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MEMORANDUM

HEAT TRANSFER IN THE TURBULENT INCOMPRESSIBLE BOUNDARY LAYER

IV - EFFECT OF LOCATION OF TRANSITION AND PREDICTION OF HEAT TRANSFER IN A KNOWN TRANSITION REGION

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SUMMARY

The effect of the location of transition on the heat transfer to the turbulent incompressible boundary layer is analyzed. The analysis indicates that considerably higher heat-transfer rates may occur for some distance downstream if the transition is very late. The results of a limited experimental investigation are in substantial agreement with the results of the analysis.

If the extent of the transition region is known, the analysis also allows adequate prediction of heat-transfer coefficients within the transition region. The nature of this analysis is such that it should predict local shear coefficients in the transition region equally well.

INTRODUCTION

This report is the last of a series of four covering a three-year investigation of heat transfer through the turbulent incompressible boundary layer with arbitrary wall-temperature variation (ref. 1). The first report describes the experimental apparatus and presents results of experiments with constant wall temperature (ref. 2). The second report gives the results of experiments and analyses for a step temperature distribution (ref. 3). In the third report the step-function analysis is used as the basis for predicting heat-transfer rates in several cases where the plate is nonisothermal (ref. 4). These predictions are compared with experiments, and good agreement is found; and a simple method for handling problems where the wall temperature varies in an arbitrary manner is presented. In the present report an analysis of the effect of the location of transition on heat transfer in the turbulent boundary layer downstream is presented and compared with experiments.

Experiments have shown that delayed transition from a laminar to a turbulent boundary layer results in higher heat-transfer rates in the

turbulent boundary layer than would be obtained if the boundary layer were turbulent from the leading edge (e.g., ref. 5). This is believed to be due to two effects. The first is a hydraulic effect; that is, the turbulent boundary layer behaves as if it had originated at some point downstream of the leading edge of the flat plate. The second is a thermal effect; that is, the heat-transfer rates in the laminar layer preceding the turbulent layer are much lower than those that would occur over the leading portion of the plate in a turbulent layer. Thus, the energy in the turbulent layer after transition is considerably less than it would have been if the layer had been turbulent from the start. Because of the combination of these effects, considerably higher heat-transfer rates may occur in the turbulent part of the boundary layer for some distance downstream if the transition Reynolds number is high.

An attempt has been made to predict analytically the effect of a late transition, based on the notion of separate contributions of hydraulic and thermal effects. The limited amount of experimental data obtained in this investigation for a variable transition Reynolds number is correlated quite adequately on the basis of this analysis. However, it was not possible to delay transition to Reynolds numbers above 400,000 in the available wind tunnel. Consequently, complete confirmation of the analysis must await the availability of experimental data for higher transition Reynolds numbers.

Since the length of the transition region is generally about the same as the length of the laminar region, estimations of heat transfer in the transition region are often desired. If the extent of the transition region is known, the local Stanton numbers in the transition and turbulent regions may be determined by employing the abrupt-transition analysis as a point concept. This concept assumes that the flow at a given point is either laminar or turbulent and that the average condition of the flow at a prescribed distance from the leading edge can be represented by the value of an intermittency factor that prescribes the fraction of the flow that is turbulent. The known expressions for laminar and turbulent heat-transfer coefficients are then assumed to hold at each point, and the mean coefficient is found as a weighted average. The heat-transfer coefficients found in this manner agree well with the data obtained with natural transition in the present investigation. It is believed that the method should work equally well for predicting local shear coefficients in the transition and turbulent regions.

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SYMBOLS

a	location of start of transition, ft
b	location of end of transition, ft
C_f	local friction factor, $\tau_w/(\rho u_\infty^2/2)$
$C_{f,lam}$	laminar friction factor
$C_{f,turb}$	turbulent friction factor
c_p	specific heat at constant pressure, Btu/(lb)(°F)
$F(Re_x;Re_s)$	$St_{turb}(Re_x;\ Re_s)/St_{turb}(Re_x;0)$
G	free-stream mass velocity, ρu_∞, lb/(hr)(ft)
h_l	local convective heat-transfer coefficient, $q_w''/\Delta t$, Btu/(hr)(sq ft)(°F)
k	thermal conductivity of fluid, Btu/(hr)(ft)(°F)
l	unheated starting length, ft
$P(s)$	probability distribution function for location of transition, $P(Re_s)$
Pr	Prandtl number, $\mu c_p/k$
q_w''	heat flux at wall, Btu/(hr)(sq ft)
Re_s	Reynolds number at abrupt transition
Re_x	Reynolds number based on x, Gx/μ
Re_ω	Reynolds number based on ω
r	location of virtual origin, ft
St	local Stanton number, h_l/Gc_p
St_{lam}	laminar Stanton number
$St(Re_x;Re_s)$	local Stanton numbers on isothermal plate for turbulent boundary layer after abrupt transition at s

St_T	$St(Re_x;0)$
St_{turb}	turbulent Stanton number
s	location of abrupt transition, ft
s_m	average location of abrupt transition, ft
T_w	absolute wall temperature, $^{\circ}R$
T_∞	absolute free-stream temperature, $^{\circ}R$
Δt	$t_w - t_\infty$, $^{\circ}F$
$\Delta t'$	Δt for fictitious part of turbulent layer, $^{\circ}F$
t_m	mean temperature of heated strip, $^{\circ}F$
Δt_m	$t_m - t_\infty$, $^{\circ}F$
t_w	wall temperature, $^{\circ}F$
t_∞	free-stream temperature, $^{\circ}F$
u_{bl}	velocity in boundary layer, ft/sec
u_∞	free-stream velocity, ft/sec
w	flow rate, lb/hr
x	distance from leading edge, ft
y	distance from wall, ft
z	coordinate normal to x and y, ft
$\beta(x)$	fraction of boundary layer at x that is turbulent, $\beta(Re_x)$
δ	boundary-layer thickness, ft
δ_a	laminar-boundary-layer thickness at a, ft
δ_b	turbulent-boundary-layer thickness at b, ft
θ	momentum thickness, $\int_0^\delta (u_{bl}/u_\infty)\left[1 - (u_{bl}/u_\infty)\right]dy$, ft

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μ	viscosity, lb/(hr)(ft)
ξ	distance from virtual origin, ft
ρ	fluid density, lb/cu ft
σ	variable of integration
τ_w	shear stress at wall, lb/sq ft
ω	standard deviation of s from mean transition-point location, ft

TRANSITION MODEL

Although a great deal is known about the heat-transfer and skin-friction characteristics of laminar and turbulent boundary layers, very little is known about heat transfer in the region of transition that exists between the turbulent and laminar regimes, or about the effect that the location of the transition region has on heat transfer and skin friction in the turbulent layer after transition. In order to estimate the effect of transition-point location on turbulent heat transfer for engineering design calculations, one of the following three assumptions has usually been made: (1) The turbulent layer behaves as if it had started at the leading edge; (2) it behaves as if it had started at a "transition point;" or (3) the turbulent layer after transition has the same thickness as the laminar layer before transition. The heat-transfer rates and friction factors determined by these approximate treatments will not be too much in error, if the point under consideration is very far downstream of the transition region and if transition ends at relatively low Reynolds numbers. In some applications, however, the transition may be quite late; not infrequently it extends to Reynolds numbers in excess of 10^7, and in such a case the late transition may cause a considerable increase in the turbulent skin friction and heat transfer for a surprisingly large distance downstream of transition. A realistic analysis of the transition effect is therefore needed; the analysis should be supported by experimental data, since assumptions will be required by the nature of the problem.

In making an analysis of the transition effect, it is important that a reasonable model of transition be employed. The recent experiments of Schubauer and Klebanoff (ref. 6) indicate that the laminar boundary layer on a flat plate becomes turbulent quite suddenly at various "spots" on the plate. The location of these spots of "abrupt" transition varies with time, but the amount of the boundary layer that is turbulent at any distance downstream of the leading edge seems to depend only on the distance. These observations suggest a model of the transition phenomena that should

be suitable as the basis for an analysis of the effect of the location of transition on heat transfer and skin friction. This model is shown by figure 1. The flow is divided into "slabs," and the flow in each "slab" is assumed to be independent of the flow in the other slabs. In each slab, transition occurs abruptly at some point, and the location of the "local abrupt transition" differs from slab to slab. However, if the slabs are made very thin, the locus of the transition points becomes a continuous curve, as is indicated by figure 1. It will be assumed that this curve is invariant in time. This assumption is in contradiction with the observations of Schubauer and Klebanoff, but since a statistical approach will be taken in the analysis no serious error should result. Well downstream of the transition locus, the effect of the variation in the location of the transition point will be smoothed out, and the local Stanton numbers and friction factors should approach the values for a boundary layer that is turbulent from the leading edge.

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By examining this model statistically, it is possible to derive expressions for the local heat-transfer rate and skin friction that apply over the entire length of the plate. The Stanton number may be expressed in terms of the Stanton number for a laminar boundary layer, the Stanton number for a turbulent boundary layer undergoing an abrupt two-dimensional transition, and the fraction of the flow that is turbulent at any point x downstream of the leading edge. A similar expression can be derived for the local friction factor.

