SUPG Finite Element Simulations of Compressible Flows for Aerothermodynamic Applications

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The physical phenomenon of interest is high-speed gas dynamics

**Physics**
- The compressible Navier-Stokes equations describe fluid flow for *all* Mach numbers
- For aerospace applications of interest the Reynolds number is almost always such that the flows are *convection dominated*
- Multiscale phenomena in the form of shock waves, boundary layers, and shear layers

**Numerics**
- Discretization of the *conservation law form* of the Navier-Stokes equations is required for convergence to physically valid solutions
- Convective terms must be treated with some form of *upwinding*
- Shocks are treated with some form of *limiting* or *shock capturing*, both of which amount to artificial diffusion which regularizes the problem
Aerodynamics

...is concerned with predicting aerodynamic forces on a vehicle which result predominantly from the surface pressure distribution, but also from viscous shear stress.

Properly characterizing the aerodynamic performance of reentry vehicles is critical for optimal trajectory design.
...is concerned with predicting the instantaneous total heat transfer rate and integrated heat load into a vehicle.

Properly characterizing this environment is crucial because it provides the design conditions for the thermal protection system:

**heat transfer rate** → thermal protection material selection

**heat load** → thermal protection material thickness
The conservation of mass, momentum, and energy for a compressible fluid may be written as

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1}
\]
\[
\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = -\nabla P + \nabla \cdot \mathbf{\tau} \tag{2}
\]
\[
\frac{\partial \rho E}{\partial t} + \nabla \cdot (\rho E \mathbf{u}) = -\nabla \cdot \mathbf{q} - P \nabla \cdot \mathbf{u} + \nabla \cdot (\mathbf{\tau} \mathbf{u}) \tag{3}
\]

where \(\rho\) is the density, \(\mathbf{u}\) is the velocity, \(E = e + \frac{u \cdot u}{2}\) is the total energy per unit mass, and \(P\) is the pressure.
The viscous stress tensor $\tau$ and the heat flux vector $q$ are defined as

$$
\tau = \mu \left( \nabla u + \nabla^T u \right) + \lambda \left( \nabla \cdot u \right) I \tag{4}
$$

$$
q = -k \nabla T \tag{5}
$$

where $\mu$ is the dynamic viscosity, $\lambda$ is the second coefficient of viscosity, $k$ is the thermal conductivity, and $T$ is the fluid temperature. The two coefficients of viscosity are related to the bulk viscosity $\kappa$ by

$$
\kappa = \frac{2}{3} \mu + \lambda \tag{6}
$$

In general, the bulk viscosity is negligible except in detailed studies of shock wave structure or for investigations of the adsorption and attenuation of acoustic waves [1]. Under this assumption, $\kappa = 0$ in Equation (6) and $\lambda$ is defined as

$$
\lambda = -\frac{2}{3} \mu \tag{7}
$$

Equation (4) with (7) is known as Stokes’ hypothesis for a Newtonian fluid [2].
Provided the transport properties may be related to the unknowns, Equations (1)–(3) form 5 equations in 7 unknowns (in three-dimensions).

For a gas in thermodynamic equilibrium any two independent properties fix the state:

\[ T = T(\rho, e), \quad P = P(\rho, e) \]

In the special case of a calorically perfect gas

\[ c_v T = e, \quad P = \rho e (\gamma - 1) \]

For the case of thermochemical nonequilibrium the equation set is enlarged to contain

- local mass balance statements for constituent species
- local energy balance/exchange statements for internal modes
In the literature equations (1)–(3) are often treated as the system\(^1\)

\[
\frac{\partial U}{\partial t} + \frac{\partial F_i}{\partial x_i} = \frac{\partial G_i}{\partial x_i}
\]

(8)

which can be written in terms of the unknowns \( U = [\rho, \rho u, \rho v, \rho w, \rho E]^T \) as

\[
\frac{\partial U}{\partial t} + A_i \frac{\partial U}{\partial x_i} = \frac{\partial}{\partial x_i} \left( K_{ij} \frac{\partial U}{\partial x_j} \right)
\]

