}\bar{L}\left[A_1(\xi+\bar{L})-\bar{L}\frac{A_2}{2}\right].$$

$$\int_{\xi}^{1}\bar{\Gamma}_1\bar{r}^2d\bar{r}=\bar{L}\sum A_n\int_{\xi}^{\pi}\sin n\theta(\xi+\bar{L}-\bar{L}\cos\theta)^2\sin\theta d\theta=$$

$$=\sum A_n\bar{L}\left\{(\xi+\bar{L})^2\int_0^{\pi}\sin n\theta\sin\theta d\theta+\bar{L}^2\int_0^{\pi}\sin n\theta\cos^2\theta\sin\theta d\theta-\right.$$

$$\left.-2(\xi+\bar{L})\bar{L}\int_0^{\pi}\sin n\theta\cdot\cos\theta\cdot\sin\theta d\theta\right\}=$$

$$=\bar{L}\left\{(\xi+\bar{L})^2A_1\frac{\pi}{2}-(\xi+\bar{L})\bar{L}A_2\frac{\pi}{2}\right\}+\bar{L}^3\sum A_n\int_0^{\pi}\sin n\theta\cos^2\theta\sin\theta d\theta.$$

But

$$\bar{L}^3\sum A_n\int_0^{\pi}\sin n\theta\cos^2\theta\sin\theta d\theta=\frac{\bar{L}^3}{2}\sum A_n\int_0^{\pi}\sin n\theta(1+\cos 2\theta)\sin\theta d\theta=$$

$$=\frac{\bar{L}^3}{2}A_1\frac{\pi}{2}+\frac{\bar{L}^3}{2}\sum A_n\int_0^{\pi}\sin n\theta\cos 2\theta\sin\ \theta d\theta=\frac{\bar{L}^3}{8}\pi(A_1+A_3).$$

Finally

$$\bar{V}\bar{v}_{a1}\int_{\xi}^{1}\bar{\Gamma}_1d\bar{r}=\bar{L}\bar{V}\bar{v}_{a1}\sum\int_0^{\pi}A_n\sin n\theta\sin\theta d\theta=\bar{L}\bar{V}\bar{v}_{a1}A_1\frac{\pi}{2}.$$

The expression for $\bar{P}_1$ assumes the form

$$\frac{\bar{P}}{k}=\bar{P}_1=(\bar{V}+\bar{v}_{a1})\frac{\pi}{2}\bar{L}\left[A_1(\xi+\bar{L})-\bar{L}\frac{A_2}{2}\right]+$$

$$+\mu_m\left[\bar{L}(\xi+\bar{L})^2A_1\frac{\pi}{2}-(\xi+\bar{L})\bar{L}^2A_2\frac{\pi}{2}+\right.$$

$$\left.+\bar{L}^3\frac{\pi}{8}(A_1+A_3)-\bar{L}\bar{V}\bar{v}_{a1}A_1\frac{\pi}{2}\right]-\bar{L}\mu_m\bar{V}\bar{v}_{a1}A_1\frac{\pi}{2}.$$

Since $\bar{v}_{a1}$ = constant it may be written

$$\bar{P}_1=(\bar{V}+\bar{v}_{a1})\ \bar{v}_{a1}\frac{\pi}{2}\bar{L}\left[A_1'(\xi+\bar{L})-\bar{L}\frac{A_2'}{2}\right]+$$

$$+\frac{\pi}{2}\bar{v}_{a1}\mu_m\left[\bar{L}(\xi+\bar{L})^2A_1'-(\xi+\bar{L})\bar{L}^2A_2'+\frac{\bar{L}^3}{4}(A_3'+A_1')\right]-\mu_m\bar{v}_{a1}^2\bar{V}\bar{L}A'_1\frac{\pi}{2}$$

or

$$\bar{P}_1=\bar{v}_{a1}^2\left\{\frac{\pi}{2}\bar{L}\left[A_1'(\xi+\bar{L})-\bar{L}\frac{A_2'}{2}\right]-\mu_m\bar{V}\bar{L}A_1'\frac{\pi}{2}\right\}+$$

$$+\bar{v}_{a1}\left\{\frac{\pi}{2}\bar{L}\bar{V}_{,}\left[A_1'(\xi+\bar{L})-\bar{L}\frac{A_2'}{2}\right]+\mu_m\frac{\pi}{2}\bar{L}\left[(\xi+\bar{L})^2A_1-\right.\right.$$

$$\left.\left.-(\xi+\bar{L})\bar{L}A_2'+\left(\frac{\bar{L}}{2}\right)^2(A_1'+A_3')\right]\right\}\equiv B_1\bar{v}_{a1}^2+B_2\bar{v}_{a1},$$

where

$$A_i' = \frac{A_i}{\bar{v}_{a1}}.$$

For the thrust $\bar{T}_1$ there is obtained

$$\bar{T}_1 = \int_\xi^1 \bar{\Gamma}_1 \left[(\bar{r} - \bar{v}_{t1}) - \mu(\bar{V} + \bar{v}_{a1})\right] d\bar{r} \approx \int_\xi^1 \bar{\Gamma}\left(\bar{r} - \frac{\bar{V}\bar{v}_{a1}}{\bar{r}}\right) d\bar{r} -$$

$$-(\bar{V} + \bar{v}_{a1})\mu_m \int_\xi^1 \bar{\Gamma} d\bar{r} = \frac{\pi}{2}\bar{L}\left[A_1(\xi + \bar{L}) - \bar{L}\frac{A_2}{2}\right] -$$

$$-(\bar{V} + \bar{v}_{a1})\mu_m \bar{L}\frac{\pi}{2}A_1 - \bar{V}\bar{v}_{a1}\int_\xi^1 \bar{\Gamma}_1 \frac{d\bar{r}}{\bar{r}}.$$