The location of transition in any slab will be denoted by s; s may be described by a probability distribution $P(s)$ where $P(s)ds$ represents the probability that transition occurs between s and $s + ds$. The probability that, at any point x, the flow in a given slab will be turbulent will be denoted by $\beta(x)$, the intermittency factor; $\beta(x)$ is also the fraction of the boundary layer that is turbulent at x. The fraction of the boundary layer that is laminar at x is taken as $1 - \beta(x)$. Since $\beta(x)$ is the probability that transition has occurred before x, $\beta(x)$ is related to $P(s)$ in the following manner:

$$\beta(x) = \int_0^x P(s)ds \tag{1}$$

Moreover, since the boundary layer eventually is entirely turbulent,

$$\beta(\infty) = 1 = \int_0^\infty P(s)ds \tag{2}$$

The mean location of transition is defined by

$$s_m = \int_0^\infty sP(s)ds \tag{3}$$

where s_m represents the average location of transition. Note that if $P(s)$ is symmetric, then s_m will be the symmetry point; and, for approximately normal distributions, the maximum value of $P(s)$ will be $P(s_m)$.

The rate of heat transfer to a turbulent slab of the boundary layer will depend on the usual flow parameters, the location of the point in question (x) and the location of the abrupt transition (s). This dependence may be indicated by writing

$$q''_{turb} = q''_{turb}(x;s) \tag{4}$$

The average heat-transfer rate for all of the turbulent slabs at any location x is defined as $\overline{q}''_{turb}(x)$. Since $q''(x;s)P(s)ds$ is the contribution to the heat flux at location x due to all slabs that went turbulent between s and $s + ds$, it follows that the total contribution to the heat flux at x from all slabs that are turbulent is

$$\beta(x)\overline{q}''_{turb}(x) = \int_0^x q''_{turb}(x;s)P(s)ds \tag{5}$$

Similarly, the average heat-transfer rate for all of the laminar slabs at any location x is

$$\overline{q}''_{lam} = q''_{lam}(x) \tag{6}$$

The total local heat-transfer rate at x for both the laminar and turbulent portions of the flow is therefore the sum of the laminar and turbulent contributions:

$$q''_w(x) = \left[1 - \beta(x)\right]\overline{q}''_{lam}(x) + \beta(x)\overline{q}''_{turb}(x;s) \tag{7}$$

By dividing by suitable constants and using equations (1) and (5), this relation may be written as

$$St(x) = \left[1 - \beta(x)\right]St_{lam} + \int_0^{Re_x} St_{turb}(Re_x;Re_s)P(Re_s)dRe_s \tag{8}$$

Similarly, for the friction factor,

$$C_f(x) = \left[1 - \beta(x)\right]C_{f,lam} + \int_0^{Re_x} C_{f,turb}(Re_x;Re_s)P(Re_s)dRe_s \tag{9}$$

Therefore, if the local Stanton numbers for a turbulent boundary layer after an abrupt transition are known, and if the fraction of the flow that is turbulent at any point x is known, the local skin friction and heat transfer for the nonabrupt transition can be determined by performing the integrations of equations (8) and (9). It should be noted that this statistical approach allows calculation of the local Stanton numbers and friction factors in all the laminar, transition, and turbulent regions of the boundary layer.

$\beta(x)$ for "Abrupt" Transition

If the boundary layer is tripped at a relatively low Reynolds number in such a way that transition occurs abruptly at the same value of x across the entire plate, $\beta(x)$ is simply a "jump function," and

$$\beta(x) = 0 \qquad x < s$$

$$\beta(x) = 1 \qquad x \geq s$$

In this case the probability density is discontinuous (a "pulse" of unit area), and the integrals must be evaluated in the Stiltjes sense, which gives

$$St(x) = St_{lam}(Re_x) \qquad x < s$$

$$St(x) = St_{turb}(Re_x;Re_s) \qquad x \geq s$$

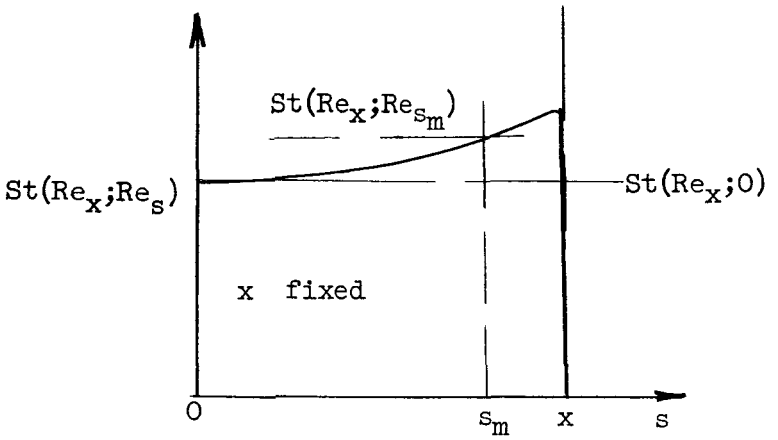
$$C_f(x) = C_{f,lam}(Re_x) \qquad x < s$$

$$C_f(x) = C_{f,turb}(Re_x;Re_s) \qquad x \geq s$$

where s is the location of the abrupt transition.

$\beta(x)$ for "Natural" Transition

The data of Schubauer and Klebanoff (ref. 6) indicate that, for natural transition, β resembles an error integral. If the parameters of the error curve were known, the integrals could, in principle, be evaluated. Unfortunately, the form of the abrupt-transition Stanton number expression is such that the integrations may not be performed in closed form. However, for values of $x > s_m$ the integrals may be approximated by replacing s by s_m in the expression for the abrupt-transition Stanton number. The justification of this is demonstrated by the following sketch:



This sketch shows the values of the Stanton numbers that would obtain at x for various locations of the abrupt transition. If x exceeds s_m, then $St(Re_x;Re_{s_m})$ lies between $St(Re_x;0)$ and $St(Re_x;Re_x)$, and a reasonable approximation is

$$\int_0^{Re_x} St_{turb}(Re_x;Re_s)P(Re_s)dRe_s \approx \int_0^{Re_x} St_{turb}(Re_x;Re_{s_m})P(Re_s)dRe_s$$

$$= St_{turb}(Re_x;Re_{s_m}) \int_0^{Re_x} P(Re_s)dRe_s$$

$$= \beta(Re_x)St_{turb}(Re_x;Re_{s_m}) \qquad x > s_m \tag{10a}$$

For values of x less than s_m this approximation has no meaning, and thus another approximation must be used. For $x < s_m$, the quantity $St_{turb}(Re_x;Re_s)$ will be approximated by

$$St(Re_x;Re_{s_m-2\omega})$$

where ω is the standard deviation of s from the mean transition-point location. This approximation will be used in the range $(s_m - 2\omega) < x < s_m$, and leads to

$$\int_0^{Re_x} St_{turb}(Re_x;Re_s)P(Re_s)dRe_s \approx \beta(x)St_{turb}(Re_x;Re_{s_m} - Re_{2\omega})$$

$$(s_m - 2\omega) < x < s_m \tag{10b}$$

In the laminar region (i.e., for $x < (s_m - 2\omega)$), β is small ($\beta < 0.03$), and thus the turbulent term may be neglected entirely. To summarize, then, the local Stanton number may therefore be approximated by

$$St(x) = St_{lam}(Re_x) \qquad x < (s_m - 2\omega) \qquad (11a)$$

$$St(x) = (1 - \beta)St_{lam}(Re_x) + \beta St_{turb}(Re_x; Re_{s_m} - Re_{2\omega}) \qquad (s_m - 2\omega) < x < s_m \qquad (11b)$$

$$St(x) = (1 - \beta)St_{lam}(Re_x) + \beta St_{turb}(Re_x; Re_{s_m}) \qquad x > s_m \qquad (11c)$$

Similar expressions for the local friction factor may be obtained:

$$C_f(x) = C_{f,lam}(Re_x) \qquad x < (s_m - 2\omega) \qquad (12a)$$

$$C_f(x) = (1 - \beta)C_{f,lam}(Re_x) + \beta C_{f,turb}(Re_x; Re_{s_m} - Re_{2\omega}) \qquad (s_m - 2\omega) < x < s_m \qquad (12b)$$

$$C_f(x) = (1 - \beta)C_{f,lam}(Re_x) + \beta C_{f,turb}(Re_x; Re_{s_m}) \qquad x > s_m \qquad (12c)$$

Assuming that the function $\beta(x)$ is known, it is necessary only to obtain expressions for the local Stanton number and friction factor for a turbulent layer that underwent transition abruptly at Re_s. An analysis leading to such expressions is presented herein.

ABRUPT-TRANSITION ANALYSIS

In the abrupt-transition analysis it is assumed that the turbulent boundary layer behaves hydrodynamically as if it "started" at some "virtual origin" (see fig. 2(a)). The location of this virtual origin, which will be denoted by r, is determined by applying the momentum and continuity theorems to the transition region (fig. 2(a)). The friction factors for the turbulent layer may then be calculated from equations for boundary layers that are turbulent from the leading edge, if the Reynolds number is based on the distance $x - r$. The local Stanton numbers for the turbulent layer could be calculated in the same manner, except that, if the plate is isothermal, the total heat transferred to the laminar layer before transition exceeds what would be transferred to the fictitious turbulent layer upstream of transition, and consequently the conservation of energy condition could not be satisfied for the transition control volume. However, by allowing the plate temperature to be higher in the fictitious portion of the turbulent layer, the energy condition can be satisfied. The "equivalent thermal problem" is shown in figure 2(b). The local Stanton numbers for this equivalent problem may be determined by employing nonisothermal heat-transfer theory (see refs. 3 and 4). This approach allows momentum, continuity, and energy conditions to be satisfied at the abrupt transition, and thus the analysis has a firm physical basis.

