A Sub-filter Scale Noise Equation for Hybrid LES Simulations

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Hybrid LES/subscale modeling approaches have an important advantage over the current noise prediction methods in that they only involve modeling of the relatively universal subscale motion and not the configuration dependent larger scale turbulence . Previous hybrid approaches use approximate statistical techniques or extrapolation methods to obtain the requisite information about the sub-filter scale motion. An alternative approach would be to adopt the modeling techniques used in the current noise prediction methods and determine the unknown stresses from experimental data. The present paper derives an equation for predicting the sub scale sound from information that can be obtained with currently available experimental procedures. The resulting prediction method would then be intermediate between the current noise prediction codes and previously proposed hybrid techniques.

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1. Introduction

Crighton (1979), in a now classical paper, makes two major criticisms of the Lilley (1974) equation approach-- which effectively treats the sound propagation as a small perturbation of a sheared mean flow. The first of these is that it provides no rational means of excluding the instability waves that eventually become infinite in any causal solution to the problem. This issue was eventually resolved by Goldstein and Leib (2005) and Goldstein and Handler (2003).The other criticism, which is dealt with in the present paper, is that the evolution time of the turbulent eddies is much greater than the time it takes a sound wave to traverse them-implying that the turbulence is effectively frozen during the passage of the sound and that there is, therefore, no particular mean flow profile that would give a true representation of the refraction effects that occur in the real flow.

This latter difficulty would not occur if the sound field were determined from a DNS (Freund, 2001) or LES solution. Unfortunately, the computer resources needed to resolve all the relevant length scales are enormous—often well beyond those available on present day machines. A reasonable compromise would be to use LES simulations with very broad filter widths. This approach was

adopted by Bastin, Lafon and Candel (1997), Bodony and Lele (2002, 2003, 2005), and others. They found that simulations of this type are quite accurate at low frequencies but significantly under predict the high frequency component of the spectrum—presumably because they do not account for the sound generated by the sub-filter scale turbulence. Bodony and Lele (2002), attempt to resolve these difficulties by adopting a "hybrid" approach that models the non-linear Reynolds stress terms in an appropriate equation for the sub-scale motion and then uses the result to calculate the missing sub-scale sound—which can be added to the LES sound field. The great attraction of this approach is that it directly calculates the configuration dependent component of the sound while modeling only the relatively universal small scale motion.

The relevant equations are obtained by dividing each of the flow variables in the Navier-Stokes equation into a component that is determined from the LES solution and a residual component that satisfies the Navier-Stokes equations with the LES equations subtracted out. By introducing new (in general non-linear) dependent variables (Goldstein, 2002, 2003) the latter equations can be rewritten in the form of the linearized Euler equations with sources that are the same as those that would be produced by external stress and heat flux perturbations. The corresponding source strengths, which depend on the non-linear sub-scale stresses, can, in principle, be modeled and the resulting linear equations can then be solved by using a Greens' function approach. The result can then be used to obtain an expression for the far field pressure autocovariance in terms of the sub-scale turbulent motion. Unfortunately it is very difficult to model the *instantaneous* subscale stresses that appear in this result.

Bodony (2004), in his outstanding Ph.D. thesis, uses an approximate deconvolution method to kinematically extrapolate information about the filtered scales to obtain the required information about the sub-filter scale motion. But, as Bodony (2004) points out, this is still too expensive to be "practical in an industrial noise prediction setting". A more practical approach might be to extend the modeling approach used in current noise prediction methods such as the MGB (Balsa, Gliebe, Kantola, Mani, Strings and Wong, 1978) and JeNo (Khavaran, and Bridges, 2004) codes that determine the unknown stresses from experimental data. The hope is that the sub-filter stresses will be much more universal than the large scale stresses that have to be modeled with these current methods and that models developed from any flow will apply to any other. This approach would be intermediate between the existing hybrid approaches of Bastin, et al.(1997), Bodony and Lele (2003) and Bodony (2004) and the current (i.e., MGB and JeNo) noise prediction methods.

But most experimental reports only document the reproducible (i.e., non-random) characteristics of these sources, such as their lower order statistics. It is therefore important to derive a formula for the subscale sound field that depends only on non-random quantities the can be measured with currently available experimental techniques. Bodony (2004) and Bodony and Lele (2003) obtain

such an equation by dividing the flow into steady and fluctuating components, deriving an inhomogeneous wave equation for the latter and then further sub-dividing the unsteady Reynolds stress source terms in that equation into filtered and subscale components- with the latter part being inputted through an appropriate model. Unfortunately this approach does not *explicitly* account for the scattering of the subscale sound by the filtered scale turbulence and therefore does not overcome Crighton's (1979) second criticism of the Lilley equation approach. This is analogous to the arguments that eventually led to the replacement of the original Lighthill (1952,1954) approach with the Lilley (1974) equation formalism. While Lighthill's equation is exact with all real flow effects, including the mean flow interaction effects, contained in the source term, it tuned out to be almost impossible to model these weak but non-local effects from experimental measurements. This should also true for the non-local scattering effects and the same arguments would dictate that they be included as part of the propagation effects.