(9)

where \( A_i = \frac{\partial F_i}{\partial U} \) is the inviscid flux Jacobian, and the viscous flux vector \( G_i \) may be written as

\[
\frac{\partial G_i}{\partial x_i} = \frac{\partial}{\partial x_i} \left( K_{ij} \frac{\partial U}{\partial x_j} \right)
\]

(10)

---

\(^1\) the notation is cumbersome, but it is fairly standard
The Streamline-Upwind Petrov-Galerkin Finite Element Method
Upwinding is required to stabilize convection-dominated flows. In SUPG schemes this is accomplished by biasing the test functions $\hat{W}$ in the upstream direction:

$$\hat{W} = W + \tau_{SUPG} A_i \frac{\partial W}{\partial x_i}$$  \hspace{1cm} (11)

Standard Galerkin treatment & SUPG-upwinding applied to (9) to produce the stabilized weak form: find $U$ such that

$$\int_{\Omega} \left[ W \cdot \left( \frac{\partial U}{\partial t} + A_i \frac{\partial U}{\partial x_i} \right) + \frac{\partial W}{\partial x_i} \cdot \left( K_{ij} \frac{\partial U}{\partial x_j} \right) \right] d\Omega$$

$$+ \sum_{e=1}^{n_{el}} \int_{\Omega_e} \tau_{SUPG} \frac{\partial W}{\partial x_k} \cdot A_k \left[ \frac{\partial U}{\partial t} + A_i \frac{\partial U}{\partial x_i} - \frac{\partial}{\partial x_i} \left( K_{ij} \frac{\partial U}{\partial x_j} \right) \right] d\Omega$$

$$= \int_{\Gamma} W \cdot g \ d\Gamma$$  \hspace{1cm} (12)

for all $W$ in an appropriate function space.

For compressible flows $\tau_{SUPG}$ is generally diagonal [3].
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(12)

for all $W$ in an appropriate function space.

For compressible flows $\tau_{SUPG}$ is generally diagonal [3].
SUPG stabilization does not yield monotone solutions. Additional treatment is needed to prevent spurious oscillations in regions of shockwaves. Hence (12) is augmented with a shock capturing term to produce the augmented weak form: find $U$ such that

$$
\int_\Omega \left[ \mathbf{W} \cdot \left( \frac{\partial U}{\partial t} + A_i \frac{\partial U}{\partial x_i} \right) + \frac{\partial \mathbf{W}}{\partial x_i} \cdot \left( K_{ij} \frac{\partial U}{\partial x_j} \right) \right] d\Omega 
+ \sum_{e=1}^{n_{el}} \int_{\Omega_e} \tau_{SUPG} \frac{\partial \mathbf{W}}{\partial x_k} \cdot A_k \left[ \frac{\partial U}{\partial t} + A_i \frac{\partial U}{\partial x_i} - \frac{\partial}{\partial x_i} \left( K_{ij} \frac{\partial U}{\partial x_j} \right) \right] d\Omega 
+ \sum_{e=1}^{n_{el}} \int_{\Omega_e} \delta \left( \frac{\partial \mathbf{W}}{\partial x_i} \cdot \frac{\partial U}{\partial x_i} \right) d\Omega = \int_{\Gamma} \mathbf{W} \cdot \mathbf{g} d\Gamma
$$

(13)

for all $\mathbf{W}$ in an appropriate function space.