Equivalent Hydrodynamic Problem

Consider a laminar boundary layer flowing over a flat plate as shown by figure 2(a). A transition region extends over a small distance dx from a, where the flow is purely laminar, to b, where a fully developed turbulent boundary layer exists. The laminar-boundary-layer thickness at point a will be denoted by δ_a, and the turbulent-boundary-layer thickness at b by δ_b. The local wall shear stress in the transition region is $\tau_w(x)$. The turbulent boundary layer is assumed to behave hydrodynamically as if it had originated at the virtual origin r; dx is small so that the shear stress may be considered constant.

Application of the momentum theorem to the control volume aa'b'b yields

$$\int_{0_{(aa')}}^{\delta_a} \rho u_{bl}^2 \, dy + \rho u_\infty^2(\delta_b - \delta_a) + u_\infty w_{a'b'} = \int_{0_{(bb')}}^{\delta_b} \rho u_{bl}^2 \, dy + \tau_w \, dx \tag{13}$$

Here $w_{a'b'}$ is the mass-flow rate across the surface a'b' that enters the control volume with a velocity component in the x-direction of u_∞. Use of the equation of continuity for the same control volume gives

$$\int_{0_{(aa')}}^{\delta_a} \rho u_{bl} \, dy + u_\infty \rho(\delta_b - \delta_a) + w_{a'b'} = \int_{0_{(bb')}}^{\delta_b} \rho u_{bl} \, dy \tag{14}$$

Multiplying equation (14) by u_∞ and combining with (13) give

$$\int_0^{\delta_b} (\rho u_{bl} u_\infty - \rho u_{bl}^2) dy - \int_0^{\delta_a} (\rho u_{bl} u_\infty - \rho u_{bl}^2) dy = \tau_w \, dx \tag{15}$$

Using the definition of the momentum thickness θ and dividing by ρu_∞^2 reduce this to

$$\theta_b - \theta_a = \frac{\tau_w}{\rho u_\infty^2} \, dx \tag{16}$$

where θ_a and θ_b are the laminar and turbulent momentum thicknesses at a and b, respectively. Since dx is small, the term involving the shear stress is of higher order and may be neglected. Thus, equation (16) reduces to

$$\theta_a = \theta_b \tag{17}$$

Equation (17) is the criterion used to define the virtual origin of the turbulent boundary layer. Both continuity and momentum are satisfied by the condition of equation (17).

The velocity profile in the laminar boundary layer may be approximated by (ref. 7, p. 69)

$$\frac{u_{bl}}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \tag{18}$$

Thus, the laminar momentum thickness at point a may be evaluated as follows:

$$\theta_a = \delta_a \int_0^1 \left(\frac{3}{2}\sigma - \frac{1}{2}\sigma^3 - \frac{9}{4}\sigma^2 + \frac{3}{2}\sigma^4 - \frac{1}{4}\sigma^6\right) d\sigma$$

$$= \frac{39}{280}\delta_a \tag{19}$$

For the turbulent boundary layer in the range of Reynolds numbers found at transition, a good approximation to the velocity profile is

$$\frac{u_{bl}}{u_\infty} = \left(\frac{y}{\delta}\right)^{1/7} \tag{20}$$

Evaluation of the turbulent momentum thickness at b gives

$$\theta_b = \delta_b \int_0^1 \left(\sigma^{1/7} - \sigma^{2/7}\right) d\sigma$$

$$= \frac{7}{72}\delta_b \tag{21}$$

Substituting equations (19) and (21) into (17) yields

$$\frac{\delta_b}{\delta_a} = 1.433 \tag{22}$$

The momentum equation of the turbulent incompressible flat-plate boundary layer may be written as

$$\frac{C_f}{2} = \frac{d}{dx}\int_0^\delta \left(\frac{u_{bl}}{u_\infty}\right)\left(1 - \frac{u_{bl}}{u_\infty}\right) dy \tag{23}$$

Substitution of the laminar velocity profile (eq. (18)) into equation (23) leads to (ref. 7, p. 69)

$$\frac{\delta_a}{a} = 4.64\mathrm{Re}_a^{-0.5} \tag{24}$$

For turbulent flow in the range $10^5 < \mathrm{Re}_x < 10^7$, the local friction factor can be given by the Blasius formula (see ref. 7, p. 117), as

$$\frac{C_f}{2} = 0.0228\mathrm{Re}_\delta^{-0.25} \tag{25}$$

Substituting equations (20) and (25) into (23) gives an expression for the turbulent-boundary-layer thickness:

$$\frac{\delta}{\xi} = 0.0376\mathrm{Re}_\xi^{-0.2} \tag{26}$$

Here ξ is measured from the virtual origin of the turbulent layer, point r. Assuming that the turbulent layer after transition behaves as if it had originated at the virtual origin r, equation (26) then gives

$$\frac{\delta_b}{b - r} = 0.376\mathrm{Re}_{b-r}^{-0.2} \tag{27}$$

By letting dx become very small, a approaches b, and, in the limit, combination of (22), (24), and (27) results in

$$\mathrm{Re}_r = \mathrm{Re}_s\left(1 - 36.3\mathrm{Re}_s^{-0.375}\right) \tag{28}$$

where $\mathrm{Re}_s = \mathrm{Re}_a = \mathrm{Re}_b$. Equation (28) defines the location of the virtual origin of the turbulent boundary layer.

Effect of Transition-Point Location on Local Turbulent Friction Factors

As shown by reference 7 (p. 117), combination of equations (20), (23), and (25) results in the following relation for the turbulent friction factor:

$$\frac{C_f(\mathrm{Re}_\xi;0)}{2} = 0.0296\mathrm{Re}_\xi^{-0.2} \tag{29}$$

where again ξ is measured from the virtual origin of the turbulent layer. Since the boundary layer behaves as if it begins at r, the friction factor at some point x may be determined from (28) and (29) as

$$\frac{C_f(Re_x;Re_s)}{2} = 0.0296Re_{x-r}^{-0.2}$$

$$= 0.0296Re_x^{-0.2}\left[1 - \frac{Re_s}{Re_x}\left(1 - 36.3Re_s^{-0.375}\right)\right]^{-0.2} \qquad (30)$$

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The term before the bracket in equation (30) is the local friction coefficient that would occur if the boundary layer had been turbulent from the leading edge of the plate, and thus the bracketed term can be viewed as a correction that incorporates the effect of the actual location of transition on skin friction. Equation (30) is shown by figure 3. Note that a considerable increase in the friction factor may be expected if the transition Reynolds number is relatively high, but that the effect is small if transition occurs close to the leading edge of the plate.

Equivalent Thermal Problem

Reference 2 shows that the local Stanton number for an isothermal plate and a purely turbulent boundary layer is given by (neglecting temperature-dependent fluid-properties effects)

$$St_T(Re_\xi;0)Pr^{0.4} = 0.0296Re_\xi^{-0.2} \qquad (31)$$

where ξ is measured from the virtual origin of the turbulent layer. The heat-transfer rate after an abrupt transition could be determined from this relation, with the Reynolds number based on the distance $x - r$, if it were not for the fact that the heat transfer over the laminar portion of the boundary layer differs from what would be transferred to the turbulent layer in the distance from the virtual origin to the transition point. As a consequence, the condition of conservation of energy for the control volume aa'b'b (for dx small) cannot be satisfied if, in the equivalent thermal problem, the plate temperature is held constant. In order to allow the energy condition to be satisfied, it is necessary to adjust the plate temperature in the fictitious turbulent region before transition. It will be assumed that the plate temperature is constant over the fictitious turbulent region, and the value of this constant will be selected so that the total energy transferred to the laminar layer before transition equals that which would be transferred to the fictitious part of the turbulent layer; that is, the part between the virtual origin and the real transition location. The heat transfer to the fictitious turbulent region may be determined from

equation (31), and the heat transfer to the laminar layer may be found from the well-known Pohlhausen solution,

$$St_{lam}Pr^{2/3} = 0.332Re_x^{-0.5} \tag{32}$$

The condition that the total heat transfers are equal may be written as

$$\int_0^s \Delta t G c_p \left(0.332Pr^{-2/3}Re_x^{-0.5}\right)dx$$

$$= \int_0^{s-r} \Delta t' G c_p \left(0.0296Pr^{-0.4}Re_\xi^{-0.2}\right)d\xi \tag{33}$$

The indicated integrations lead to the result that

$$\frac{\Delta t'}{\Delta t} = 1.012Pr^{-0.27} \tag{34}$$

Note that, for fluids with Prandtl numbers less than unity, the temperature difference in the hypothetical turbulent region between the virtual origin and the transition point is greater than the actual temperature difference, indicating that the total heat transfer in the laminar portion is greater than in the hypothetical turbulent region. This is possible because the fictitious turbulent region is much shorter than the actual laminar region.

