Explicit Von Neumann Stability Conditions for the $c-\tau$ Scheme—A Basic Scheme in the Development of the CE-SE Courant Number Insensitive Schemes

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The c-τ scheme—A basic scheme in the development of the CE-SE Courant number insensitive schemes

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Abstract

As part of the continuous development of the space-time conservation element and solution element (CE-SE) method, recently a set of so called “Courant number insensitive schemes” has been proposed. The key advantage of these new schemes is that the numerical dissipation associated with them generally does not increase as the Courant number decreases. As such, they can be applied to problems with large Courant number disparities (such as what commonly occurs in Navier-Stokes problems) without incurring excessive numerical dissipation.

A basic scheme in the development of the Courant number insensitive schemes is the so called “c-τ scheme”. It is a solver of the PDE

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0$$

where \(a \neq 0\) is a constant. At each space-time staggered mesh points \((j, n)\), the c-τ scheme is formed by

$$u_j^n = \frac{1}{2} \left\{ (1 + \nu)u_{j-1/2}^{n-1/2} + (1 - \nu)u_{j+1/2}^{n-1/2} + (1 - \nu^2) \left[ (u_x)_{j-1/2}^{n-1/2} - (u_x)_{j+1/2}^{n-1/2} \right] \right\}$$

and

$$(u_x)_j^n = \frac{1}{2(1 + \tau)} \left[ u_{j+1/2}^{n-1/2} - (1 + 2\nu - \tau)(u_x)_{j+1/2}^{n-1/2} - u_{j-1/2}^{n-1/2} - (1 - 2\nu - \tau)(u_x)_{j-1/2}^{n-1/2} \right]$$

Here: (i) \(u_j^n\) and \((u_x)_j^n\), respectively, denote the numerical analogues of \(u\) and \((\Delta x/4)\partial u/\partial x\) at the mesh point \((j, n)\); (ii) \(\nu \overset{\text{def}}{=} a\Delta t/\Delta x\) is the Courant number; and (iii) \(\tau\) is an adjustable parameter \(\neq -1\).

Because the c-τ scheme is formed by two rather complicated equations involving two parameters \(\nu\) and \(\tau\), it were not expected that its von Neumann stability conditions could be cast into an explicit analytical form. Against this expectation, it will be shown rigorously in this paper that, based on the von Neumann analysis, the c-τ scheme is stable if and only if

$$\nu^2 \leq 1, \quad \tau \geq \tau_o(\nu^2), \quad \text{and} \quad (\nu^2, \tau) \neq (1, 1)$$

where

$$\tau_o(x) \overset{\text{def}}{=} \begin{cases} 0 & \text{if} \quad x = 0 \\ \frac{4 - x - 2\sqrt{2(2 - x - x^2)}}{x} & \text{if} \quad 0 < x \leq 3/11 \\ \frac{x - 1 + \sqrt{1 - 2x + 5x^2}}{2x} & \text{if} \quad 3/11 \leq x \leq 1 \end{cases}$$

Note that the current stability conditions are in complete agreement with those generated numerically and reported earlier.

In addition, it will be shown that: (i) \(\tau_o(x)\) is continuous at \(x = 0\); (ii) \(\tau_o(x)\) is consistently defined at \(x = 3/11\); (iii)

$$\lim_{x \rightarrow 0^+} \tau_o'(x) = \lim_{x \rightarrow 1^+} \tau_o'(x) = 121/90$$

where \(\tau_o'(x) \overset{\text{def}}{=} d\tau_o(x)/dx\); (iv) \(\tau_o(x)\) is strictly monotonically increasing in the interval \(0 < x < 1\); and (v)

$$x < \tau_o(x) < \sqrt{x}, \quad 0 < x < 1$$
1. Introduction

As part of the continuous development of the space-time conservation element and solution element (CE-SE) method [1–11], recently a set of so called “Courant number insensitive schemes” has been reported in [9–11]. The key advantage of these new schemes is that the numerical dissipation associated with them generally does not increase as the Courant number decreases. As such, they can be applied to problems with large Courant number disparities (such as what commonly occurs in Navier-Stokes problems) without incurring excessive numerical dissipation.

A basic scheme in the development of the Courant number insensitive schemes is the so called “c-τ scheme” [11]. It is a solver of the PDE

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \quad (1.1)$$

where $a \neq 0$ is a constant. Consider Fig. 1 and let $\Omega$ denote the set of all space-time staggered mesh points (dots in Fig. 1), where $n = 0, \pm 1/2, \pm 1, \pm 3/2, \pm 2, \ldots$, and, for each $n, j = n \pm 1/2, n \pm 3/2, n \pm 5/2, \ldots$. Then, at each $(j, n) \in \Omega$, the c-τ scheme is formed by

$$u^n = \frac{1}{2} \left\{ (1 + \nu) u^{n-1/2}_{j-1/2} + (1 - \nu) u^{n-1/2}_{j+1/2} + (1 - \nu^2) \left[ (\bar{u}_{x})^{n-1/2}_{j-1/2} - (\bar{u}_{x})^{n-1/2}_{j+1/2} \right] \right\} \quad (1.2)$$

and

$$(u_{x})^{n}_{j} = \frac{1}{2(1 + \tau)} \left[ u^{n-1/2}_{j+1/2} - (1 + 2\nu - \tau)(\bar{u}_{x})^{n-1/2}_{j+1/2} - u^{n-1/2}_{j-1/2} - (1 - 2\nu - \tau)(\bar{u}_{x})^{n-1/2}_{j-1/2} \right] \quad (1.3)$$

Here: (i) $u^n$ and $(\bar{u}_{x})^n_j$, respectively, denote the numerical analogues of $u$ and $(\Delta x/4)\partial u/\partial x$ at the mesh point $(j, n)$; (ii)

$$\nu \overset{\text{def}}{=} \frac{a \Delta t}{\Delta x} \quad (1.4)$$

is the Courant number; and (iii) $\tau$ is an adjustable parameter $\neq -1$. It is shown in [12] that Eqs. (1.2) and (1.3) are consistent with a pair of PDEs with Eq. (1.1) being one of them.

Because the c-τ scheme is formed by two rather complicated equations involving two parameters $\nu$ and $\tau$, it was not expected that its von Neumann stability conditions could be cast into an explicit analytical form. But to the contrary, it will be shown rigorously in this paper that, based on the von Neumann analysis, the c-τ scheme is stable if and only if

$$\nu^2 \leq 1, \quad \tau \geq \tau_o(\nu^2), \quad \text{and} \quad (\nu^2, \tau) \neq (1, 1) \quad (1.5)$$

where

$$\tau_o(x) \overset{\text{def}}{=} \begin{cases} 0 & \text{if } x = 0 \\ \frac{4 - x - 2 \sqrt{2(2 - x - x^2)}}{x} & \text{if } 0 < x \leq 3/11 \\ \frac{x - 1 + \sqrt{1 - 2x + 5x^2}}{2x} & \text{if } 3/11 \leq x \leq 1 \end{cases} \quad (1.6)$$
Figure 1.—A space-time mesh.
Note that the current stability conditions are in complete agreement with those generated numerically and reported earlier in [11].

In addition, it will be shown that: (i) \( \tau_o(x) \) is continuous at \( x = 0 \); (ii) \( \tau_o(x) \) is consistently defined at \( x = 3/11 \); (iii)

\[
\lim_{x \to 3/11^-} \tau_o'(x) = \lim_{x \to 3/11^+} \tau_o'(x) = 121/90 \tag{1.7}
\]

where \( \tau_o'(x) \) is defined as \( d\tau_o(x)/dx \); (iv) \( \tau_o(x) \) is strictly monotonically increasing in the interval \( 0 < x < 1 \); and (v)

\[
x < \tau_o(x) < \sqrt{x}, \quad 0 < x < 1 \tag{1.8}
\]

Eqs. (1.5) and (1.8) coupled with the facts that \( \tau_o(0) = 0 \) and \( \sqrt{\nu^2} = |\nu| \) imply that the \( c-\tau \) scheme is stable if

\[
\tau = |\nu| < 1 \tag{1.9}
\]

On the other hand, Eqs. (1.5) and (1.8) imply that the \( c-\tau \) scheme is unstable for the cases (i)

\[
\nu^2 > 1 \tag{1.10}
\]

and (ii)

\[
\tau = \nu^2 \quad \text{and} \quad 0 < \nu^2 < 1 \tag{1.11}
\]

Note that, for a reason explained in [9,11], the special \( c-\tau \) scheme with Eq. (1.9) is a Courant number insensitive solver for Eq. (1.1).

The rest of the paper is outlined as follows. For any pair of \( \nu \) and \( \tau \), and any phase angle \( \theta \), the amplification matrix \( Q(\nu, \tau, \theta) \) that arises from the von Neumann stability analysis is presented in Sec. 2 (see Eq. (2.8)). The definition of stability (Definition 1) is then given in the same section in terms of the behaviors of \( [Q(\nu, \tau, \theta)]^m, -\pi < \theta \leq \pi \), as the integer \( m \to +\infty \). In Sec. 3, Theorems 1 and 2 are introduced to link stability with the spectral radii \( \rho(Q(\nu, \tau, \theta)) \) of \( Q(\nu, \tau, \theta), -\pi < \theta \leq \pi \). Based on the preliminaries given in Secs. 2 and 3, the main results are given in Sec. 4. Specifically, Sec. 4 begins with Theorem 3, in which the necessary and sufficient stability conditions are expressed implicitly in terms of a requirement on \( \rho(Q(\nu, \tau, \theta)), -\pi < \theta \leq \pi \). It is then followed by a systematic and rigorous effort to obtain the explicit solution to the above implicit conditions. Finally, conclusions and discussions are presented in Sec. 5. Moreover, to give the reader extra confidence on the main results established analytically in Theorems 34 and 35, these theorems are further validated numerically in Appendices A and B, respectively.
2. von Neumann Stability Analysis

For any \((j, n) \in \Omega\), let

\[
\mathbf{q}(j, n) \defeq \begin{pmatrix} u_n^j \\ (u_x)^n_j \end{pmatrix}\tag{2.1}
\]

\[
Q_+(\nu, \tau) \defeq \frac{1}{2} \begin{pmatrix} 1 + \nu & 1 - \nu^2 \\ -1 & 1 - 2\nu - \tau \\ 1 + \tau & 1 + \tau \end{pmatrix}\tag{2.2}
\]

and

\[
Q_-(\nu, \tau) \defeq \frac{1}{2} \begin{pmatrix} 1 - \nu & -(1 - \nu^2) \\ 1 & 1 + 2\nu - \tau \\ 1 + \tau & 1 + \tau \end{pmatrix}\tag{2.3}
\]

where

\[
1 + \tau \neq 0\tag{2.4}
\]

is assumed. Then Eqs. (1.2) and (1.3) can be expressed as

\[
\mathbf{q}(j, n) = Q_+ \mathbf{q}(j - 1/2, n - 1/2) + Q_- \mathbf{q}(j + 1/2, n - 1/2)\tag{2.5}
\]

Hereafter \(Q_+(\nu, \tau)\) and \(Q_-(\nu, \tau)\) may be abbreviated as \(Q_+\) and \(Q_-\), respectively.

To study the stability of the \(c-\tau\) scheme using the von Neumann analysis [1], for all \((j, n) \in \Omega\), let

\[
\tilde{\mathbf{q}}(j, n) = \mathbf{q}^*(n, \theta) e^{ij\theta}\tag{2.6}
\]

Here (i) \(i \defeq \sqrt{-1}\), (ii) \(-\infty < \theta < +\infty\), is the phase angle variation per \(\Delta x\), and (iii) \(\mathbf{q}^*(n, \theta)\) is a \(2 \times 1\) column matrix. Substituting Eq. (2.6) into Eq. (2.5) and using Eq. (2.4), one has

\[
\mathbf{q}^*(n + 1/2, \theta) = Q(\nu, \tau, \theta) \mathbf{q}^*(n, \theta)\tag{2.7}
\]

where \(n = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots\), and

\[
Q(\nu, \tau, \theta) \defeq e^{-i\theta/2}Q_+(\nu, \tau) + e^{i\theta/2}Q_-\tag{2.8}
\]

\[
= \begin{pmatrix} \cos(\theta/2) - i\nu \sin(\theta/2) & -i(1 - \nu^2) \sin(\theta/2) \\ \frac{i\sin(\theta/2)}{1 + \tau} & \frac{(1 - \tau) \cos(\theta/2) + 2i\nu \sin(\theta/2)}{1 + \tau} \end{pmatrix}
\]

Because of Eq. (2.7), \(Q(\nu, \tau, \theta)\) is referred to as the amplification matrix of the \(c-\tau\) scheme per marching step (or per \(\Delta t/2\)). Also, by using Eq. (2.7), one has

\[
\mathbf{q}^*(n + m/2, \theta) = [Q(\nu, \tau, \theta)]^m \mathbf{q}^*(n, \theta)\tag{2.9}
\]
where \( m = 1, 2, 3, \ldots \) and \( n = 0, \pm 1/2, \pm 1, \pm 3/2, \ldots \).

As a result of Eq. (2.9), we have Definition 1.

**Definition 1.** The c-\( \tau \) scheme is said to be stable with respect to a given ordered pair \((\nu, \tau)\) if, for every \( \theta \), \(-\infty < \theta < +\infty\), all elements of the matrix \([Q(\nu, \tau, \theta)]^m\) associated with this pair remain bounded as the positive integer \( m \to +\infty \). On the other hand, the scheme is said to be unstable with respect to a given \((\nu, \tau)\) if, for any \( \theta \), \(-\infty < \theta < +\infty\), at least one element of the matrix \([Q(\nu, \tau, \theta)]^m\) associated with this \((\nu, \tau)\) becomes unbounded as \( m \to +\infty \). Hereafter, a given \((\nu, \tau)\) is said to be c-\( \tau \) stable (unstable) if the c-\( \tau \) scheme is stable (unstable) with respect to this \((\nu, \tau)\).

Note that: (i) Eq. (2.8) implies that, for any integer \( \ell \),
\[
Q(\nu, \tau, \theta + 2\ell \pi) = (-1)^\ell Q(\nu, \tau, \theta)
\]
and (ii) for any \( \theta \), \(-\infty < \theta < +\infty\), there are a \( \theta' \), \(-\pi < \theta' \leq \pi\) and an integer \( \ell \) such that \( \theta = \theta' + 2\ell \pi \). As such, Definitions 1 is equivalent to the simplified form in which the original range of \( \theta \), i.e., \(-\infty < \theta < +\infty\), is replaced by
\[
-\pi < \theta \leq \pi
\]
Hereafter, the simplified form of Definition 1 is assumed.

Given Definition 1, it will be shown in this paper that a given \((\nu, \tau)\) is c-\( \tau \) stable if and only if it satisfies Eq. (1.5). As a first step, in Sec. 3 we will answer the following question: For any given ordered set \((\nu, \tau, \theta)\), what are the requirements the matrix \(Q(\nu, \tau, \theta)\) must meet so that all elements of the matrix \([Q(\nu, \tau, \theta)]^m\) will remain bounded as \( m \to +\infty \)?
3. Two Matrix Theorems

Let $M$ be any $N \times N$ matrix with real or complex elements. By definition, the eigenspace of $M$ is the vector space spanned by its eigenvectors. Let the dimension of this eigenspace be denoted by $N'$. Then $1 \leq N' \leq N$. The matrix is said to be (i) nondefective if $N' = N$ and (ii) defective if $N' < N$ [13].

Hereafter let $N = 2$. Then the eigenvalues $\lambda_1$ and $\lambda_2$ of the matrix $M$ are the two roots of a quadratic characteristic equation. Moreover, we have Theorem 1.

**Theorem 1.** The matrix $M$ is defective if and only if (i) $\lambda_1 = \lambda_2$, and (ii) $M \neq \lambda_c I$, where $I$ is the $2 \times 2$ identity matrix and $\lambda_c$ is the common value of $\lambda_1$ and $\lambda_2$.

**Proof.** Let $\vec{b}_1$ and $\vec{b}_2$ be two nonnull $2 \times 1$ column matrices with

$$M\vec{b}_\ell = \lambda_\ell \vec{b}_\ell, \quad \ell = 1, 2 \quad (3.1)$$

Then, for each $\ell$, $\vec{b}_\ell$ is an eigenvector of $M$ with the eigenvalue $\lambda_\ell$. In case that $\lambda_1 \neq \lambda_2$, it is known that $\vec{b}_1$ and $\vec{b}_2$ are linearly independent [13]. Thus $N' = 2$ and $M$ is nondefective.

Next let $\lambda_1 = \lambda_2$ and $M$ be nondefective. Then $N' = 2$, i.e., there exist two linearly independent $2 \times 1$ column matrices $\vec{b}_1$ and $\vec{b}_2$ that satisfy Eq. (3.1). Let

$$\vec{b}_\ell = \begin{pmatrix} b_{1\ell} \\ b_{2\ell} \end{pmatrix}, \quad \ell = 1, 2 \quad (3.2)$$

and

$$B \stackrel{\text{def}}{=} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \quad (3.3)$$

Then, because $\lambda_1 = \lambda_2$, Eq. (3.1) can be expressed as

$$(M - \lambda_c I)B = 0 \quad (3.4)$$

where $\lambda_c$ is the common value of $\lambda_1$ and $\lambda_2$. Because $\vec{b}_1$ and $\vec{b}_2$ are linearly independent, $B$ is nonsingular [13]. Thus, $B^{-1}$, the inverse of $B$, must exist. Multiplying the expressions on the two sides of Eq. (3.4) from the right with $B^{-1}$ leads to the conclusion that $M - \lambda_c I = 0$, i.e., $M = \lambda_c I$.

Conversely let $M = \lambda_c I$ where $\lambda_c$ is any scalar. Then it can be shown easily that (i) $\lambda_1 = \lambda_2 = \lambda_c$, and (ii) any $2 \times 1$ nonnull column matrix is an eigenvector of $M$. The conclusion (ii) implies that $N' = 2$ and thus $M$ is nondefective.

It has been shown that: (i) $M$ is nondefective if $\lambda_1 \neq \lambda_2$; and (ii) in case that $\lambda_1 = \lambda_2$, $M$ is nondefective if and only if $M = \lambda_c I$ (i.e., $M$ is defective if and only if $M \neq \lambda_c I$) where $\lambda_c$ is the common value of $\lambda_1$ and $\lambda_2$. Thus the proof is completed. **QED.**
Next let (i) \( m \) be an integer \( > 0 \); and (ii) \( \rho(M) \) be the spectral radius of \( M \), i.e.,

\[
\rho(M) \overset{\text{def}}{=} \max\{ |\lambda_1|, |\lambda_2| \}
\] (3.5)

Then we have Theorem 2.

**Theorem 2.** Every element of \( M^m \) will remain bounded as \( m \to +\infty \) if and only if

\[
\rho(M) \begin{cases} 
\leq 1 & \text{if } M \text{ is nondefective} \\
< 1 & \text{if } M \text{ is defective}
\end{cases}
\] (3.6)

**Proof.** According to the Jordan canonical form theorem [13], there exists a nonsingular \( 2 \times 2 \) matrix \( S \) such that

\[
M = SA S^{-1}
\] (3.7)

Here (i) \( S^{-1} \) is the inverse of \( S \); (ii)

\[
\Lambda \overset{\text{def}}{=} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{if } M \text{ is nondefective}
\] (3.8)

and (iii)

\[
\Lambda \overset{\text{def}}{=} \begin{pmatrix} \lambda_c & 1 \\ 0 & \lambda_c \end{pmatrix} \quad \text{if } M \text{ is defective}
\] (3.9)

Note that \( \lambda_c \) in Eq. (3.9) is the common value of \( \lambda_1 \) and \( \lambda_2 \) in the defective case.

By using Eqs. (3.8) and (3.9), one has: (i)

\[
\Lambda^m = \begin{pmatrix} \lambda_1^m & 0 \\ 0 & \lambda_2^m \end{pmatrix} \quad \text{if } M \text{ is nondefective}
\] (3.10)

and (ii)

\[
\Lambda^m = \begin{pmatrix} \lambda_c^m & m\lambda_c^{m-1} \\ 0 & \lambda_c^m \end{pmatrix} \quad \text{if } M \text{ is defective}
\] (3.11)

Because (i) Eq. (3.7) implies that

\[
M^m = SA^m S^{-1}
\] (3.12)

and (ii) Eq. (3.12) is equivalent to

\[
\Lambda^m = S^{-1} M^m S
\] (3.13)
one can infer from Eq. (3.10) that, for the nondefective case, every element of $M^m$ will remain bounded as $m \to +\infty$ if and only if

$$\rho(M) \leq 1 \quad \text{(the nondefective case)} \quad (3.14)$$

On the other hand, for the defective case, by using (i) $\rho(M) = |\lambda_c|$, and (ii)

$$\lim_{m \to +\infty} |m\lambda_c^{m-1}| = \begin{cases} 
0 & \text{if } |\lambda_c| < 1 \\
+\infty & \text{if } |\lambda_c| \geq 1
\end{cases} \quad (3.15)$$

Eqs. (3.11)–(3.13) imply that, for the defective case, every element of $M^m$ will remain bounded as $M \to +\infty$ if and only if

$$\rho(M) < 1 \quad \text{(the defective case)} \quad (3.16)$$

Because Eq. (3.6) is the combined form of Eqs. (3.14) and (3.16), the proof is completed. QED.

At this juncture, note that the term $|m\lambda_c^{m-1}|$ grows linearly with $m$ as $m \to +\infty$ if $|\lambda_c| = 1$. Thus, for the defective case with $|\lambda_c| = 1$, the growth rate of the magnitude of any element of $M^m$ as $m \to +\infty$ is very low compared with the exponential growth rate associated with a nondefective or defective case with $\rho(M) > 1$. The implication of this observation will be addressed later.
4. Main Results

An immediate result of Definition 1 and Theorem 2 is Theorem 3.

**Theorem 3.** A given \((\nu, \tau)\) is \(c\)-\(\tau\) stable if and only if the condition

\[
\rho(Q(\nu, \tau, \theta)) \begin{cases} 
\leq 1 & \text{if } Q(\nu, \tau, \theta) \text{ is nondefective} \\
< 1 & \text{if } Q(\nu, \tau, \theta) \text{ is defective}
\end{cases}
\]  

(4.1)

associated with the given \((\nu, \tau)\) is met for all \(\theta, -\pi < \theta \leq \pi\).

Two immediate results of Theorem 3 are Theorems 4 and 5.

**Theorem 4.** A necessary condition for any given \((\nu, \tau)\) to be \(c\)-\(\tau\) stable is

\[
\rho(Q(\nu, \tau, \theta)) \leq 1, \quad -\pi < \theta \leq \pi
\]  

(4.2)

**Theorem 5.** In case that

\[
\rho(Q(\nu, \tau, \theta)) \neq 1
\]  

(4.3)

for all defective \(Q(\nu, \tau, \theta) (-\pi < \theta \leq \pi)\) associated with a given \((\nu, \tau)\), Eq. (4.2) is also a sufficient condition for this \((\nu, \tau)\) to be \(c\)-\(\tau\) stable.