But by the mean value theorem

$$\int_\xi^1 \bar{\Gamma}_1 \frac{d\bar{r}}{\bar{r}} = \bar{\Gamma}_1{}^* \int_\xi^1 \frac{d\bar{r}}{\bar{r}} = \bar{\Gamma}_1{}^* \ln\frac{1}{\xi} < \bar{\Gamma}_{1\,max} \ln\frac{1}{\xi},$$

where $\bar{\Gamma}_1{}^*$ is a certain intermediate value of $\bar{\Gamma}_1$. Thus

$$\bar{T}_1 = \frac{\pi}{2}\bar{L}\left[A_1(\xi + \bar{L}) - \bar{L}\frac{A_2}{2}\right] - \mu_m(\bar{V} + \bar{v}_{a1})\bar{L}A_1\frac{\pi}{2} - \bar{V}\bar{v}_{a1}\bar{\Gamma}^* \ln\frac{1}{\xi}.$$

Now redesign the propeller 3CMB–1 with $\varphi = 34^\circ$, leaving the shape of the blades and camber the same. For $\bar{V}$ then take the value $\bar{V} = 0.4$, for the mean value of the interference coefficient $h_m = 0.04$[1] so that $\bar{V}_i = 0.384$. For μ_m now take 0.025. The power of this propeller from a preliminary computation was equal to $\bar{P} = 0.00129$ and therefore $\bar{P}_1 = 0.00043$. Now find A_1', A_2', and A_3'. For this purpose now make use of the formula (CAHI Report No. 324).

$$\bar{w}_{n1} = \frac{\bar{v}_{a1}}{\cos\beta} = \frac{\pi}{\bar{L}\sin\theta}\left[A_1(\sin\theta + C_1) + A_2(2\sin 2\theta + C_2) + A_3(3\sin 3\theta + C_3)\right].$$

[1] The propeller 3CMB–1 was located ahead of a body of revolution the ratio of which midsection to the area swept out by the effective part of the blade was equal to 0.18. The coefficient h_m was taken for the half body characterized by the same ratio. It may be noted that the problem was solved also with variable h and it was found from the mean value theorem that $h(\xi) = 0.045$ and therefore $1 - h = 0.955$.

writing it for three sections $\bar{r} = 0.4, 0.6, 0.8$ in the form:

$$\left(\theta_2 = \frac{\pi}{3}\right): \quad \frac{\sqrt{0,4^2+0,4^2}}{0,4}\frac{\bar{L}}{\pi}\frac{\sqrt{3}}{2} = A_1'\left(\frac{\sqrt{3}}{2}+C_{12}\right)+$$

$$+A_2'\left(\sqrt{3}+C_{22}\right)+A_3'C_{32}$$

$$\left(\theta_3 = \frac{\pi}{2}\right); \quad \frac{\sqrt{0,4^2+0,6^2}}{0,6}\frac{\bar{L}}{\pi} = A_1'(1+C_{13})+A_2'C_{23}+A_3'(-3+C_{33})$$

$$\left(\theta_4 = \frac{2}{3}\pi\right); \quad \frac{\sqrt{0,4^2+0,8^2}}{0,8}\frac{\bar{L}}{\pi}\frac{\sqrt{3}}{2} = A_1'\left(\frac{\sqrt{3}}{2}+C_{14}\right)+$$

$$+A_2'\left(-\sqrt{3}+C_{24}\right)+A_3'C_{34},$$

where

$$A_i' = \frac{A_i}{\bar{v}_{a1}}.$$

The values of C_{ik} are taken from the curves (fig. 2), no account being taken of the effect of the so-called logarithmic term (see reference 3), because in obtaining C_{ik} the charts of T. Moriya were used in which this term was likewise not taken into account near the singular point ($\bar{r} = \bar{r}'$).

After substituting C_{ik} now the following system of equations is obtained:

$$1,055A_1'+1,76A_2'+0,01A_3' = 0,156$$
$$1,21A_1'+0,1A_2'-3,04A_3' = 0,153$$
$$1,025A_1'-1,7A_2'-0,16A_3' = 0,123$$

Solving this system there is obtained[1]

$$A_1' = 0.1346, \ A_2' = 0.0087, \ A\ ' = 0.00357.$$

We further find B_1 and B_2. Setting $\xi = 0.2$, then $\bar{L} = 0.4$, $\xi + \bar{L} = 0.6$ and

$$A_1'\ (\xi + \bar{L}) = 0.1346 \quad 0.6 = 0.08076$$

$$\bar{L}\ \frac{A_2'}{2} = 0.2 \quad 0.0087 = 0.00174$$

[1]The computations are approximate.