The present paper (which is base on the conceptually simpler double decomposition formulation) is an attempt to eliminate this problem by exploiting the statistically independence (i.e., the decoupling) of the filtered motion in an LES simulation from the detailed sub-scale fluctuations. The only coupling that can occur in these simulations is through the filtered Reynolds stress—which is eventually modeled in terms of the filtered variables and their derivatives. The LES solution is therefore calculated from a closed set of equations that only involve the filtered variables. In the "hybrid" approach, the deviation of the sub-scale stresses from the filtered stresses, say $\sigma'_{\mu j}$, also has to be modeled. The present approach assumes that the model will be constructed from spatially filtered experimental data (which can be obtained from PIV measurements). Unlike the LES simulation, the instantaneous random subscale fluctuations in the experiment will eventually produce $O(1)$ changes in the large scale motion (due to the chaotic nature of the solutions to the N-S equations) and the entire experimental flow should therefore be uncorrelated (in the statistical sense that covariances of various quantities will vanish) with the LES simulation where the randomness comes from the random initial conditions-- which are themselves uncorrelated with the initial conditions in the experiment (Lesier, 1987, p. 230 and Pope, 2000, pp.612-613). It therefore seems reasonable to model $\sigma'_{\mu j}$ as if it were statistically independent of the LES solution. Notice that this does not imply that the expectation or average value of this quantity should be independent of the LES solution.

The main purpose of this note is to show that this type of modeling leads to an expression for the far field pressure autocovariance that depends only on the overall correlation of the subscale stresses, which is a non-random quantity of the type that is usually reported in the experimental literature and not on their instantaneous values, which is not. The present approach also provides a framework for the systematic introduction of additional approximations to further simplify the results—with the most significant of these arising from the near

isotropy of the subscale turbulence. The most general subscale turbulence correlation tensor has 45 independent components, but the isotropy assumption reduces that to 7. And if retarded time variations (i.e., variations of the propagation factor over the correlation volume of the turbulence) are also neglected, only two of those actually enter the formula for the far field pressure autocovariance. It would be very difficult to argue for the neglect of retarded time effects if the non local effects had to be included as part of the source term.

The relevant sub-filter scale acoustic equations are introduced in section 2 and a formal Greens' function solution is obtained in section 3. This solution is then used (in section 4) to derive a formula for the acoustic pressure autocovariance and the statistical independence assumption/model is then used to eliminate the dependence on the instantaneous subscale fluctuations and thereby obtain a formula that involves only commonly reported statistical quantities. The result is simplified by neglecting retarded time variations and an isotropic turbulence model is introduced in section 5 to simplify it even further.

2. Sub-filter scale equations in vector form

The pressure p, density ρ and velocity v_i in any flow can be decomposed into filtered $\hat{\rho}, \hat{p}$ and Favre (1969) filtered $\tilde{v}_i \equiv \widehat{\rho v_i} / \hat{\rho}$ components that satisfy the LES equations and residual components

$$\rho' = \rho - \hat{\rho}, \quad v_i' = v_i - \tilde{v}_i, \quad p' = p - \hat{p} \tag{1.1}$$

that satisfy the 5 formally linear equations (Goldstein, 2002, 2003)

$$L_{\mu\nu} u_\nu = s_\mu \quad \text{for} \quad \mu, \nu = 1,2,3,4,5 \tag{1.2}$$

where

$$\{u_\nu\} \equiv \{\rho v_i', \pi', \rho'\} \equiv \{\rho v_1', \rho v_2', \rho v_3', \pi', \rho'\} \qquad \pi' \equiv p' - \frac{(\gamma - 1)}{2} \sigma_{jj}' \tag{1.3}$$

and the first order linear operator $L_{\mu\nu}$ is defined by

$$L_{\mu\nu} \equiv \delta_{\mu\nu} \tilde{D}_0 + \delta_{\nu 4} \partial_\mu + \partial_\nu \left(\widetilde{c^2} \delta_{\mu 4} + \delta_{\mu 5} \right) + K_{\mu\nu} \tag{1.4}$$

with

$$K_{\mu\nu} \equiv \partial_\nu \tilde{v}_\mu - \frac{1}{\bar{\rho}} \frac{\partial \tilde{\theta}_{\mu j}}{\partial x_j} \delta_{\nu 5} - (\gamma - 1) \left(\frac{\partial \tilde{v}_j}{\partial x_j} \delta_{\nu 4} - \frac{1}{\bar{\rho}} \frac{\partial \tilde{\theta}_{\nu j}}{\partial x_j} \right) \delta_{\mu 4} \quad i, j = 1,2,3; \tag{1.5}$$

and

$$\tilde{D}_0 \equiv \frac{\partial}{\partial t} + \frac{\partial}{\partial x_j} \tilde{v}_j, \quad \tilde{\theta}_{ij} \equiv \delta_{ij} \bar{p} - \tilde{\sigma}_{ij}, \quad \partial_\mu \equiv \frac{\partial}{\partial x_j}, \quad \mu = j = 1,2,3;$$
$$\tilde{v}_\mu, \tilde{\theta}_{\mu j}, \partial_\mu = 0 \; otherwise \tag{1.6}$$