$\delta$ in this work is modified from [4, 5]. Note that discretizations of (13) reduce to $O(h)$ in regions of appreciable $\delta$. 
The shock capturing operator, $\delta$, was adapted for a system of conservation variables by LeBeau and Tezduyar [4, 5, 6] from the original definition employed by Hughes et al. for the case of entropy variables [7, 8]. A modified form is employed in the present work and is defined as

$$\delta = \left[ \frac{\left\| \frac{\partial U}{\partial t} + A_i \frac{\partial U}{\partial x_i} - \frac{\partial}{\partial x_i} \left( K_{ij} \frac{\partial U}{\partial x_j} \right) \right\|_{A_0^{-1}}}{\left\| \nabla \xi \cdot \nabla U \right\|_{A_0^{-1}} + \left\| \nabla \eta \cdot \nabla U \right\|_{A_0^{-1}} + \left\| \nabla \zeta \cdot \nabla U \right\|_{A_0^{-1}}} \right]^{1/2}$$  \hspace{1cm} (14)$$

where $(\xi, \eta, \zeta)$ are the canonical reference element coordinates and $A_0^{-1}$ is the mapping from conservation to entropy variables.

In this work the numerator is modified from its original form to retain consistency with (9) in the case of transient, viscous flows.
Inviscid Flux Discretization

Expand $U(x, t)$ and $F_i(x, t)$ in terms of standard Lagrange finite element basis functions:

\[ U(x, t) = \sum_j \phi_j(x) U(x_j, t) \]  \hspace{1cm} (15)

\[ F_i(x, t) = \sum_j \phi_j(x) F_i(x_j, t) \]  \hspace{1cm} (16)

where $U(x_j, t)$ and $F_i(x_j, t) = A_i(U(x_j, t)) U(x_j, t)$ are the nodal solution inviscid flux values at time $t$. A standard piecewise linear Lagrange basis is chosen for $\{\phi\}$ (yielding a nominally 2nd–order accurate scheme).

This approach is in contrast to previous SUPG discretizations for compressible flows. [4, 5, 3, 9]

\[ F_i(x, t) = \sum_j \phi_j(x) F_i(x_j, t) = \sum_j \phi_j(x) A_i(U(x_j, t)) U(x_j, t) \]  \hspace{1cm} (17)

contrasts to the typical approach in which

\[ F_i(x, t) = A_i(U(x, t)) U(x, t) \]  \hspace{1cm} (18)

where $U(x, t)$ is interpolated from nodal values as in (15).
Expanding $U(x, t)$ and $F_i(x, t)$ in terms of standard Lagrange finite element basis functions:

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$$F_i(x, t) = \sum_j \phi_j(x) F_i(x_j, t)$$

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This approach is in contrast to previous SUPG discretizations for compressible flows. [4, 5, 3, 9]

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contrasts to the typical approach in which

$$F_i(x, t) = A_i(U(x, t)) U(x, t) \quad (18)$$

where $U(x, t)$ is interpolated from nodal values as in (15).
Consider a one-dimensional, inviscid, normal shock at Mach 5. For this simple case the governing equations reduce to

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = 0 \\
\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x} (\rho u^2 + P) = 0 \\
\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x} (\rho u H) = 0
\]

and, at steady state, become

\[
\frac{\partial}{\partial x} (\rho u) = \frac{\partial}{\partial x} (\rho u^2 + P) = \frac{\partial}{\partial x} (\rho u H) \equiv 0 \quad (19)
\]

which implies that \( \rho u, \rho u^2 + P, \) and \( \rho u H \) are all constant.
Inviscid Flux Discretization

Comparison between reconstructed and interpolated inviscid flux discretizations
The semidiscrete weak form in equation (13) is discretized in time using a backwards finite difference scheme. Both first and second-order accurate in time schemes may be derived from Taylor series expansions in time about $U_{n+1}$:

$$U_n = U_{n+1} + \frac{\partial U_{n+1}}{\partial t} (t_n - t_{n+1}) + \frac{\partial^2 U_{n+1}}{\partial t^2} \frac{(t_n - t_{n+1})^2}{2} + O\left((t_n - t_{n+1})^3\right)$$

$$U_{n-1} = U_{n+1} + \frac{\partial U_{n+1}}{\partial t} (t_{n-1} - t_{n+1}) + \frac{\partial^2 U_{n+1}}{\partial t^2} \frac{(t_{n-1} - t_{n+1})^2}{2} + O\left((t_{n-1} - t_{n+1})^3\right)$$