Subtracting the lower from the upper equation there is obtained 0.07902. Further

$$\mu_m \overline{V} A_1' = 0.025 \times 0.4 \times 0.1346 = 0.001346$$

Subtracting the latter figure from 0.07902 there is obtained 0.07777. Multiplying the latter by $\frac{\pi}{2}$ 0.4 = 0.628 there is obtained $B_1 = 0.0489$. Now proceed to the computation of B_2.

$$(\xi + \overline{L}) \overline{V}_i = 0.6 \times 0.384 = 0.2304$$

$$\mu_m \left[(\xi + \overline{L})^2 + \left(\frac{\overline{L}}{2}\right)^2 \right] = 0.025 \; (0.36 + 0.04)$$

$$= 0.025 \times 0.4 = 0.01$$

The coefficient before A_1' is equal to 0.2404.

$$A_1' \times 0.2404 = 0.0325$$

$$1/2 \; \overline{V}_i = 0.192$$

$$\mu_m (\xi + \overline{L}) = 0.025 \times 0.6 = 0.015$$

$$- \overline{L} \left[\frac{\overline{V}_i}{2} + \mu_m (\xi + \overline{L}) \right] A_2' = - 0.207 \times 0.4 \times 0.0087 = -0.00072$$

$$\mu_m \left(\frac{\overline{L}}{2}\right)^2 A_3' = 0.025 \times 0.04 \times 0.00357 = 0.00000357$$

hence

$$B_2 = 0{,}0201.$$

There is obtained:

$$0{,}0489\bar{v}^2_{a\,1} + 0{,}0201\bar{v}_{a\,1} - 0{,}00043 = 0$$

or

$$\bar{v}^2_{a1} + 0{,}412\bar{v}_{a\,1} - 0{,}0088 = 0.$$

hence, the positive root will be

$$\bar{v}_{a\,1} \approx 0{,}0204$$

and therefore

$$A_1 = 0{,}00275$$

$$A_2 = 0{,}0001775$$

$$A_3 = 0{,}00007275.$$

Now proceed to the determination of the thrust. According to the general formula then find

$$\overline{T}_1 = \frac{\pi}{2}\overline{L}\,(0{,}00275\cdot 0{,}6 - 0{,}2\cdot 0{,}0001775) - 0{,}025\cdot 0{,}4204\cdot 0{,}628\cdot 0{,}00275 -$$

$$- 0{,}4\cdot 0{,}0204\overline{\Gamma}_1{}^* \ln 5$$

or

$$\overline{T}_1 = 0{,}628\,(0{,}00165 - 0{,}0000355) - 0{,}00001815 - 0{,}0131\overline{\Gamma}^*,$$

hence

$$\overline{T}_1 = 0{,}000997 - 0{,}0131\overline{\Gamma}_1{}^*.$$

The value of $\overline{\Gamma}_1{}^*$ is evidently less than $(\overline{\Gamma}_1)_{max}$ which on account of the smallness of A_2 and A_3 may be assumed equal to $A_1 = 0.00275$ and this gives

$$\overline{T}_1 = 0{,}000997 - 0{,}000036 = 0{,}000961$$

and therefore

$$\eta_{ef} = \frac{0{,}384\cdot 0{,}000961}{0{,}00043} \approx 0{,}86.$$

Now compute the values of the circulation at the different radii making use of the formula

$$\overline{\Gamma}_1 = \sum A_n \sin n\theta.$$

There is obtained

TABLE I

$\bar{r}$	0.25	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
$\overline{\Gamma}_1$	0.00156	0.00207	0.00253	0.00269	0.00267	0.00252	0.00222	0.00171	0.00126

In a check computation of the propeller 3CMB-1 for $\overline{V}_1 = 0.384$ there is obtained a propeller efficiency equal to 0.818 which result agreed with the experiment. It may be expected that the foregoing propeller computation should give good agreement with the experiment.

TABLE II

$\bar{r}$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\bar{\Gamma}_1$	0.00207	0.00253	0.00269	0.00267	0.00252	0.00222	0.00171
$\mu(\bar{V}+\bar{v}_{a1})$				0.01055			
$\bar{v}_{t1}=\bar{v}_{a1}\frac{\bar{V}}{\bar{r}}$	0.0272	0.0204	0.0163	0.0136	0.01165	0.0104	0.00906
$\bar{r}-\bar{v}_{t1}$	0.2728	0.3796	0.4737	0.5864	0.6884	0.7896	0.89
$(\bar{r}-\bar{v}_{t1})-\mu(V+\bar{v}_{a1})$	0.2622	0.369	0.4631	0.5758	0.6778	0.779	0.8894
$1-h$	0.818	0.9036	0.9443	0.9656	0.9774	0.9843	0.9887
$\bar{T}_1{}'_{ef}$	0.000445	0.000844	0.00118	0.001485	0.00167	0.0017	0.00148
			$\bar{T}_{1ef}=0.000922$				
			$\eta_{ef}=\frac{0.000925\cdot 0.4}{0.00043}=0.86$				

It is noted that it is possible to replace κ by κ_m under the integral signs of the power and thrust since, as computations have shown, the values of κ_m are very near the values which are obtained for κ^* in applying the mean-value theorem. In determining the true angles of attack of the blade elements it is necessary to make use of the true values of κ. In table II there is given the computation of the value of the effective propeller thrust for a given circulation with variable κ. As may be seen from this computation the value of the effective efficiency of the propeller rotating in front of the fuselage was the same as in the case of the computation with κ replaced by κ_m. Figure 3 shows the curves of the distribution of $T_1{}'_{ef}$ along the propeller radius for the case $\bar{v}_{a1}$ = constant and for the preliminary computation of the propeller 3CMB-1 (curves II and I).