The source function s_μ is defined by

$$s_\mu \equiv \frac{\partial e_{\mu j}}{\partial x_j} + \delta_{\mu 4} (\gamma - 1) e_{ij} \frac{\partial \tilde{v}_i}{\partial x_j} \tag{1.7}$$

with

$$e_{\mu j} = \sigma'_{\mu j} - \frac{(\gamma - 1)}{2} \delta_{\mu j} \sigma'_{kk} \tag{1.8}$$

$$\sigma'_{\mu j} \equiv -\left(\rho v'_\mu v'_j + \tilde{\sigma}_{\mu j} \right), \qquad v'_4 \equiv -(\gamma - 1) h'_0 \tag{1.9}$$

$$\tilde{\sigma}_{ij} \equiv -\bar{\rho} \left(\widetilde{v_i v_j} - \tilde{v}_i \tilde{v}_j \right), \quad \tilde{\sigma}_{4j} \equiv -(\gamma - 1) \left[\bar{\rho} \left(\widetilde{h_0 v_j} - \tilde{h}_0 \tilde{v}_j \right) + \frac{1}{2} \tilde{\sigma}_{ii} \tilde{v}_j + \tilde{\sigma}_{ij} \tilde{v}_i \right] \tag{1.10}$$

and

$$h_0 \equiv h + \frac{1}{2} v^2, \qquad h'_0 \equiv h' + \frac{1}{2} v'^2 \tag{1.11}$$

where $\widetilde{c^2} = \gamma R \tilde{T}$ is the Favre filtered square sound speed and, as usual, the viscous terms, which are believed to play an insignificant role in the sound generation process, have been neglected.

3. Vector Greens' function solution

These equations can be solved in terms of the vector Greens' function $g_{\nu\sigma}(x,t|y,\tau)$ which satisfies the inhomogeneous linear equations

$$\left(L_{\mu\nu} \right)_{x,t} g_{\nu\sigma}(x,t|y,\tau) = \delta_{\mu\sigma} \delta(x - y) \delta(t - \tau) \tag{2.1}$$

with delta-function type source term together with the causality condition

$$g_{\nu\mu}\left(x,t\middle|y,\tau\right)=0 \qquad for \quad t<\tau \tag{2.2}$$

to obtain

$$u_\nu\left(x,t\right)=\int_V\int_{-\infty}^{\infty} g_{\nu\mu}\left(x,t\middle|y,\tau\right)s_\mu\left(y,\tau\right)dyd\tau=-\int_V\int_{-\infty}^{\infty}\tilde{\gamma}_{\nu j\mu}\left(x,t\middle|y,\tau\right)e_{\mu j}\left(y,\tau\right)dyd\tau \tag{2.3}$$

where

$$\tilde{\gamma}_{\nu j\mu}\left(x,t\middle|y,\tau\right)\equiv\frac{\partial g_{\nu\mu}\left(x,t\middle|y,\tau\right)}{\partial y_j}-\left(\gamma-1\right)\frac{\partial\tilde{v}_\mu}{\partial y_j}g_{\nu 4}\left(x,t\middle|y,\tau\right) \tag{2.4}$$

From the acoustics perspective the primary interest is in the 4th (i.e., the pressure-like) component of (3.3) which only involves the 4th component vector Greens' function $g_{4\nu}\left(x,t\middle|y,\tau\right)$. The latter quantity can, in principle, be found by solving be solving the system (3.1). But since this consists of 15 first order equations, it turns out to be simpler to compute the adjoint Greens' function $g^a{}_{\nu 4}\left(y,\tau\middle|x,t\right)$, which was introduced into the Aeroacoustics literature by Tam and Auriault (1998) to study sound propagation through steady flows and used effectively by Bodony (2004) to implement his hybrid approach. Equations (A-1) to (A-4) show that it is related to $g_{4\nu}\left(x,t\middle|y,\tau\right)$ by the reciprocity relation

$$g^a{}_{\nu 4}\left(y,\tau\middle|x,t\right)=g_{4\nu}\left(x,t\middle|y,\tau\right) \tag{2.5}$$

and satisfies the system

$$\left(L^a_{\mu\nu}\right)_{y,\tau}\; g^a_{\nu 4}\left(y,\tau\middle|x,t\right)=\delta_{\mu 4}\delta\left(x-y\right)\delta\left(t-\tau\right) \tag{2.6}$$

of five first order equations which are given in component form by (A-5a,b,c). These equations suggest that $\breve{g}_{\nu 4}\left(y\middle|x;\tau,t-\tau\right)\equiv g^a_{\nu 4}\left(y,\tau\middle|x,t\right)$ will be a stationary random function of its third argument, τ, which results from the time dependence of the coefficients in $\left(L^a_{\mu\nu}\right)_{y,\tau}$ that are determined from the LES solution and are,

therefore, random functions of time. (Notice that this argument would not be needed if the LES solution were steady.)

