A Simple Derivation of the Acoustic Boundary Condition in the Presence of Flow

F. Farassat

NASA Langley Research Center

Hampton, Virginia 23681

M.H. Dunn

Old Dominion University

Norfolk, Virginia 23508

In modeling the noise propagation in a duct of a ducted fan engine, the authors needed the boundary condition satisfied by the acoustic pressure on a lined duct wall in the presence of flow. The result in the form used by the authors was derived by M.K. Myers [1] who presented a formal derivation of a related result by K. Taylor [2]. A good review of the subject can be found in reference 3. Here a brief and simple derivation of the acoustic boundary condition of Myers is given. The main difference in the derivation from that in reference [1] is that the Gaussian coordinate system $\left(q^1, q^2, q^3\left(q^1, q^2, t\right)\right)$ is used to specify the instantaneous position, i.e. Lagrangian variable, of the point on the mean position of the wall with curvilinear (Gaussian) coordinates $\left(q^1, q^2\right)$. Myers uses a locally orthogonal coordinate system which is somewhat less specific than what is used here.

One assumes that one has a base or background flow with velocity $\vec{u}_0(\vec{x})$ which is time independent perturbed by a small velocity distribution $\varepsilon\vec{u}_1(\vec{x},t)$ where $0 < \varepsilon << 1$. One also assumes that the wall boundary's mean position S_0 is independent of time and specified by the position vector $\vec{x}_0\left(q^1, q^2\right)$ where $\left(q^1, q^2\right)$ are the curvilinear (Gaussian) coordinates on the boundary surface. The position of the time dependent boundary S is given by

$$\vec{x}(q^1,q^2,q^3,t)=\vec{x}_0(q^1,q^2)+\varepsilon q^3(q^1,q^2,t)\vec{n}_0(q^1,q^2), \tag{1}$$

where $\vec{n}_0$ is the local unit normal to S_0 and εq^3 is the distance along the normal $\vec{n}_0$ from S_0 to S at (q^1,q^2,t). The fundamental physical requirement at the boundary is

$$(\vec{u}_0+\varepsilon\vec{u}_1)\cdot\vec{n}_0=\varepsilon\frac{\partial q^3}{\partial t}. \tag{2}$$

This is the instantaneous equality of the normal fluid velocity and the surface velocity. Note that the symbol ε will be retained for order of magnitude comparison for now.

The left side of equation (2) will now be expanded as follows and equated to the right side:

$$\begin{aligned}&\left[\vec{u}_0(\vec{x}_0+\varepsilon q^3\vec{n}_0)+\varepsilon\vec{u}_1(\vec{x}_0+\varepsilon q^3\vec{n}_0)\right]\cdot\vec{n}_0\\&=\vec{u}_0(\vec{x}_0)\cdot\vec{n}_0+\varepsilon\left[q^3\vec{n}_0\cdot\nabla\vec{u}_0(\vec{x}_0)+\vec{u}_1(\vec{x}_0)\right]\cdot\vec{n}_0=\varepsilon\frac{\partial q^3}{\partial t}\end{aligned}. \tag{3}$$

The equality of the zeroth and first order terms from both sides gives

$$\vec{u}_0(\vec{x}_0)\cdot\vec{n}_0=0 \tag{4}$$

and

$$q^3\vec{n}_0\cdot\left[\vec{n}_0\cdot\nabla\vec{u}_0(\vec{x}_0)\right]+\vec{u}_1(\vec{x}_0)\cdot\vec{n}_0=\frac{\partial q^3}{\partial t}. \tag{5}$$

Equation (4) tells us that the normal velocity based on the mean flow is zero on the mean surface. Equation (5) can be written as

$$\vec{u}_1(\vec{x}_0)\cdot\vec{n}_0=\frac{\partial q^3}{\partial t}-q^3\vec{n}_0\cdot\left[\vec{n}_0\cdot\nabla\vec{u}_0(\vec{x}_0)\right]. \tag{6}$$

One now assumes $\varepsilon=1$ and thus $|\vec{u}_1|<<|\vec{u}_0|$ and q^3 is the local normal distance between S_0 and S as a function of time. Note that $\partial q^3/\partial t$ is the local normal velocity of S in terms of the Lagrangian variables (q^1,q^2). Using Eulerian variables $\vec{x}_0$, one can define a new function $g(\vec{x}_0,t)$ such that

$q^3 = g(\vec{x}_0, t)$. Note that since q^3 is the normal distance between S_0 and S, one has $|\nabla g| = 1$. One notes that

$$\frac{\partial q^3}{\partial t}(q^1, q^2) = \frac{\partial g}{\partial t} + \vec{u}_0(\vec{x}_0)\cdot\nabla g\,. \tag{7}$$

Using this result in equation (6) gives equation (11) of Myers [1]:

$$\vec{u}_1(\vec{x}_0)\cdot\vec{n}_0 = \frac{\partial g}{\partial t} + \vec{u}_0(\vec{x}_0)\cdot\nabla g - g\vec{n}_0\cdot\left[\vec{n}_0\cdot\nabla\vec{u}_0(\vec{x}_0)\right], \tag{8}$$

which is the condition that the perturbation velocity $\vec{u}_1$ must satisfy on the mean surface S_0.

One now derives the liner boundary condition based on equation (8). For a time harmonic disturbance proportional to $e^{i\omega t}$, the complex acoustic pressure p and g are related to each other by the following relation on S_0:

$$g = -\frac{p}{i\omega Z}, \tag{9}$$

where Z is the complex normal impedance. Using this result in equation (8) gives

$$\vec{u}_1\cdot\vec{n}_0 = -\frac{p}{Z} - \frac{1}{i\omega}\vec{u}_0\cdot\nabla\left(\frac{p}{Z}\right) + \frac{p}{i\omega Z}\vec{n}_0\cdot(\vec{n}_0\cdot\nabla\vec{u}_0), \tag{10}$$

which is the liner boundary condition, equation (15), in Myers [1]. Equation (10) is implemented in a ducted fan noise prediction code developed for NASA Langley Research Center by the authors [4].

References

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