Now there will be shown in what sense the term "distribution" of the effective thrust over the propeller radius must be understood. The elementary propeller thrust, as is known, is expressed by the formula

$$d\bar{T}=\bar{\Gamma}(\bar{r}-\bar{v}_{t1})\,d\bar{r}.$$

The elementary useful power imparted by the propeller to the airplane is equal to (see sec. 4):

$$d\overline{P}_p \approx (\overline{V} - \overline{v}_i{}')\,\overline{\Gamma}\,(\overline{r} - \overline{v}_{t1})\,d\overline{r}$$

where $\overline{v}_i{}'$ is the change in velocity of the flow due to the fuselage at radius $\overline{r}$. The useful power of the entire propeller will be equal to

$$\overline{P}_p \quad \overline{V}\int_{\xi}^{1} (1 - h')\,\overline{\Gamma}\,(\overline{r} - \overline{v}_{t1})\,d\overline{r}$$

In correspondence with the foregoing formula it is convenient to speak of an effective thrust determined by the formula

$$\overline{T}_{ef} = \frac{\overline{P}_p}{\overline{V}} = \int_{\xi}^{1} \kappa'\,(\overline{r})\;\overline{\Gamma}\,(\overline{r} - \overline{v}_{t1})\,d\overline{r}$$

and also of an elementary effective thrust. The magnitude $\kappa' = 1 - h'$ should be thought of as taken in the propeller-fuselage system. Since the effect of κ is not large then take for κ' its value for the isolated fuselage. In the process of computing the propeller by the element method it is necessary to determine the distribution of the effective thrust over the blades so that by planimetering the magnitude of T_{ef} may then be found.

DESIGN OF PROPELLER FOR THE CASE $\overline{w}_1$ = const

It has been shown that the velocity induced by the helical vortices at the propeller blades is expressed by

$$\overline{w}_{n1} = \frac{\pi}{\overline{\overline{L}}\sin\theta}\sum_{n}\left(A_n\,n\sin n\theta + A_n\,C_n(\theta)\right) \qquad (a)$$

where $C_n(\theta)$ are the coefficients expressing the effect of the helical vortices. For the case $\overline{w}_1$ = constant the foregoing formula may be written as

$$\frac{\pi}{\overline{L}} \frac{1}{\sin\theta} \sum_n (A_n' \, n \sin n\theta + A_n' \, C_n) = \cos\beta \qquad (b)$$

where

$$A_n' = \frac{A_n}{\overline{w}_1}$$

For the sections $\overline{r}_2 = 0.4$, $\overline{r}_3 = 0.6$, $\overline{r}_4 = 0.8$ to which correspond the values $\theta_2 = \pi/3$, $\theta_3 = \pi/2$, $\theta_4 = \frac{2}{3}\pi$ the following system of equations is obtained (the values of C_{ik} are taken from the curves of fig. 2 for $\overline{V} = 0.4$):

$$1.055 \, A'_1 + 1.76 \, A'_2 + 0.01 \, A'_3 = 0.078$$

$$1.21 \, A'_1 + 0.1 \, A'_2 - 3.04 \, A'_3 = 0.106$$

$$1.025 \, A'_1 - 1.7 \, A'_2 - 0.16 \, A'_3 = 0.0985$$

The solution of this system gives approximately

$$A'_1 = 0.08495$$

$$A'_2 = -0.006568$$

$$A'_3 = -0.00125$$

The values of $\overline{\Gamma}_1{}^* = \frac{\overline{\Gamma}_1}{\overline{w}_1}$ are found by the formula

$$\overline{\Gamma}_1{}^* = \sum_n A'_n \sin n\theta$$

and are given in table III.

TABLE III

$\overline{r}$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$_1^*$	0	0.04867	0.0678	0.08	0.08625	0.0864	0.0792	0.06167

There is no need to set up and solve the system of equations each time for various values of $\overline{V}$ since such systems may be set up and solved once for all and tables and charts set up for $\overline{\Gamma}_1{}^*$ (fig. 4). The latter figure gives the curves drawn from the values of C_{ik} taken for two-blade propellers. Since the effect of C_{ik} is not large (especially at large values of $\overline{V}$) and the chief component in formula (b) is the first term, not depending on the number of blades, these curves may be used, as computations have shown, also for three-blade propellers. The foregoing substitution may lead only to an insignificant change in the tip losses. For the case $\overline{w}_1$ = constant the expression for the power may be written as

$$\overline{P} = k \int_{\xi}^{1} \overline{\Gamma}_1{}^* \, \overline{w}_1 \left\{ \overline{V} - \overline{v}_i + \overline{w}_1 \frac{\overline{r}^2}{\overline{V}^2 + \overline{r}^2} + \mu \left(\overline{r} - \overline{w}_1 \frac{\overline{r}\,\overline{V}}{\overline{V}^2 + \overline{r}^2} \right) \right\} \overline{r}\, d\overline{r}$$