It therefore follows from (2.7), (2.8), (3.4) and (3.5) that the 4th component of (3.3) can be written as

$$\pi'(\boldsymbol{x},t) = -\int_V \int_{-\infty}^{\infty} \gamma_{j\mu}\left(\boldsymbol{y}|\boldsymbol{x}:\tau,t-\tau\right)\sigma'_{\mu j}\left(\boldsymbol{y},\tau\right)d\boldsymbol{y}d\tau \tag{2.7}$$

where like, $\breve{g}_{v4}\left(\boldsymbol{y}|\boldsymbol{x}:\tau,t-\tau\right) \equiv g^a_{v4}\left(\boldsymbol{y},\tau|\boldsymbol{x},t\right)$, the propagation factor

$$\gamma_{j\mu}\left(\boldsymbol{y}|\boldsymbol{x}:\tau,t-\tau\right) \equiv -\lambda_{j\mu}\left(\boldsymbol{y}|\boldsymbol{x}:\tau,t-\tau\right) + \frac{(\gamma-1)}{2}\delta_{j\mu}\lambda_{kk}\left(\boldsymbol{y}|\boldsymbol{x}:\tau,t-\tau\right) \tag{2.8}$$

along with

$$\lambda_{j\mu}\left(\boldsymbol{y}|\boldsymbol{x}:\tau,t-\tau\right) \equiv \begin{cases} \lambda_{jk}\left(\boldsymbol{y}|\boldsymbol{x}:\tau,t-\tau\right) & \mu = k = 1,2,3 \\ \dfrac{\partial g^a_{44}\left(\boldsymbol{y},\tau|\boldsymbol{x},t\right)}{\partial y_j} & \mu = 4 \end{cases} \tag{2.9}$$

and the six independent components

$$\lambda_{jk}\left(\boldsymbol{y}|\boldsymbol{x}:\tau,t-\tau\right) \equiv \frac{1}{2}\left[\frac{\partial g^a_{4k}\left(\boldsymbol{y},\tau|\boldsymbol{x},t\right)}{\partial y_j} + \frac{\partial g^a_{4j}\left(\boldsymbol{y},\tau|\boldsymbol{x},t\right)}{\partial y_k}\right] - \frac{(\gamma-1)}{2}\left(\frac{\partial \tilde{v}_k}{\partial y_j} + \frac{\partial \tilde{v}_j}{\partial y_k}\right) g^a_{44}\left(\boldsymbol{y},\tau|\boldsymbol{x},t\right) \tag{2.10}$$

of $\lambda_{jk}\left(\boldsymbol{y}|\boldsymbol{x}:\tau,t-\tau\right)$ should be random functions of their third arguments (which arise from the third argument of $\breve{g}_{v4}\left(\boldsymbol{y}|\boldsymbol{x}:\tau,t-\tau\right)$ as well as the τ-dependence of $\partial\tilde{v}_k / \partial y_j$).

4. Pressure autocovariance

Since $\pi'(x,t) \to p'(x,t) \quad as\, x \to \infty$, the far field pressure autocovariance

$$\overline{\Pi(x,\tau)} \equiv \lim_{x\to\infty}\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} p'(x,t)p'(x,t+\tau)\,dt\right] \tag{3.1}$$

becomes

$$\begin{aligned}\overline{\Pi} &= \frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty}\iiint_V \Gamma_{i\sigma j\mu}\left(y,\eta \mid x:t',t'',\tau+t''-t'\right)\sigma'_{\sigma i}(y,t')\,\sigma'_{\mu j}(y+\eta,t'')\,dy\,d\eta\,dt'\,dt'' \\ &= \frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty}\iiint_V \Gamma_{i\sigma j\mu}\left(y,\eta \mid x:t',t''+t',\tau+t''\right)\sigma'_{\sigma i}(y,t')\,\sigma'_{\mu j}(y+\eta,t''+t')\,dy\,d\eta\,dt'\,dt''\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}\Gamma_{j\sigma l\mu}\left(y,\eta \mid x:t',t'',\tau+t''-t'\right) &\equiv \int_{-\infty}^{\infty}\gamma_{j\sigma}\left(y|x:t',t+\tau-t'\right)\gamma_{l\mu}\left(y+\eta|x:t'',t-t''\right)dt \\ &= \int_{-\infty}^{\infty}\gamma_{j\sigma}\left(y|x:t',t+\tau+t''-t'\right)\gamma_{l\mu}\left(y+\eta|x:t'',t\right)dt\end{aligned} \tag{3.3}$$

is a stationary random function of its fourth and fifth arguments.
$\Gamma_{i\sigma j\mu}\left(y,\eta \mid x:t',t''+t',\tau+t''\right)$ and

$$R_{\sigma i\mu j}(y;\eta,t'',t') \equiv \sigma'_{\sigma i}(y,t')\,\sigma'_{\mu j}(y+\eta,t''+t') \tag{3.4}$$