which, upon substituting $t_{n+1} - t_n \equiv \Delta t_{n+1}$ and $t_{n+1} - t_{n-1} = \Delta t_{n+1} + \Delta t_n$, becomes

$$U_n = U_{n+1} - \frac{\partial U_{n+1}}{\partial t} \Delta t_{n+1} + \frac{\partial^2 U_{n+1}}{\partial t^2} \frac{\Delta t_{n+1}^2}{2} - O\left(\Delta t_{n+1}^3\right)$$

$$U_{n-1} = U_{n+1} - \frac{\partial U_{n+1}}{\partial t} (\Delta t_{n+1} + \Delta t_n) + \frac{\partial^2 U_{n+1}}{\partial t^2} \frac{(\Delta t_{n+1} + \Delta t_n)^2}{2} - O\left((\Delta t_{n+1} + \Delta t_n)^3\right)$$
The semidiscrete weak form in equation (13) is discretized in time using a backwards finite difference scheme. Both first and second-order accurate in time schemes may be derived from Taylor series expansions in time about \( U_{n+1} \):

\[
U_n = U_{n+1} + \frac{\partial U_{n+1}}{\partial t} (t_n - t_{n+1}) + \frac{\partial^2 U_{n+1}}{\partial t^2} \frac{(t_n - t_{n+1})^2}{2} + \mathcal{O} ((t_n - t_{n+1})^3)
\]

\[
U_{n-1} = U_{n+1} + \frac{\partial U_{n+1}}{\partial t} (t_{n-1} - t_{n+1}) + \frac{\partial^2 U_{n+1}}{\partial t^2} \frac{(t_{n-1} - t_{n+1})^2}{2} + \mathcal{O} ((t_{n-1} - t_{n+1})^3)
\]

which, upon substituting \( t_{n+1} - t_n \equiv \Delta t_{n+1} \) and \( t_{n+1} - t_{n-1} = \Delta t_{n+1} + \Delta t_n \), becomes

\[
U_n = U_{n+1} - \frac{\partial U_{n+1}}{\partial t} \Delta t_{n+1} + \frac{\partial^2 U_{n+1}}{\partial t^2} \frac{\Delta t_{n+1}^2}{2} - \mathcal{O} (\Delta t_{n+1}^3)
\]

\[
U_{n-1} = U_{n+1} - \frac{\partial U_{n+1}}{\partial t} (\Delta t_{n+1} + \Delta t_n) + \frac{\partial^2 U_{n+1}}{\partial t^2} \frac{(\Delta t_{n+1} + \Delta t_n)^2}{2} - \mathcal{O} ((\Delta t_{n+1} + \Delta t_n)^3)
\]
Time Discretization

The semidiscrete weak form in equation (13) is discretized in time using a backwards finite difference scheme. Both first and second-order accurate in time schemes may be derived from Taylor series expansions in time about \( U_{n+1} \):

\[
\begin{align*}
U_n &= U_{n+1} + \frac{\partial U_{n+1}}{\partial t} (t_n - t_{n+1}) + \frac{\partial^2 U_{n+1}}{\partial t^2} \left( \frac{(t_n - t_{n+1})^2}{2} + O((t_n - t_{n+1})^3) \right) \\
U_{n-1} &= U_{n+1} - \frac{\partial U_{n+1}}{\partial t} (t_{n-1} - t_{n+1}) + \frac{\partial^2 U_{n+1}}{\partial t^2} \left( \frac{(t_{n-1} - t_{n+1})^2}{2} + O((t_{n-1} - t_{n+1})^3) \right)
\end{align*}
\]

which, upon substituting \( t_{n+1} - t_n \equiv \Delta t_{n+1} \) and \( t_{n+1} - t_{n-1} = \Delta t_{n+1} + \Delta t_n \), becomes