Local Stanton Numbers After Abrupt Transition

The heat-transfer rates downstream of the abrupt transition may be determined by applying the methods of superposition described in reference 4 to the equivalent thermal problem. Reference 4 shows that the local heat-transfer rate due to a series of steps in the wall temperature may be determined simply by summing the heat-transfer rates due to each step. Reference 3 shows that the heat transfer to a turbulent boundary layer for a plate that is unheated from the leading edge to l and maintained at constant temperature thereafter is given by

$$\frac{St}{St_T} = \left[1 - \left(\frac{l}{\xi}\right)^{9/10}\right]^{-1/9} \tag{35}$$

where ξ is measured from the leading edge. Thus, addition of the heat-transfer rate due to a step of height $\Delta t'$ at the fictitious leading edge and the heat-transfer rate due to a step of height $(\Delta t - \Delta t')$ at a distance $(s - r)$ downstream of the fictitious leading edge yields the following result:

$$St\ \Delta t = 0.0296Pr^{-0.4}Re_{x-r}^{-0.2}\ \Delta t'$$

$$+ 0.0296Pr^{-0.4}Re_{x-r}^{-0.2}\left[1 - \left(\frac{s - r}{x - r}\right)^{9/10}\right]^{-1/9}(\Delta t - \Delta t') \qquad (36)$$

By using equation (28) to eliminate r, equation (36) may be reduced to

$$\frac{St(Re_x;Re_s)}{St_T(Re_x;0)} = \left[1 - \frac{Re_s}{Re_x}\left(1 - 36.3Re_s^{-0.375}\right)\right]^{-0.2}$$

$$\times \left(1.012Pr^{-0.27} + \left(1 - 1.012Pr^{-0.27}\right)\right.$$

$$\left.\times \left\{1 - \left[\frac{36.3\ \frac{Re_s}{Re_x}\ Re_s^{-0.375}}{1 - \frac{Re_s}{Re_x}\left(1 - 36.3Re_s^{-0.375}\right)}\right]^{9/10}\right\}^{1/9}\right)$$

$$= F(Re_x;Re_s) \qquad (37)$$

where $St(Re_x;Re_s)$ is the local Stanton number for the turbulent part of a boundary layer that is laminar from $x = 0$ to $x = s$ and undergoes abrupt transition to a turbulent boundary layer at s; and $St_T(Re_x;0)$ is the local Stanton number for a boundary layer that is turbulent from the leading edge, where $x = 0$ (as given by eq. (31) or an equivalent expression). Note that, as Re_s/Re_x approaches zero, the local Stanton number approaches the value for a boundary layer that is entirely turbulent, and this is the correct limiting behavior. At the transition point the analysis indicates that the local Stanton number is $-\infty$, and this is due to the discontinuity in the wall temperature in the equivalent thermal problem. Equation (37) is plotted for $Pr = 0.7$ in figure 4. Note that there is a considerable overshoot of the asymptotic curve (eq. (31)) if the transition occurs at a high Reynolds number; the local heat-transfer coefficient may be nearly twice as high as the values predicted by the usual turbulent equation for $Re_s = 10^8$. At high transition Reynolds numbers the effect of transition location is very important and the increase in Stanton numbers persists well downstream of transition. It is probable that the increase in the local Stanton numbers and friction factors is less if the transition occurs over a longer region, but it is evident that the effects of transition location are too large to be overlooked in many applications. The abrupt-transition analysis provides an

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"upper limit" on these effects and is also useful in its own right in evaluating experiments where the boundary layer has been deliberately "tripped" in a two-dimensional manner.

RESULTS AND DISCUSSION

Comparison of Abrupt-Transition Analysis with Heat-Transfer Experiments

Using the flat-plate test equipment described in reference 2, a series of test runs were made in which abrupt transitions were stimulated at Reynolds numbers ranging from 0.11×10^5 to 2.6×10^5. In these runs transition was stimulated by a 1/2-inch-wide strip of fine (414 grit) emery cloth, which was cemented to the plate at the desired point. A series of tests with different grits indicated that the 414-grit cloth had no appreciable "roughness effect" on the turbulent heat transfer downstream of the trip. In addition, one run was made in which the boundary layer was allowed to undergo natural transition. In this run transition was not abrupt but started at a Reynolds number of about 2×10^5 and appeared to be complete by a Reynolds number of 6×10^5. The data from these tests are shown by figure 5 and are tabulated in table I. Reference 2 shows that effects of temperature-dependent fluid properties may be lumped into a temperature ratio and that the local Stanton number for a purely turbulent flow may be determined from

$$St_T Pr^{0.4} = 0.0296 Re_x^{-0.2} \left(\frac{T_w}{T_\infty}\right)^{-0.4} \tag{38}$$

where the fluid properties are to be evaluated at the free-stream temperature. Thus, the data are plotted in the form $St(T_w/T_\infty)^{0.4}$ against Re_x. Note that the data do tend to overshoot the limiting relation (38) and that the effect is greatest for the most delayed transitions. These are characteristics predicted by the foregoing abrupt-transition analysis.

The experimental data from these runs are compared with the analysis in figure 6. The analysis appears to correlate the abrupt-transition data very well. The natural-transition data were corrected with the abrupt-transition analysis, based on a transition Reynolds number of 4×10^5, which is approximately in the middle of the transition region. Only the fully turbulent data ($Re_x > 6\times10^5$) are shown in figure 6 for the natural-transition run. In spite of the fact that the transition is not abrupt,

the correction of the natural-transition data by the abrupt-transition analysis appears to be adequate. Thus, it seems that the abrupt-transition analysis may be used in estimating the effect of transition-point location for natural transitions if the Re_s is evaluated somewhere in the middle of the transition region.

Comparison of Natural-Transition Heat-Transfer Data with Statistical Predictions

By use of the abrupt-transition analysis (eq. (37)), the Pohlhausen laminar solution (eq. (32)), and the statistical prediction of the local Stanton numbers given by equations (11), the local Stanton numbers for the natural-transition run were predicted. The function $\beta(Re_x)$ was taken as an error integral, as is suggested by the results of Schubauer and Klebanoff (ref. 6):

$$\beta(Re_x) = \int_{-\infty}^{Re_x} \frac{1}{Re_\omega\sqrt{2\pi}} \exp\left[\frac{-(Re_x - Re_{s_m})^2}{2Re_\omega^2}\right] dRe_x$$

Examination of the heat-transfer data for the natural-transition run indicates that transition begins to be important at about $Re_x = 2\times10^5$ and that the boundary layer is almost entirely turbulent by $Re_x = 6\times10^5$. Thus the mean, Re_{s_m}, is about 4×10^5. By taking Re_ω as 10^5, the value of β is about 0.025 at $Re_x = 2\times10^5$ and about 0.975 at $Re_x = 6\times10^5$. As this is the approximate extent of the transition region, Re_ω was taken as 10^5 in the predictions for the local Stanton number. The data are compared with the statistical predictions in figure 7. The predictions appear to be quite good, even in the transition region. Since the abrupt-transition analysis indicates that the local turbulent Stanton number at the transition is $-\infty$, the predictions have singularities at $Re_{2\omega}$ and Re_{s_m} (see eqs. (11a), (11b), and (37)). Since this is an artificiality introduced by the approximations of equations (10), the predictions are not shown in the neighborhood of the singularities. The influence of the singularity extends only over a small region, however, as may be seen by examination of figure 4. Thus, it seems that the statistical model, together with the abrupt-transition analysis and the distribution function β, provides a method for calculating the heat transfer in the transition region and the effect of the location of transition on the turbulent heat-transfer rates if the location and approximate extent of the transition region are known.

CONCLUDING REMARKS

An analysis has been made that allows the prediction of the effect of the location of transition on the heat transfer to the turbulent boundary layer. The analysis assumes that the boundary layer undergoes an abrupt transition at some point downstream of the leading edge. The "patching" of the laminar and turbulent layers at the transition "point" is such that the momentum, continuity, and energy theorems are all satisfied. The abrupt-transition data of the present investigation are in good agreement with the analysis. Unfortunately, these data were obtained over only a limited range of transition Reynolds numbers, and thus the analysis lacks experimental confirmation in the range where the effect is largest. The abrupt-transition analysis may be used to predict the local turbulent Stanton numbers after a natural transition if the "transition Reynolds number" is evaluated in the middle of the transition region. If the extent of the transition region is known, it may be assumed that the function $\beta(Re_x)$ is an error integral; then the statistical treatment may be used to calculate the heat transfer to both the transition and turbulent portions of the boundary layer. In this manner an estimate of the local heat-transfer coefficient can be made that is in good agreement with the experimental data for all types of flow.

Stanford University,
Stanford, Calif., October 22, 1957.