are, therefore, both stationary random functions of t'. They should be considered to be uncorrelated in the present approach because, as note in the Introduction, the filtered motion, which is determined from an LES solution to a closed set of equations that only involve the filtered variables, should be statistically independent of (i.e., decoupled from) the detailed sub-scale motion determined from actual experimental data because, unlike the LES simulation, the experimental subscale fluctuations eventually produce $O(1)$ changes in the large scale motion (due to the chaotic nature of the solutions to the N-S equations). This causes the entire experimental flow to be uncorrelated with the LES simulation where the randomness comes from the random initial conditions--which are themselves uncorrelated with the initial conditions in the experiment (Lesier, 1987, p. 230 and Pope, 2000, pp.612-613). The only coupling that can

occur is through the filtered Reynolds stress—which is eventually modeled in the LES simulation. It is therefore reasonable to require that any model used to represent the deviation $\sigma'_{\mu j} \equiv -\left(\rho v'_\mu v'_j + \tilde{\sigma}_{\mu j}\right)$ of the sub-filter Reynolds stresses from the filtered stresses $\tilde{\sigma}_{\mu j}$ be statistically independent of the filtered motion (and consequently with the coefficients of $L^a_{\mu\nu}$). This means that the covariance

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Gamma''_{i\sigma j\mu}\left(y,\eta \mid x:t',t''+t',\tau+t''\right) R''_{\sigma i\mu j}\left(y;\eta,t'',t'\right) dt'$$

where

$$\begin{aligned} \Gamma''_{i\sigma j\mu}\left(y,\eta \mid x:t',t''+t',\tau+t''\right) \equiv \\ \Gamma_{i\sigma j\mu}\left(y,\eta \mid x:t',t''+t',\tau+t''\right) - \overline{\Gamma}_{i\sigma j\mu}\left(y,\eta \mid x:t'',\tau+t''\right) \end{aligned} \tag{3.5}$$

$$R''_{\sigma i\mu j}\left(y;\eta,t'',t'\right) \equiv R_{\sigma i\mu j}\left(y;\eta,t'',t'\right) - \overline{R}_{\sigma i\mu j}\left(y;\eta,t''\right) \tag{3.6}$$

$$\overline{\Gamma}_{i\sigma j\mu}\left(y,\eta \mid x:t'',\tau+t''\right) \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Gamma_{i\sigma j\mu}\left(y,\eta \mid x:t',t''+t',\tau+t''\right) dt' \tag{3.7}$$

and

$$\overline{R}_{\sigma i\mu j}\left(y;\eta,t''\right) \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{\sigma i\mu j}\left(y;\eta,t'',t'\right) dt' \tag{3.8}$$

should equal zero. It therefore, follows that $\overline{\Pi(x,\tau)}$ can be written as an integral

$$\overline{\Pi(x,\tau)} = \int_{-\infty}^{\infty} \iiint_V \overline{\Gamma}_{i\sigma j\mu}\left(y,\eta \mid x:t'',\tau+t''\right) \overline{R}_{\sigma i\mu j}\left(y;\eta,t''\right) dy\, d\eta\, dt'' \tag{3.9}$$

over the product of two non-random functions.

The first of these, which can be expressed as the integral

$$\overline{\Gamma}_{i\sigma j\mu}\left(y,\eta \mid x:t'',\tau+t''\right) = \int_{-\infty}^{\infty} \overline{\gamma}_{i\sigma j\mu}\left(y,\eta \mid x:t'',t+\tau+t'',t\right) dt \qquad (3.10)$$

over time of the correlation

$$\overline{\gamma}_{i\sigma j\mu}\left(y,\eta \mid x:t'',t+\tau+t'',t\right) \equiv \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\gamma_{i\sigma}\left(y\middle|x:t',t+\tau+t''\right)\gamma_{l\mu}\left(y+\eta\middle|x:t'+t'',t\right)dt' \qquad (3.11)$$

of the two random propagation factors $\gamma_{i\sigma}\left(y\middle|x:t',t+\tau+t''-t'\right)$ and $\gamma_{l\mu}\left(y+\eta\middle|x:t'',t\right)$, can be thought of as an expected or mean propagation factor. It can be determined as part of the LES computation. The second factor, which, can be thought of as a source function, describes the low order statistics of the unknown component of the sub-scale fluctuations. It has to be modeled, *but it should be much easier to do this than to model the instantaneous values of the sub-scale fluctuations (which would have to be done if the source and propagation fluctuations were not de-correlated).* Notice that this de-correlation only implies that the fluctuation, $\Gamma''_{i\sigma j\mu}$, (and not $\overline{\Gamma}_{i\sigma j\mu}$ itself) be independent of the resolved scales. It may, therefore, be appropriate to paramatize the model for this quantity and determine the parameters from the LES calculation.

It is useful to introduce the pressure autocovariance

$$\overline{\Pi\left(x\middle|y,\tau\right)} \equiv \int_{V}\int_{-\infty}^{\infty}\overline{\Gamma}_{i\sigma j\mu}\left(y,\eta \mid x:t'',\tau+t''\right)\ \overline{R}_{\sigma i\mu j}\left(y;\eta,t''\right)\ d\eta\, dt'' \qquad as\, x\to\infty \qquad (3.12)$$

at the observation point x due to a unit volume of subscale turbulence at the source point y - which only involves the statistical correlation $\overline{R}_{\sigma i\mu j}\left(y;\eta,t''\right)$ of the subscale velocities and not their instantaneous values.