or

$$P_1 = \overline{w}_1{}^2 \int_{\xi}^{1} \overline{\Gamma}_1{}^*(\overline{r} - \mu \overline{V}) \frac{\overline{r}^2 d\overline{r}}{\overline{r}^2 + \overline{V}^2} + \overline{w}_1 \int_{\xi}^{1} \overline{\Gamma}_1{}^* \overline{r} (\overline{\overline{V}}_i + \mu \overline{r}) d\overline{r} \equiv A\overline{w}_1{}^2 + B\overline{w}_1$$

where $\overline{\overline{V}}_i = \overline{V} - \overline{v}_i$. Replacing under the integral signs μ_m for μ then compute the coefficients A and B. The computations of these coefficients are given in table V. After planimetering there are obtained for A and B the values

$$A = 0.02307; \quad B = 0.01289$$

and therefore

$$\overline{w}_1{}^2 + 0.559\ \overline{w}_1 = 0.01865$$

(for $\overline{F}_1$ then take the same value as in the preceding example). Solving the obtained quadratic equation a positive root equal to $\overline{w}_1 = 0.0316$ is obtained

TABLE IV.- VALUES OF $\overline{\Gamma}_1{}^*$ FOR $k = 2$ AND $\xi = 0.2$

$\overline{r}\Rightarrow$ $\overline{V}\Downarrow$	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	0.03935	0.0447	0.04539	0.04546	0.04539	0.0447	0.03935
.2	.05401	.0676	.07337	.07525	.07357	.068	.05445
.3	.05535	.0726	.082	.08562	.08396	.0762	.05932
.4	.04867	.0678	.08	.08625	.0864	.0792	.06167
.5	.044823	.0636	.0761	.0831	.08389	.0775	.06062
.6	.0425	.0606	.0732	.079	.078	.0706	.055

The value of this root permits finding the value of $\overline{T}_1$ by the formula:

$$\overline{T}_1 = \overline{w}_1 \int_{\xi}^{1} \Gamma_1{}^*(\overline{r} - \mu\overline{V}) - \overline{w}_1{}^2 \int_{\xi}^{1} \Gamma_1{}^*(\overline{V}_i + \mu\overline{r})\frac{\overline{r}\,d\overline{r}}{\overline{r}^2+\overline{V}^2} \equiv B_1\overline{w}_1 - A_1\overline{w}_1{}^2$$

The computation of A_1 and B_1 is clear from table V in which are also computed the values $T'_{1\,ef.}$ The computation gives $\eta_{ef} = 0.88$. It is seen that the computation for the case $\overline{w}_1$ = constant is sufficiently simple. It may still further be simplified by computing in advance the values A, B, A_1, and B_1 as may be done by replacing

κ by κ_m under the integral signs. This substitution practically has no effect on the final results. It may be noted also that the magnitude $\bar{\Gamma}_1{}^*$ may also be computed in advance and for it curves and tables may be set up. In determining $\bar{\Gamma}_1{}^*$, now start from the Betz formula

$$\bar{\Gamma}_1{}^* = \frac{\kappa^*}{k} \frac{\bar{r}^2 \bar{V}}{\bar{r}^2 + \bar{V}^2}, \tag{c}$$

where κ^* is the Prandtl correction or the Goldstein correction. In employing this formula it is necessary to assume that $\xi = 0$. If a second approximation is desired, it is necessary throughout to replace $\bar{V}$ by $\bar{V} + \bar{w}_1{}'$ and write the expression for $\bar{\Gamma}_1{}^*$ in the form $\bar{V}_1 = \bar{V}(1-h)$

TABLE V

$\bar{r}$	0,3	0,4	0,5	0,6	0,7	0,8	0,9
$\bar{\Gamma}'_1$	0,04867	0,0678	0,08	0,08625	0,0864	0,0792	0,06167
$\bar{r}^2$	0,09	0,16	0,25	0,36	0,49	0,64	0,81
$\bar{V}^2$	0,16	0,16	0,16	0,16	0,16	0,16	0,16
$\bar{r}^2+\bar{V}^2$	0,25	0,32	0,41	0,52	0,65	0,80	0,97
$\bar{r}\,(\bar{r}^2+\bar{V}^2)$	1.2	1,25	1,22	1,154	1,078	1,00	0,928
$\mu_m \bar{V}$	0,01	0,01	0,01	0,01	0,01	0,01	0,01
$\bar{r}-\mu\bar{V}$	0,29	0,39	0,49	0,59	0,69	0,79	0,89
$1-h$	0,818	0,9036	0,9443	0,9656	0,9774	0,9843	0,9887
$\bar{V}(1-h)$	0,327	0,3602	0,377	0,386	0,391	0,393	0,395
$\mu_m \bar{r}$	0,0075	0,01	0,0125	0,015	0,0175	0,02	0,0225
$\bar{\Gamma}'_1(\bar{r}-\mu_m\bar{V})\frac{\bar{r}^2}{\bar{r}^2+\bar{V}^2}$	0,00507	0,01323	0,0239	0,0351	0,045	0,05	0,0458
$\bar{V}_i+\mu\bar{r}$	0,3345	0,37	0,3895	0,401	0.4085	0,413	0,4175
$\bar{\Gamma}'_1\bar{r}(\bar{V}_i+\mu_m\bar{r})$	0,00488	0,01005	0,01558	0,0207	0,0247	0,0262	0,0232
$\bar{w}_1\bar{\Gamma}'_1(\bar{r}-\mu_m\bar{V})$	0,000447	0,000836	0,001239	0,00161	0,00188	0,001975	0,001735
$w_1{}^2\bar{\Gamma}'_1\cdot\frac{\bar{r}(\bar{V}_i+\mu\bar{r})}{\bar{r}^2+\bar{V}^2}$	0,0000196	0,0000314	0,000038	0,00004	0,0000376	0,0000328	0,0000239
$\bar{\Gamma}'_1$	0,0004274	0,0008046	0,001201	0,00157	0,0018424	0,0019422	0,0017111
$\bar{\Gamma}'_1(1-h)$[1]	0,00035	0,000726	0,00113	0,001516	0,0018	0,00191	0,001695