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TABLE I. - EXPERIMENTAL HEAT-TRANSFER DATA

Strip	G, $\frac{lb}{(hr)(sq\ ft)}$ $\times 10^{-3}$	Δt_m, ^oF	q_w'', $\frac{Btu}{(hr)(sq\ ft)}$	h_l, $\frac{Btu}{(hr)(sq\ ft)(^oF)}$	St $\times 10^3$	$St\left(\frac{T_w}{T_\infty}\right)^{0.4}$ $\times 10^3$	Re_x $\times 10^{-6}$	G, $\frac{lb}{(hr)(sq\ ft)}$ $\times 10^{-3}$	Δt_m, ^oF	q_w'', $\frac{Btu}{(hr)(sq\ ft)}$	h_l, $\frac{Btu}{(hr)(sq\ ft)(^oF)}$	St $\times 10^3$	$St\left(\frac{T_w}{T_\infty}\right)^{0.4}$ $\times 10^3$	Re_x $\times 10^{-6}$
	Natural transition; t_∞ = 75.4° F; ρ_∞ = 0.0750 lb/cu ft							Re_B = 0.11×10^5; t_∞ = 65.3° F; ρ_∞ = 0.0765 lb/cu ft						
2	10.6	23.9	97.2	4.07	1.70	1.73	0.082	11.7	16.6	165	9.95	3.53	3.57	0.092
3	10.6	24.4	75.7	3.10	1.21	1.24	.136	11.7	16.3	147	9.04	3.22	3.26	.151
4	10.6	24.6	66.8	2.72	1.07	1.08	.186	11.7	16.3	138	8.48	3.02	3.06	.206
5	10.6	24.2	66.1	2.73	1.07	1.09	.236	11.7	16.3	129	7.94	2.82	2.88	.264
6	10.6	24.7	70.6	2.86	1.12	1.14	.286	11.7	16.2	124	7.65	2.72	2.76	.319
7	10.6	24.8	88.6	3.58	1.41	1.44	.334	11.8	16.1	120	7.48	2.64	2.68	.377
8	10.6	24.5	110.0	4.49	1.76	1.79	.385	11.7	16.3	118	7.24	2.56	2.59	.433
9	10.6	23.8	136.1	5.77	2.27	2.31	.434	11.7	16.3	121	7.40	2.62	2.67	.489
10	10.6	24.2	147.9	6.11	2.41	2.45	.485	11.8	16.4	111	6.74	2.38	2.41	.546
11	10.6	24.1	151.5	6.29	2.48	2.52	.533	11.8	16.4	108	6.61	2.34	2.37	.601
12	10.6	24.0	155.0	6.46	2.53	2.58	.587	11.8	16.4	112	6.86	2.42	2.45	.660
13	10.6	23.9	155.5	6.51	2.55	2.60	.637	11.8	16.7	110	6.57	2.32	2.35	.715
14	10.6	24.2	156.7	6.48	2.55	2.60	.684	11.8	16.6	107	6.46	2.23	2.31	.771
15	10.6	24.2	151.5	6.26	2.61	2.66	.732	11.8	16.4	109	6.67	2.35	2.38	.827
16	10.6	24.4	156.4	6.37	2.51	2.55	.784	11.8	16.4	107	6.54	2.31	2.34	.884
17	10.6	24.7	150.9	6.11	2.55	2.59	.831	11.8	16.9	108	6.38	2.26	2.29	.939
18	10.6	24.5	144.3	5.89	2.32	2.36	.883	11.8	16.9	105	6.20	2.19	2.22	.995
19	10.6	24.4	139.6	5.78	2.28	2.32	.933	11.7	16.9	102	6.01	2.13	2.16	1.048
20	10.5	24.6	141.5	5.75	2.27	2.32	.976	11.8	16.7	101	6.07	2.14	2.17	1.104
21	10.5	25.0	141.8	5.67	2.24	2.29	1.028	11.7	16.8	104	6.17	2.19	2.22	1.160
22	10.5	25.3	135.5	5.36	2.12	2.16	1.078	11.7	16.6	93	5.58	1.98	2.02	1.214
23	10.5	25.1	132.6	5.29	2.09	2.13	1.127	11.7	16.9	99	5.84	2.07	2.10	1.270
	Re_B = 0.92×10^5; t_∞ = 69.9° F; ρ_∞ = 0.0755 lb/cu ft							Re_B = 1.44×10^5; t_∞ = 71.3° F; ρ_∞ = 0.0756 lb/cu ft						
2	11.8	16.3	96.5	5.92	2.10	2.12	0.092	11.7	16.7	72.7	4.35	1.55	1.56	0.092
3	11.8	16.1	171.3	10.64	3.77	3.82	.151	11.7	17.5	50.9	2.91	1.04	1.05	.150
4	11.7	16.1	137.7	8.55	3.04	3.08	.206	11.7	16.3	159.2	9.77	3.48	3.53	.205
5	11.8	16.1	131.8	8.18	2.88	2.92	.264	11.7	16.5	149.2	9.04	3.22	2.36	.261
6	11.8	16.2	125.4	7.75	2.73	2.78	.320	11.7	16.7	138.9	8.32	2.97	3.00	.316
7	11.8	16.1	120.2	7.48	2.64	2.67	.376	11.7	16.7	132.7	7.95	2.82	2.86	.373
8	11.8	16.2	116.9	7.21	2.56	2.59	.429	11.7	16.6	124.6	7.50	2.67	2.70	.427
9	11.7	16.3	121.0	7.42	2.64	2.67	.485	11.7	16.6	127.9	7.70	2.74	2.77	.484
10	11.8	16.4	112.0	6.83	2.42	2.45	.542	11.7	17.0	117.1	6.89	2.45	2.48	.540
11	11.8	16.4	109.8	6.70	2.36	2.39	.600	11.7	16.8	116.0	6.90	2.45	2.48	.595
12	11.7	16.5	113.1	6.87	2.44	2.47	.652	11.7	17.0	115.8	6.82	2.42	2.45	.650
13	11.8	16.7	111.1	6.66	2.35	2.38	.712	11.8	17.2	117.4	6.83	2.41	2.44	.710
14	11.8	16.4	107.5	6.56	2.32	2.36	.765	11.8	17.2	113.8	6.62	2.35	2.38	.765
15	11.8	16.5	103.6	6.27	2.21	2.24	.824	11.7	17.5	109.6	6.25	2.21	2.24	.820
16	11.8	16.5	103.9	6.30	2.22	2.45	.881	11.8	17.1	112.6	6.58	2.23	2.35	.880
17	11.8	16.5	105.0	6.36	2.24	2.27	.938	11.8	17.3	109.3	6.32	2.23	2.26	.935
18	11.8	16.5	102.7	6.22	2.19	2.22	.992	11.8	17.3	111.5	6.44	2.27	2.30	.991
19	11.8	16.6	99.7	6.01	2.13	2.16	1.042	11.8	17.4	108.1	6.21	2.19	2.22	1.048
20	11.7	16.6	102.3	6.17	2.19	2.22	1.092	11.7	17.4	104.4	6.00	2.13	2.16	1.093
21	11.8	16.8	102.0	6.08	2.15	2.18	1.158	11.7	17.5	107.6	6.14	2.18	2.21	1.150
22	11.8	16.7	94.3	5.65	2.00	2.03	1.210	11.7	17.7	99.8	5.61	1.99	2.01	1.208
23	11.8	16.8	93.7	5.58	1.98	2.00	1.266	11.7	18.0	103.0	5.77	2.05	2.07	1.264
	Re_B = 2.00×10^5; t_∞ = 71.5° F; ρ_∞ = 0.0758 lb/cu ft							Re_B = 2.60×10^5; t_∞ = 64.0° F; ρ_∞ = 0.0766 lb/cu ft						
2	11.7	17.8	76.2	4.38	1.53	1.55	0.091	11.7	18.2	78.0	4.28	1.52	1.55	0.092
3	11.8	18.4	58.2	3.16	1.12	1.14	.150	11.7	18.3	60.9	3.33	1.17	1.21	.151
4	11.8	18.6	38.7	2.08	.74	.75	.206	11.7	18.1	53.6	2.96	1.05	1.07	.208
5	11.8	17.3	165.4	9.57	3.39	3.44	.262	11.7	18.3	50.5	2.76	.98	.99	.264
6	11.8	17.4	151.0	8.68	3.07	3.16	.317	11.7	17.9	153.0	8.55	3.04	3.09	.318
7	11.8	17.7	144.0	8.13	2.87	2.91	.373	11.7	17.9	155.2	8.68	3.08	3.12	.375
8	11.8	17.8	138.9	7.80	2.76	2.80	.428	11.7	18.1	147.5	8.15	2.89	2.94	.432
9	11.8	18.1	135.5	7.48	2.55	2.56	.484	11.7	18.1	146.3	8.09	2.88	2.92	.487
10	11.9	18.1	122.2	6.76	2.38	2.41	.544	11.7	18.5	134.5	7.27	2.58	2.62	.544
11	11.8	17.9	120.7	6.74	2.37	2.41	.598	11.7	18.3	130.7	7.14	2.54	2.52	.600
12	11.8	18.0	121.2	6.74	2.36	2.41	.656	11.8	18.5	131.3	7.10	2.51	2.55	.662
13	11.8	18.3	120.4	6.58	2.32	2.36	.711	11.7	18.6	130.2	7.00	2.49	2.53	.710
14	11.8	18.1	118.4	6.55	2.32	2.35	.764	11.7	18.6	129.5	6.96	2.47	2.51	.768
15	11.8	18.3	116.4	6.36	2.24	2.27	.823	11.7	18.5	123.0	6.65	2.36	2.40	.824
16	11.9	18.0	111.8	6.21	2.18	2.22	.881	11.7	18.4	122.8	6.67	2.37	2.41	.880
17	11.8	18.2	113.2	6.23	2.19	2.22	.935	11.7	18.4	118.9	6.46	2.30	2.33	.935
18	11.8	18.4	110.6	6.01	2.13	2.16	.985	11.7	18.5	116.2	6.28	2.13	2.16	.990
19	11.8	18.2	106.0	5.83	2.06	2.09	1.040	11.7	18.4	112.6	6.11	2.17	2.20	1.049
20	11.8	18.1	103.7	5.72	2.03	2.06	1.092	11.7	18.4	109.7	5.95	2.12	2.14	1.110
21	11.8	18.2	105.7	5.80	2.06	2.08	1.151	11.7	18.3	112.5	6.15	2.18	2.20	1.160
22	11.8	18.4	102.1	5.55	1.96	1.99	1.210	11.7	18.2	102.6	5.64	2.00	2.03	1.216
23	11.8	18.7	104.8	5.60	1.98	2.01	1.262	11.7	18.5	105.3	5.70	2.01	2.06	1.270