From Theorem 3, it becomes clear that a thorough stability study of the \(c\)-\(\tau\) scheme requires a systematic investigation of the matrix \(Q(\nu, \tau, \theta)\) and its eigenvalues over the entire range of \(\nu, \tau, \text{and } \theta\). In the following, first we shall try to narrow down the possible \((\nu, \tau)\) that are \(c\)-\(\tau\) stable by ruling out those that fail to satisfy Eq. (4.2).

Let \(\text{det}(M)\) denote the determinant of any square matrix \(M\). Then any eigenvalue \(\lambda\) of \(Q(\nu, \tau, \theta)\) satisfies the characteristic equation \(\text{det} (Q(\nu, \tau, \theta) - \lambda I) = 0\), i.e.,

\[
(1 + \tau)\lambda^2 - [2\tau \cos(\theta/2) - i\nu(3 + \tau) \sin(\theta/2)] \lambda
- (1 - \tau) \cos^2(\theta/2) - (1 + \nu^2) \sin^2(\theta/2) - i\nu(1 + \tau) \sin(\theta/2) \cos(\theta/2) = 0
\]  

(4.4)

Let

\[
X(\nu, \tau, \theta) \overset{\text{def}}{=} 4 \cos^2(\theta/2) + [4(1 + \tau) - \nu^2(\tau^2 + 2\tau + 5)] \sin^2(\theta/2)
\]  

(4.5)

and

\[
Y(\nu, \tau, \theta) \overset{\text{def}}{=} 4\nu(1 - \tau) \sin(\theta/2) \cos(\theta/2)
\]  

(4.6)

Then, with the aid of Eq. (2.4), Eq. (4.4) implies that \(\lambda = \lambda_+(\nu, \tau, \theta)\) or \(\lambda = \lambda_-(\nu, \tau, \theta)\) where

\[
\lambda_{\pm}(\nu, \tau, \theta) \overset{\text{def}}{=} \frac{2\tau \cos(\theta/2) - i\nu(3 + \tau) \sin(\theta/2) \pm \sqrt{X + iY}}{2(1 + \tau)}, \quad 1 + \tau \neq 0
\]  

(4.7)
Hereafter $X(\nu, \tau, \theta)$ and $Y(\nu, \tau, \theta)$ may be abbreviated as $X$ and $Y$, respectively. Because the range of the phase angle $\phi$ in the polar form of the principal square root $\sqrt{X + iY}$ is $-\pi/2 < \phi \leq \pi/2$, it can be shown that

$$\sqrt{X + iY} = \frac{1}{\sqrt{2}} \left[ \sqrt{\sqrt{X^2 + Y^2} + X + i \text{sign}(Y)} \sqrt{\sqrt{X^2 + Y^2} - X} \right] \quad (4.8)$$

where

$$\text{sign}(Y) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } Y \geq 0 \\ -1 & \text{if } Y < 0 \end{cases} \quad (4.9)$$

With the aid of Eq. (4.8), Eq. (4.7) implies that

$$\lambda_{\pm}(\nu, \tau, \theta) = \frac{1}{2(1 + \tau)} \left\{ \frac{1}{\sqrt{2}} \sqrt{\sqrt{X^2 + Y^2} + X} \right. \\
- i \left[ \nu(3 + \tau) \sin(\theta/2) \mp \frac{1}{\sqrt{2}} \text{sign}(Y) \sqrt{\sqrt{X^2 + Y^2} - X} \right] \right\} \quad (1 + \tau \neq 0) \quad (4.10)$$

Next Eq (4.10) is used to yield

$$2(1 + \tau)^2 (|\lambda_+|^2 + |\lambda_-|^2) = 4\tau^2 \cos^2(\theta/2) + \nu^2 (3 + \tau)^2 \sin^2(\theta/2) + \sqrt{X^2 + Y^2} \quad (4.11)$$

and

$$(1 + \tau)^2 |\lambda_+|^2 |\lambda_-|^2 = (1 - \tau)^2 \cos^4(\theta/2) + (1 + \nu^2)^2 \sin^4(\theta/2)$$
$$+ (2 - 2\tau + 3\nu^2 + \tau^2 \nu^2) \sin^2(\theta/2) \cos^2(\theta/2) \quad (4.12)$$

For simplicity, hereafter $\lambda_+(\nu, \tau, \theta)$ and $\lambda_-(\nu, \tau, \theta)$ may be abbreviated as $\lambda_+$ and $\lambda_-$, respectively. Next, let

$$s \overset{\text{def}}{=} \sin^2(\theta/2), \quad -\pi < \theta \leq \pi \quad (4.13)$$

Then

$$\cos^2(\theta/2) = 1 - s \quad (4.14)$$

and, corresponding to the domain $-\pi < \theta \leq \pi$, the range of $s$ is

$$0 \leq s \leq 1 \quad (4.15)$$

Next, let

$$D(\nu, \tau, s) \overset{\text{def}}{=} 2(1 - \nu^2)(\tau^2 - \nu^2)s^2 + [4\tau + (\tau^2 - 6\tau - 3)\nu^2] s + 4, \quad 0 \leq s \leq 1 \quad (4.16)$$

$$E(\nu, \tau, s) \overset{\text{def}}{=} \left[ 16\tau^2 - 8(\tau^3 + 4\tau^2 + \tau + 2)\nu^2 + (\tau^2 + 2\tau + 5)^2 \nu^4 \right] s^2$$
$$+ 8 [4\tau + (\tau^2 - 6\tau - 3)\nu^2] s + 16, \quad 0 \leq s \leq 1 \quad (4.17)$$
and
\[ F(\nu, \tau, s) \equiv (1 - \nu^2)(\nu^2 - \tau^2)s^2 - [2\tau(1 - \tau) + (3 + \tau^2)\nu^2] s + 4\tau, \quad 0 \leq s \leq 1 \quad (4.18) \]

Then, by using Eqs. (4.5), (4.6), and (4.11)–(4.14), it can be shown that
\[
E(\nu, \tau, s) = [X(\nu, \tau, \theta)]^2 + [Y(\nu, \tau, \theta)]^2 \geq 0 \quad (4.19)
\]
\[
D(\nu, \tau, s) - \sqrt{E(\nu, \tau, s)} = 2(1 + \tau)^2 (1 - |\lambda_+|^2) (1 - |\lambda_-|^2) \quad (4.20)
\]

and
\[
F(\nu, \tau, s) = (1 + \tau)^2 (1 - |\lambda_+|^2|\lambda_-|^2) \quad (4.21)
\]

As a preliminary to the future development, let
\[
H(\nu, \tau, s) \equiv [D(\nu, \tau, s)]^2 - E(\nu, \tau, s) \quad (4.22)
\]

Then Eqs. (4.16) and (4.17) imply that
\[
H(\nu, \tau, s) = 4(1 - \nu^2)s^2G(\nu, \tau, s) \quad (4.23)
\]
where
\[
G(\nu, \tau, s) \equiv (1 - \nu^2)(\tau^2 - \nu^2)s^2 + (\tau^2 - \nu^2) \left[ \nu^2\tau^2 + (4 - 6\nu^2)\tau - 3\nu^2 \right] s
\]
\[+ 4\tau \left[ \nu^2\tau^2 + (1 - \nu^2)\tau - \nu^2 \right], \quad 0 \leq s \leq 1 \quad (4.24)\]

With the above preparations, we have Theorem 6.

**Theorem 6.** (A) For any \((\nu, \tau)\), the condition Eq. (4.2) is equivalent to the conditions
\[
D(\nu, \tau, s) \geq 0, \quad 0 \leq s \leq 1 \quad (4.25)
\]
\[
H(\nu, \tau, s) \geq 0, \quad 0 \leq s \leq 1 \quad (4.26)
\]

and
\[
F(\nu, \tau, s) \geq 0, \quad 0 \leq s \leq 1 \quad (4.27)
\]

(B) Eqs. (4.25)–(4.27) are necessary conditions for any \((\nu, \tau)\) to be \(c-\tau\) stable.

**Proof.** Part B is an immediate result of part A and Theorem 4. Thus only part A needs to be proved. To proceed, note that \(|\lambda_+| \leq 1\) and \(|\lambda_-| \leq 1\) if and only if (i)
\[
(1 - |\lambda_+|^2) (1 - |\lambda_-|^2) \geq 0
\]
and (ii)
\[
(1 - |\lambda_+|^2|\lambda_-|^2) \geq 0,
\]
Thus, by using Eqs. (3.5), (2.4), (4.15), (4.20), and (4.21), it is easy to see that Eq. (4.2) is equivalent to Eq. (4.27) and

$$D(\nu, \tau, s) - \sqrt{E(\nu, \tau, s)} \geq 0, \quad 0 \leq s \leq 1$$  \hspace{1cm} (4.28)

As a result, to complete the proof, one needs only to show that Eqs. (4.25) and (4.26) is equivalent to Eq. (4.28).

To proceed, for simplicity, in the following $D(\nu, \tau, s)$, $E(\nu, \tau, s)$, $F(\nu, \tau, s)$, $G(\nu, \tau, s)$, and $H(\nu, \tau, s)$ may be abbreviated as $D$, $E$, $F$, $G$, and $H$, respectively. By using the fact that $E \geq 0$ (see Eq. (4.19)), it is easy to show that the condition $D - \sqrt{E} \geq 0$ implies that (i) $D \geq 0$ and (ii)

$$D^2 - E = (D + \sqrt{E})(D - \sqrt{E}) \geq 0$$  \hspace{1cm} (4.29)

Thus, with the aid of Eq. (4.22), one concludes that Eq. (4.28) implies both Eqs. (4.25) and (4.26).

To show that Eqs. (4.25) and (4.26) imply Eq. (4.28), note that

$$D - \sqrt{E} = D \geq 0 \quad \text{if} \quad D \geq 0 \quad \text{and} \quad E = 0$$  \hspace{1cm} (4.30)

Moreover, because $D + \sqrt{E} > 0$ if $D \geq 0$ and $E > 0$, one has

$$D - \sqrt{E} = \frac{D^2 - E}{D + \sqrt{E}} \geq 0 \quad \text{if} \quad D \geq 0, \quad D^2 - E \geq 0, \quad \text{and} \quad E > 0$$  \hspace{1cm} (4.31)

Thus, with the aid of Eqs. (4.19), (4.22), (4.30) and (4.31), one concludes that Eqs. (4.25) and (4.26) indeed imply Eq. (4.28).  \textbf{QED.}

At this juncture note that, given any $(\nu, \tau)$, $D(\nu, \tau, 0)$, $F(\nu, \tau, 0)$ and $G(\nu, \tau, s)$ are all quadratic polynomials in $s$ and thus their minimum values in the interval $0 \leq s \leq 1$ are easy to evaluate. As will be shown, this makes the analytical study of Eqs. (4.25)–(4.27) a relatively simple one. This is very fortunate because, according to Theorem 6, these equations play key roles in the current stability study.

To proceed, note that an immediate result of Theorem 6 is Theorem 7.

**Theorem 7.** (i) $D(\nu, \tau, 0) \geq 0$, (ii) $D(\nu, \tau, 1) \geq 0$, (iii) $F(\nu, \tau, 0) \geq 0$, (iv) $F(\nu, \tau, 1) \geq 0$, (v) $H(\nu, \tau, 0) \geq 0$, and (vi) $H(\nu, \tau, 1) \geq 0$ are all necessary conditions for a given $(\nu, \tau)$ to be $c$-$\tau$ stable.

To study conditions (i)–(vi) referred to above, Eqs. (4.16) (4.18), (4.23), and (4.24) are used to yield

$$D(\nu, \tau, 0) = 4$$  \hspace{1cm} (4.32)

$$D(\nu, \tau, 1) = (2 - \nu^2)\tau^2 + 2(2 - 3\nu^2)\tau + 2\nu^4 - 5\nu^2 + 4$$  \hspace{1cm} (4.33)

$$F(\nu, \tau, 0) = 4\tau$$  \hspace{1cm} (4.34)
\[ F(\nu, \tau, 1) = (2 + \tau + \nu^2)(\tau - \nu^2) \]  
\[ H(\nu, \tau, 0) = 0 \]  
and  
\[ H(\nu, \tau, 1) = 4(1 - \nu^2)(\tau - \nu^2)^2 \left[ (2 + \tau)^2 - \nu^2 \right] \]

According to Eqs. (4.32) and (4.36), conditions (i) and (v) referred to in Theorem 7 are satisfied automatically. The significance of other conditions will be partially addressed in the following Theorems 8–11.

**Theorem 8.** \( F(\nu, \tau, 0) \geq 0 \) and \( F(\nu, \tau, 1) \geq 0 \) if and only if \( \tau \geq \nu^2 \).

**Proof.** According to Eq. (4.34), \( F(\nu, \tau, 0) \geq 0 \) if and only if \( \tau \geq 0 \). With the aid of Eq. (4.35) and the fact that \( 2 + \tau + \nu^2 > 0 \) if \( \tau \geq 0 \), one concludes that \( F(\nu, \tau, 0) \geq 0 \) and \( F(\nu, \tau, 1) \geq 0 \) imply \( \tau \geq \nu^2 \). Conversely, it is easy to see that \( F(\nu, \tau, 0) \geq 0 \) and \( F(\nu, \tau, 1) \geq 0 \) imply \( \tau \geq \nu^2 \). QED.

**Theorem 9.** Let \( \tau \geq \nu^2 \). Then \( H(\nu, \tau, 1) > 0 \) if and only if \( \tau > \nu^2 \) and \( \nu^2 < 1 \).

**Proof.** With the aid of the assumption \( \tau \geq \nu^2 \) and Eq. (4.37), \( H(\nu, \tau, 1) > 0 \) implies (i) \( \tau > \nu^2 \) and (ii)  
\[ (\nu^2 - 1)[\nu^2 - (2 + \tau)^2] > 0 \]  
Because \( \tau > \nu^2 \) implies \( \tau > 0 \) and thus \( \nu^2 - 1 > \nu^2 - (2 + \tau)^2 \), conditions (i) and (ii) imply either (a) \( \nu^2 < 1 \) or (b) \( \nu^2 > (2 + \tau)^2 \). Case (b) can be ruled out because it along with condition (i) implies \( \tau > (2 + \tau)^2 \), a result inconsistent with \( \tau > 0 \) which follows from condition (i). Thus \( H(\nu, \tau, 1) > 0 \) implies \( \tau > \nu^2 \) and \( \nu^2 < 1 \), if \( \tau \geq \nu^2 \) is assumed.

Conversely, because \( (2 + \tau)^2 > \tau > \nu^2 \) if \( \tau > \nu^2 \), Eq. (4.37) implies that \( H(\nu, \tau, 1) > 0 \) if \( \tau > \nu^2 \) and \( \nu^2 < 1 \). Thus the proof is completed. QED.

**Theorem 10.** Let \( \tau \geq \nu^2 \). Then \( H(\nu, \tau, 1) = 0 \) if and only if at least one of the two cases: (i) \( \tau = \nu^2 \) and (ii) \( \nu^2 = 1 \), is true.

**Proof.** Eq. (4.37) implies that \( H(\nu, \tau, 1) = 0 \) if and only if at least one of the three cases: (i) \( \nu^2 = 1 \), (ii) \( \tau = \nu^2 \), and (iii) \( \nu^2 = (2 + \tau)^2 \), is true. Case (iii) can be ruled out because it along with the assumption \( \tau \geq \nu^2 \) implies \( \tau \geq (2 + \tau)^2 \), a result inconsistent with \( \tau \geq 0 \) (which follows from \( \tau \geq \nu^2 \)). Thus the proof is completed. QED.

**Theorem 11.** Let \( \tau = \nu^2 \). Then \( D(\nu, \tau, 1) \geq 0 \) if and only if \( \nu^2 \leq 1 \).

**Proof.** Let \( \tau = \nu^2 \). Then Eq. (4.33) implies that  
\[ D(\nu, \tau, 1) = (1 - \tau)(\tau^2 + 3\tau + 4) \]  
(\( \tau = \nu^2 \))  
(4.39)

With the aid of Eq. (4.39) and the fact that  
\[ \tau^2 + 3\tau + 4 = (\tau + 3/2)^2 + 7/4 \geq 7/4, \quad -\infty < \tau < +\infty \]  
(4.40)
it is easy to see that, assuming $\tau = \nu^2$, $D(\nu, \tau, 1) \geq 0$ if and only if $\nu^2 \leq 1$. QED.

According to Theorems 8–10, the conditions (i) $F(\nu, \tau, 0) \geq 0$, (ii) $F(\nu, \tau, 1) \geq 0$, and (iii) $H(\nu, \tau, 1) \geq 0$ require that $\tau = \nu^2$ if the conditions $\tau \geq \nu^2$ and $\nu^2 \leq 1$ are not satisfied simultaneously. On the other hand, according to Theorem 11, the condition $D(\nu, \tau, 1) \geq 0$ requires that $\nu^2 \leq 1$ for the case $\tau = \nu^2$. Thus one has Theorem 12.

**Theorem 12.** The conditions (i) $D(\nu, \tau, 1) \geq 0$, (ii) $F(\nu, \tau, 0) \geq 0$, (iii) $F(\nu, \tau, 1) \geq 0$, and (iv) $H(\nu, \tau, 1) \geq 0$ require that $\tau \geq \nu^2$ and $\nu^2 \leq 1$. As such, Theorem 7 implies that

$$\tau \geq \nu^2 \quad \text{and} \quad \nu^2 \leq 1 \quad (4.41)$$

are necessary conditions for a given $(\nu, \tau)$ to be c-$\tau$ stable.

In the following, it will be shown that only a subset of those $\tau$ and $\nu$ that satisfy the necessary conditions Eq. (4.41) will also satisfy the sufficient conditions for stability. As a prerequisite, we shall first study the conditions under which the matrix $Q(\nu, \tau, \theta)$ is defective if $\tau$ and $\nu$ satisfy Eq. (4.41). We begin with Theorem 13.

**Theorem 13.** Let $\tau \geq \nu^2$ and $\nu^2 \leq 1$. Then $Q(\nu, \tau, \theta)$ is defective if and only if

$$4(1 + \tau) = \nu^2(\tau^2 + 2\tau + 5) \quad (4.42)$$

and

$$\cos(\theta/2) = 0 \quad (4.43)$$

**Proof.** Assuming $\tau \geq \nu^2$ and $\nu^2 \leq 1$, first we will show that

$$\lambda_+ (\nu, \tau, \theta) = \lambda_- (\nu, \tau, \theta) \quad (4.44)$$

if and only if Eqs. (4.42) and (4.43) are satisfied. According to Eq. (4.10), Eq. (4.44) is equivalent to

$$\sqrt{X^2 + Y^2} + X = 0 \quad \text{and} \quad \sqrt{X^2 + Y^2} - X = 0 \quad (4.45)$$

Thus Eq. (4.44) is true if and only if

$$X = Y = 0 \quad (4.46)$$

According to Eq. (4.6), $Y = 0$ if and only if at least one of the four cases: (a) $\nu = 0$, (b) $\tau = 1$, (c) $\sin(\theta/2) = 0$, and (d) $\cos(\theta/2) = 0$, is true. For case (a) $\nu = 0$, Eqs. (4.5) and the assumption $\tau \geq \nu^2$ imply that

$$X = 4 \left[1 + \tau \sin^2(\theta/2)\right] \geq 4 \quad (\nu = 0) \quad (4.47)$$

Thus case (a) is incompatible with Eq. (4.46).
For case (b) \( \tau = 1 \), Eq. (4.5) implies that
\[
X = 4\cos^2(\theta/2) + 8(1 - \nu^2)\sin^2(\theta/2) \quad (\tau = 1)
\] (4.48)

Using the assumption \( \nu^2 \leq 1 \), Eq. (4.48) implies that, for case (b), \( X = 0 \) if and only if \( \nu^2 = 1 \) and \( \cos(\theta/2) = 0 \).

Because \( \cos^2(\theta/2) = 1 \) if \( \sin(\theta/2) = 0 \), Eq. (4.5) implies that \( X = 4 \) if \( \sin(\theta/2) = 0 \). Thus case (c) is incompatible with Eq. (4.46).

Because \( \sin^2(\theta/2) = 1 \) if \( \cos(\theta/2) = 0 \), Eq. (4.5) implies that
\[
X = 0 \quad \text{if} \quad \cos(\theta/2) = 0.
\] (4.49)

if \( \cos(\theta/2) = 0 \). Thus, for case (d), \( X = 0 \) if and only if Eq. (4.42) is satisfied.

Assuming \( \tau \geq \nu^2 \) and \( \nu^2 \leq 1 \), it has been shown that \( X = Y = 0 \) if and only if at least one of the following two conditions: (i)
\[
\tau = 1, \quad \nu^2 = 1, \quad \text{and} \quad \cos(\theta/2) = 0 \quad \text{(i.e., case (b))}
\]

and (ii)
\[
\cos(\theta/2) = 0 \quad \text{and} \quad 4(1 + \tau) = \nu^2(\tau^2 + 2\tau + 5) \quad \text{(i.e., case (d))}
\]
is met. Because \( \tau = 1 \) and \( \nu^2 = 1 \) form a special solution of Eq. (4.42), condition (i) is only a special case of condition (ii). Thus, assuming \( \tau \geq \nu^2 \) and \( \nu^2 \leq 1 \), Eq. (4.44) (which is equivalent to \( X = Y = 0 \)) is true if and only if Eqs. (4.42) and (4.43) are satisfied. Moreover, with the aid of Eq. (2.8) and the fact that \( \sin(\theta/2) = \pm 1 \) if \( \cos(\theta/2) = 0 \), Eq. (4.43) also implies that one of the off-diagonal elements of \( Q(\nu, \tau, \theta) \) does not vanish and thus \( Q(\nu, \tau, \theta) \) is not a multiple of \( I \). According to Theorem 1, \( Q(\nu, \tau, \theta) \) is defective if and only if (i) Eq. (4.44) is true and (ii) \( Q(\nu, \tau, \theta) \) is not a multiple of \( I \). Thus the current theorem is proved. QED.

An immediate result of Theorem 13 is Theorem 14.

**Theorem 14.** The matrix \( Q(\nu, \tau, \theta) \) is defective if \( \tau = \nu^2 = 1 \) and \( \cos(\theta/2) = 0 \).

To proceed, we will establish Theorem 15.