[1]Curve III on fig. 3.

$$\bar{\Gamma}_1{}^* = \frac{\kappa^*}{k} \frac{\bar{r}^2 (\bar{V} + \bar{w}'_1)}{\bar{r}^2 + (\bar{V} + \bar{w}'_1)^2} \qquad (d)$$

where $\bar{w}_1{}'$ is taken from the first approximation.

In conclusion it is pointed out that the advantage of the method of trigonometric series as compared with the method using formula (c) lies in the fact that with the aid of the former before designing the propeller it is possible to carry out a preliminary computation of one of the series production propellers available corresponding to the given conditions and estimate to what extent the theory deviates from experiment. After a preliminary computation it is then possible to proceed with the design of the propeller with the aid of the method presented in the foregoing. Present day series propellers deviate from the optimum apparently to such a small extent that the preliminary computation and the design propor will refer to almost the identical conditions and there is little probability that the theory should in the final design give a different degree of accuracy than in the preliminary computation.

CONCLUSIONS

On the basis of the method of trigonometric series a rational design of propellers was found possible making use of the variation conditions the correctness of which was shown in the paper. An illustrative computation showed that (1) Propellers with w_1 = constant are suitable for design and have high efficiency, as corresponds with the theoretical assumptions. (2) Propellers with v_{a_1} = constant are also suitable for design and give good efficiency which is, however, less than those under (1). (3) The method of trigonometric series permits carrying out a preliminary check computation of existing propellers after which the design is improved. (4) The method of Ritz may be used in solving the variational problem but it is not a rational method.

Translation by S. Reiss,
National Advisory Committee
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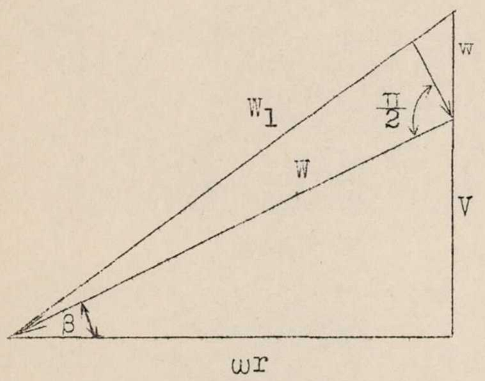