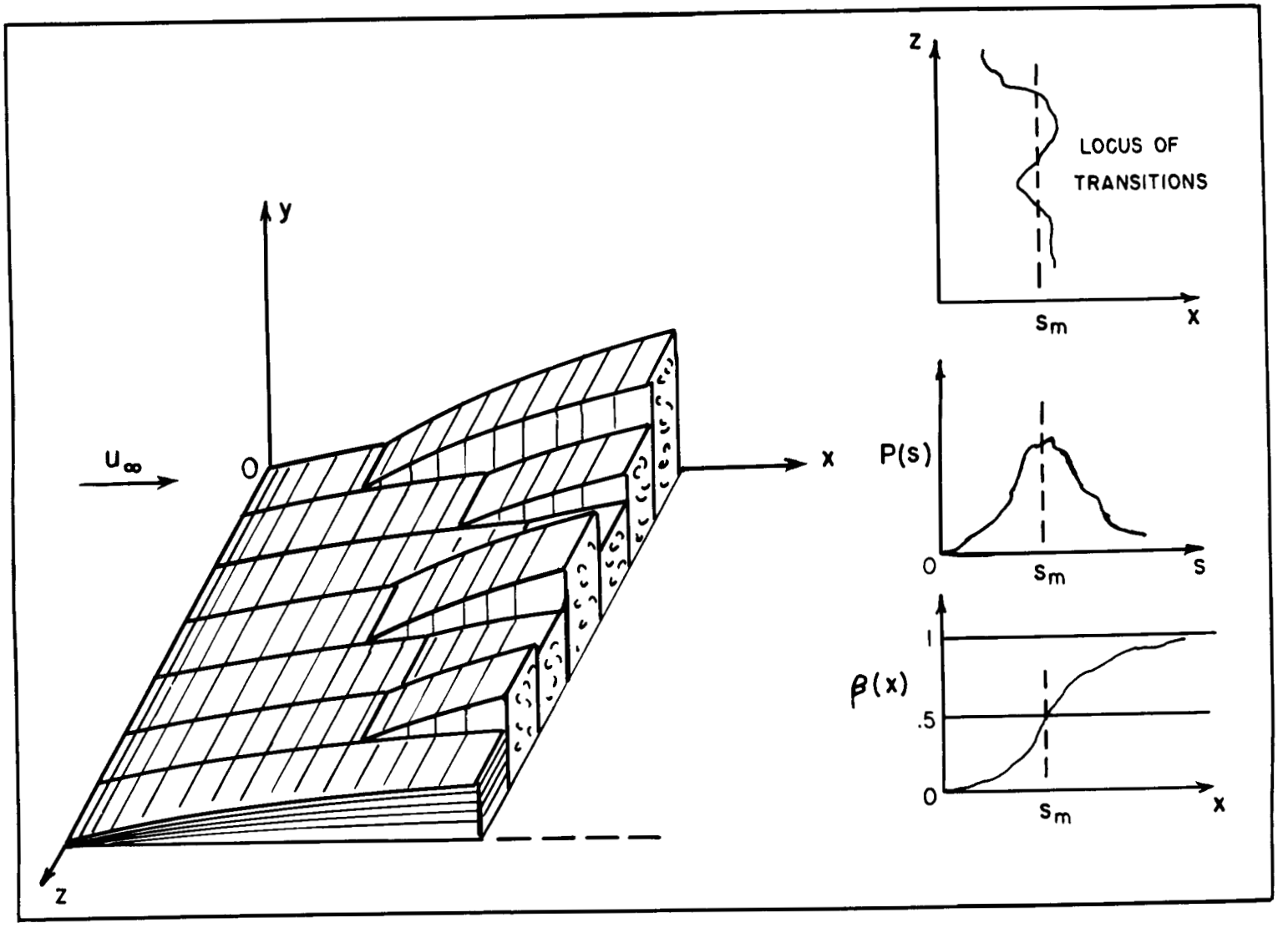
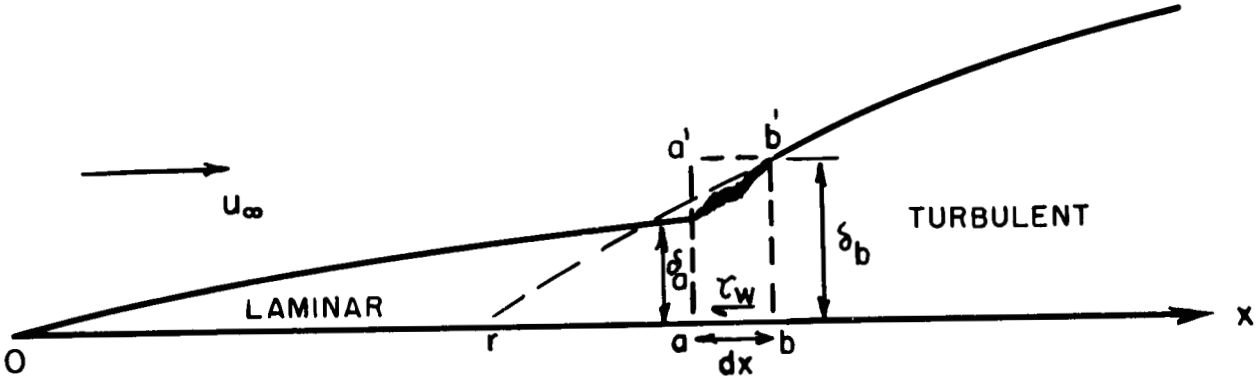
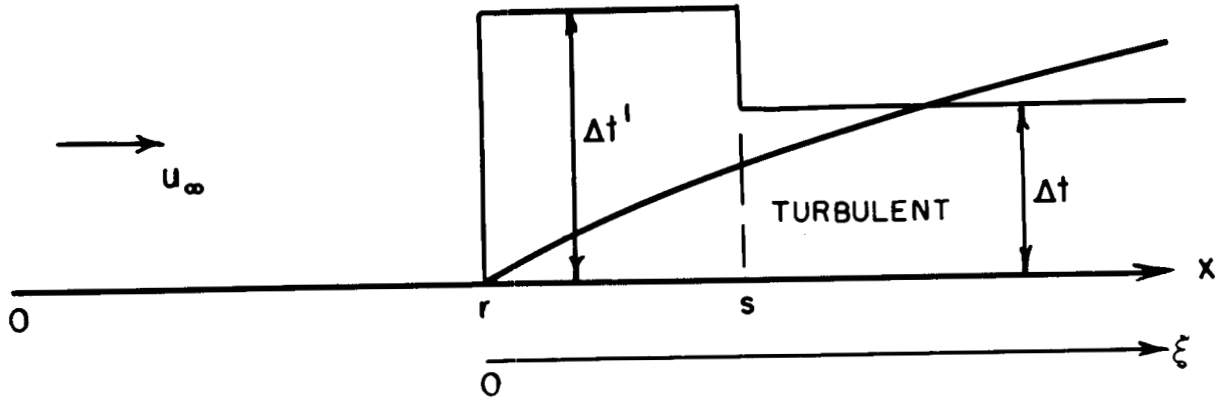


Figure 1. - Transition model.



(a) Equivalent hydrodynamic problem.



(b) Equivalent thermal problem.

Figure 2. - Equivalent problems.