**Theorem 15.** Let \( Q(\nu, \tau, \theta) \) be defective with \( \tau \geq \nu^2 \) and \( \nu^2 \leq 1 \). Then the special case
\[
\rho(Q(\nu, \tau, \theta)) = 1 \quad (4.50)
\]
occurs if and only if
\[
\tau = \nu^2 = 1, \quad \text{and} \quad \cos(\theta/2) = 0 \quad (4.51)
\]

**Proof.** As a preliminary, first we will deduce several results from the current basic assumption, i.e., \( Q(\nu, \tau, \theta) \) is defective with \( \tau \geq \nu^2 \) and \( \nu^2 \leq 1 \). According to Theorem 13
and its proof, Eqs. (4.42), (4.43), and (4.46) follow immediately from the basic assumption. Also, by using Eq. (4.42) and the fact that
\[ \tau^2 + 2\tau + 5 = (1 + \tau)^2 + 4 \geq 4, \quad -\infty < \tau < +\infty \]  
(4.52)
one concludes that
\[ \nu^2 = \frac{4(1 + \tau)}{\tau^2 + 2\tau + 5} \]  
(4.53)
Moreover, because \( \sin(\theta/2) = \pm 1 \) if \( \cos(\theta/2) = 0 \), with the aid of Eqs. (4.43) and (4.46), Eq. (4.10) implies that
\[ \rho(Q(\nu, \tau, \theta)) = \left| \frac{\nu(3 + \tau)}{2(1 + \tau)} \right| \]  
(4.54)
Next assume Eq. (4.50). Because \( 3 + \tau > 0 \) (which follows from the assumption \( \tau \geq \nu^2 \)), Eqs. (4.50) and (4.54) imply that
\[ \nu^2 = \frac{4(1 + \tau)^2}{(3 + \tau)^2} \]  
(4.55)
Eliminating \( \nu^2 \) from Eqs. (4.53) and (4.55) and using the basic assumption Eq. (2.4) (which is consistent with the current assumption \( \tau \geq \nu^2 \)), one has
\[ \tau^3 + 2\tau^2 + \tau - 4 \equiv (\tau - 1)(\tau^2 + 3\tau + 4) = 0 \]  
(4.56)
Eq. (4.56) coupled with Eq. (4.40) implies that \( \tau = 1 \). In turn, by using either Eq. (4.53) or Eq. (4.55), one has \( \nu^2 = 1 \) as a result of \( \tau = 1 \). Because Eq. (4.43) (i.e., \( \cos(\theta/2) = 0 \)) is a result of the basic assumption, it has been shown that Eq. (4.51) follows from the basic assumption and Eq. (4.50).

Conversely, with the aid of (i) Theorem 1, and (ii) Eqs. (2.8) and (3.5), it can be shown by direct substitution that both the basic assumption and Eq. (4.50) are valid for the special case Eq. (4.51). Thus the proof is completed. QED.

Next we have Theorem 16.

**Theorem 16.** A given \((\nu, \tau)\) satisfies Eq. (4.2) and yet is c-\(\tau\) unstable if and only if \(\tau = \nu^2 = 1\).

**Proof.** Theorems 6 and 12 imply that Eq. (4.41) is a result of Eq. (4.2). Thus, according to Theorems 5 and 15, \(\tau = \nu^2 = 1\) if \((\nu, \tau)\) satisfies Eq. (4.2) and is also c-\(\tau\) unstable.

Conversely, Theorem 6 coupled with Eqs. (4.16), (4.18), and (4.23) implies that any \((\nu, \tau)\) with \(\tau = \nu^2 = 1\) satisfies Eq. (4.2). Moreover, according to Theorems 3, 14 and 15, such a \((\nu, \tau)\) is also c-\(\tau\) unstable. Thus the proof is completed. QED.

At this juncture, note that Theorems 14 and 15 state that, for the special case Eq. (4.51), \(Q(\nu, \tau, \theta)\) is defective with \(\rho(Q(\nu, \tau, \theta)) = 1\). Thus, according to a comment made following Eq. (3.16), for this special case, the magnitude of any element in
[Q(\nu, \tau, \theta)]^m will grow not faster than linearly with \(m\). Because round-off errors associated with a modern computer are in the order of 10^{-10} or less, the instability associated with this special case generally is very mild and may not be detected even after billions of time steps have elapsed.

Next, by combining Theorems 6, 12 and 16, one arrives at Theorem 17.

**Theorem 17.** A given (\nu, \tau) which does not satisfy Eq. (4.41) is \(c\)-\tau unstable. On the other hand, a given (\nu, \tau) which satisfies Eq. (4.41) is \(c\)-\tau stable if and only if (i) it satisfies Eqs. (4.25)–(4.27); and (ii) it does not belong to the special case \(\tau = \nu^2 = 1\).

Compared to those given in Theorem 3, the necessary and sufficient stability conditions given in Theorem 17 are much more explicit and easier to handle. As such, this theorem will be used repeatedly in the rest of the development. In particular, it will be used to establish Theorem 18.

**Theorem 18.** The \(c\)-\tau scheme is stable for any one of the following special cases: (a) \(\nu = 0\) and \(\tau \geq 0\); (b) \(\nu^2 = 1\) and \(\tau > 1\); and (c) \(0 < \nu^2 < 1\) and \(\tau = |\nu|\).

**Proof.** Let \(0 \leq s \leq 1\) throughout this proof. Then, with the aid of Eqs. (4.16), (4.18), (4.23), and (4.24), for case (a) \(\nu = 0\) and \(\tau \geq 0\), one has

\[
D(\nu, \tau, s) = D(0, \tau, s) = 2[(1 + \tau s)^2 + 1] \geq 4
\]

(4.57)

\[
F(\nu, \tau, s) = F(0, \tau, s) = \tau(2 - s)(2 + \tau s) \geq 0
\]

(4.58)

and

\[
H(\nu, \tau, s) = H(0, \tau, s) = 4s^2\tau^2(2 + \tau s)^2 \geq 0
\]

(4.59)

Because \(\nu = \pm 1\) if \(\nu^2 = 1\), for case (b) \(\nu^2 = 1\) and \(\tau > 1\), one has

\[
D(\nu, \tau, s) = D(\pm 1, \tau, s) = (1 - \tau)^2 s + 4(1 - s) > 0
\]

(4.60)

\[
F(\nu, \tau, s) = F(\pm 1, \tau, s) = (1 - \tau)^2 s + 4(\tau - s) > 0
\]

(4.61)

and

\[
H(\nu, \tau, s) = H(\pm 1, \tau, s) = 0
\]

(4.62)

Because \(0 < \nu^2 < 1\) and \(\tau = |\nu|\) if and only if \(\nu = \pm \tau\) and \(0 < \tau < 1\), for case (c) \(0 < \nu^2 < 1\) and \(\tau = |\nu|\), one has

\[
D(\nu, \tau, s) = D(\pm \tau, \tau, s) = \tau(1 - \tau)(8 + 5\tau - \tau^2)s + 4(1 - \tau s) > 0
\]

(4.63)

\[
F(\nu, \tau, s) = F(\pm \tau, \tau, s) = \tau(1 - \tau)(\tau^2 + \tau + 2)s + 4\tau(1 - s) > 0
\]

(4.64)

and

\[
H(\nu, \tau, s) = H(\pm \tau, \tau, s) = 16\tau^2(1 - \tau^2)^2(1 - \tau)s^2 \geq 0
\]

(4.65)
Obviously cases (a) and (b) are special cases of the more general case defined by Eq. (4.41). Moreover, because $\nu^2 < |\nu|$ if $0 < \nu^2 < 1$, case (c) is also a special case of the more general case. In addition, none of cases (a)–(c) contains the special case $\tau = \nu^2 = 1$.

With the aid of these observations and Eqs. (4.57)–(4.65), Theorem 18 follows directly from Theorem 17. QED.

Next let

$$\Psi \overset{\text{def}}{=} \{(\nu, \tau) | 0 < \nu^2 < 1, \tau \geq \nu^2 \text{ and } \tau^2 \neq \nu^2\}$$ \hspace{1cm} (4.66)

$$\Psi_- \overset{\text{def}}{=} \{(\nu, \tau) | 0 < \nu^2 < 1, \tau \geq \nu^2 \text{ and } \tau^2 < \nu^2\}$$ \hspace{1cm} (4.67)

and

$$\Psi_+ \overset{\text{def}}{=} \{(\nu, \tau) | 0 < \nu^2 < 1, \tau \geq \nu^2 \text{ and } \tau^2 > \nu^2\}$$ \hspace{1cm} (4.68)

Then $\Psi_-$ and $\Psi_+$ are disjoint, and

$$\Psi = \Psi_+ \cup \Psi_-$$ \hspace{1cm} (4.69)

Moreover, we have Theorems 19 and 20.

**Theorem 19.** Excluding the four special cases addressed in Theorems 16 and 18, $\Psi$ is the set of all other $(\nu, \tau)$ that satisfy the necessary stability conditions $\tau \geq \nu^2$ and $\nu^2 \leq 1$ given in Theorem 12.

**Proof.** Note that (i) $\tau = |\nu| > \nu^2$ if $0 < \nu^2 < 1$ and $\tau = |\nu|$; (ii) $\tau^2 = \nu^2$ if $\tau = |\nu|$, (iii) $\tau = |\nu|$ if $\tau \geq \nu^2$ and $\tau^2 = \nu^2$, and (iv) $\tau = \tau^2 = \nu^2$ implies either $\tau^2 = \nu^2 = 0$ or $\tau^2 = \nu^2 = 1$. Items (i)–(iii) imply that $0 < \nu^2 < 1$ and $\tau = |\nu|$ (which is case (c) in Theorem 18) if and only if $0 < \nu^2 < 1$ and $\tau > \nu^2$, and $\tau^2 = \nu^2$. On the other hand, item (iv) implies that the case with both $0 < \nu^2 < 1$ and $\tau = \tau^2 = \nu^2$ does not exist. The proof follows from the above two observations and the facts that (i) $\tau \geq \nu^2 = 0$ if and only if $\nu = 0$ and $\tau \geq 0$, and (ii) $\tau \geq \nu^2 = 1$ if and only if either (a) $\tau = \nu^2 = 1$ or (b) $\nu^2 = 1$ and $\tau > 1$. QED.

**Theorem 20.** Eq. (4.68) is equivalent to

$$\Psi_+ = \{(\nu, \tau)|0 < \nu^2 < 1, \tau > \nu^2 \text{ and } \tau^2 > \nu^2\}$$ \hspace{1cm} (4.70)

**Proof.** Note that (i) $\nu^4 > \nu^2$ if $\tau = \nu^2$ and $\tau^2 > \nu^2$, and (ii) the relations $\nu^4 > \nu^2$ and $0 < \nu^2 < 1$ are contradictory. Thus the case with $0 < \nu^2 < 1$, $\tau = \nu^2$, and $\tau^2 > \nu^2$ does not exist, i.e., Eq. (4.68) is equivalent to Eq. (4.70). QED.

To proceed, we will establish Theorems 21 and 22.

**Theorem 21.** Let $(\nu, \tau) \in \Psi$. Then

$$D(\nu, \tau, s) > 0, \hspace{1cm} 0 \leq s \leq 1$$ \hspace{1cm} (4.71)
Proof. As a preliminary, note that Eq. (4.33) implies that

\[ D(\nu, \tau, 1) = (2 - \nu^2) \left[ \left( \tau + \frac{2 - 3\nu^2}{2 - \nu^2} \right)^2 + \frac{2(1 - \nu^2)(\nu^4 + \nu^2 + 2)}{(2 - \nu^2)^2} \right], \quad \nu^2 \neq 2 \quad (4.72) \]

Thus

\[ D(\nu, \tau, 1) > 0 \quad \text{if} \quad \nu^2 < 1 \quad (4.73) \]

Let \((\nu, \tau) \in \Psi_-.\) Then Eqs. (4.16) and (4.67) imply that

\[ \left[ \frac{\partial^2 D(\nu, \tau, s)}{\partial s^2} \right]_{\nu, \tau} = 4(1 - \nu^2)(\tau^2 - \nu^2) < 0 \quad ((\nu, \tau) \in \Psi_-) \quad (4.74) \]

i.e., for any given \((\nu, \tau) \in \Psi_-\), the relation between the function \(D(\nu, \tau, s)\) and \(s\) is represented by a curve which is concave downward on the \(s-D\) plane. Thus

\[ \min_{0 \leq s \leq 1} D(\nu, \tau, s) = \min\{D(\nu, \tau, 0), D(\nu, \tau, 1)\} \quad ((\nu, \tau) \in \Psi_-) \quad (4.75) \]

By using Eqs. (4.32) and (4.73), Eq. (4.75) implies that

\[ D(\nu, \tau, s) > 0, \quad 0 \leq s \leq 1 \quad ((\nu, \tau) \in \Psi_-) \quad (4.76) \]

Next let \((\nu, \tau) \in \Psi_+.\) Then, by using Eq. (4.68) (in particular the facts that \(\nu^2 < 1\) and \((1 - \nu^2)(\tau^2 - \nu^2) > 0\)), Eq. (4.16) implies that

\[
D(\nu, \tau, s) \geq \left[ 4\tau + (\tau^2 - 6\tau - 3\nu^2) \right] s + 4 \geq \left[ 4\tau \nu^2 + (\tau^2 - 6\tau - 3\nu^2) \right] s + 4 \\
= (1 - \tau)^2 \nu^2 s + 4(1 - \nu^2 s) > 0, \quad 0 \leq s \leq 1 \quad ((\nu, \tau) \in \Psi_+) \quad (4.77)
\]

It has been shown that \(D(\nu, \tau, s) > 0, 0 \leq s \leq 1,\) for both case (a) \((\nu, \tau) \in \Psi_-\) and case (b) \((\nu, \tau) \in \Psi_+.\) Because \(\Psi = \Psi_- \cup \Psi_+\), the proof is completed. QED.

Theorem 22. Let \((\nu, \tau) \in \Psi.\) Then

\[ F(\nu, \tau, s) \geq 0, \quad 0 \leq s \leq 1 \quad (4.78) \]

Proof. Let \((\nu, \tau) \in \Psi_+.\) Then Eqs. (4.18) and (4.68) imply that

\[ \left[ \frac{\partial^2 F(\nu, \tau, s)}{\partial s^2} \right]_{\nu, \tau} = 2(1 - \nu^2)(\nu^2 - \tau^2) < 0 \quad ((\nu, \tau) \in \Psi_+) \quad (4.79) \]

i.e., for any given \((\nu, \tau) \in \Psi_+,\) the relation between the function \(F(\nu, \tau, s)\) and \(s\) is represented by a curve which is concave downward on the \(s-F\) plane. Thus

\[ \min_{0 \leq s \leq 1} F(\nu, \tau, s) = \min\{F(\nu, \tau, 0), F(\nu, \tau, 1)\} \quad ((\nu, \tau) \in \Psi_+) \quad (4.80) \]
By using Eqs. (4.34), (4.35) and (4.70), Eq. (4.80) implies that

\[ F(\nu, \tau, s) > 0, \quad 0 \leq s \leq 1 \quad ((\nu, \tau) \in \Psi_+) \]  

(4.81)

Next let \((\nu, \tau) \in \Psi_-\). Then, by using Eq. (4.67) (in particular the facts that \((1 - \nu^2)(\nu^2 - \tau^2) > 0\) and \(0 < \tau < |\nu| < 1\)), Eq. (4.18) implies that

\[
\left[ \frac{\partial F(\nu, \tau, s)}{\partial s} \right]_{\nu, \tau} = 2(1 - \nu^2)(\nu^2 - \tau^2)s - [2\tau(1 - \tau) + (3 + \tau^2)\nu^2]
\]
\[
\leq 2(1 - \nu^2)(\nu^2 - \tau^2) - [2\tau(1 - \tau) + (3 + \tau^2)\nu^2]
\]
\[
= -2(1 - \nu^2)\tau^2 - 2\nu^4 - 2\tau(1 - \tau) - (1 + \tau^2)\nu^2 < 0,
\]
\[
0 \leq s \leq 1 \quad ((\nu, \tau) \in \Psi_-)
\]

(4.82)

Thus, for any given \((\nu, \tau) \in \Psi_-\), the relation between \(F\) and \(s\) is represented by a curve on the \(s\)-\(F\) plane which has a negative slope in the interval \(0 \leq s \leq 1\). In turn, this fact coupled with Eqs. (4.35) and (4.67) implies that

\[ F(\nu, \tau, s) \geq F(\nu, \tau, 1) \geq 0, \quad 0 \leq s \leq 1 \quad ((\nu, \tau) \in \Psi_-) \]  

(4.83)

It has been shown that \(F(\nu, \tau, s) \geq 0, 0 \leq s \leq 1\), for both case (a) \((\nu, \tau) \in \Psi_+\) and case (b) \((\nu, \tau) \in \Psi_-\). Because \(\Psi = \Psi_- \cup \Psi_+\), the proof is completed. \textbf{QED}.

According to Theorems 21 and 22, Eqs. (4.25) and (4.27) are satisfied by all \((\nu, \tau) \in \Psi\). Thus, Theorem 17 implies that a given \((\nu, \tau) \in \Psi\) is \(c\)-\(\tau\) stable if and only if it satisfies Eq. (4.26). Thus, with the aid of Eqs. (4.23) and (4.66), one arrives at Theorem 23.

\textbf{Theorem 23.} For any given \((\nu, \tau) \in \Psi\), Eq. (4.26) is equivalent to

\[ \inf_{0 < s \leq 1} G(\nu, \tau, s) \geq 0 \]  

(4.84)

where the expression on the left side of the sign “\(\geq\)” denotes the infimum (i.e., the greatest lower bound) of \(G(\nu, \tau, s)\) in the interval \(0 < s \leq 1\). As such, a given \((\nu, \tau) \in \Psi\) is \(c\)-\(\tau\) stable if and only if it satisfies Eq. (4.84).

Because of Theorem 23, in the following we shall focus on finding those \((\nu, \tau) \in \Psi\) that satisfy Eq. (4.84).

To proceed, first we will establish Theorem 24.

\textbf{Theorem 24.} For any given \((\nu, \tau) \in \Psi\), let

\[ s_o(\nu, \tau) \overset{\text{def}}{=} \frac{\nu^2\tau^2 + (4 - 6\nu^2)\tau - 3\nu^2}{2(1 - \nu^2)(\nu^2 - \tau^2)} \]  

(4.85)
Let \( s_o(\nu, \tau) \) be abbreviated as \( s_o \). Then

\[
\inf_{0 < s \leq 1} G(\nu, \tau, s) = \begin{cases} 
G(\nu, \tau, s_o) & \text{if } 0 < s_o < 1 \\
G(\nu, \tau, 1) & \text{if } s_o \geq 1 \\
G(\nu, \tau, 0) & \text{if } s_o \leq 0
\end{cases}
\]

(4.86)

**Proof.** To facilitate the proof, the domain of the function \( G \) defined in Eq. (4.24) will be extended to \( -\infty < s < +\infty \). As such, for any given \((\nu, \tau) \in \Psi \) and any \( s \) with \( -\infty < s < +\infty \), one has

\[
\left[ \frac{\partial G(\nu, \tau, s)}{\partial s} \right]_{\nu, \tau} = 2(1 - \nu^2)(\tau^2 - \nu^2)^2 [s - s_o(\nu, \tau)]
\]

(4.87)

and

\[
\left[ \frac{\partial^2 G(\nu, \tau, s)}{\partial s^2} \right]_{\nu, \tau} = 2(1 - \nu^2)(\tau^2 - \nu^2)^2 > 0
\]

(4.88)

Thus, for any given \((\nu, \tau) \in \Psi \), (i) the relation between the function \( G(\nu, \tau, s) \) and \( s \) is represented by a curve which is concave upward on the \( s-G \) plane, and thus the absolute minimum of \( G \) in the interval \( -\infty < s < +\infty \) occurs at where \( \partial G/\partial s = 0 \), i.e.,

\[
s = s_o(\nu, \tau)
\]

(4.89)

(ii) \( G \) is strictly monotonically decreasing in the interval \( s < 1 \) if \( s_o \geq 1 \); and (iii) \( G \) is strictly monotonically increasing in the interval \( s > 0 \) if \( s_o \leq 0 \). In addition, for any given \((\nu, \tau) \), because \( G \) is a continuous function of \( s \) in the interval \( -\infty < s < +\infty \), one also has (iv)

\[
\lim_{s \to 0^+} G(\nu, \tau, s) = G(\nu, \tau, 0)
\]

(4.90)

Eq. (4.86) is a direct result of (i)–(vi). QED.

With the aid of Theorem 24, the bulk of the remainder of the paper will be devoted to answer a key question, i.e., given any \( \nu \) with \( 0 < \nu^2 < 1 \) (which is required by the condition \((\nu, \tau) \in \Psi \)), what is the range of \( \tau \) that will satisfy Eq. (4.84) and the rest of the condition \((\nu, \tau) \in \Psi \) (i.e., \( \tau \geq \nu^2 \) and \( \tau^2 \neq \nu^2 \))?

To proceed, let

\[
I_{\pm}(x) \overset{\text{def}}{=} \frac{3x - 2 \pm 2\sqrt{3x^2 - 3x + 1}}{x}, \quad 0 < x < 1
\]

(4.91)

and (iii)

\[
J_{\pm}(x) \overset{\text{def}}{=} \frac{3x - 2 \pm \sqrt{2(x^3 - x + 2)}}{2 - x}, \quad 0 < x < 1
\]

(4.92)
Hereafter, for any function $f(x)$, as usual $\sqrt{f(x)}$ denotes the principal square root of $f(x)$. As such $\sqrt{f(x)} \geq 0$ if $f(x) \geq 0$. Given Eqs. (4.91) and (4.92), one can establish Theorem 25.

**Theorem 25.** In the domain $0 < x < 1$, we have

$$I_+(x) > 0 \quad (0 < x < 1) \quad (4.93)$$

$$I_-(x) < 0 \quad (0 < x < 1) \quad (4.94)$$

$$J_+(x) > 0 \quad (0 < x < 1) \quad (4.95)$$

and

$$J_-(x) < 0 \quad (0 < x < 1) \quad (4.96)$$

**Proof.** Because

$$4(3x^2 - 3x + 1) = (3x - 2)^2 + 3x^2 \quad (4.97)$$

one has

$$2\sqrt{3x^2 - 3x + 1} > |3x - 2|, \quad x \neq 0 \quad (4.98)$$

Eqs. (4.93) and (4.94) follow directly from Eqs. (4.91) and (4.98).

Next because

$$2(x^3 - x + 2) = (3x - 2)^2 + 2x(x - 2) \left(x - \frac{5}{2}\right) \quad (4.99)$$

one has

$$\sqrt{2(x^3 - x + 2)} > |3x - 2|, \quad 0 < x < 2 \quad (4.100)$$

Eqs. (4.95) and (4.96) follow directly from Eqs. (4.92) and (4.100). QED.

With the above preparations and the understanding that hereafter the symbol “$\Leftrightarrow$” may be used to take the place of the statement “if and only if”, Theorem 26 can now be presented.