Figure 1

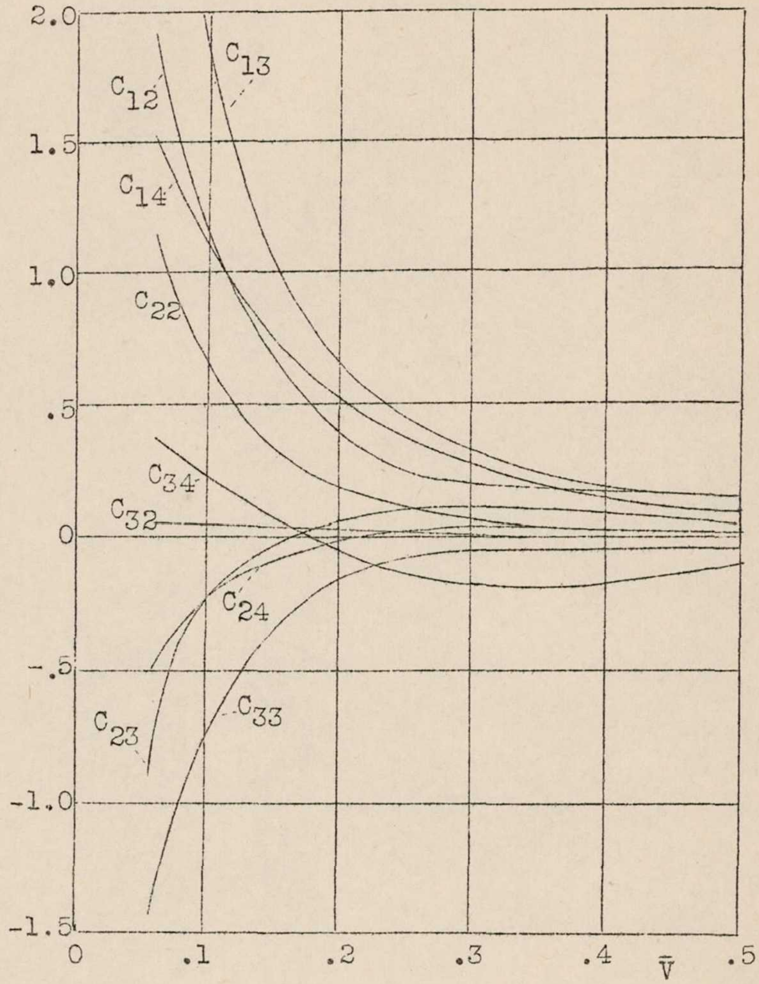

Figure 2

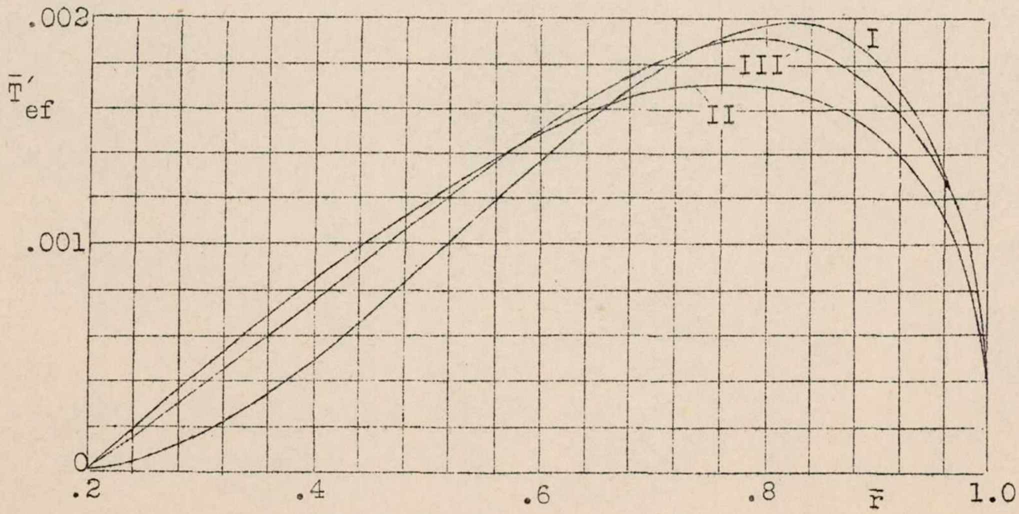

Figure 3

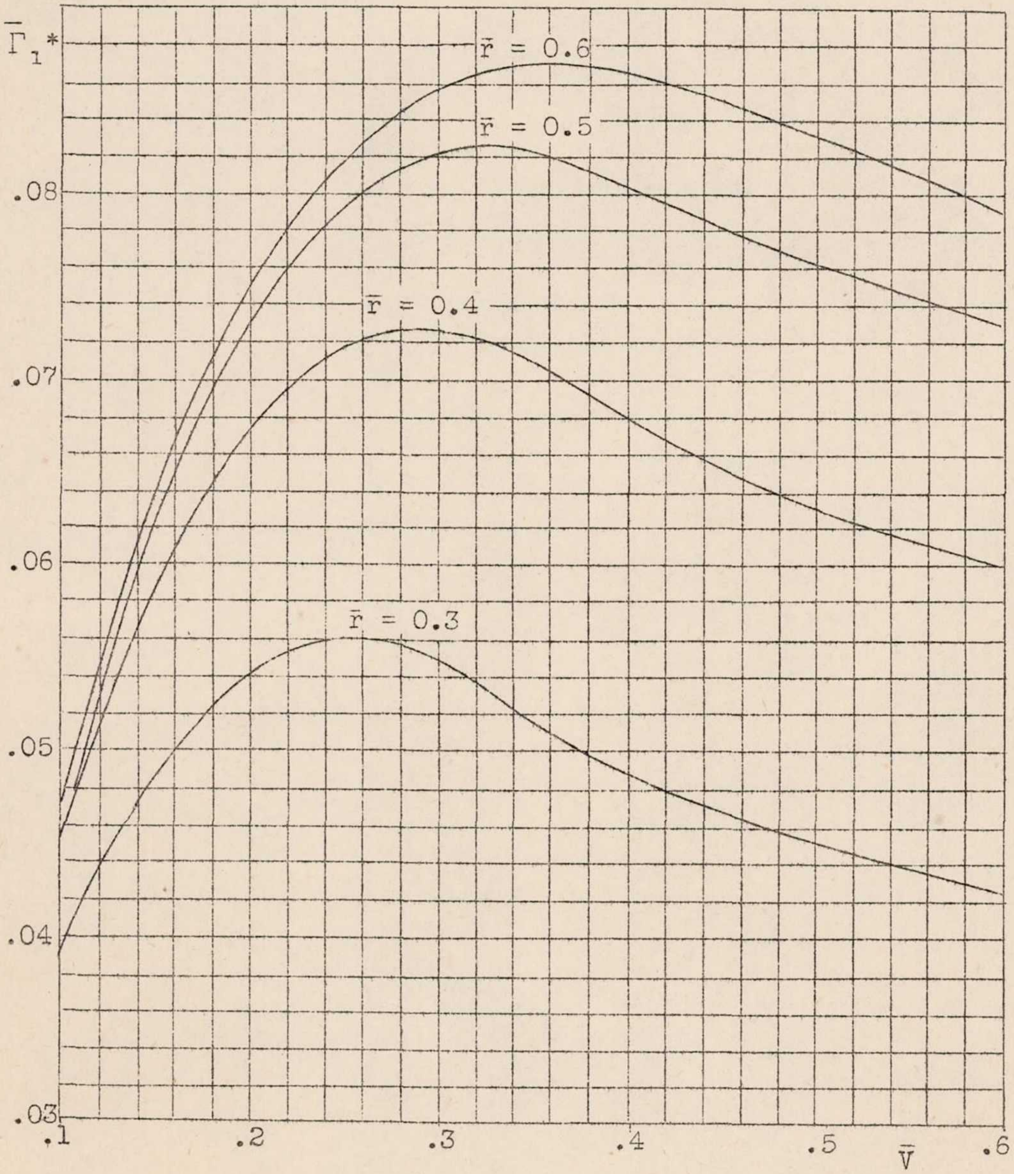
$\bar{\Gamma}_1^*$
$\bar{r}$ = 0.6
$\bar{r}$ = 0.5
$\bar{r}$ = 0.4
$\bar{r}$ = 0.3
.08
.07
.06
.05
.04
.03
.1
.2
.3
.4
.5
$\bar{V}$
.6