But computation of the far field pressure still requires a great deal of information about these quantities (which cannot be obtained from the LES solution). This requirement would certainly be reduced if the correlation volume of the subscale turbulence were small compared to the characteristic length scale over which the filtered scale turbulence, and therefore the propagation factor $\overline{\Gamma}_{i\sigma j\mu}\left(y,\eta \mid x:t'',\tau+t''\right)$, varied. The latter could then be treated as a constant relative to the η-integration causing the result to depend only on the integral of $\overline{R}_{\sigma i\mu j}\left(y;\eta,t''\right)$ over the separation variable η, which, in turn, depends only on the decay time t'' at any given source point y. Since most of the non-local scattering effects are already accounted for by the propagation factor, this is likely to be the case for variations perpendicular to the mean flow direction, but not necessarily in the in the mean flow direction itself. However, as Lighthill

(1952, 1954) pointed out in a slightly different context, the streamwise decay rate should be much more rapid in a reference frame moving with the subscale convection velocity, $U_c(y)$, of the turbulence. Ffowcs Williams (1963) showed that this idea can best be implemented by introducing a moving frame correlation, say

$$\bar{R}^m_{\sigma i\mu j}(y;\xi,t'') \equiv \bar{R}_{\sigma i\mu j}(y;\eta,t'') \tag{3.13}$$

where

$$\xi \equiv \eta - U_c(y)\tau \tag{3.14}$$

before integrating the turbulence correlation over the separation vector. Introducing this into (4.12) and changing the integration variable to ξ and neglecting variation of $\bar{\Gamma}_{i\sigma j\mu}$ over the correlation volume yields

$$\overline{\Pi(x|y,\tau)} \equiv \int_{-\infty}^{\infty} \bar{\Gamma}_{i\sigma j\mu}\left(y, U_c t'' \,|\, x:t'', \tau+t''\right) \left[\int_V \bar{R}^m_{\sigma i\mu j}(y;\xi,t'')\, d\xi\right] dt'' \quad as\ x \to \infty \tag{3.15}$$

Enthalpy fluctuations are thought to be negligible in cold air-jets (Lilley,1996). We therefore neglect these quantities, for simplicity. In which case (3.15) becomes

$$\overline{\Pi(x|y,\tau)} \equiv \int_{-\infty}^{\infty} \bar{\Gamma}_{ijkl}\left(y, U_c t'' \,|\, x:t'', \tau+t''\right) \left[\int_V \bar{R}^m_{jilk}(y;\xi,t'')\, d\xi\right] dt'' \quad as\ x \to \infty \tag{3.16}$$

Bodony's (2004) LES computations imply that the convection velocity can be expected to vary with the source location y and to be scale dependent for the larger turbulent scales, but should become much more constant and scale independent for the smaller scales. We therefore expect the subscale convection velocity $U_c(y)$ to exhibit similar behavior.

Obviously

$$\overline{\Pi(x,\tau)} = \int_V \overline{\Pi(x|y,\tau)}\,dy \tag{3.17}$$

5. Isotropic subscale turbulence

The 81 component tensor $\overline{R}^m_{jilk}(y;\xi,t'')$ has 45 independent components. Fortunately, the small scale turbulence should be quite isotropic in the moving frame (but, unlike the medium scales, not even approximately quasi-normal). Batchelor (1953) points out that the most general 4th order isotropic tensor is of the form

$$\begin{aligned}\overline{R}^m_{ijkl} &= A\xi_i\xi_j\xi_k\xi_l + B\xi_i\xi_j\delta_{kl} + C\xi_i\xi_k\delta_{jl} + D\xi_i\xi_l\delta_{jk} + E\xi_j\xi_k\delta_{il} + F\xi_j\xi_l\delta_{ik} + G\xi_k\xi_l\delta_{ij} \\ &+ H\delta_{ij}\delta_{kl} + I\delta_{ik}\delta_{jl} + J\delta_{il}\delta_{jk}\end{aligned} \tag{4.1}$$

where $A,B,...J$ are functions of $y,|\xi|,t$. But it follows from (4.4) and (2.9) and (2.10) that

$$\overline{R}^m_{ijkl} = \overline{R}^m_{jikl} = \overline{R}^m_{ijlk} \tag{4.2}$$

and, therefore, that

$$\begin{aligned}\overline{R}^m_{ijkl} &= A\xi_i\xi_j\xi_k\xi_l + B\xi_i\xi_j\delta_{kl} + +G\xi_k\xi_l\delta_{ij} + C\xi_i\left(\xi_k\delta_{jl} + \xi_l\delta_{jk}\right) + E\beta\xi_j\left(\xi_k\delta_{il} + \xi_l\delta_{ik}\right) \\ &+ H\delta_{ij}\delta_{kl} + I\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\end{aligned} \tag{4.3}$$

which can be integrated over the unit sphere to show that

$$\int_V \overline{R}^m_{jilk}(y;\xi,t'')\,d\xi = \mathbb{I}_1\delta_{ij}\delta_{kl} + \mathbb{I}_2\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{4.4}$$