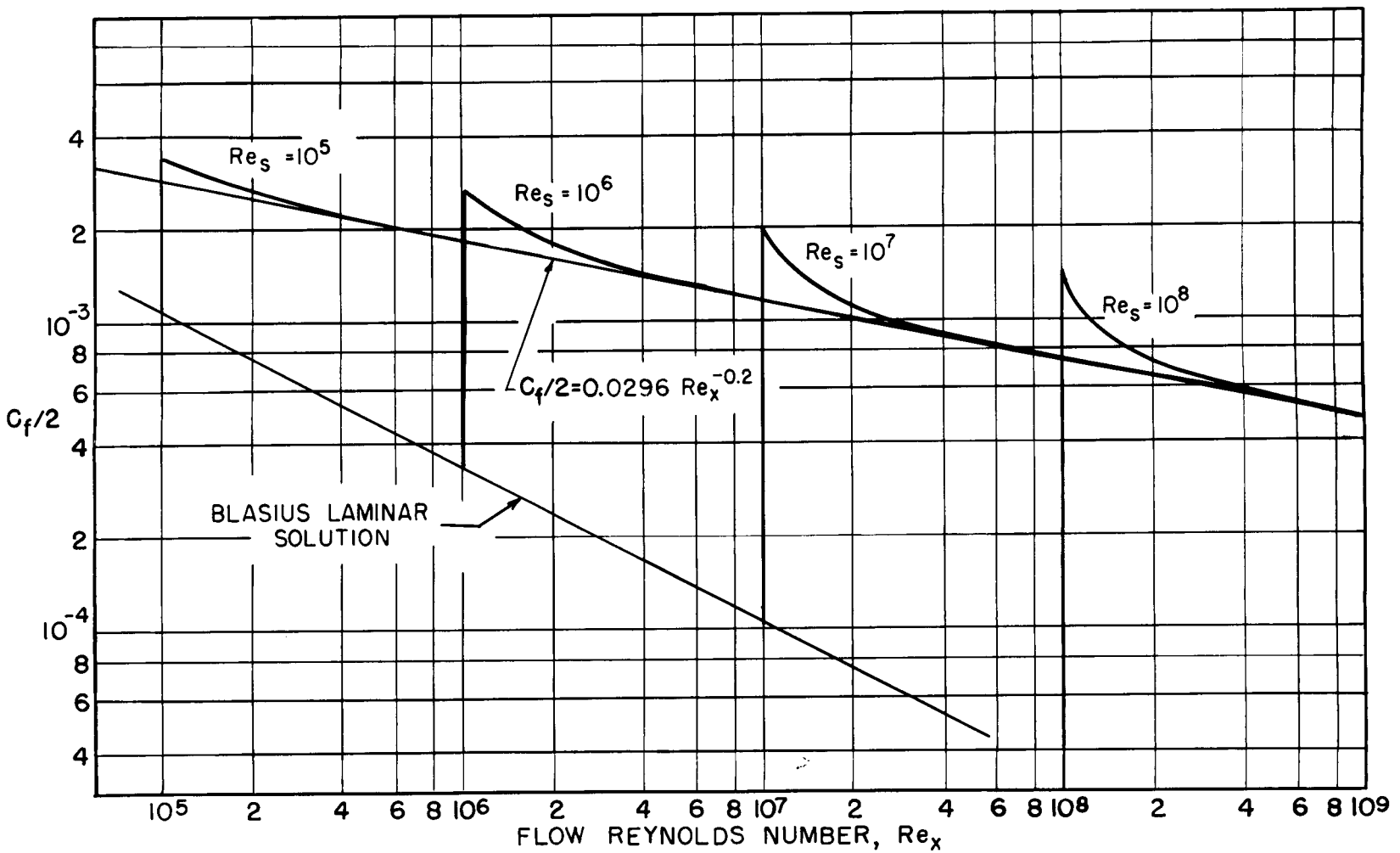


Figure 3. - Effect of delayed transition on turbulent friction factors.

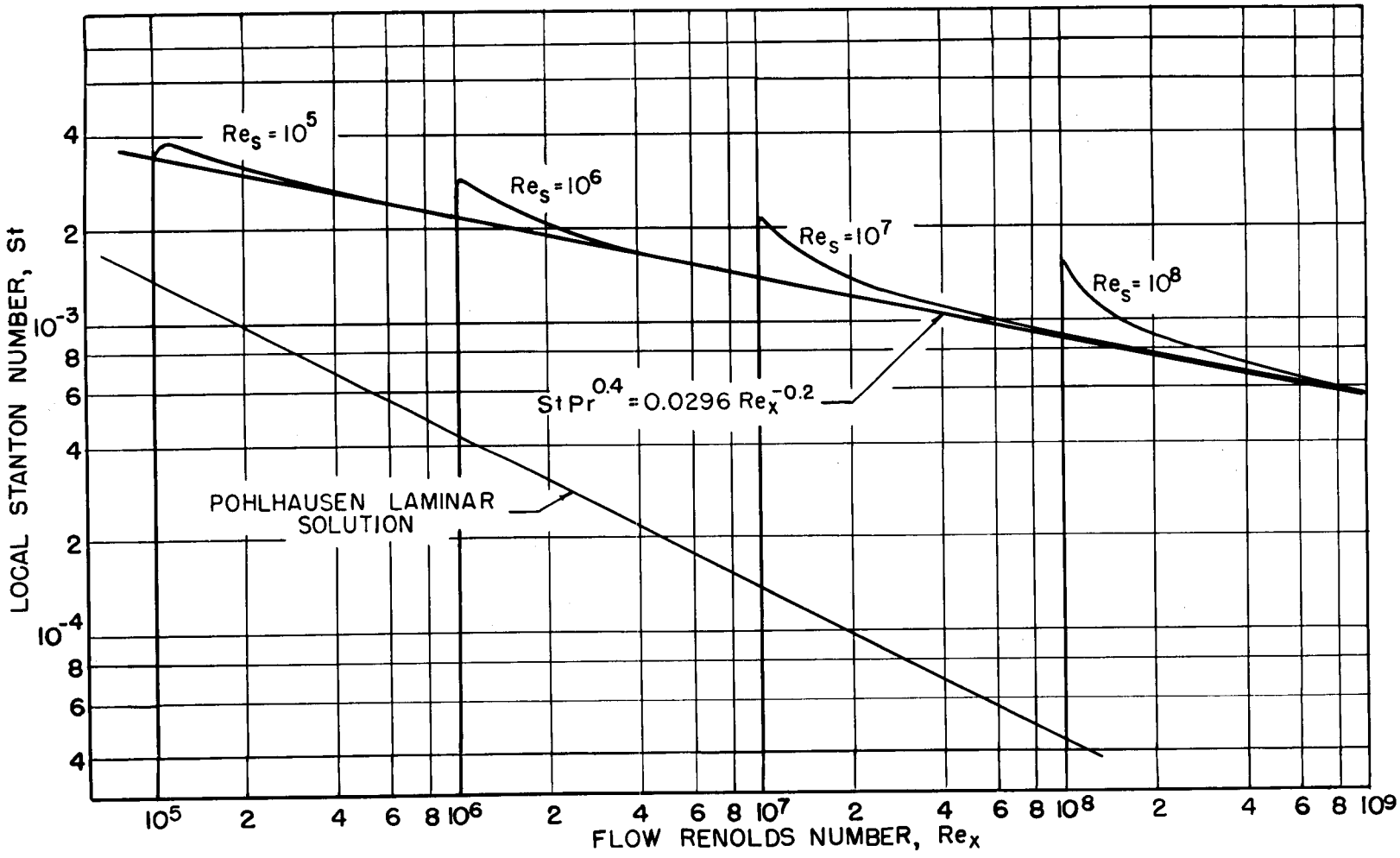


Figure 4. - Effect of delayed transition on turbulent heat transfer. Prandtl number, 0.7.

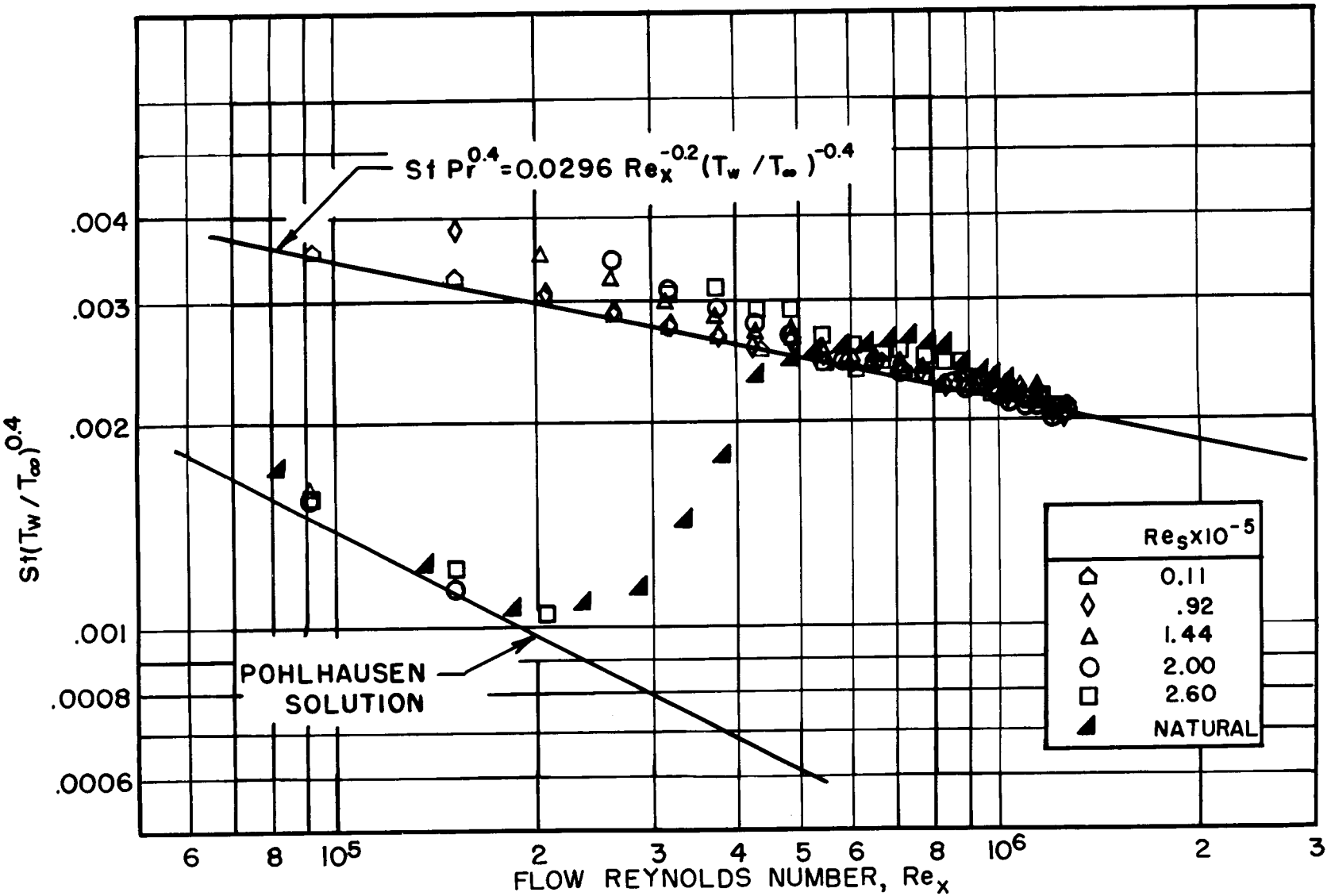


Figure 5. - Effect of location of transition on turbulent heat transfer. Prandtl number, 0.7.