**Theorem 26.** (A) For any $(\nu, \tau) \in \Psi_-$, we have

$$s_o(\nu, \tau) \begin{cases} > 0 & \Leftrightarrow \tau > I_+(\nu^2) \\ = 0 & \Leftrightarrow \tau = I_+(\nu^2) \quad ((\nu, \tau) \in \Psi_-) \\ < 0 & \Leftrightarrow \tau < I_+(\nu^2) \end{cases} \quad (4.101)$$

and

$$s_o(\nu, \tau) \begin{cases} > 1 & \Leftrightarrow \tau > J_+(\nu^2) \\ = 1 & \Leftrightarrow \tau = J_+(\nu^2) \quad ((\nu, \tau) \in \Psi_-) \\ < 1 & \Leftrightarrow \tau < J_+(\nu^2) \end{cases} \quad (4.102)$$
On the other hand, (B) for any \((\nu, \tau) \in \Psi_+\), we have
\[
s_o(\nu, \tau) \begin{cases} 
> 0 & \iff \tau < I_+(\nu^2) \\
= 0 & \iff \tau = I_+(\nu^2) \quad ((\nu, \tau) \in \Psi_+) \\
< 0 & \iff \tau > I_+(\nu^2)
\end{cases}
\]
and
\[
s_o(\nu, \tau) \begin{cases} 
> 1 & \iff \tau < J_+(\nu^2) \\
= 1 & \iff \tau = J_+(\nu^2) \quad ((\nu, \tau) \in \Psi_+) \\
< 1 & \iff \tau > J_+(\nu^2)
\end{cases}
\]

Proof. As a preliminary, note that
\[
\nu^2 \tau^2 + (4 - 6\nu^2)\tau - 3\nu^2 = \nu^2 [\tau - I_+(\nu^2)] [\tau - I_-(\nu^2)] \quad (0 < \nu^2 < 1)
\]
In addition, because \(\tau \geq \nu^2\) and \(0 < \nu^2 < 1\) if \((\nu, \tau) \in \Psi\), Eq. (4.94) implies that
\[
\tau - I_-(\nu^2) > 0, \quad (\nu, \tau) \in \Psi
\]
Because the expression on the left side of Eq. (4.105) is the numerator of the fraction on the right side of Eq. (4.85), Eq. (4.101) now follows from Eqs. (4.85), (4.105) and (4.106), and the fact that \(0 < \nu^2 < 1\) and \(\nu^2 - \tau^2 > 0\) if \((\nu, \tau) \in \Psi\).

To prove Eq. (4.102), note that Eq. (4.85) implies that, for any \((\nu, \tau) \in \Psi\),
\[
s_o(\nu, \tau) - 1 = \frac{(2 - \nu^2)\tau^2 + (4 - 6\nu^2)\tau - \nu^2(5 - 2\nu^2)}{2(1 - \nu^2)(\nu^2 - \tau^2)}
\]
Also one has
\[
(2 - \nu^2)\tau^2 + (4 - 6\nu^2)\tau - \nu^2(5 - 2\nu^2) = (2 - \nu^2) [\tau - J_+(\nu^2)] [\tau - J_-(\nu^2)] \quad (0 < \nu^2 < 1)
\]
In addition, because \(\tau \geq \nu^2\) and \(0 < \nu^2 < 1\) if \((\nu, \tau) \in \Psi\), Eq. (4.96) implies that
\[
\tau - J_-(\nu^2) > 0, \quad (\nu, \tau) \in \Psi
\]
Because the expression on the left side of Eq. (4.108) is the numerator of the fraction on the right side of Eq. (4.107), Eq. (4.102) now follows from Eqs. (4.107)–(4.109), and the fact that \(0 < \nu^2 < 1\) and \(\nu^2 - \tau^2 > 0\) if \((\nu, \tau) \in \Psi\).

This finishes the proof of part A. Part B can be proved using a line of logic identical to that used to prove part A. The only difference that sets part B apart from part A is that \(\nu^2 - \tau^2 < 0\) for the case \((\nu, \tau) \in \Psi_+\) while \(\nu^2 - \tau^2 > 0\) for the case \((\nu, \tau) \in \Psi_-\). QED.
Next, note that Eq. (4.24) yields

\[
G(\nu, \tau, 1) = (\tau - \nu^2)^2 \left[(2 + \nu^2)^2 - \nu^2\right]
\]

(4.110)

and

\[
G(\nu, \tau, 0) = 4\tau \left[\nu^2 \tau^2 + (1 - \nu^2)\tau - \nu^2\right]
\]

(4.111)

In addition, for any \((\nu, \tau) \in \Psi\), Eqs. (4.24) and (4.85) also yield

\[
G(\nu, \tau, s_o) = -\frac{\nu^2(1 + \tau)^2 \left[\nu^2 \tau^2 + 2(\nu^2 - 4)\nu + 9\nu^2\right]}{4(1 - \nu^2)}
\]

(4.112)

An immediate result of Eqs. (4.66) and (4.110) is Theorem 27.

**Theorem 27.** For any \((\nu, \tau) \in \Psi\), we have

\[
G(\nu, \tau, 1) \geq 0 \quad ((\nu, \tau) \in \Psi)
\]

(4.113)

Next let

\[
K_{\pm}(x) \overset{\text{def}}{=} \frac{x - 1 \pm \sqrt{1 - 2x + 5x^2}}{2x}, \quad 0 < x < 1
\]

(4.114)

Then one has Theorems 28 and 29.

**Theorem 28.** In the domain \(0 < x < 1\), we have

\[
K_+(x) > 0 \quad (0 < x < 1)
\]

(4.115)

and

\[
K_-(x) < 0 \quad (0 < x < 1)
\]

(4.116)

**Proof.** Because

\[
1 - 2x + 5x^2 = (x - 1)^2 + 4x^2
\]

(4.117)

one has

\[
\sqrt{1 - 2x + 5x^2} > |x - 1|, \quad x \neq 0
\]

(4.118)

Eqs. (4.115) and (4.116) follow directly from Eqs. (4.114) and (4.118). QED.

**Theorem 29.** For any \((\nu, \tau) \in \Psi\), we have

\[
G(\nu, \tau, 0) \geq 0 \iff \tau \geq K_+(\nu^2) \quad ((\nu, \tau) \in \Psi)
\]

(4.119)

**Proof.** Note that

\[
4\tau \left[\nu^2 \tau^2 + (1 - \nu^2)\tau - \nu^2\right] = 4\tau \nu^2 \left[\tau - K_+(\nu^2)\right] \left[\tau - K_+(\nu^2)\right], \quad 0 < \nu^2 < 1
\]

(4.120)
In addition, because \( \tau \geq \nu^2 \) and \( 0 < \nu^2 < 1 \) if \( (\nu, \tau) \in \Psi \), Eq. (4.116) implies that
\[
\tau - K_-(\nu^2) > 0, \quad (\nu, \tau) \in \Psi \tag{4.121}
\]
Eq. (4.119) now follows from Eqs. (4.111), (4.120) and (4.121), and the fact that \( \tau \geq \nu^2 \) and \( 0 < \nu^2 < 1 \) if \((\nu, \tau) \in \Psi\). QED.

Next let
\[
L_{\pm}(x) \overset{\text{def}}{=} \frac{4 - x \pm 2\sqrt{2(2 - x - x^2)}}{x}, \quad 0 < x < 1 \tag{4.122}
\]
Then one has Theorems 30 and 31.

**Theorem 30.** In the domain \( 0 < x < 1 \), we have
\[
L_+(x) > L_-(x) > 0 \quad (0 < x < 1) \tag{4.123}
\]

**Proof.** Note that (i)
\[
2 - x - x^2 = -(x + 2)(x - 1) > 0, \quad -2 < x < 1 \tag{4.124}
\]
and (ii)
\[
(4 - x)^2 - \left[ 2\sqrt{2(2 - x - x^2)} \right]^2 = 9x^2 > 0, \quad x \neq 0 \tag{4.125}
\]
Thus
\[
4 - x = |4 - x| > 2\sqrt{2(2 - x - x^2)} > 0, \quad 0 < x < 1 \text{ or } -2 < x < 0 \tag{4.126}
\]
Eq. (4.123) is a result of Eqs. (4.122) and (4.126). QED.

**Theorem 31.** For any \((\nu, \tau) \in \Psi\), we have
\[
G(\nu, \tau, s_o) \geq 0 \iff L_-(\nu^2) \leq \tau \leq L_+(\nu^2) \quad ((\nu, \tau) \in \Psi) \tag{4.127}
\]

**Proof.** Note that
\[
\nu^2\tau^2 + 2(\nu^2 - 4)\tau + 9\nu^2 = \nu^2 \left[ \tau - L_+(\nu^2) \right] \left[ \tau - L_-(\nu^2) \right] \quad (0 < \nu^2 < 1) \tag{4.128}
\]
Because \( 1 + \tau > 0, \nu^2 > 0 \), and \( 1 - \nu^2 > 0 \) if \((\nu, \tau) \in \Psi\), Eqs. (4.112) and (4.128) imply that
\[
G(\nu, \tau, s_o) \geq 0 \iff \left[ \tau - L_+(\nu^2) \right] \left[ \tau - L_-(\nu^2) \right] \leq 0 \quad ((\nu, \tau) \in \Psi) \tag{4.129}
\]
if \((\nu, \tau) \in \Psi\). Because 0 < \(\nu^2 < 1\) if \((\nu, \tau) \in \Psi\), Eq. (4.127) now follows from Eq. (4.129) and a result of Eq. (4.123), i.e.,

\[
[\tau - L_-(\nu^2)] > [\tau - L_+(\nu^2)], \quad 0 < \nu^2 < 1
\]

(4.130)

QED.

With the above preliminaries, one can establish Theorem 32.

**Theorem 32.** (A) Let \((\nu, \tau) \in \Psi_-.\) Then \((\nu, \tau)\) is c-\(\tau\) stable if and only if it satisfies one of the three mutually exclusive sets of conditions specified, respectively, in Eqs. (4.131)–(4.133):

\[
\tau \geq J_+(\nu^2)
\]

(4.131)

\[
K_+(\nu^2) \leq \tau \leq I_+(\nu^2)
\]

(4.132)

and

\[
I_+(\nu^2) < \tau < J_+(\nu^2) \quad \text{and} \quad L_-(\nu^2) \leq \tau \leq L_+(\nu^2)
\]

(4.133)

(B) Let \((\nu, \tau) \in \Psi_+\). Then \((\nu, \tau)\) is c-\(\tau\) stable if and only if it satisfies one of the three mutually exclusive sets of conditions specified, respectively, in Eqs. (4.134)–(4.136):

\[
\tau \leq J_+(\nu^2)
\]

(4.134)

\[
\tau \geq I_+(\nu^2) \quad \text{and} \quad \tau \geq K_+(\nu^2)
\]

(4.135)

and

\[
J_+(\nu^2) < \tau < I_+(\nu^2) \quad \text{and} \quad L_-(\nu^2) \leq \tau \leq L_+(\nu^2)
\]

(4.136)

**Proof.** Let

\[
\Psi_{-}^{(\alpha)} \overset{\text{def}}{=} \{(\nu, \tau)|(\nu, \tau) \in \Psi_- \text{ and } s_o(\nu, \tau) \geq 1\}
\]

(4.137)

\[
\Psi_{-}^{(\beta)} \overset{\text{def}}{=} \{(\nu, \tau)|(\nu, \tau) \in \Psi_- \text{ and } s_o(\nu, \tau) \leq 0\}
\]

(4.138)

\[
\Psi_{-}^{(\gamma)} \overset{\text{def}}{=} \{(\nu, \tau)|(\nu, \tau) \in \Psi_- \text{ and } 0 < s_o(\nu, \tau) < 1\}
\]

(4.139)

\[
\Psi_{+}^{(\alpha)} \overset{\text{def}}{=} \{(\nu, \tau)|(\nu, \tau) \in \Psi_+ \text{ and } s_o(\nu, \tau) \geq 1\}
\]

(4.140)

\[
\Psi_{+}^{(\beta)} \overset{\text{def}}{=} \{(\nu, \tau)|(\nu, \tau) \in \Psi_+ \text{ and } s_o(\nu, \tau) \leq 0\}
\]

(4.141)

and

\[
\Psi_{+}^{(\gamma)} \overset{\text{def}}{=} \{(\nu, \tau)|(\nu, \tau) \in \Psi_+ \text{ and } 0 < s_o(\nu, \tau) < 1\}
\]

(4.142)

Because \(\Psi_-\) and \(\Psi_+\) are mutually exclusive, the above definitions imply that (i) \(\Psi_{-}^{(\alpha)}\), \(\Psi_{-}^{(\beta)}\), \(\Psi_{-}^{(\gamma)}\), \(\Psi_{+}^{(\alpha)}\), \(\Psi_{+}^{(\beta)}\), and \(\Psi_{+}^{(\gamma)}\) are mutually exclusive; (ii)

\[
\Psi_- = \Psi_{-}^{(\alpha)} \cup \Psi_{-}^{(\beta)} \cup \Psi_{-}^{(\gamma)}
\]

(4.143)
and (iii)

\[ \Psi_+ = \Psi_+^{(\alpha)} \cup \Psi_+^{(\beta)} \cup \Psi_+^{(\gamma)} \]  

Moreover, by using Theorem 26, Eqs. (4.137)–(4.142) imply

\[ \Psi_-^{(\alpha)} = \{ (\nu, \tau) | (\nu, \tau) \in \Psi_- \text{ and } \tau \geq J_+(\nu^2) \} \]  

\[ \Psi_-^{(\beta)} = \{ (\nu, \tau) | (\nu, \tau) \in \Psi_- \text{ and } \tau \leq I_+(\nu^2) \} \]  

\[ \Psi_-^{(\gamma)} = \{ (\nu, \tau) | (\nu, \tau) \in \Psi_- \text{ and } I_+(\nu^2) < \tau < J_+(\nu^2) \} \]  

\[ \Psi_+^{(\alpha)} = \{ (\nu, \tau) | (\nu, \tau) \in \Psi_+ \text{ and } \tau \leq J_+(\nu^2) \} \]  

\[ \Psi_+^{(\beta)} = \{ (\nu, \tau) | (\nu, \tau) \in \Psi_+ \text{ and } \tau \geq I_+(\nu^2) \} \]  

and

\[ \Psi_+^{(\gamma)} = \{ (\nu, \tau) | (\nu, \tau) \in \Psi_+ \text{ and } J_+(\nu^2) < \tau < I_+(\nu^2) \} \]  

respectively.

To proceed, note that:

(a) With the aid of (i) Eqs. (4.137) and (4.140), and (ii) Theorems 24 and 27, Theorem 23 implies that a given \((\nu, \tau)\) is always c-\(\tau\) stable.

(b) With the aid of (i) Eqs. (4.138) and (4.141), and (ii) Theorems 24 and 29, Theorem 23 implies that a given \((\nu, \tau)\) is c-\(\tau\) stable if and only if

\[ \tau \geq K_+(\nu^2) \]  

(4.151)

(c) With the aid of (i) Eqs. (4.139) and (4.142), and (ii) Theorems 24 and 31, Theorem 23 implies that a given \((\nu, \tau)\) is c-\(\tau\) stable if and only if

\[ L_-(\nu^2) \leq \tau \leq L_+(\nu^2) \]  

(4.152)

Theorem 32 now follows from Eqs. (4.143)–(4.150) and the facts presented in the above items (a)–(c). QED.

In principle, the question of whether a given \((\nu, \tau)\) is c-\(\tau\) stable can now be answered by using Theorems 12, 16, 18, 19, and 32. However, in its current complicated form, Theorem 32 is difficult to use. Fortunately, Theorem 32 can be simplified greatly and, in fact, the stability condition for the c-\(\tau\) scheme can be cast into a rather simple explicit form. To obtain this simple form, we begin with Theorem 33.

**Theorem 33.** We have: (A)

\[ (\nu, \tau) \in \Psi_- \iff 0 < \nu^2 < 1 \text{ and } \nu^2 \leq \tau < \sqrt{\nu^2} \]  

(4.153)
(B) $\Psi_-$ is not empty; and (C)

$$ (\nu, \tau) \in \Psi_+ \quad \Leftrightarrow \quad 0 < \nu^2 < 1 \text{ and } \tau > \sqrt{\nu^2} $$

(4.154)

Proof. Because (i) $-\sqrt{\nu^2} < \tau < \sqrt{\nu^2}$ if $\tau^2 < \nu^2$, and (ii) $\tau^2 < \nu^2$ if $0 \leq \tau < \sqrt{\nu^2}$, part A is an immediate result of Eq. (4.67). Part B follows from the trivial fact that $\nu^2 < \sqrt{\nu^2}$ if $0 < \nu^2 < 1$. To prove part C, note that (i) $\tau > 0$ if $\nu^2 > 0$ and $\tau \geq \nu^2$, and (ii) $\tau > \sqrt{\nu^2}$ if $\tau > 0$ and $\tau^2 > \nu^2$. Thus Eq. (4.70) implies that $0 < \nu^2 < 1$ and $\tau > \sqrt{\nu^2}$ if $(\nu, \tau) \in \Psi_+$. Conversely, because (i) $\sqrt{\nu^2} > \nu^2$ if $0 < \nu^2 < 1$; (ii) $\tau > \nu^2$ if $\tau > \sqrt{\nu^2}$ and $\sqrt{\nu^2} > \nu^2$; and (iii) $\tau^2 > \nu^2$ if $\tau > \sqrt{\nu^2}$, one concludes that $(\nu, \tau) \in \Psi_+$ if $0 < \nu^2 < 1$ and $\tau > \sqrt{\nu^2}$.

QED.

Next let

$$ c_1 \overset{\text{def}}{=} 3 - 2\sqrt{2} $$

(4.155)

$$ c_2 \overset{\text{def}}{=} 3/11 $$

(4.156)

$$ c_3 \overset{\text{def}}{=} (41 - 7\sqrt{33})/2 $$

(4.157)

and

$$ c_4 \overset{\text{def}}{=} \left[ \left( \sqrt{\frac{1664}{27}} + \frac{181}{27} \right)^{\frac{1}{2}} - \left( \sqrt{\frac{1664}{27}} - \frac{181}{27} \right)^{\frac{1}{2}} - \frac{2}{3} \right]^2 $$

(4.158)

We have (i) $c_1 \approx 0.172$, $c_2 \approx 0.273$, $c_3 \approx 0.394$ and $c_4 \approx 0.530$, and (ii)

$$ 0 < c_1 < c_2 < c_3 < c_4 < 1 $$

(4.159)

With the above preparations, we have Theorem 34.

**Theorem 34.** (A) In the domain $0 < x < 1$, $I_+(x)$, $J_+(x)$, $K_+(x)$, and $L_-(x)$ are strictly monotonically increasing while $L_+(x)$ is strictly monotonically decreasing; (B) we have

$$ I_+(x) < x < K_+(x) < L_-(x) < J_+(x) < \sqrt{x} < L_+(x), \quad 0 < x < c_1 $$

(4.160)

$$ I_+(x) = x < K_+(x) < L_-(x) < J_+(x) < \sqrt{x} < L_+(x), \quad x = c_1 $$

(4.161)

$$ x < I_+(x) < K_+(x) < L_-(x) < J_+(x) < \sqrt{x} < L_+(x), \quad c_1 < x < c_2 $$

(4.162)

$$ x < I_+(x) = K_+(x) = L_-(x) < J_+(x) < \sqrt{x} < L_+(x), \quad x = c_2 $$

(4.163)

$$ x < K_+(x) < L_-(x) < I_+(x) < J_+(x) < \sqrt{x} < L_+(x), \quad c_2 < x < c_3 $$

(4.164)

$$ x < K_+(x) < L_-(x) < I_+(x) = J_+(x) = \sqrt{x} < L_+(x), \quad x = c_3 $$

(4.165)

$$ x < K_+(x) < L_-(x) < \sqrt{x} < J_+(x) < I_+(x) < L_+(x), \quad c_3 < x < c_4 $$

(4.166)
\[ x < K_+(x) < L_-(x) = \sqrt{x} < J_+(x) < I_+(x) < L_+(x), \quad x = c_4 \quad (4.167) \]

and
\[ x < K_+(x) < \sqrt{x} < L_-(x) < J_+(x) < I_+(x) < L_+(x), \quad c_4 < x < 1 \quad (4.168) \]

(C)
\[ K'_+(c_2) = L'_-(c_2) = 121/90 \quad (4.169) \]

where \( K'_+(x) \overset{\text{def}}{=} dK_+(x)/dx \) and \( L'_-(x) \overset{\text{def}}{=} dL_-(x)/dx \); and (D)
\[ \lim_{x \to 0^+} L_-(x) = 0 \quad \text{and} \quad \lim_{x \to 1^-} K_+(x) = 1 \quad (4.170) \]

In order not to interrupt the current stream of development, the lengthy proof for Theorem 34 will be provided later in the paper. Here, with the aid of this theorem, we shall establish a simplified form of the stability condition for the \( c \)-\( \tau \) scheme as given in Theorem 35.

**Theorem 35.** Let

\[ \tau_o(x) \overset{\text{def}}{=} \begin{cases} 
0 & \text{if } x = 0 \\
L_-(x) & \text{if } 0 < x \leq 3/11 \\
K_+(x) & \text{if } 3/11 \leq x < 1 \\
1 & \text{if } x = 1 
\end{cases} \quad (4.171) \]

and
\[ \Gamma_o \overset{\text{def}}{=} \{(\nu, \tau)|\nu^2 \leq 1, \tau \geq \tau_o(\nu^2) \text{ and } (\nu^2, \tau) \neq (1, 1)\} \quad (4.172) \]

Then: (A) \( \tau_o(x) \) is continuous at \( x = 0 \) and \( x = 1 \); (B) \( \tau_o(x) \) is consistently defined at \( x = 3/11 \); (C)
\[ \lim_{x \to 3/11^-} \tau'_o(x) = \lim_{x \to 3/11^+} \tau'_o(x) = 121/90 \quad (4.174) \]

where \( \tau'_o(x) \overset{\text{def}}{=} d\tau_o(x)/dx \); (D) \( \tau_o(x) \) is strictly monotonically increasing in the interval \( 0 < x < 1 \); (E)
\[ x < \tau_o(x) < \sqrt{x}, \quad 0 < x < 1 \quad (4.175) \]

(F) a given \((\nu, \tau)\) is \( c \)-\( \tau \) stable if and only if \((\nu, \tau) \in \Gamma_o\); and (G) a given \((\nu, \tau)\) satisfies Eq. (4.2) if and only if \((\nu, \tau) \in \Gamma\).

**Proof.** Part A is a result of Eqs. (4.170) and (4.171). Part B follows from the fact that \( L_-(3/11) = K_+(3/11) = 1/3 \). Part C follows from Eqs. (4.156) and (4.169). Part D
is a result of part A of Theorem 34, and parts B and C of the current theorem. Part E is a result of Eqs. (4.160)–(4.168) and (4.171).

To prove part F, one needs to show that: (i) \((\nu, \tau) \in \Gamma_o\) for any \((\nu, \tau)\) that is \(c\tau\) stable; and (ii) \((\nu, \tau) \not\in \Gamma_o\) for any \((\nu, \tau)\) that is \(c\tau\) unstable. Here whether any particular \((\nu, \tau)\) is \(c\tau\) stable is determined using Theorems 12, 16, 18, 19, and 35.

To proceed, let
\[
\Phi_1 \overset{\text{def}}{=} \{ (\nu, \tau) | \nu^2 > 1 \text{ or } \tau < \nu^2 \leq 1 \} \tag{4.176}
\]
\[
\Phi_2 \overset{\text{def}}{=} \{ (\nu, \tau) | \tau = \nu^2 = 1 \} \tag{4.177}
\]
\[
\Phi_3 \overset{\text{def}}{=} \{ (\nu, \tau) | \nu^2 = 0 \text{ and } \tau \geq 0 \} \tag{4.178}
\]
\[
\Phi_4 \overset{\text{def}}{=} \{ (\nu, \tau) | \nu^2 = 1 \text{ and } \tau > 1 \} \tag{4.179}
\]
\[
\Phi_5 \overset{\text{def}}{=} \{ (\nu, \tau) | 0 < \nu^2 < 1 \text{ and } \tau = |\nu| \} \tag{4.180}
\]

With the aid Theorem 19, it is seen that \(\Psi_-, \Psi_+, \) and the five sets defined above are inclusive and yet mutually exclusive, i.e., any \((\nu, \tau)\) belongs to one and only one of these sets. To facilitate the proof, \(\Psi_- \) and \(\Psi_+ \), respectively, will be further divided into several disjoint subsets to be defined immediately.