Figure 4a

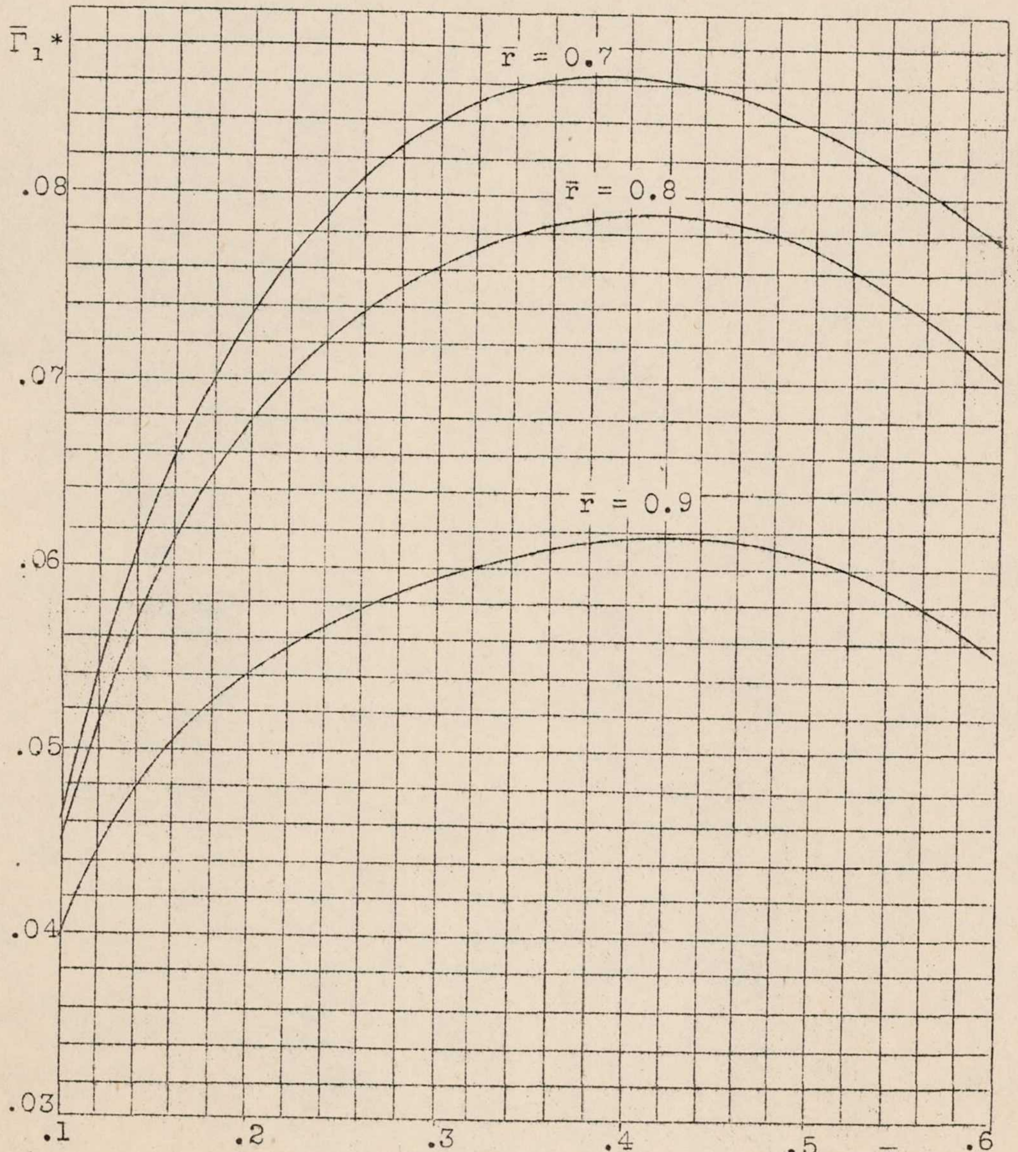

Figure 4b