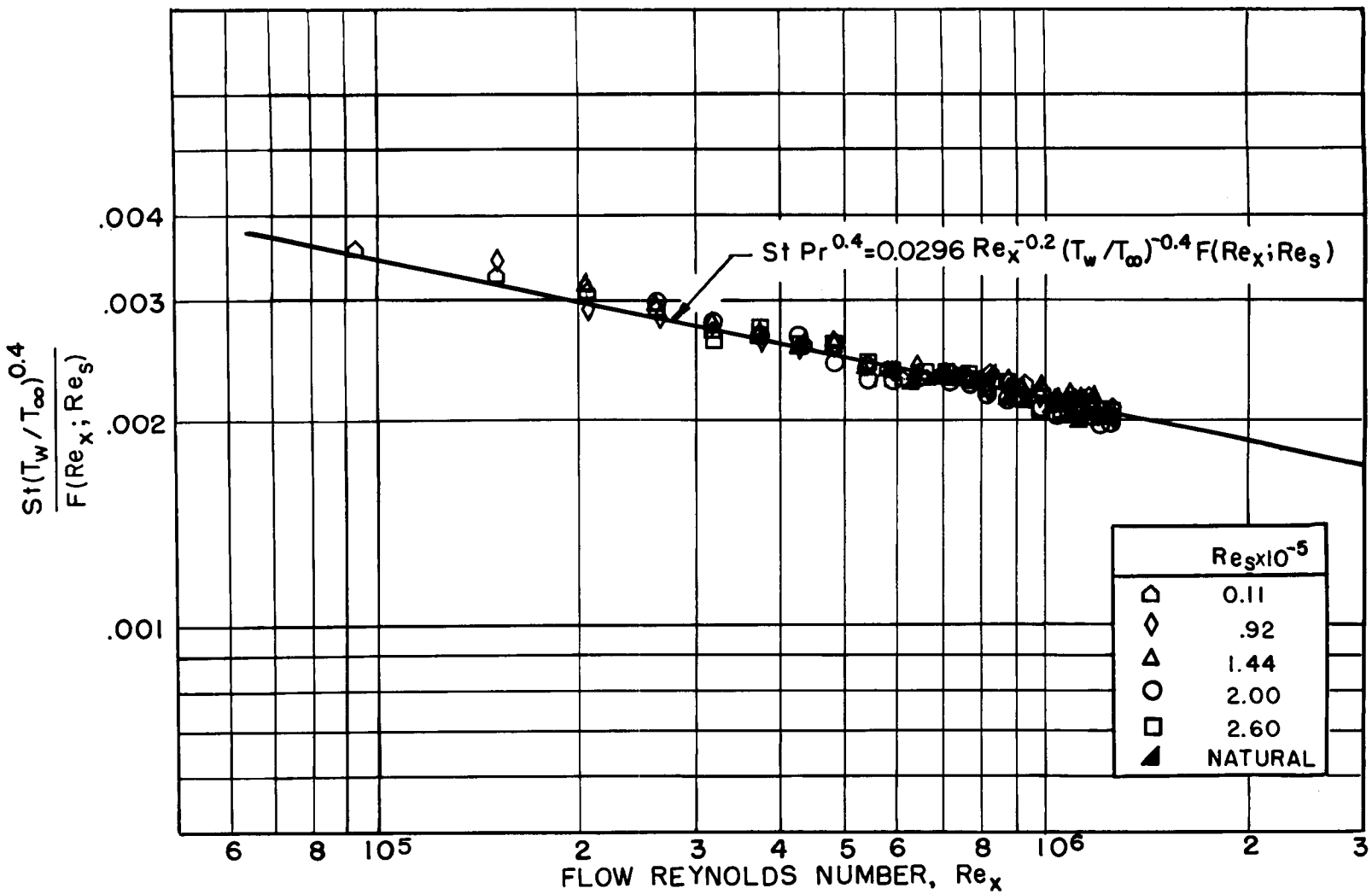


Figure 6. - Comparison of data with abrupt-transition analysis. Prandtl number, 0.7.

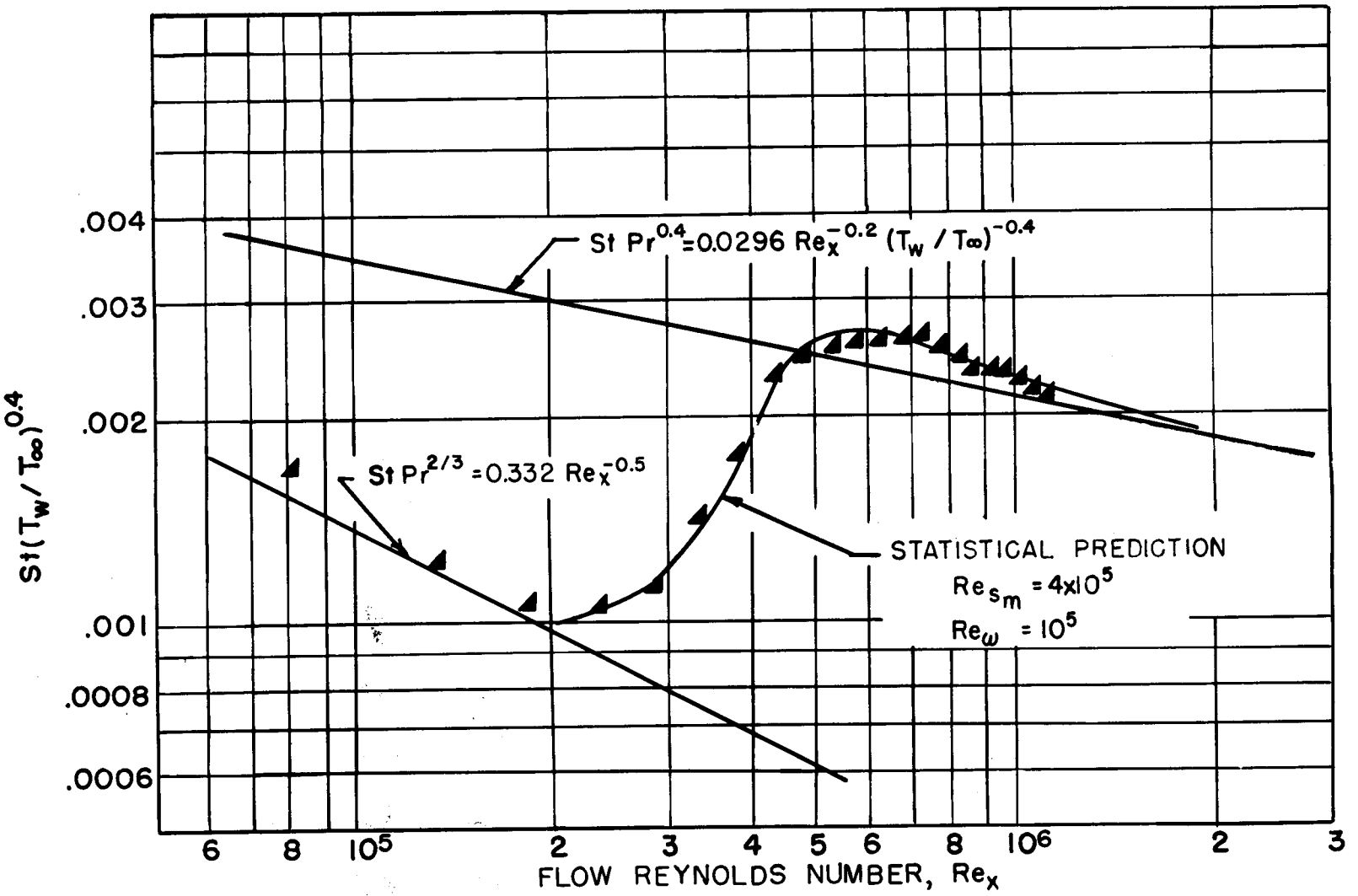


Figure 7. - Comparison of natural-transition data with statistical predictions. Prandtl number, 0.7.

NASA - Langley Field, Va.