Let
\[
\Psi_-^{(1)} \overset{\text{def}}{=} \{ (\nu, \tau) | 0 < \nu^2 < c_2 \text{ and } \nu^2 \leq \tau < \sqrt{\nu^2} \} \tag{4.181}
\]
\[
\Psi_-^{(2)} \overset{\text{def}}{=} \{ (\nu, \tau) | \nu^2 = c_2 \text{ and } \nu^2 \leq \tau < \sqrt{\nu^2} \} \tag{4.182}
\]
\[
\Psi_-^{(3)} \overset{\text{def}}{=} \{ (\nu, \tau) | c_2 < \nu^2 < c_3 \text{ and } \nu^2 \leq \tau < \sqrt{\nu^2} \} \tag{4.183}
\]

and
\[
\Psi_-^{(4)} \overset{\text{def}}{=} \{ (\nu, \tau) | c_3 \leq \nu^2 < 1 \text{ and } \nu^2 \leq \tau < \sqrt{\nu^2} \} \tag{4.184}
\]

Because \((\nu, \tau) \in \Psi_- \Leftrightarrow 0 < \nu^2 < 1 \) and \(\nu^2 \leq \tau < \sqrt{\nu^2} \) (see Theorem 33), one concludes that (i) \(\Psi_-^{(\ell)}, \ell = 1, 2, 3, 4, \) are nonempty disjoint subsets of \(\Psi_- \), and (ii)
\[
\Psi_- = \bigcup_{\ell=1}^{4} \Psi_-^{(\ell)} \tag{4.185}
\]

Next let
\[
\Psi_+^{(1)} \overset{\text{def}}{=} \{ (\nu, \tau) | 0 < \nu^2 \leq c_3 \text{ and } \tau > \sqrt{\nu^2} \} \tag{4.186}
\]

and
\[
\Psi_+^{(2)} \overset{\text{def}}{=} \{ (\nu, \tau) | c_3 < \nu^2 < 1 \text{ and } \tau > \sqrt{\nu^2} \} \tag{4.187}
\]

Because \((\nu, \tau) \in \Psi_+ \Leftrightarrow 0 < \nu^2 < 1 \text{ and } \tau > \sqrt{\nu^2} \) (see Theorem 33), one concludes that (i) \(\Psi_+^{(1)} \) and \(\Psi_+^{(2)} \), are nonempty disjoint subsets of \(\Psi_+ \), and (ii)
\[
\Psi_+ = \Psi_+^{(1)} \cup \Psi_+^{(2)} \tag{4.188}
\]
From the above discussion, the sets (i) \( \Phi_\ell, \ell = 1, 2, 3, 4, 5 \); (ii) \( \Psi^{(\ell)}_-, \ell = 1, 2, 3, 4 \); and (iii) \( \Psi^{(1)}_+ \) and \( \Psi^{(2)}_+ \), are inclusive and yet mutually exclusive, i.e., any \((\nu, \tau)\) must belong to one and only one of these sets. Part F will be proved by showing that it is valid over each of these sets in the following case-by-case discussions:

1. \((\nu, \tau) \in \Phi_1\). According to Theorem 12, any \((\nu, \tau) \in \Phi_1\) is \(c-\tau\) unstable. Thus part F is true over \(\Phi_1\) if one can show that \((\nu, \tau) \notin \Gamma_o\) if \((\nu, \tau) \in \Phi_1\). Because \((\nu, \tau) \notin \Gamma_o\) if \(\nu^2 > 1\) (see Eq. (4.172)), the proof for case 1 is completed if one can show that

\[
(\nu, \tau) \notin \Gamma_o\ 	ext{if} \tau < \nu^2 \leq 1.
\]

To proceed, note that Eq. (4.175) and the facts that \(\tau_o(0) = 0\) and \(\tau_o(1) = 1\) imply that

\[
\nu^2 \leq \tau_o(\nu^2), \quad \nu^2 \leq 1
\]  

(4.189)

Thus \(\tau < \tau_o(\nu^2)\) if \(\tau < \nu^2 \leq 1\). As a result of Eq. (4.172), this in turn implies that \((\nu, \tau) \notin \Gamma_o\) if \(\tau < \nu^2 \leq 1\). As such part F is true over \(\Phi_1\).

2. \((\nu, \tau) \in \Phi_2\). According to Theorem 16, any \((\nu, \tau) \in \Phi_2\) is \(c-\tau\) unstable. Also, according to Eq. (4.172), \((\nu, \tau) \notin \Gamma_o\) if \((\nu, \tau) \in \Phi_2\). Thus part F is true over \(\Phi_2\).

3. \((\nu, \tau) \in \Phi_3\). According to Theorem 18, any \((\nu, \tau) \in \Phi_3\) is \(c-\tau\) stable. Because \(\tau_o(0) = 0\), Eq. (4.172) implies that \((\nu, \tau) \in \Gamma_o\) if \((\nu, \tau) \in \Phi_3\). Thus part F is true over \(\Phi_3\).

4. \((\nu, \tau) \in \Phi_4\). According to Theorem 18, any \((\nu, \tau) \in \Phi_4\) is \(c-\tau\) stable. Because \(\tau_o(1) = 1\), Eq. (4.172) implies that \((\nu, \tau) \in \Gamma_o\) if \((\nu, \tau) \in \Phi_4\). Thus part F is true over \(\Phi_4\).

5. \((\nu, \tau) \in \Phi_5\). According to Theorem 18, any \((\nu, \tau) \in \Phi_5\) is \(c-\tau\) stable. On the other hand, Eqs. (4.175) implies that

\[
\tau_o(\nu^2) < \sqrt{\nu^2}, \quad 0 < \nu^2 < 1
\]  

(4.190)

i.e., \(\tau_o(\nu^2) < \sqrt{\nu^2} = |\nu|\) if \(0 < \nu^2 < 1\). This coupled with Eq. (4.172) implies that \((\nu, \tau) \in \Gamma_o\) if \((\nu, \tau) \in \Phi_5\). Thus part F is true over \(\Phi_5\).

6. \((\nu, \tau) \in \Psi^{(1)}_-\). For this case, we have (i) \(0 < \nu^2 < c_2\), and (ii) \(\nu^2 \leq \tau < \sqrt{\nu^2}\). To proceed, Note that Eqs. (4.160)–(4.162) imply that

\[
I_+(\nu^2) < K_+(\nu^2), \quad 0 < \nu^2 < c_2
\]  

(4.191)

\[
\nu^2 < L_-(\nu^2) < J_+(\nu^2) < \sqrt{\nu^2}, \quad 0 < \nu^2 < c_2
\]  

(4.192)

and

\[
I_+(\nu^2) < L_-(\nu^2) < J_-(\nu^2) < L_+(\nu^2), \quad 0 < \nu^2 < c_2
\]  

(4.193)

Because Eq. (4.191) contradicts Eq. (4.132), Eq. (4.132) cannot be satisfied by any \((\nu, \tau) \in \Psi^{(1)}_-\). Moreover, by using Eq. (4.192), it can be shown that

\[
\Psi^{(1)}_- = \Psi^{(1,1)}_- \cup \Psi^{(1,2)}_- \cup \Psi^{(1,3)}_-
\]  

(4.194)
Thus any $(\nu, \tau) \in \Psi^{(1)}_{-}$ must fall into one and only one of the following three sub-cases: 
(i) $(\nu, \tau) \in \Psi^{(1),1}_{-}$, (ii) $(\nu, \tau) \in \Psi^{(1),2}_{-}$, and (iii) $(\nu, \tau) \in \Psi^{(1),3}_{-}$.

Let $(\nu, \tau) \in \Psi^{(1),1}_{-}$. By using the relation $L_{-}(\nu^2) < J_{+}(\nu^2)$ which follows from Eq. (4.192) or Eq. (4.193), it is seen that Eq. (4.131) cannot be true for the current sub-case where $\nu^2 \leq \tau < L_{-}(\nu^2)$. Also, the second part of Eq. (4.133), i.e., $L_{-}(\nu^2) \leq \tau < J_{+}(\nu^2)$, cannot be true for the sub-case. Moreover, for a reason given earlier, Eq. (4.132) also cannot be true for the sub-case. According to part A of Theorem 32, the above results imply that any $(\nu, \tau) \in \Psi^{(1),1}_{-}$ is $c$-$\tau$ unstable. On the other hand, because $\tau_o(\nu^2) = L_{-}(\nu^2)$ if $0 < \nu^2 < c_2$ (see Eqs. (4.156) and (4.171)), one concludes that $\tau < \tau_o(\nu^2)$ and thus $(\nu, \tau) \notin \Gamma_o$ if $(\nu, \tau) \in \Psi^{(1),1}_{-}$. As such it has been shown that part F is true over $\Psi^{(1),1}_{-}$.

Let $(\nu, \tau) \in \Psi^{(1),2}_{-}$. It follows from Eq. (4.193) that Eq. (4.133) is satisfied by any $(\nu, \tau)$ with $L_{-}(\nu^2) \leq \tau < J_{+}(\nu^2)$. According to part A of Theorem 32 and Eq. (4.196), this implies that any $(\nu, \tau)$ in the current sub-case is $c$-$\tau$ stable. On the other hand, because $\tau_o(\nu^2) = L_{-}(\nu^2)$ if $0 < \nu^2 < c_2$, one concludes that $\tau \geq \tau_o(\nu^2)$ and thus $(\nu, \tau) \in \Gamma_o$ if $(\nu, \tau) \in \Psi^{(1),2}_{-}$. As such, it has been shown that part F is true over $\Psi^{(1),2}_{-}$.

Let $(\nu, \tau) \in \Psi^{(1),3}_{-}$. Obviously Eq. (4.131) is true for the current sub-case where $J_{+}(\nu^2) \leq \tau < \sqrt{\nu^2}$. According to part A of Theorem 32, this implies that any $(\nu, \tau)$ in the current sub-case is $c$-$\tau$ stable. On the other hand, because (i) $\tau_o(\nu^2) = L_{-}(\nu^2)$ if $0 < \nu^2 < c_2$, and (ii) the relation $L_{-}(<\nu^2) < J_{+}(\nu^2)$ is a part of Eq. (4.193), one concludes that $\tau > \tau_o(\nu^2)$ and thus $(\nu, \tau) \in \Gamma_o$ if $(\nu, \tau) \in \Psi^{(1),3}_{-}$. As such, it has been shown that part F is true over $\Psi^{(1),3}_{-}$.

It has been shown that part F is true over each of the three nonempty disjoint sets $\Psi^{(1),1}_{-}$, $\Psi^{(1),2}_{-}$, and $\Psi^{(1),3}_{-}$. Eq. (4.194) now implies that part F is true over $\Psi^{(1)}_{-}$.

7. $(\nu, \tau) \in \Psi^{(2)}_{-}$ For this case, we have (i) $\nu^2 = c_2$, and (ii) $\nu^2 \leq \tau < \sqrt{\nu^2}$. To proceed, note that Eqs. (4.163) implies that 

$$
\nu^2 < I_{+}(\nu^2) = K_{+}(\nu^2) = L_{-}(\nu^2) < J_{+}(\nu^2) < \sqrt{\nu^2} < L_{+}(\nu^2), \quad \nu^2 = c_2 \tag{4.198}
$$

With the aid of Eq. (4.198), it can be shown that 

$$
\Psi^{(2)}_{-} = \Psi^{(2),1}_{-} \cup \Psi^{(2),2}_{-} \cup \Psi^{(2),3}_{-} \cup \Psi^{(2),4}_{-} \tag{4.199}
$$
where $\Psi^{(2,1)}_-$, $\Psi^{(2,2)}_-$, $\Psi^{(2,3)}_-$, and $\Psi^{(2,4)}_-$ are nonempty disjoint sets defined by

\[
\Psi^{(2,1)}_- \overset{\text{def}}{=} \{ (\nu, \tau) | \nu^2 = c_2 \text{ and } \nu^2 \leq \tau < L_-(\nu^2) \} \tag{4.200}
\]

\[
\Psi^{(2,2)}_- \overset{\text{def}}{=} \{ (\nu, \tau) | \nu^2 = c_2 \text{ and } \tau = L_-(\nu^2) \} \tag{4.201}
\]

\[
\Psi^{(2,3)}_- \overset{\text{def}}{=} \{ (\nu, \tau) | \nu^2 = c_2 \text{ and } L_-(\nu^2) < \tau < J_+(\nu^2) \} \tag{4.202}
\]

and

\[
\Psi^{(2,4)}_- \overset{\text{def}}{=} \{ (\nu, \tau) | \nu^2 = c_2 \text{ and } J_+(\nu^2) \leq \tau < \sqrt{\nu^2} \} \tag{4.203}
\]

Thus any $(\nu, \tau) \in \Psi^{(2)}_-$ must fall into one and only one of the following four sub-cases:

(i) $(\nu, \tau) \in \Psi^{(2,1)}_-$, (ii) $(\nu, \tau) \in \Psi^{(2,2)}_-$, (iii) $(\nu, \tau) \in \Psi^{(2,3)}_-$, and (iv) $(\nu, \tau) \in \Psi^{(2,4)}_-$. Let $(\nu, \tau) \in \Psi^{(2,1)}_-$. By using the relation $L_-(\nu^2) < J_+(\nu^2)$ which follows from Eq. (4.198), it is seen that Eq. (4.131) cannot be true for the current sub-case where $\nu^2 \leq \tau < L_-(\nu^2)$. Moreover, by using the relation $I_+(\nu^2) = K_+(\nu^2) = L_-(\nu^2)$ which also follows from Eq. (4.198), it is seen that Eq. (4.132) also cannot be true for the sub-case. In addition, the second part of Eq. (4.133) also cannot be true for the sub-case. According to part A of Theorem 32, this implies that any $(\nu, \tau) \in \Psi^{(2,1)}_-$ is $c$-$\tau$ unstable. On the other hand, because $\tau_o(\nu^2) = L_-(\nu^2)$ if $\nu^2 = c_2$, one concludes that $\tau < \tau_o(\nu^2)$ and thus $(\nu, \tau) \notin \Gamma_o$ if $(\nu, \tau) \in \Psi^{(2,1)}_-$. As such it has been shown that part F is true over $\Psi^{(2,1)}_-$. Let $(\nu, \tau) \in \Psi^{(2,2)}_-$. By using the relation $I_+(\nu^2) = K_+(\nu^2) = L_-(\nu^2)$ which follows from Eq. (4.198), it is seen that Eq. (4.132) is true for the current sub-case where $\tau = L_-(\nu^2)$. According to part A of Theorem 32, this implies that any $(\nu, \tau) \in \Psi^{(2,2)}_-$ is $c$-$\tau$ stable. On the other hand, because $\tau_o(\nu^2) = L_-(\nu^2)$ if $\nu^2 = c_2$, one concludes that $\tau = \tau_o(\nu^2)$ and thus $(\nu, \tau) \in \Gamma_o$ if $(\nu, \tau) \in \Psi^{(2,2)}_-$. As such it has been shown that part F is true over $\Psi^{(2,2)}_-$. Let $(\nu, \tau) \in \Psi^{(2,3)}_-$. By using the relation $I_+(\nu^2) = L_-(\nu^2) < J_+(\nu^2)$ which follows from Eq. (4.198), it is seen that Eq. (4.133) is true for the current case where $L_-(\nu^2) < \tau < J_+(\nu^2)$. According to part A of Theorem 32, this implies that any $(\nu, \tau) \in \Psi^{(2,3)}_-$ is $c$-$\tau$ stable. On the other hand, because $\tau_o(\nu^2) = L_-(\nu^2)$ if $\nu^2 = c_2$, one concludes that $\tau > \tau_o(\nu^2)$ and thus $(\nu, \tau) \in \Gamma_o$ if $(\nu, \tau) \in \Psi^{(2,3)}_-$. As such it has been shown that part F is true over $\Psi^{(2,3)}_-$. Let $(\nu, \tau) \in \Psi^{(2,4)}_-$. Obviously Eq. (4.131) is true for the current sub-case where $J_+(\nu^2) \leq \tau < \sqrt{\nu^2}$. According to part A of Theorem 32, this implies that any $(\nu, \tau)$ in the current sub-case is $c$-$\tau$ stable. On the other hand, because (i) $\tau_o(\nu^2) = L_-(\nu^2)$ if $\nu^2 = c_2$, and (ii) the relation $L_-(\nu^2) < J_+(\nu^2)$ is a part of Eq. (4.198), one concludes that $\tau > \tau_o(\nu^2)$ and thus $(\nu, \tau) \in \Gamma_o$ if $(\nu, \tau) \in \Psi^{(2,4)}_-$. As such, it has been shown that part F is true over $\Psi^{(2,4)}_-$.  

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It has been shown that part F is true over each of the four nonempty disjoint sets \( \Psi_1^{(2,1)}, \Psi_1^{(2,2)}, \Psi_1^{(2,3)}, \) and \( \Psi_1^{(2,4)} \). Eq. (4.199) now implies that part F is true over \( \Psi_1^{(2)} \).

8. \([\nu, \tau] \in \Psi_1^{(3)} \). For this case, we have (i) \( c_2 < \nu^2 < c_3 \), and (ii) \( \nu^2 \leq \tau < \sqrt{\nu^2} \). To proceed, Note that Eqs. (4.164) implies that

\[
\nu^2 < K_+(\nu^2) < L_-(\nu^2) < I_+(\nu^2) < J_+(\nu^2) < \sqrt{\nu^2} < L_+(\nu^2), \quad c_2 < \nu^2 < c_3
\]

With the aid of Eq. (4.204), it can be shown that

\[
\Psi_1^{(3)} = \Psi_1^{(3,1)} \cup \Psi_1^{(3,2)} \cup \Psi_1^{(3,3)} \cup \Psi_1^{(3,4)}
\]

where \( \Psi_1^{(3,1)}, \Psi_1^{(3,2)}, \Psi_1^{(3,3)}, \) and \( \Psi_1^{(3,4)} \) are nonempty disjoint sets defined by

\[
\Psi_1^{(3,1)} \overset{\text{def}}{=} \{(\nu, \tau)|c_2 < \nu^2 < c_3 \text{ and } \nu^2 \leq \tau < K_+(\nu^2)\}
\]

\[
\Psi_1^{(3,2)} \overset{\text{def}}{=} \{(\nu, \tau)|c_2 < \nu^2 < c_3 \text{ and } K_+(\nu^2) \leq \tau \leq I_+(\nu^2)\}
\]

\[
\Psi_1^{(3,3)} \overset{\text{def}}{=} \{(\nu, \tau)|c_2 < \nu^2 < c_3 \text{ and } I_+(\nu^2) < \tau < J_+(\nu^2)\}
\]

and

\[
\Psi_1^{(3,4)} \overset{\text{def}}{=} \{(\nu, \tau)|c_2 < \nu^2 < c_3 \text{ and } J_+(\nu^2) \leq \tau < \sqrt{\nu^2}\}
\]

Thus any \((\nu, \tau) \in \Psi_1^{(3)}\) must fall into one and only one of the following four sub-cases:

(i) \((\nu, \tau) \in \Psi_1^{(3,1)}\), (ii) \((\nu, \tau) \in \Psi_1^{(3,2)}\), (iii) \((\nu, \tau) \in \Psi_1^{(3,3)}\), and (iv) \((\nu, \tau) \in \Psi_1^{(3,4)}\).

Let \((\nu, \tau) \in \Psi_1^{(3,1)}\). By using the relation \( K_+(\nu^2) < J_+(\nu^2) \) which follows from Eq. (4.204), it is seen that Eq. (4.131) cannot be true for the current sub-case where \( \nu^2 \leq \tau < K_+(\nu^2) \). Moreover, obviously Eq. (4.132) is also not true for the sub-case. In addition, by using the relation \( K_+(\nu^2) < L_-(\nu^2) < I_+(\nu^2) \) which also follows from Eq. (4.204), one concludes that Eq. (4.133) also cannot be true for the sub-case. According to part A of Theorem 32, the above results imply that any \((\nu, \tau) \in \Psi_1^{(3,1)}\) is \( c \)-\( \tau \) unstable. On the other hand, because \( \tau_o(\nu^2) = K_+(\nu^2) \) if \( c_2 < \nu^2 < c_3 \), one concludes that \( \tau < \tau_o(\nu^2) \) and thus \((\nu, \tau) \notin \Gamma_o \) if \((\nu, \tau) \in \Psi_1^{(3,1)}\). As such it has been shown that part F is true over \( \Psi_1^{(3,1)}\).

Let \((\nu, \tau) \in \Psi_1^{(3,2)}\). Obviously Eq. (4.132) is true for the current sub-case where \( K_+(\nu^2) \leq \tau \leq I_+(\nu^2) \). According to part A of Theorem 32, this implies that any \((\nu, \tau) \in \Psi_1^{(3,2)}\) is \( c \)-\( \tau \) stable. On the other hand, because \( \tau_o(\nu^2) = K_+(\nu^2) \) if \( c_2 < \nu^2 < c_3 \), one concludes that \( \tau \geq \tau_o(\nu^2) \) and thus \((\nu, \tau) \in \Gamma_o \) if \((\nu, \tau) \in \Psi_1^{(3,2)}\). As such it has been shown that part F is true over \( \Psi_1^{(3,2)}\).

Let \((\nu, \tau) \in \Psi_1^{(3,3)}\). By using the relation \( L_-(\nu^2) < I_+(\nu^2) < J_+(\nu^2) < L_+(\nu^2) \) which follows from Eq. (4.204), it is seen that Eq. (4.133) is true for the current case
where \( I_+ (\nu^2) < \tau < J_+ (\nu^2) \). According to part A of Theorem 32, this implies that any \((\nu, \tau) \in \Psi_\neg(3,3)\) is \(c\)–\(\tau\) stable. On the other hand, because (i) \( \tau_o (\nu^2) = K_+ (\nu^2) \)
if \( c_2 < \nu^2 < c_3 \), and (ii) the relation \( K_+ (\nu^2) < I_+ (\nu^2) \) is a part of Eq. (4.204), one concludes that \( \tau > \tau_o (\nu^2) \) and thus \((\nu, \tau) \in \Gamma_o \) if \((\nu, \tau) \in \Psi_\neg(3,3)\). As such it has been shown that part F is true over \( \Psi_\neg(3,3) \).

Let \((\nu, \tau) \in \Psi_\neg(3,4)\). Obviously Eq. (4.131) is true for the current sub-case where \( J_+ (\nu^2) \leq \tau < \sqrt{\nu^2} \). According to part A of Theorem 32, this implies that any \((\nu, \tau) \) in the current sub-case is \(c\)–\(\tau\) stable. On the other hand, because (i) \( \tau_o (\nu^2) = K_+ (\nu^2) \)
if \( c_2 < \nu^2 < c_3 \), and (ii) the relation \( K_+ (\nu^2) < J_+ (\nu^2) \) is a part of Eq. (4.204), one concludes that \( \tau > \tau_o (\nu^2) \) and thus \((\nu, \tau) \in \Gamma_o \) if \((\nu, \tau) \in \Psi_\neg(3,4)\). As such, it has been shown that part F is true over \( \Psi_\neg(3,4) \).