where

$$\begin{aligned}\mathbb{I}_1(y;t'') &\equiv 4\pi\int_0^\infty\left[\frac{1}{15}A\xi^4 + \frac{1}{3}(B+G)\xi^2 + H\right]\xi^2 d\xi \\ &= \int_V \overline{R}^m_{1111}(y;\xi,t'')\,d\xi - 2\int_V \overline{R}^m_{1212}(y;\xi,t'')\,d\xi\end{aligned} \tag{4.5}$$

and

$$\mathbb{I}_2(y;t'') \equiv 4\pi\int_0^\infty\left[\frac{1}{15}A\xi^4 + \frac{1}{3}(C+E)\xi^2 + I\right]\xi^2 d\xi = \int_V \overline{R}^m_{1212}(y;\xi,t'')\,d\xi \tag{4.6}$$

Substituting this into (4.16) and using (3.6)-(3.8), (4.10) and (4.11) shows that

$$\overline{\Pi(x|y,\tau)} = \int_{-\infty}^{\infty}\Bigg[\bar{G}_1(y|\,x:t'',\tau+t'')\int_V \bar{R}^m_{1111}(y;\xi,t'')\,d\xi + \bar{G}_2(y|\,x:t'',\tau+t'')\int_V \bar{R}^m_{1212}(y;\xi,t'')\,d\xi\Bigg]dt'' \tag{4.7}$$

where

$$\bar{G}_1(y,|\,x:t'',\tau+t'') \equiv \bar{\Gamma}_{iijj}(y,U_c t''|\,x:t'',\tau+t'') = \left(3\frac{\gamma-1}{2}-1\right)^2\int_{-\infty}^{\infty}\bar{\lambda}_{iijj}(y|\,x:t'',t+\tau+t'',t)\,dt \tag{4.8}$$

$$\bar{G}_2(y|\,x:t'',\tau+t'') \equiv 2\left[\bar{\Gamma}_{ijij}(y,U_c t''|\,x:t'',\tau+t'') - \bar{\Gamma}_{iijj}(y,U_c t''|\,x:t'',\tau+t'')\right]$$

$$= 2\left\{\int_{-\infty}^{\infty}\bar{\lambda}_{ijij}(y|\,x:t'',t+\tau+t'',t)\,dt - \left[6\left(\frac{\gamma-1}{2}\right)^2 - 2\frac{\gamma-1}{2}+1\right]\int_{-\infty}^{\infty}\bar{\lambda}_{iijj}(y|\,x:t'',t+\tau+t'',t)\,dt\right\} \tag{4.9}$$

and

$$\bar{\lambda}_{ijkl}(y|\,x:t'',t+\tau+t'',t) \equiv \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\lambda_{ij}(y|x:t',t+\tau+t'')\lambda_{kl}(y+U_c t''|x:t'+t'',t)dt' \tag{4.10}$$

The latter quantity and, therefore, the $\bar{G}_k, k=1,2,$ can be computed from the six independent components of $\lambda_{jk}(y|x:\tau,t-\tau)$ which are given in terms of the 4th component adjoint vector Greens' function $g^a{}_{v4}(y,\tau|x,t)$ by (3.10).

6 Concluding remarks

A previously derived equation for the sound generation by the sub-filter turbulence scales in a large eddy simulation (Goldstein, 2002, 2003) is used to obtain a formula for the acoustic pressure autocovariance in terms of the sub-

scale turbulence correlation tensor by exploiting the statistical independence of the filtered and sub-filter scale motion. The sub-scale acoustic radiation can therefore be calculated by introducing appropriate models for this relatively universal correlation while determining the configuration dependent large scale sound directly from the LES solution. This approach is in accord with recent speculation that there are two distinct sources of jet noise—one of which is associated with the large scale motion and the other with the small scale motion (Tam, 1998). The approach also accounts for the scattering of the sub-scale sound by the large scale motion-an effect that was emphasized by Crighton (1979).

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Appendix A Adjoint equations

The adjoint Greens' function $g^a{}_{\nu\sigma}\left(y,\tau|x,t\right)$ satisfies the adjoint equation (Morse and Feshbach, 1953, Tam and Auriault, 1998, Bodony and Lele, 2002)

$$\left(L^a_{\mu\nu}\right)_{y,\tau} g^a_{\nu\sigma}\left(y,\tau|x,t\right) = \delta_{\mu\sigma}\delta\left(x-y\right)\delta\left(t-\tau\right) \tag{A-1}$$

where

$$L^a_{\mu\nu} \equiv -\delta_{\nu\mu}\frac{\tilde{D}}{D\tau} - \left(\gamma-1\right)\delta_{\mu 4}\partial_\nu + \left(\frac{\widetilde{c^2}}{\gamma-1}\delta_{\nu 4} + \delta_{\nu 5}\right)\partial_\mu + K_{\nu\mu} \tag{A-2}$$

and

$$\frac{\tilde{D}}{D\tau} \equiv \frac{\partial}{\partial\tau} + \tilde{v}_i\left(y,\tau\right)\frac{\partial}{\partial y_i} \tag{A-3}$$

and is related to the direct Greens' function $g_{\sigma\nu}\left(x,t|y,\tau\right)$ by the reciprocity

relation

$$g^a{}_{\nu\sigma}\left(y,\tau|x,t\right) = g_{\sigma\nu}\left(x,t|y,\tau\right) \tag{A-4}$$