It has been shown that part F is true over each of the four nonempty disjoint sets \( \Psi_\neg(3,1), \Psi_\neg(3,2), \Psi_\neg(3,3), \) and \( \Psi_\neg(3,4) \). Eq. (4.205) now implies that part F is true over \( \Psi_\neg(3) \).

9. \((\nu, \tau) \in \Psi_\neg(4)\). For this case, we have (i) \( c_3 \leq \nu^2 < 1 \), and (ii) \( \nu^2 \leq \tau < \sqrt{\nu^2} \). To proceed, Note that Eqs. (4.165)–(4.168) implies that

\[
\nu^2 < K_+ (\nu^2) < \sqrt{\nu^2} \leq J_+ (\nu^2) \leq I_+ (\nu^2) < L_+ (\nu^2), \quad c_3 \leq \nu^2 < 1 \tag{4.210}
\]

With the aid of Eq. (4.210), it can be shown that

\[
\Psi_\neg(4) = \Psi_\neg(4,1) \cup \Psi_\neg(4,2) \tag{4.211}
\]

where \( \Psi_\neg(4,1) \) and \( \Psi_\neg(4,2) \) are nonempty disjoint sets defined by

\[
\Psi_\neg(4,1) \ \text{def} \ \{ (\nu, \tau) | c_3 \leq \nu^2 < 1 \ \text{and} \ \nu^2 \leq \tau < K_+ (\nu^2) \} \tag{4.212}
\]

and

\[
\Psi_\neg(4,2) \ \text{def} \ \{ (\nu, \tau) | c_3 \leq \nu^2 < 1 \ \text{and} \ K_+ (\nu^2) \leq \tau < \sqrt{\nu^2} \} \tag{4.213}
\]

Thus any \((\nu, \tau) \in \Psi_\neg(4)\) must fall into one and only one of the following two sub-cases: (i) \((\nu, \tau) \in \Psi_\neg(4,1)\) and (ii) \((\nu, \tau) \in \Psi_\neg(4,2)\).

Let \((\nu, \tau) \in \Psi_\neg(4,1)\). By using the relation \( K_+ (\nu^2) < J_+ (\nu^2) \leq I_+ (\nu^2) \) which follows from Eq. (4.210), it is seen that none of Eqs. (4.131)–(4.133) is true for the current sub-case where \( \nu^2 \leq \tau < K_+ (\nu^2) \). According to part A of Theorem 32, this implies that any \((\nu, \tau) \in \Psi_\neg(4,1)\) is \(c\)–\(\tau\) unstable. On the other hand, because \( \tau_o (\nu^2) = K_+ (\nu^2) \)
if \( c_3 \leq \nu^2 < 1 \), one concludes that \( \tau < \tau_o (\nu^2) \) and thus \((\nu, \tau) \notin \Gamma_o \) if \((\nu, \tau) \in \Psi_\neg(4,1)\). As such it has been shown that part F is true over \( \Psi_\neg(4,1) \).

Let \((\nu, \tau) \in \Psi_\neg(4,2)\). By using the relation \( K_+ (\nu^2) < \sqrt{\nu^2} \leq I_+ (\nu^2) \) which follows from Eq. (4.210), it is seen that Eq. (4.132) is true for the current sub-case where
\[ K_+(\nu^2) \leq \tau < \sqrt{\nu^2} \]. According to part A of Theorem 32, this implies that any \((\nu, \tau) \in \Psi^{(4,2)}_+\) is \(c\)-\(\tau\) stable. On the other hand, because \(\tau_o(\nu^2) = K_+(\nu^2)\) if \(c_3 \leq \nu^2 < 1\), one concludes that \(\tau \geq \tau_o(\nu^2)\) and thus \((\nu, \tau) \in \Gamma_o\) if \((\nu, \tau) \in \Psi^{(4,2)}_-\). As such it has been shown that part F is true over \(\Psi^{(4,2)}_-\).

It has been shown that part F is true over each of the two nonempty disjoint sets \(\Psi^{(4,1)}_-\) and \(\Psi^{(4,2)}_-\). Eq. (4.211) now implies that part F is true over \(\Psi^{(4)}_-\).

10. \((\nu, \tau) \in \Psi^{(1)}_+\). For this case, we have (i) \(0 < \nu^2 \leq c_3\), and (ii) \(\tau > \sqrt{\nu^2}\). To proceed, Note that Eqs. (4.166)–(4.165) imply that

\[
I_+(\nu^2) \leq \sqrt{\nu^2}, \quad 0 < \nu^2 \leq c_3 \tag{4.214}
\]

and

\[
K_+(\nu^2) < \sqrt{\nu^2}, \quad 0 < \nu^2 \leq c_3 \tag{4.215}
\]

By using Eqs. (4.214) and (4.215), one concludes that Eq. (4.135) is true for the current case where \(\tau > \sqrt{\nu^2}\). According to part B of Theorem 32, this implies that any \((\nu, \tau)\) in the current case is \(c\)-\(\tau\) stable. On the other hand, because (i) \(\tau_o(\nu^2) = L_-(\nu^2)\) if \(0 < \nu^2 \leq c_2\), and (ii) \(\tau_o(\nu^2) = K_+(\nu^2)\) if \(c_2 \leq \nu^2 \leq c_3\), Eqs. (4.215) and (4.216) imply that \(\tau > \tau_o(\nu^2)\) and thus \((\nu, \tau) \in \Gamma_o\) if \((\nu, \tau) \in \Psi^{(1)}_+\). As such, it has been shown that part F is true over \(\Psi^{(1)}_+\).

11. \((\nu, \tau) \in \Psi^{(2)}_+\). For this case, we have (i) \(c_3 < \nu^2 < 1\), and (ii) \(\tau > \sqrt{\nu^2}\). To proceed, Note that Eqs. (4.166)–(4.168) imply that

\[
K_+(\nu^2) < \sqrt{\nu^2} < J_+(\nu^2) < I_+(\nu^2), \quad c_3 < \nu^2 < 1 \tag{4.217}
\]

and

\[
L_-(\nu^2) < J_+(\nu^2) < I_+(\nu^2) < L_+(\nu^2), \quad c_3 < \nu^2 < 1 \tag{4.218}
\]

By using Eq. (4.217), one has

\[
\Psi^{(2)}_+ = \Psi^{(2,1)}_+ \cup \Psi^{(2,2)}_+ \cup \Psi^{(2,3)}_+ \tag{4.219}
\]

where \(\Psi^{(2,1)}_+, \Psi^{(2,2)}_+,\) and \(\Psi^{(2,3)}_+\) are nonempty disjoint sets defined by

\[
\Psi^{(2,1)}_+ \overset{\text{def}}{=} \{(\nu, \tau) | c_3 < \nu^2 < 1 \text{ and } \sqrt{\nu^2} < \tau \leq J_+(\nu^2)\} \tag{4.220}
\]

\[
\Psi^{(2,2)}_+ \overset{\text{def}}{=} \{(\nu, \tau) | c_3 < \nu^2 < 1 \text{ and } J_+(\nu^2) < \tau < I_+(\nu^2)\} \tag{4.221}
\]

and

\[
\Psi^{(2,3)}_+ \overset{\text{def}}{=} \{(\nu, \tau) | c_3 < \nu^2 < 1 \text{ and } \tau \geq I_+(\nu^2)\} \tag{4.222}
\]
Thus any \((\nu, \tau) \in \Psi_+^{(2)}\) must fall into one and only one of the following three sub-cases:

(i) \((\nu, \tau) \in \Psi_+^{(2,1)}\), (ii) \((\nu, \tau) \in \Psi_+^{(2,2)}\), and (iii) \((\nu, \tau) \in \Psi_+^{(2,3)}\).

Let \((\nu, \tau) \in \Psi_+^{(2,1)}\). Eq. (4.134) is true for any \((\nu, \tau)\) in the current sub-case where \(\sqrt{\nu^2} < \tau \leq I_+(\nu^2)\). According to part B of Theorem 32, this implies that the any \((\nu, \tau) \in \Psi_+^{(2,1)}\) is \(c\)-\(\tau\) stable. On the other hand, because (i) \(\tau_o(\nu^2) = K_+(\nu^2)\) if \(c_3 < \nu^2 < 1\), and (ii) the relation \(K_+(\nu^2) < \sqrt{\nu^2}\) is a part of Eq. (4.217), one concludes that \(\tau > \tau_o(\nu^2)\) and thus \((\nu, \tau)\) \(\in \Gamma_o\) if \((\nu, \tau) \in \Psi_+^{(2,1)}\). As such it has been shown that part F is true over \(\Psi_+^{(2,1)}\).

Let \((\nu, \tau) \in \Psi_+^{(2,2)}\). By using Eq. (4.218), one concludes that Eq. (4.136) is true for the current case where \(J_+(\nu^2) < \tau < I_+(\nu^2)\). According to part B of Theorem 32, this implies that any \((\nu, \tau) \in \Psi_+^{(2,2)}\) is \(c\)-\(\tau\) stable. On the other hand, because (i) \(\tau_o(\nu^2) = K_+(\nu^2)\) if \(c_3 < \nu^2 < 1\), and (ii) the relation \(K_+(\nu^2) < I_+(\nu^2)\) is a part of Eq. (4.217), one concludes that \(\tau > \tau_o(\nu^2)\) and thus \((\nu, \tau) \in \Gamma_o\) if \((\nu, \tau) \in \Psi_+^{(2,2)}\). As such it has been shown that part F is true over \(\Psi_+^{(2,2)}\).

Let \((\nu, \tau) \in \Psi_+^{(2,3)}\). By using the relation \(K_+(\nu^2) < I_+(\nu^2)\) which follows from Eq. (4.217), one concludes that Eq. (4.135) is true for the current sub-case where \(\tau \geq I_+(\nu^2)\). According to part B of Theorem 32, this implies that any \((\nu, \tau) \in \Psi_+^{(2,3)}\) is \(c\)-\(\tau\) stable. On the other hand, because (i) \(\tau_o(\nu^2) = K_+(\nu^2)\) if \(c_3 < \nu^2 < 1\), and (ii) the relation \(K_+(\nu^2) < I_+(\nu^2)\) is a part of Eq. (4.217), one concludes that \(\tau > \tau_o(\nu^2)\) and thus \((\nu, \tau) \in \Gamma_o\) if \((\nu, \tau) \in \Psi_+^{(2,3)}\). As such, it has been shown that part F is true over \(\Psi_+^{(2,3)}\).

It has been shown that part F is true over each of the three nonempty disjoint sets \(\Psi_+^{(2,1)}, \Psi_+^{(2,2)}, \) and \(\Psi_+^{(2,3)}\). Eq. (4.219) now implies that part F is true over \(\Psi_+^{(2)}\).

It has been established that part F is true over each of the sets mentioned in the paragraph immediately following Eq. (4.188). Because any \((\nu, \tau)\) must belong to one and only one of these sets, the proof of part F is completed.

Finally, with the aid of Theorems 4 and 16, one can obtain part G from part F. QED.

As promised earlier, a proof for Theorem 34 will be provided in the remainder of the paper. As a preliminary, we have Theorem 36.

**Theorem 36.** In the domain \(0 < x < 1\), (A) \(I_+(x)\), \(J_+(x)\), \(K_+(x)\), and \(L_-(x)\) are strictly monotonically increasing while \(L_+(x)\) is strictly monotonically decreasing. Moreover, we have (B)

\[
\begin{align*}
3 &> I_+(x) > 0, \quad 0 < x < 1 \\
3 &> J_+(x) > 0, \quad 0 < x < 1 \\
1 &> K_+(x) > 0, \quad 0 < x < 1 \\
3 &> L_-(x) > 0, \quad 0 < x < 1
\end{align*}
\]
and

\[ L_+(x) > 3, \quad 0 < x < 1 \quad (4.227) \]

**Proof.** Let \( f'(x) \stackrel{\text{def}}{=} df(x)/dx \) for any function \( f \) of \( x \). Then (i) Eqs. (4.91) and (4.98) imply that

\[ I_+'(x) = \frac{3x - 2 + 2\sqrt{3x^2 - 3x + 1}}{x^2\sqrt{3x^2 - 3x + 1}} > 0, \quad 0 < x < 1 \quad (4.228) \]

(ii) Eqs. (4.92) and (4.100) imply that

\[
J_+'(x) = \frac{-x^3 + 6x^2 - x + 2 + 4\sqrt{2(x^3 - x + 2)}}{(2 - x)^2\sqrt{2(x^3 - x + 2)}} = \frac{x^2(1 - x) + 5x^2 + 1 + (1 - x) + 4\sqrt{2(x^3 - x + 2)}}{(2 - x)^2\sqrt{2(x^3 - x + 2)}} > 0, \quad 0 < x < 1
\]

(iii) Eqs. (4.114) and (4.118) imply that

\[ K_+'(x) = \frac{\sqrt{1 - 2x + 5x^2} - (1 - x)}{2x^2\sqrt{1 - 2x + 5x^2}} > 0, \quad 0 < x < 1 \quad (4.230) \]

(iv) Eqs. (4.122) and (4.126) imply that

\[ L_-'(x) = \frac{2\left[4 - x - 2\sqrt{2(2 - x - x^2)}\right]}{x^2\sqrt{2(2 - x - x^2)}} > 0, \quad 0 < x < 1 \quad (4.231) \]

and (v) Eqs. (4.122) and (4.126) imply that

\[ L_+'(x) = -\frac{2\left[4 - x + 2\sqrt{2(2 - x - x^2)}\right]}{x^2\sqrt{2(2 - x - x^2)}} < 0, \quad 0 < x < 1 \quad (4.232) \]

Thus part A is true.

Moreover, by using (i) Eqs. (4.91), (4.92), (4.114), and (4.122), and (ii) L’hopital’s rule, one has (i)

\[
\lim_{x \to 1^-} I_+(x) = \lim_{x \to 1^-} J_+(x) = \lim_{x \to 1^-} L_-(x) = \lim_{x \to 1^-} L_+(x) = 3, \quad \text{and} \quad \lim_{x \to 1^-} K_+(x) = 1
\]

\[
(4.233)
\]

(ii)

\[
\lim_{x \to 0^+} I_+(x) = \lim_{x \to 0^+} \left(3 + \frac{6x - 3}{\sqrt{3x^2 - 3x + 1}}\right) = 3 + (-3) = 0
\]

(4.234)
(iii) \[
\lim_{x \to 0^+} J_+(x) = 0
\] (4.235)

(iv) \[
\lim_{x \to 0^+} K_+(x) = \lim_{x \to 0^+} \frac{1}{2} \left( 1 + \frac{5x-1}{\sqrt{1-2x+5x^2}} \right) = \frac{1}{2} (1 - 1) = 0
\] (4.236)

and (v) \[
\lim_{x \to 0^+} L_-(x) = \lim_{x \to 0^+} \left[ -1 + \frac{2(1+2x)}{\sqrt{2(2-x-x^2)}} \right] = -1 + 1 = 0
\] (4.237)

part B now follows from Part A and Eqs. (4.233)–(4.237). QED

An immediate result of Theorem 36 and the fact that \(0 < x < \sqrt{x} < 1\) if \(0 < x < 1\) is given in Theorem 37.

**Theorem 37.** We have
\[
x < \sqrt{x} < L_+(x), \quad I_+(x) < L_+(x), \quad J_+(x) < L_+(x),
\]
\[
K_+(x) < L_+(x) \quad \text{and} \quad L_-(x) < L_+(x), \quad 0 < x < 1
\] (4.238)

Theorem 37 is but one of many algebraic relations that are needed in the proof of Theorem 34. Note that, in establishing other needed relations, several inequalities that involve the four principal square roots that appear in the definitions of \(I_\pm(x), J_\pm(x), K_\pm(x),\) and \(L_\pm(x),\) i.e.,
\[
\sqrt{3x^2 - 3x + 1} > 0, \quad -\infty < x < +\infty
\] (4.239)
\[
\sqrt{2(x^3 - x + 2)} > 0, \quad 0 < x < 2
\] (4.240)
\[
\sqrt{1 - 2x + 5x^2} > 0, \quad -\infty < x < +\infty
\] (4.241)

and
\[
\sqrt{2(2-x-x^2)} > 0, \quad -2 < x < 1
\] (4.242)

(which follow from Eqs. (4.97), (4.100), (4.117), and (4.124), respectively) will be used repeatedly. Also to be used often is the following algebraic property:

**Property I.** Let \(a \geq 0\) and \(b \geq 0.\) Then
\[
a^2 - b^2 \begin{cases} > 0 & \iff a - b > 0 \\ = 0 & \iff a - b = 0 \\ < 0 & \iff a - b < 0 \end{cases}
\] (4.243)
With the above preparations, a set of relations will be given in Theorems 38–48.

**Theorem 38.** We have

\[
x - I_+(x) = \begin{cases} 
> 0 & \text{if } 0 < x < 3 - 2\sqrt{2} \\
= 0 & \text{if } x = 3 - 2\sqrt{2} \\
< 0 & \text{if } 3 - 2\sqrt{2} < x < 1
\end{cases}
\]  

(4.244)

**Proof.** Let \(0 < x < 1\) throughout the proof. Then Eq. (4.91) implies that

\[
x - I_+(x) = \frac{x^2 - 3x + 2 - 2\sqrt{3x^2 - 3x + 1}}{x}
\]  

(4.245)

With the aid of Property I, Eq. (4.244) is a result of Eq. (4.245) and the following relations:

(i) Eq. (4.239); (ii)

\[x^2 - 3x + 2 = (x - 1)(x - 2) > 0\]  

(4.246)

(iii)

\[(x^2 - 3x + 2)^2 - \left(2\sqrt{3x^2 - 3x + 1}\right)^2 = x^2(x^2 - 6x + 1)\]  

(4.247)

and (iv) \(0 < 3 - 2\sqrt{2} < 1 < 3 + 2\sqrt{2}\). **QED.**

**Theorem 39.** We have

\[x < K_+(x), \quad 0 < x < 1\]  

(4.248)

**Proof.** Let \(0 < x < 1\) throughout the proof. Then Eq. (4.114) implies that

\[K_+(x) - x = \frac{\sqrt{1 - 2x + 5x^2} - (2x^2 - x + 1)}{2x}\]  

(4.249)

With the aid of Property I, Eq. (4.248) is a result of Eq. (4.249) and the following relations:

(i) Eq. (4.241); (ii)

\[2x^2 - x + 1 = 2(x - 1/4)^2 + 7/8 \geq 7/8\]  

(4.250)

and (iii)

\[
\left(\sqrt{1 - 2x + 5x^2}\right)^2 - (2x^2 - x + 1)^2 = 4x^3(1 - x) > 0
\]  

(4.251)

**QED.**
**Theorem 40.** Let $c_3$ be the constant defined in Eq. (4.157). Then

\[
\sqrt{x} - I_+(x) \begin{cases} 
>0 & \text{if } 0 < x < c_3 \\
=0 & \text{if } x = c_3 \\
<0 & \text{if } c_3 < x < 1
\end{cases}
\]  \tag{4.252}

**Proof.** Let $0 < x < 1$ throughout the proof. Then Eq. (4.91) implies that

\[
\sqrt{x} - I_+(x) = \frac{x\sqrt{x} - 3x + 2 - 2\sqrt{3x^2 - 3x + 1}}{x}
\]  \tag{4.253}

With the aid of Property I, Eq. (4.252) is a result of Eq. (4.253) and the following relations:

(i) Eq. (4.239); (ii) \(x\sqrt{x} - 3x + 2 = (1 - \sqrt{x})[1 + 2\sqrt{x} + (1 - x)] > 0\)  \tag{4.254}

(iii)

\[
(x\sqrt{x} - 3x + 2)^2 - (2\sqrt{3x^2 - 3x + 1})^2 = x^3 - 6x^{5/2} - 3x^2 + 4x^{3/2}
\]

\[
= x^{3/2}(\sqrt{x} + 1) \left(\sqrt{x} - \frac{7 + \sqrt{33}}{2}\right) \left(\sqrt{x} - \frac{7 - \sqrt{33}}{2}\right)
\]  \tag{4.255}

(iv) \(0 < (7 - \sqrt{33})/2 < 1 < (7 + \sqrt{33})/2\); and (v) \(c_3 = \left[(7 - \sqrt{33})/2\right]^2\). QED.

**Theorem 41.** Let $c_3$ be the constant defined in Eq. (4.157). Then

\[
\sqrt{x} - J_+(x) \begin{cases} 
>0 & \text{if } 0 < x < c_3 \\
=0 & \text{if } x = c_3 \\
<0 & \text{if } c_3 < x < 1
\end{cases}
\]  \tag{4.256}

**Proof.** Let $0 < x < 1$ throughout the proof. Then Eq. (4.92) implies that

\[
\sqrt{x} - J_+(x) = \frac{-x\sqrt{x} - 3x + 2\sqrt{x} + 2 - \sqrt{2(x^3 - x + 2)}}{2 - x}
\]  \tag{4.257}

With the aid of Property I, Eq. (4.256) is a result of Eq. (4.257) and the following relations:

(i) Eq. (4.240); (ii) \(-x\sqrt{x} - 3x + 2\sqrt{x} + 2 = (1 - \sqrt{x})(x + 4\sqrt{x} + 2) > 0\)  \tag{4.258}
(iii)

\[
(-x\sqrt{x} - 3x + 2\sqrt{x} + 2)^2 - \left[\sqrt{2(x^3 - x^2 + 2)}\right]^2 \\
= -x^3 + 6x^{5/2} + 5x^2 - 16x^{3/2} - 6x + 8\sqrt{x} \\
= \sqrt{x}(2 - x)(\sqrt{x} + 1)\left(\sqrt{x} - \frac{7 + \sqrt{33}}{2}\right)\left(\sqrt{x} - \frac{7 - \sqrt{33}}{2}\right)
\]

(iv) \(0 < (7 - \sqrt{33})/2 < 1 < (7 + \sqrt{33})/2\); and (v) \(c_3 = \left[(7 - \sqrt{33})/2\right]^2\).

**QED.**

**Theorem 42.** We have

\[K_+(x) < \sqrt{x}, \quad 0 < x < 1\]  

**Proof.** Let \(0 < x < 1\) throughout the proof. Then Eq. (4.114) implies that

\[\sqrt{x} - K_+(x) = \frac{2x\sqrt{x} - x + 1 - \sqrt{1 - 2x + 5x^2}}{2x}\]  

With the aid of Property I, Eq. (4.260) is a result of Eq. (4.261) and the following relations:

(i) Eq. (4.241); (ii) \(2x\sqrt{x} - x + 1 = 2x\sqrt{x} + (1 - x) > 0\)  

and (iii)

\[(2x\sqrt{x} - x + 1)^2 - \left(\sqrt{1 - 2x + 5x^2}\right)^2 = 4x\sqrt{x}(1 - x)(1 - \sqrt{x}) > 0\]

**QED.**

**Theorem 43.** Let \(c_4\) be the constant defined in Eq. (4.158). Then

\[
\sqrt{x} - L_-(x) \begin{cases} 
> 0 & \text{if } 0 < x < c_4 \\
= 0 & \text{if } x = c_4 \\
< 0 & \text{if } c_4 < x < 1
\end{cases}
\]  

**Proof.** Unless specified otherwise. Let \(0 < x < 1\) in this proof. Then Eq. (4.122) implies that

\[
\sqrt{x} - L_-(x) = \frac{2\sqrt{2(2 - x - x^2)} - (4 - x - x\sqrt{x})}{x}
\]
To proceed, note that
\[
\left[2\sqrt{2(2-x-x^2)}\right]^2 - (4-x-x\sqrt{x})^2 = -x\sqrt{x}g(x)
\]  
(4.266)
where
\[
g(x) \overset{\text{def}}{=} x\sqrt{x} + 2x + 9\sqrt{x} - 8, \quad x \geq 0
\]  
(4.267)
Because (i)
\[
g'(x) = 3\sqrt{x}/2 + 2 + 9/(2\sqrt{x}) = 3/(2\sqrt{x}) \left[ (\sqrt{x} + 2/3)^2 + 23/9 \right] > 0, \quad x > 0
\]  
(4.268)
and (ii)
\[
g(0) = -8 \quad \text{and} \quad g(1) = 4
\]  
(4.269)
one concludes that \(g(x)\) is strictly monotonically increasing in the interval \(0 < x < 1\) and
there is one and only one real root of \(g(x) = 0\) in this interval. By using the standard
formula for the roots of a cubic equation, it can be shown that this root is given by \(x = c_4\). Moreover, Eqs. (4.268) and (4.269) imply that: (i) \(g(x) < 0\) if \(0 < x < c_4\); (ii) \(g(x) = 0\) if \(x = c_4\); and (iii) \(g(x) > 0\) if \(c_4 < x < 1\). As such Eq. (4.266) implies that
\[
\begin{aligned}
\left[2\sqrt{2(2-x-x^2)}\right]^2 - (4-x-x\sqrt{x})^2 &= \begin{cases} 
> 0 & \text{if } 0 < x < c_4 \\
= 0 & \text{if } x = c_4 \\
< 0 & \text{if } c_4 < x < 1
\end{cases}
\end{aligned}
\]  
(4.270)
With the aid of Property I, Eq. (4.264) is a result of Eqs. (4.265) and (4.270), and the the
following relations: (i) Eq. (4.242); and (ii)
\[
4 - x - x\sqrt{x} = 2 + (1-x) + (1-x\sqrt{x}) > 2, \quad 0 < x < 1
\]  
(4.271)
QED.

**Theorem 44.** We have
\[
L_-(x) < J_+(x), \quad 0 < x < 1
\]  
(4.272)

**Proof.** Let \(0 < x < 1\) throughout this proof. Then Eqs. (4.92) and (4.122) imply that
\[
J_+(x) - L_-(x) = \frac{x\sqrt{2(x^3-x+2)} + 2(2-x)\sqrt{2(2-x-x^2)} - (8-4x-2x^2)}{x(2-x)}
\]  
(4.273)
Let
\[
\beta(x) \overset{\text{def}}{=} x\sqrt{2(x^3-x+2)} + 2(2-x)\sqrt{2(2-x-x^2)} + (8-4x-2x^2)
\]  
(4.274)
and
\[ \beta_{\pm}(x) \overset{\text{def}}{=} 4\sqrt{(x^3 - x + 2)(2 + x)} \pm \sqrt{1 - x}(8 + 3x - x^2) \] (4.275)

Then (i)
\[
[J_+(x) - L_-(x)]\beta(x)
\]
\[
= 8x(2 - x)\sqrt{(x^3 - x + 2)(1 - x)(2 + x)} + 2x^5 - 12x^4 + 6x^3 + 36x^2 - 32x
\]
\[
x(2 - x)
\]
\[
= 8x(2 - x)\sqrt{(x^3 - x + 2)(1 - x)(2 + x)} - 2x(1 - x)(2 - x)(8 + 3x - x^2)
\]
\[
x(2 - x)
\]
\[
= 2\sqrt{1 - x}\beta_-(x)
\]

and (ii)
\[ \beta_-(x)\beta_+(x) = x^5 + 9x^4 + 31x^3 + 39x^2 + 16x \] (4.277)

Thus
\[
[J_+(x) - L_-(x)]\beta(x)\beta_+(x) = 2\sqrt{1 - x}(x^5 + 9x^4 + 31x^3 + 39x^2 + 16x)
\] (4.278)

By using (i) Eqs. (4.240) and (4.242); (ii) \( \sqrt{(x^3 - x + 2)(2 + x)} > 0 \); (iii)
\[ 8 - 4x - 2x^2 = 2 + 4(1 - x) + 2(1 - x^2) > 2 \] (4.279)

and (iv)
\[ 8 + 3x - x^2 = 7 + 3x + (1 - x^2) > 7 \] (4.280)

it follows from Eqs. (4.274) and (4.275) that
\[ \beta(x) > 0 \quad \text{and} \quad \beta_+(x) > 0, \quad 0 < x < 1 \] (4.281)

Eq. (4.272) is a result of Eq. (4.281) and the fact that the expression on the right side of Eq. (4.278) is positive everywhere in the interval \( 0 < x < 1 \). QED

**Theorem 45.** We have
\[
K_+(x) - I_+(x) \begin{cases} > 0 \quad \text{if} \quad 0 < x < 3/11 \\ = 0 \quad \text{if} \quad x = 3/11 \\ < 0 \quad \text{if} \quad 3/11 < x < 1 \end{cases}
\] (4.282)
Proof. Eqs. (4.91) and (4.114) imply that

\[ K_+(x) - I_+(x) = \frac{(3 - 5x) - (4\sqrt{3x^2 - 3x + 1} - \sqrt{1 - 2x + 5x^2})}{2x}, \quad 0 < x < 1 \]  \hspace{1cm} (4.283)

By using (i) Eqs. (4.239) and (4.241), and (ii)

\[ \left(4\sqrt{3x^2 - 3x + 1}\right)^2 - \left(\sqrt{1 - 2x + 5x^2}\right)^2 = 43x^2 - 46x + 15 \]  \hspace{1cm} (4.284)

\[ = 43[(x - 23/43)^2 + 116/(43)^2] \geq 116/43, \quad -\infty < x < \infty \]

an application of Property I leads to the conclusion

\[ 4\sqrt{3x^2 - 3x + 1} - \sqrt{1 - 2x + 5x^2} > 0, \quad -\infty < x < +\infty \]  \hspace{1cm} (4.285)

Moreover, we have

\[ 3 - 5x \begin{cases} \leq 0 & \text{if } x \geq 3/5 \\ > 0 & \text{if } x < 3/5 \end{cases} \]  \hspace{1cm} (4.286)

Combining Eqs. (4.283), (4.285) and (4.286), one has

\[ K_+(x) - I_+(x) < 0, \quad 3/5 \leq x < 1 \]  \hspace{1cm} (4.287)

To proceed, let

\[ \xi(x) \overset{\text{def}}{=} \frac{1}{2} \left(3 - 5x + 4\sqrt{3x^2 - 3x + 1} - \sqrt{1 - 2x + 5x^2}\right), \quad 0 < x < 1 \]  \hspace{1cm} (4.288)

and

\[ \xi_\pm(x) \overset{\text{def}}{=} 2\sqrt{(3x^2 - 3x + 1)(1 - 2x + 5x^2)} \pm (7x^2 - 5x + 2), \quad 0 < x < 1 \]  \hspace{1cm} (4.289)

Then Eqs. (4.285) and (4.286) imply that

\[ \xi(x) > 0, \quad 0 < x < 3/5 \]  \hspace{1cm} (4.290)

In addition, by using (i) \( \sqrt{(3x^2 - 3x + 1)(1 - 2x + 5x^2)} > 0, \quad -\infty < x < +\infty \) (which follows from Eqs. (4.239) and (4.241)); and (ii)

\[ 7x^2 - 5x + 2 = 7 \left[(x - 5/14)^2 + 31/196\right] \geq 31/28, \quad -\infty < x < +\infty \]  \hspace{1cm} (4.291)

one has

\[ \xi_+(x) > 0, \quad 0 < x < 1 \]  \hspace{1cm} (4.292)
Combining Eqs. (4.290) and (4.292), one arrives at the conclusion:

\[ \xi(x) \xi_+(x) > 0, \quad 0 < x < 3/5 \]  
(4.293)

Next, Eqs. (4.283), (4.288), and (4.289) imply that (i)

\[ [K_+(x) - I_+(x)] \xi(x) = \frac{\xi_-(x)}{x}, \quad 0 < x < 1 \]  
(4.294)

and (ii)

\[ \xi_-(x) \xi_+(x) = 11x^4 - 14x^3 + 3x^2 = 11x^2(x - 1)(x - 3/11), \quad 0 < x < 1 \]  
(4.295)

Thus

\[ [K_+(x) - I_+(x)] \xi(x) \xi_+(x) = 11x(x - 1)(x - 3/11), \quad 0 < x < 1 \]  
(4.296)

It follows from Eqs. (4.293) and (4.296) that

\[
K_+(x) - I_+(x) = \begin{cases} 
> 0 & \text{if } 0 < x < 3/11 \\
= 0 & \text{if } x = 3/11 \\
< 0 & \text{if } 3/11 < x < 3/5 
\end{cases} 
\]  
(4.297)

Eq. (4.282) is an immediate result of Eqs. (4.287) and (4.297). \textbf{QED.}

\textbf{Theorem 46.} We have

\[
L_-(x) - I_+(x) = \begin{cases} 
> 0 & \text{if } 0 < x < 3/11 \\
= 0 & \text{if } x = 3/11 \\
< 0 & \text{if } 3/11 < x < 1 
\end{cases} 
\]  
(4.298)

\textbf{Proof.} Eqs. (4.91) and (4.122) imply that

\[ L_-(x) - I_+(x) = \frac{2}{x} \left( 3 - 2x - \sqrt{2(2 - x - x^2)} - \sqrt{3x^2 - 3x + 1} \right), \quad 0 < x < 1 \]  
(4.299)

By using Eq. (4.299) and the definitions

\[ \mu(x) \overset{\text{def}}{=} \frac{1}{2} \left( 3 - 2x + \sqrt{2(2 - x - x^2)} + \sqrt{3x^2 - 3x + 1} \right), \quad 0 < x < 1 \]  
(4.300)
and
\[ \mu_{\pm}(x) \overset{\text{def}}{=} 3x^2 - 7x + 4 \pm 2 \sqrt{2(2 - x - x^2)(3x^2 - 3x + 1)}, \quad 0 < x < 1 \] (4.301)

one has
\[ [L_-(x) - I_+(x)] \mu(x) = \frac{\mu_-(x)}{x}, \quad 0 < x < 1 \] (4.302)

and
\[ \mu_-(x) \mu_+(x) = 33x^4 - 42x^3 + 9x^2 = 33x^2(x - 1)(x - 3/11), \quad 0 < x < 1 \] (4.303)

In turn, Eqs. (4.302) and (4.303) imply that
\[ [L_-(x) - I_+(x)] \mu(x) \mu_+(x) = 33x(x - 1)(x - 3/11), \quad 0 < x < 1 \] (4.304)

By using (i) Eqs. (4.239) and (4.242); (ii) \( 3 - 2x > 0 \) if \( x < 3/2 \); and (iii)
\[ 3x^2 - 7x + 4 = 3(x - 1)(x - 4/3) > 0, \quad x < 1 \text{ or } x > 4/3 \] (4.305)

Eqs. (4.300) and (4.301) imply that
\[ \mu(x) > 0 \quad \text{and} \quad \mu_+(x) > 0, \quad 0 < x < 1 \] (4.306)

Eq. (4.298) is an immediate result of Eqs. (4.304) and (4.306). QED.

**Theorem 47.** We have
\[ L_-(x) - K_+(x) \begin{cases} > 0 & \text{if} \quad 0 < x < 3/11 \text{ or } 3/11 < x < 1 \\ = 0 & \text{if} \quad x = 3/11 \end{cases} \] (4.307)

**Proof.** Eqs. (4.114) and (4.122) imply that
\[ L_-(x) - K_+(x) = \frac{9 - 3x - 4 \sqrt{2(2 - x - x^2)} - \sqrt{1 - 2x + 5x^2}}{2x}, \quad 0 < x < 1 \] (4.308)

By using Eq. (4.308) and the definitions
\[ \psi(x) \overset{\text{def}}{=} \frac{1}{2} \left( 9 - 3x + 4 \sqrt{2(2 - x - x^2)} + \sqrt{1 - 2x + 5x^2} \right), \quad 0 < x < 1 \] (4.309)

and
\[ \psi_{\pm}(x) \overset{\text{def}}{=} 9x^2 - 5x + 4 \pm 2 \sqrt{2(2 - x - x^2)(1 - 2x + 5x^2)}, \quad 0 < x < 1 \] (4.310)
one has
\[ [L_-(x) - K_+(x)] \psi(x) = \frac{\psi_-(x)}{x} , \quad 0 < x < 1 \] (4.311)
and
\[ \psi_-(x) \psi_+(x) = 121x^4 - 66x^3 + 9x^2 = 121x^2(x - 3/11)^2 , \quad 0 < x < 1 \] (4.312)

In turn, Eqs. (4.311) and (4.312) imply that
\[ [L_-(x) - K_+(x)] \psi(x) \psi_+(x) = 121x(x - 3/11)^2 , \quad 0 < x < 1 \] (4.313)

By using (i) Eqs. (4.241) and (4.242); (ii) \( 9 - 3x > 0 \) if \( x < 3 \); and (iii)
\[ 9x^2 - 5x + 4 = 9[(x - 5/18)^2 + 119/324] \geq 119/36 , \quad -\infty < x < +\infty \] (4.314)

Eqs. (4.309) and (4.310) imply that
\[ \psi(x) > 0 \quad \text{and} \quad \psi_+(x) > 0 , \quad 0 < x < 1 \] (4.315)

Eq. (4.307) is an immediate result of Eqs. (4.313) and (4.315). \textbf{QED}.

\textbf{Theorem 48.} Let \( c_3 \) be the constant defined in Eq. (4.157). Then we have
\[ J_+(x) - I_+(x) \begin{cases} > 0 & \text{if} \quad 0 < x < c_3 \\
= 0 & \text{if} \quad x = c_3 \\
< 0 & \text{if} \quad c_3 < x < 1 \end{cases} \] (4.316)

\textbf{Proof.} Eqs. (4.91) and (4.92) imply that
\[ J_+(x) - I_+(x) = \frac{6x^2 - 10x + 4 - \left[2(2 - x)\sqrt{3x^2 - 3x + 1} - x\sqrt{2(x^3 - x + 2)}\right]}{x(2 - x)} \]
\[ , \quad 0 < x < 1 \] (4.317)

To proceed, note that Eq. (4.239) implies that
\[ 2(2 - x)\sqrt{3x^2 - 3x + 1} > 0 , \quad x < 2 \] (4.318)

Also Eq. (4.240) implies that
\[ x\sqrt{2(x^3 - x + 2)} > 0 , \quad 0 < x < 2 \] (4.319)
Moreover, we have

\[
\left[2(2 - x)\sqrt{3x^2 - 3x + 1}\right]^2 - \left[x\sqrt{2(x^3 - x + 2)}\right]^2
= -2x^5 + 12x^4 - 58x^3 + 96x^2 - 64x + 16
= 2(1 - x)(x^4 - 5x^3 + 24x^2 - 24x + 8)
= 2(1 - x)\left[x^2(x^2 - 5x + 6) + 2(9x^2 - 12x + 4)\right]
= 2(1 - x)\left[x^2(x - 2)(x - 3) + 2(3x - 2)^2\right] > 0, \quad 0 < x < 1
\]

With the aid of Eqs. (4.318)–(4.320), an application of Property I leads to the conclusion

\[
2(2 - x)\sqrt{3x^2 - 3x + 1} - x\sqrt{2(x^3 - x + 1)} > 0, \quad 0 < x < 1
\]

Next note that

\[
x(2 - x) > 0, \quad 0 < x < 2
\]

and

\[
6x^2 - 10x + 4 = 6(x - 1)(x - 2/3) \begin{cases} 
\leq 0 & \text{if } 2/3 \leq x \leq 1 \\
> 0 & \text{if } x < 2/3 \text{ or } x > 1
\end{cases}
\]

By combining Eq. (4.317) with Eqs. (4.321)–(4.323), one concludes that

\[
J_+(x) - I_+(x) < 0, \quad 2/3 \leq x < 1
\]

To study the case where $0 < x < 2/3$, let

\[
\eta(x) \overset{\text{def}}{=} \frac{1}{2}\left[ 6x^2 - 10x + 4 + 2(2 - x)\sqrt{3x^2 - 3x + 1} - x\sqrt{2(x^3 - x + 2)} \right], \quad 0 < x < 1
\]

and

\[
\eta_\pm(x) \overset{\text{def}}{=} 2\sqrt{2}\sqrt{(3x^2 - 3x + 1)(x^3 - x + 2)} \pm (-x^3 + 10x^2 - 9x + 4), \quad 0 < x < 1
\]

By using Eqs. (4.321) and (4.323), Eq. (4.325) implies that

\[
\eta(x) > 0, \quad 0 < x < 2/3
\]

Moreover, because (i) $\sqrt{(3x^2 - 3x + 1)(x^3 - x + 2)} > 0, \quad 0 < x < 2$ (see Eqs. (4.239) and (4.240)); and (ii)

\[
-x^3 + 10x^2 - 9x + 4 = x^2(1 - x) + (3x - 3/2)^2 + 7/4 > 7/4, \quad 0 < x < 1
\]
Eq. (4.326) implies that
\[ \eta_+(x) > 0, \quad 0 < x < 1 \]  
(4.329)
Next, by using Eqs. (4.157), (4.317), (4.325) and (4.326), it can be shown that (i)
\[
\begin{align*}
&[J_+(x) - I_+(x)] \eta(x) \\
&= 2\sqrt{2} x(2 - x) \sqrt{(3x^2 - 3x + 1)(x^3 - x + 2) - (x^5 - 12x^4 + 29x^3 - 22x^2 + 8x)} \\
&= 2\sqrt{2} x(2 - x) \sqrt{(3x^2 - 3x + 1)(x^3 - x + 2) - x(2 - x)(-x^3 + 10x^2 - 9x + 4)} \\
&= \eta_-(x), \quad 0 < x < 1
\end{align*}
\]  
(4.330)
and (ii)
\[
\begin{align*}
\eta_-(x) \eta_+(x) &= -x^6 + 44x^5 - 142x^4 + 172x^3 - 89x^2 + 16x = x(1 - x)^3(x^2 - 41x + 16) \\
&= x(1 - x)^3(x - c_3) \left(x - \frac{41 + 7\sqrt{33}}{2}\right), \quad 0 < x < 1
\end{align*}
\]  
(4.331)
In turn, Eqs. (4.330) and (4.331) imply that
\[
\begin{align*}
&[J_+(x) - I_+(x)] \eta(x) \eta_+(x) = x(1 - x)^3(x - c_3) \left(x - \frac{41 + 7\sqrt{33}}{2}\right), \quad 0 < x < 1
\end{align*}
\]  
(4.332)
With the aid of Eqs. (4.327), (4.329) and (4.332), and the relation
\[
0 < c_3 < 2/3 < 1 < \frac{41 + 7\sqrt{33}}{2}
\]  
(4.333)
one concludes that
\[
J_+(x) - I_+(x) \begin{cases} 
> 0 & \text{if } 0 < x < c_3 \\
= 0 & \text{if } x = c_3 \\
< 0 & \text{if } c_3 < x < 2/3
\end{cases}
\]  
(4.334)
Eq. (4.316) now is an immediate result of Eqs. (4.324) and (4.334). QED.

With the above preparations, Theorem 34 can now be proved. Part A is identical to part A of Theorem 36. Part B can be shown using Theorems 38–48 and the two relations
\[
\sqrt{x} < L_+(x) \quad \text{and} \quad I_+(x) < L_+(x), \quad 0 < x < 1
\]  
(4.335)
which form a part of Theorem 37. Part C follows from Eqs. (4.230), (4.231), and (4.156). Part D was shown in Eqs. (4.233) and (4.237). QED.

Finally, note that none of the relations

\[ x < J_+(x), \quad x < L_-(x), \quad \text{and} \quad K_+(x) < J_+(x), \quad 0 < x < 1 \]  \hspace{1cm} (4.336)

appears in Theorems 37–48. However, they can be shown using Theorem 34. As such, they can be considered as results of Theorems 38–48 and the relations Eq. (4.335).
5. Conclusions and Discussions

With the aid of many unexpected mathematical simplifications that occur along the way, it has been shown in Sec. 4 that there is an explicit analytical solution to the implicit stability conditions stated in Theorem 3. The first and perhaps the most important “break” encountered is the simple relation Eq. (4.23), i.e., \( H(\nu, \tau, s) \), a quartic polynomial in \( s \), is equal to the product of \( 4(1 - \nu^2)s^2 \) and \( G(\nu, \tau, s) \), a quadratic polynomial in \( s \). Without Eq. (4.23) and the fortunate fact that both \( D(\nu, \tau, s) \) and \( F(\nu, \tau, s) \) are also quadratic polynomials in \( s \), the relatively straightforward study of the necessary stability conditions Eqs. (4.25)–(4.27) (Theorem 6) as presented in Sec. 4 would have become much more complicated.

Moreover, the fact that \( F(\nu, \tau, 1) \) and \( H(\nu, \tau, 1) \) can be cast into the simple factorized forms Eqs. (4.35) and (4.37), respectively, are instrumental in the successful effort to establish Eq. (4.41) as necessary conditions for stability (Theorem 12).

With the aid of Theorems 13–15, it was shown that the special case in which \((\nu, \tau)\) satisfies Eq. (4.2) and yet is c-\( \tau \) unstable occurs if and only if \( \tau = \nu^2 = 1 \) (Theorem 16). Using Theorem 16, Theorem 17 was then established to provide a set of necessary and sufficient stability conditions much more explicit and easier to handle than those given originally in Theorem 3. Based on Theorem 17, it was then shown that the c-\( \tau \) scheme is stable if (a) \( \nu = 0 \) and \( \tau \geq 0 \); or (b) \( \nu^2 = 1 \) and \( \tau > 1 \); or (c) \( 0 < \nu^2 < 1 \) and \( \tau = |\nu| \) (Theorem 18).

Excluding the four special cases addressed in Theorems 16 and 18, the set \( \Psi \) defined in Eq. (4.66) is the set of all other \((\nu, \tau)\) that satisfy the necessary stability conditions \( \tau \geq \nu^2 \) and \( \nu^2 \leq 1 \) (Theorem 19). To facilitate the development, \( \Psi \) is divided into two disjoint subsets \( \Psi_- \) and \( \Psi_+ \), which are defined in Eq. (4.66) and (4.67).

It turns out that Eqs. (4.25) and (4.27) are satisfied by all \((\nu, \tau) \in \Psi \) (Theorems 21 and 22). Thus, according to Theorem 17, a given \((\nu, \tau) \in \Psi \) is c-\( \tau \) stable if and only if it satisfies Eq. (4.26). As such, one arrives at the conclusion that a given \((\nu, \tau) \in \Psi \) is c-\( \tau \) stable if and only if it satisfies Eq. (4.84) (Theorem 23). This necessary and sufficient stability condition obviously is even simpler than those given in Theorem 17.