When written out in full the system (3.6) becomes (Bodony and Lele, 2003)

$$-\frac{\tilde{D}g_{i4}^{a}}{D\tau}+g_{j4}^{a}\frac{\partial\tilde{v}_j}{\partial y_i}+\frac{\widetilde{c^2}}{\gamma-1}\frac{\partial g_{44}^{a}}{\partial y_i}+\frac{\gamma-1}{\bar{\rho}}\frac{\partial\tilde{\Theta}_{ij}}{\partial y_j}g_{44}^{a}+\frac{\partial g_{54}^{a}}{\partial y_i}=0 \tag{A-5a}$$

$$-\frac{\tilde{D}g_{44}^{a}}{D\tau}-(\gamma-1)\left(\frac{\partial g_{i4}^{a}}{\partial y_i}+g_{44}^{a}\frac{\partial\tilde{v}_j}{\partial y_j}\right)=\delta(\boldsymbol{x}-\boldsymbol{y})\delta(t-\tau) \tag{A-5b}$$

$$-\frac{\tilde{D}g_{54}^{a}}{D\tau}-\frac{1}{\bar{\rho}}\frac{\partial\tilde{\Theta}_{ij}}{\partial y_j}g_{i4}^{a}=0 \tag{A-5c}$$

Then when $x, y\to\infty$, $\widetilde{c^2}\to c_0^2=\text{constant}$ and

$$-\frac{\partial g_{i4}^{a}}{\partial\tau}+\frac{c_0^2}{\gamma-1}\frac{\partial g_{44}^{a}}{\partial y_i}=0 \tag{A-6a}$$

$$-\frac{\partial g_{44}^{a}}{\partial\tau}-(\gamma-1)\frac{\partial g_{i4}^{a}}{\partial y_i}=\delta(\boldsymbol{x}-\boldsymbol{y})\delta(t-\tau) \tag{A-6b}$$

$$\frac{\partial g_{54}^{a}}{\partial\tau}=0 \tag{A-6c}$$

which shows that g_{44}^{a} satisfies the inhomogeneous wave equation

$$-\frac{\partial^2 g_{44}^{a}}{\partial\tau^2}+c_0^2\frac{\partial^2 g_{44}^{a}}{\partial y_i\partial y_i}=\delta(\boldsymbol{x}-\boldsymbol{y})\frac{\partial}{\partial\tau}\delta(t-\tau) \tag{A-7}$$

The relevant solution which satisfies the causality condition

$$g_{44}^{a}(\boldsymbol{y},\tau|\boldsymbol{x},t)=0 \text{ for } \quad t<\tau \tag{A-8}$$

is

$$g_{44}^{a}\left(\boldsymbol{y},\tau\left|\boldsymbol{x},t\right.\right)=\frac{1}{4\pi\left|\boldsymbol{x}-\boldsymbol{y}\right|c_0^2}\frac{\partial}{\partial\tau}\delta\left(\tau-t+\frac{\left|\boldsymbol{x}-\boldsymbol{y}\right|}{c_0}\right) \tag{A-9}$$

So that when $x\equiv\left|\boldsymbol{x}\right|\to\infty$

$$g_{44}^{a}\to\frac{1}{4\pi x c_0^2}\frac{\partial}{\partial\tau}\delta\left(\tau-\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{xc_0}-t+\frac{x}{c_0}\right) \tag{A-10}$$

Notice that the far field adjoint Greens' function can be calculated directly by allowing $x\equiv\left|\boldsymbol{x}\right|\to\infty$ in (A-7). . Smoother boundary conditions can be obtained by putting

$$g_{44}^{a}\left(\boldsymbol{y},\tau\left|\boldsymbol{x},t\right.\right)\equiv\frac{1}{4\pi x c_0^2}\frac{\partial^2}{\partial\tau^2}G_4^{a}\left(\boldsymbol{y},\tau\left|\boldsymbol{x},t\right.\right) \tag{A-11}$$

which satisfies

$$\left(L_{\mu\nu}^{a}\right)_{\mathrm{y},\tau}\frac{\partial^2}{\partial\tau^2}G_4^{a}\left(\boldsymbol{y},\tau\left|\boldsymbol{x},t\right.\right)\ =0 \tag{A-12}$$

together with the smoother boundary condition

$$G_4^{a}\left(\boldsymbol{y},\tau\left|\boldsymbol{x},t\right.\right)\to H\left(\tau-\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{xc_0}-t+\frac{x}{c_0}\right)\quad\text{as } x\equiv\left|\boldsymbol{x}\right|\to\infty \tag{A-13}$$

where H denotes the Heaviside unit function. It may also be convenient to divide the solution into incident and scattered components.

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