With the aid of Theorems 24–31, for the set \( \Psi \), we are able to obtain the explicit solution to the necessary and sufficient stability condition Eq. (4.84) in the form given in Theorem 32. The functions \( I_+(x) \), \( J_+(x) \), \( K_+(x) \), \( L_+(x) \), and \( L_-(x) \), \( 0 < x < 1 \), that appear in Theorem 32 are defined in Eqs. (4.91), (4.92), (4.114), and (4.122).

In principle, whether a given \((\nu, \tau)\) is c-\( \tau \) stable can be determined by using Theorems 12, 16, 18, 19, and 32. However, by using the alternative definitions of \( \Psi_- \) and \( \Psi_+ \) given in Theorem 33, and the ordering properties Eqs. (4.160)–(4.168) given in Theorem 34, it was shown that Theorems 12, 16, 18, 19, and 32 can be combined and turned into the simple explicit form of necessary and sufficient stability conditions given in Theorem 35.

Finally note that the proof of the ordering properties Eqs. (4.160)–(4.168) is hinged on the rather incredible facts that the 4–6th order polynomials in \( x \) or \( \sqrt{x} \) that appear in Eqs. (4.247), (4.251), (4.255), (4.259), (4.263), (4.295), (4.303), (4.312), and (4.331) all can be factorized and studied analytically.
References


Appendix A. Numerical Validation of Theorem 34
implicit real*8(a-h,o-z)

program "ineqs.for".

this program is used to verify numerically the inequalities
equations (4.160) - (4.168) (see theorem 34).

*** the functions I-plus, K-plus, L-minus, and L-plus are undefined
*** at x=0.d0. thus, instead of being evaluated at x=0.d0, these
*** functions will be evaluated at x=ep where ep is a very small
*** positive number.

*** at x=1.d0, 2.d0*(2.d0-x-x**2)=0.d0. because of round-off errors,
*** the value of this expression may become negative when x is very
*** close to 1.d0. as such the square root of this expression and
*** therefore the functions L-minus and L-plus may be undefined
*** computationally when x is too close to 1.d0. thus, instead of
*** being evaluated at x=1.d0, the functions will be evaluated at
*** x=1.d0-eq where eq is a very small positive number.

*** n1 = number of uniform sub-intervals in (0,c1).
*** n2 = number of uniform sub-intervals in (c1,c2).
*** n3 = number of uniform sub-intervals in (c2,c3).
*** n4 = number of uniform sub-intervals in (c3,c4).
*** n5 = number of uniform sub-intervals in (c4,1).

srt(x)=dsqrt(x)
fip(x)=(3.d0*x-2.d0+2.d0*dsqrt(3.d0*x**2-3.d0*x+1.d0))/x
fjp(x)=(3.d0*x-2.d0+dsqrt(2.d0*(x**3-x+2.d0)))/(2.d0-x)
fkp(x)=(x-1.d0+dsqrt(5.d0*x**2-2.d0*x+1.d0))/(2.d0*x)
flm(x)=(4.d0-x-2.d0*dsqrt(2.d0*(2.d0-x-x**2)))/x
flp(x)=(4.d0-x+2.d0*dsqrt(2.d0*(2.d0-x-x**2)))/x

n1=17
n2=10
n3=12
n4=14
n5=47
ep=1.d-7
eq=1.d-12
one=1.d0-eq
n5m=n5-1

c1=3.d0-2.d0*dsqrt(2.d0)
c2=3.d0/11.d0
c3=(41.d0-7.d0*dsqrt(33.d0))/2.d0
c4=(dexp((1.d0/3.d0)*dlog(dsqrt(1664.d0/27.d0)+181.d0/27.d0))
   * -dexp((1.d0/3.d0)*dlog(dsqrt(1664.d0/27.d0)-181.d0/27.d0))
   * -2.d0/3.d0)**2

c
open (unit=8,file='ineqs.txt')
write (8,1)
write (8,2)
write (8,3) n1,n2,n3,n4,n5
write (8,4) ep,eq
write (8,2)
write (8,10) c1,c2,c3,c4
write (8,2)

dx1=c1/dfloat(n1)
dx2=(c2-c1)/dfloat(n2)
dx3=(c3-c2)/dfloat(n3)
dx4=(c4-c3)/dfloat(n4)
dx5=(1.d0-c4)/dfloat(n5)
c
write (8,20) ep
write (8,30) fip(ep),ep,fkp(ep),flm(ep)
write (8,40) fjp(ep),srt(ep),flp(ep)
write (8,2)
x=0.d0
do 100 i=1,n1
x=x+dx1
write (8,20) x
write (8,30) fip(x),x,fkp(x),flm(x)
write (8,40) fjp(x),srt(x),flp(x)
100   continue
write (8,2)
x=c1
do 200 i=1,n2
x=x+dx2
write (8,20) x
write (8,50) x,fip(x),fkp(x),flm(x)
write (8,40) fjp(x),srt(x),flp(x)
200   continue
write (8,2)
x=c2
do 300 i=1,n3
x=x+dx3
write (8,20) x
write (8,60) x,fkp(x),flm(x),fip(x)
write (8,40) fjp(x),srt(x),flp(x)
300   continue
write (8,2)
x=c3
do 400 i=1,n4
x=x+dx4
write (8,20) x
write (8,70) x,fkp(x),srt(x),flm(x)
write (8,80) fjp(x),fip(x),flp(x)
400   continue
write (8,2)
x=c4
do 500 i=1,n5m
x=x+dx5
write (8,20) x
write (8,90) x,fkp(x),srt(x),flm(x)
write (8,80) fjp(x),fip(x),flp(x)
500   continue
write (8,2)
write (8,20) one
write (8,90) one,fkp(one),srt(one),flm(one)
write (8,80) fjp(one),fip(one),flp(one)
close (unit=8)
1     format (' ***** The output for the code "ineqs.for". **********')
2     format (' ******************************************************')
3     format (' n1 =',i3,' n2 =',i3,' n3 =',i3,' n4 =',i3,' n5 =',i3)
4     format (' ep =',gl14.7,' eq =',gl14.7)
10    format (' c1 =',gl14.7,' c2 =',gl14.7,' c3 =',gl14.7,' c4 =',gl14.7)
20    format (' x =',gl14.7)
30    format (' fip =',gl14.7,' x =',gl14.7,' fkp =',gl14.7,' flm =',gl14.7)
40 format (' fjp =',gl4.7,' srt =',gl4.7,' flp =',gl4.7)
50 format (' x =',gl4.7,' fip =',gl4.7,' fkp =',gl4.7,' flm =',gl4.7)
60 format (' x =',gl4.7,' fkp =',gl4.7,' flm =',gl4.7,' fip =',gl4.7)
70 format (' x =',gl4.7,' fkp =',gl4.7,' flm =',gl4.7,' srt =',gl4.7)
80 format (' fjp =',gl4.7,' fip =',gl4.7,' flp =',gl4.7)
90 format (' x =',gl4.7,' fkp =',gl4.7,' srt =',gl4.7,' flm =',gl4.7)
stop
end
***** The output for the code "ineqs.for".  ****************************

n1 = 17  n2 = 10  n3 = 12  n4 = 14  n5 = 47
ep = 0.1000000E-06  eq = 0.1000000E-11

-----------------------------------------------
c1 = 0.1715729     c2 = 0.2727273     c3 = 0.3940307     c4 = 0.5302216

-----------------------------------------------
x = 0.1000000E-06
fip = 0.7549517E-07  x = 0.1000000E-06  fkp = 0.9936496E-07  flm = 0.1154632E-06
fp = 0.1250000E-06  srt = 0.3162278E-03  flp = 0.8000000E+08

-----------------------------------------------
x = 0.1009252E-01
fip = 0.7685593E-02  x = 0.1009252E-01  fkp = 0.4199420E-01  flm = 0.5751692E-01
fjp = 0.1267669E-01  srt = 0.1420741     flp =  394.3102
fp = 0.1546633E-01  srt = 0.196.1206

-----------------------------------------------
x = 0.2018504E-01
fip = 0.1561019E-01  x = 0.2018504E-01  fkp = 0.2059214E-01  flm = 0.2282467E-01
fjp = 0.2547771E-01  srt = 0.1420741     flp =  394.3102
fp = 0.3127491E-01  srt = 0.196.1206

-----------------------------------------------
x = 0.3027757E-01
fip = 0.2378408E-01  x = 0.3027757E-01  fkp = 0.3119254E-01  flm = 0.3432655E-01
fjp = 0.3840651E-01  srt = 0.1740045     flp =  262.1877
fp = 0.4037009E-01  srt = 0.2246388

-----------------------------------------------
x = 0.4037009E-01
fip = 0.3221806E-01  x = 0.4037009E-01  fkp = 0.4199420E-01  flm = 0.4589012E-01
fjp = 0.3840651E-01  srt = 0.2246388
fp = 0.5046261E-01  srt = 0.2246388

-----------------------------------------------
x = 0.5046261E-01
fip = 0.4023491E-01  x = 0.5046261E-01  fkp = 0.5299516E-01  flm = 0.5751692E-01
fjp = 0.4646170E-01  srt = 0.2246388
fp = 0.6055513E-01  srt = 0.2246388

-----------------------------------------------
x = 0.6055513E-01
fip = 0.4991230E-01  x = 0.6055513E-01  fkp = 0.6419281E-01  flm = 0.6920851E-01
fjp = 0.5750954E-01  srt = 0.2460795
fp = 0.7064765E-01  srt = 0.2460795

-----------------------------------------------
x = 0.7064765E-01
fip = 0.5919705E-01  x = 0.7064765E-01  fkp = 0.7558387E-01  flm = 0.8096653E-01
fjp = 0.7779544E-01  srt = 0.2657963
fp = 0.8790909E-01  srt = 0.2657963

-----------------------------------------------
x = 0.8790909E-01
fip = 0.6879090E-01  x = 0.8790909E-01  fkp = 0.8716441E-01  flm = 0.9279262E-01
fjp = 0.1050943     srt = 0.2841482
fp = 0.9083270E-01  srt = 0.2841482

-----------------------------------------------
x = 0.9083270E-01
fip = 0.7870770E-01  x = 0.9083270E-01  fkp = 0.892977E-01  flm = 0.7046885
fjp = 0.1188673     srt = 0.3103846
fp = 0.1009252

-----------------------------------------------
x = 0.1009252
fip = 0.8896199E-01  x = 0.1009252  fkp = 0.1108746  flm = 0.1166559
fjp = 0.1327948     srt = 0.2460795
fp = 0.1110177

-----------------------------------------------
x = 0.1110177
fip = 0.9956905E-01  x = 0.1110177  fkp = 0.1229927  flm = 0.1286967
fjp = 0.1468811     srt = 0.3331933
fp = 0.1211103

-----------------------------------------------
x = 0.1211103
fip = 0.1105449  x = 0.1211103  fkp = 0.1352774  flm = 0.1408127
fjp = 0.1611304     srt = 0.3480090
fp = 0.1312028

-----------------------------------------------
x = 0.1312028
fip = 0.1219063  x = 0.1312028  fkp = 0.1477212  flm = 0.1530058
fjp = 0.1753472     srt = 0.3622193
fp = 0.1412953

-----------------------------------------------
x = 0.1412953
fip = 0.1336709  x = 0.1412953  fkp = 0.1603157  flm = 0.1652779
fjp = 0.1901361     srt = 0.3758927
fp = 0.1513878

-----------------------------------------------
x = 0.1513878
fip = 0.1458572  x = 0.1513878  fkp = 0.1730522  flm = 0.1776312
fjp = 0.2049017     srt = 0.3890859
fp = 0.1614804

-----------------------------------------------
x = 0.1614804
fip = 0.1584845  x = 0.1614804  fkp = 0.1859211  flm = 0.1900676
fjp = 0.2198489     srt = 0.4018462
fp = 0.1715729

-----------------------------------------------
x = 0.1715729
fip = 0.1715729  x = 0.1715729  fkp = 0.1989124  flm = 0.2025894
fjp = 0.2349824     srt = 0.4142136
fp = 0.442483
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<td></td>
</tr>
</tbody>
</table>

**ineqs.txt**
ineqs.txt

\[ f_{jp} = 0.6068962 \quad s_{rt} = 0.6196145 \quad f_{lp} = 18.34702 \]
\[ x = 0.3940307 \]
\[ f_{jp} = 0.4925164 \quad f_{lp} = 0.5056950 \quad f_{ip} = 0.6277187 \]
\[ s_{rt} = 0.6277187 \]
\[ x = 0.4037587 \]
\[ f_{kp} = 0.5046894 \quad f_{lm} = 0.5204302 \quad s_{rt} = 0.6354201 \]
\[ f_{jp} = 0.6480814 \quad f_{ip} = 0.6878137 \quad f_{lp} = 17.79729 \]
\[ x = 0.4134866 \]
\[ f_{kp} = 0.5167423 \quad f_{lm} = 0.5353209 \quad s_{rt} = 0.653348 \]
\[ f_{jp} = 0.6687705 \quad f_{ip} = 0.7192926 \quad f_{lp} = 16.81234 \]
\[ x = 0.4232145 \]
\[ f_{kp} = 0.5286703 \quad f_{lm} = 0.5503721 \quad s_{rt} = 0.659351 \]
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\[ x = 0.4329424 \]
\[ f_{kp} = 0.5404690 \quad f_{lm} = 0.5655889 \quad s_{rt} = 0.6653348 \]
\[ f_{jp} = 0.7111613 \quad f_{ip} = 0.7851000 \quad f_{lp} = 15.91262 \]
\[ x = 0.4426703 \]
\[ f_{kp} = 0.5521347 \quad f_{lm} = 0.5809764 \quad s_{rt} = 0.6726056 \]
\[ f_{jp} = 0.7328803 \quad f_{ip} = 0.8194089 \quad f_{lp} = 15.08699 \]
\[ x = 0.4523983 \]
\[ f_{kp} = 0.5636642 \quad f_{lm} = 0.6065403 \quad s_{rt} = 0.6797986 \]
\[ f_{jp} = 0.7549601 \quad f_{ip} = 0.8546332 \quad f_{lp} = 14.69900 \]
\[ x = 0.4621262 \]
\[ f_{kp} = 0.5750545 \quad f_{lm} = 0.6228265 \quad s_{rt} = 0.6869164 \]
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\[ f_{kp} = 0.5863032 \quad f_{lm} = 0.6422209 \quad s_{rt} = 0.6939611 \]
\[ f_{jp} = 0.8002391 \quad f_{ip} = 0.8907504 \quad f_{lp} = 13.96757 \]
\[ x = 0.4815820 \]
\[ f_{kp} = 0.5974081 \quad f_{lm} = 0.6606804 \quad s_{rt} = 0.7009351 \]
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\[ f_{jp} = 0.8711002 \quad f_{ip} = 1.166769 \quad f_{lp} = 12.96878 \]
\[ x = 0.5107658 \]
\[ f_{kp} = 0.6298458 \quad f_{lm} = 0.6939742 \quad s_{rt} = 0.7214525 \]
\[ f_{jp} = 0.8955453 \quad f_{ip} = 1.043517 \quad f_{lp} = 12.65907 \]
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\[ x = 0.5302216 \]
\[ f_{kp} = 0.6507306 \quad f_{lm} = 0.7281632 \quad s_{rt} = 0.7281632 \]
\[ f_{jp} = 0.9457355 \quad f_{ip} = 1.124304 \quad f_{lp} = 12.06277 \]
\[ x = 0.5402169 \]
\[ f_{kp} = 0.6612284 \quad s_{rt} = 0.7349945 \quad f_{lp} = 0.7460974 \]
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\[ x = 0.5502122 \]
\[ f_{kp} = 0.6715692 \quad s_{rt} = 0.7417629 \quad f_{lp} = 0.7642956 \]
\[ f_{jp} = 0.9991881 \quad f_{ip} = 1.209809 \quad f_{lp} = 11.77555 \]
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\[ f_{jp} = 1.026661 \quad f_{ip} = 1.253358 \quad f_{lp} = 11.49765 \]
\[ x = 0.5702028 \]
\[ f_{kp} = 0.6917814 \quad s_{rt} = 0.7551177 \quad f_{lp} = 0.8015269 \]
\[ f_{jp} = 1.054648 \quad f_{ip} = 1.297353 \quad f_{lp} = 11.22857 \]
\[ x = 0.5801981 \]
\[ f_{kp} = 0.7016542 \quad s_{rt} = 0.7617073 \quad f_{lp} = 0.8205830 \]
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ineqs.txt
fjp =  3.000000  fip =  3.000000  flp =  3.000005
\end{verbatim}
Appendix B. Numerical Validation of Theorem 35
implicit real*8(a-h,o-z)
complex*16 a,b,c,cdsqrt,x1,x2,x3,dcmplx

Program "ctausc.for".

This program is used to verify numerically the assertion made in part G of Theorem 35.

The critical value of tau, i.e., tauo(nu**2), is evaluated for each given value of nu (the Courant number).

Given any (nu,tau), the spectral radius of the amplification matrix is a function of the phase angle theta. The least upper bound (denoted by "am") of the spectral radii over the range -pi < theta .le. pi is evaluated for each given (nu,tau).

When nu is replaced by -nu, each of the two resulting amplification factors (defined in Eq. (4.7)) becomes the complex conjugate of that before sign-change. Thus the spectral radius does not change as nu is replaced by -nu. For this reason, the range of nu can be limited to nu. ge. 0.

When theta is replaced by -theta, each of the two resulting amplification factors also becomes the complex conjugate of that before sign-change. Thus the range of theta can be limited to 0 .le. theta .le. pi.

Theorems 16 and 18 imply that the least upper bound am = 1 if nu = 1 and tau .ge. 1 (Note: According to Eq. (4.7), the value of the principal amplification factor = 1 when theta = 0. Thus am .ge. 1 for any (nu,tau). In turn, this implies that am = 1 for any (nu,tau) which meets the condition Eq. (4.2)). Moreover, Theorems 6 and 12 imply that am > 1 if nu > 1 regardless the value assumed by tau. Thus numerical results may not be consistent with theoretical predictions at the singular case nu = 1 if round-off errors are not controlled carefully. For this reason, a statement "if (dabs(x-1.d0).lt.ep) x=1.d0" is added in the code to insure that the value of x is really "1" as intended. Here ep (>0) is an input parameter and assumes to be very small.

x = nu.

z = The phase angle theta of a Fourier component.

nx = number of the values of nu.

tau = number of the values of tau with tau>tauo (tau<tauo) for each value of nu. Here tauo is the critical value of tau associated with a given value of nu. Because the case with tau=tauo is always considered, there are (2*nt+1) values of tau associated with each value of nu, i.e.,
tauo*(1-dt*nt), tauo*(1-dt*(nt-1)), ..., tauo*(1-dt), tauo, tauo*(1+dt), ..., tauo*(1+dt*(nt-1)), tauo*(1+dt*n).

nz = number of the intervals over the domain 0 .le. theta .le. pi.

xs = The initial value of nu.

fkp(s) = (s-1.d0+dsqrt(5.d0*s**2-2.d0*s+1.d0))/(2.d0*s)
\[
flm(s) = \frac{(4.0\cdot s - 2.0\cdot s^2 + \sqrt{2.0\cdot (2.0\cdot s - s^2)})}{s}
\]

\[\pi = 3.1415926535897932d0\]
\[\text{nx} = 25\]
\[\text{nt} = 5\]
\[\text{nz} = 1000\]
\[\text{xs} = 0.0d0\]
\[\text{dx} = 5.0d-2\]
\[\text{dt} = 1.0d-4\]
\[\text{ep} = 1.0d-7\]
\[c_2 = \frac{3.0d0}{11.0d0}\]
\[dz = \pi / \text{dfloat(nz)}\]
\[x = \text{xs}\times\text{dx}\]
\[\text{nzp} = \text{nz}\times1\]
\[\text{ts} = 1.0d0 - \text{dt}\times\text{dfloat(nt+1)}\]
\[\text{nt2p} = \text{nt}\times2\times1\]

```
c
open (unit=8, file='ctausc.txt')
write (8,10)
write (8,15)
write (8,20) nx,nt,nz
write (8,30) xs,dx,dt,ep
write (8,15)
do 200 i = 1,nx
  x = x+dx
  if (dabs(x-1.0d0) .lt. ep) x = 1.0d0
  xx = x**2
  if (xx .eq. 0.0d0) tauo = 0.0d0
  if (xx .gt. 0.0d0 .and. xx .le. c2) tauo = flm(xx)
  if (xx .gt. c2) tauo = fkp(xx)
  tau = tauo\times ts
dtau = tauo\times dt
do 200 j = 1,nt2p
tau = tau+dtau
  am = 0.0d0
  z = -dz
  do 100 k = 1,nzp
    z = z+dz
    z1 = dcos(z/2.0d0)
    z2 = dsin(z/2.0d0)
    ar = 1.0d0+tau
    ai = 0.0d0
    br = -2.0d0\times tau\times z1
    bi = x\times (3.0d0+tau)\times z2
    cr = -((1.0d0 - tau)\times z1**2 + (1.0d0 + x**2)\times z2**2)
    ci = -x\times (1.0d0 + tau)\times z1\times z2
    a = dcplx(ar,ai)
b = dcplx(br,bi)
c = dcplx(cr,ci)
x1 = (-b + cdsqrt(b**2 - 4.0d0\times a\times c))/(2.0d0\times a)
x2 = (-b - cdsqrt(b**2 - 4.0d0\times a\times c))/(2.0d0\times a)
a1 = cdabs(x1)
a2 = cdabs(x2)
am = dmax1(a1,a2,am)
100 continue
write (8,40) x,tauo,tau,am
200 continue
close (unit=8)
10 format ('' ***** The output for the code "ctausc.for". *****'')
15 format ('' ********************************************'')
```
20 format ("nx =",i4,"nt =",i4,"nz =",i4)
30 format ("xs =",g14.7,"dx =",g14.7,"dt =",g14.7,"ep =",g14.7)
40 format ("nu =",g11.4,"tauo =",g14.7,"tau =",g14.7,
   * "am =",g21.14)
   stop
   end
ctauscBText.txt

***** The output for the code "ctausc.for". ******
******************************************************************************
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xs = 0.000000 dx = 0.5000000E-01 dt = 0.1000000E-03 ep = 0.1000000E-06
******************************************************************************
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Explicit Von Neumann Stability Conditions for the c-τ Scheme—A Basic Scheme in the Development of the CE-SE Courant Number Insensitive Schemes

Sin-Chung Chang

National Aeronautics and Space Administration
John H. Glenn Research Center at Lewis Field
Cleveland, Ohio 44135–3191

As part of the continuous development of the space-time conservation element and solution element (CE-SE) method, recently a set of so called “Courant number insensitive schemes” has been proposed. The key advantage of these new schemes is that the numerical dissipation associated with them generally does not increase as the Courant number decreases. As such, they can be applied to problems with large Courant number disparities (such as what commonly occurs in Navier-Stokes problems) without incurring excessive numerical dissipation.