\[
\begin{align*}
U_n &= U_{n+1} + \frac{\partial U_{n+1}}{\partial t} (\Delta t_{n+1}) + \frac{\partial^2 U_{n+1}}{\partial t^2} \left( \frac{(\Delta t_{n+1})^2}{2} + O((\Delta t_{n+1})^3) \right) \\
U_{n-1} &= U_{n+1} - \frac{\partial U_{n+1}}{\partial t} (\Delta t_{n+1} + \Delta t_n) + \frac{\partial^2 U_{n+1}}{\partial t^2} \left( \frac{(\Delta t_{n+1} + \Delta t_n)^2}{2} + O((\Delta t_{n+1} + \Delta t_n)^3) \right)
\end{align*}
\]
Which can be rewritten for \( \frac{\partial U_{n+1}}{\partial t} \) as:

\[
\frac{\partial U_{n+1}}{\partial t} = \frac{U_{n+1}}{\Delta t_{n+1}} - \frac{U_n}{\Delta t_{n+1}} + \frac{\partial^2 U_{n+1}}{\partial t^2} \frac{\Delta t_{n+1}}{2} - O\left(\Delta t_{n+1}^2\right) \tag{20}
\]

\[
\frac{\partial U_{n+1}}{\partial t} = \frac{U_{n+1}}{\Delta t_{n+1} + \Delta t_n} - \frac{U_{n-1}}{\Delta t_{n+1} + \Delta t_n} + \frac{\partial^2 U_{n+1}}{\partial t^2} \frac{(\Delta t_{n+1} + \Delta t_n)}{2} \nonumber \\
- O\left((\Delta t_{n+1} + \Delta t_n)^2\right) \tag{21}
\]
The familiar backwards Euler time discretization follows directly from (20) by recognizing

$$\frac{\partial U_{n+1}}{\partial t} = \frac{U_{n+1} - U_n}{\Delta t_{n+1}} + O(\Delta t_{n+1})$$

(22)

which provides a first-order in time approximation upon neglecting the $O(\Delta t_{n+1})$ term.
A linear combination of \(\left(1 + \frac{\Delta t_{n+1}}{\Delta t_n}\right)\times(20)\) and \(-\frac{\Delta t_{n+1}}{\Delta t_n}\times(21)\) can be used to annihilate the leading \(\frac{\partial^2 U_{n+1}}{\partial t^2}\) term to create a backwards, second-order accurate approximation to \(\frac{\partial U_{n+1}}{\partial t}\).

This approximation, along with (22), can be generalized in the form

\[
\frac{\partial U_{n+1}}{\partial t} = \alpha_t U_{n+1} + \beta_t U_n + \gamma_t U_{n-1} + O\left(\Delta t_{n+1}^p\right)
\]

(23)

to yield either a first or second-order accurate scheme. The weights \(\alpha_t, \beta_t,\) and \(\gamma_t\) are given below for \(p = 1\) and \(p = 2\).

<table>
<thead>
<tr>
<th>(p)</th>
<th>(\alpha_t)</th>
<th>(\beta_t)</th>
<th>(\gamma_t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\frac{1}{\Delta t_{n+1}})</td>
<td>(-\frac{1}{\Delta t_{n+1}})</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>(-\beta_t - \gamma_t)</td>
<td>(-\left[\frac{1}{\Delta t_{n+1}} + \frac{1}{\Delta t_n}\right])</td>
<td>(\frac{\Delta t_{n+1}}{\Delta t_n(\Delta t_{n+1} + \Delta t_n)})</td>
</tr>
</tbody>
</table>
The simulation begins with the domain initialized to freestream conditions everywhere and a user-specified initial time step $\Delta t_0$ is used to advance the solution. The time step then grows geometrically with the relative change in the unsteady residual measured over $k$ time steps. Explicitly,

$$\Delta t_{n+1} = \Delta t_{n-k} \max \left( \left[ \frac{R_{n-k}}{R_n} \right]^r, 1 \right)$$

$$\Delta t_{n+1} = \min (\Delta t_{n+1}, \Delta t_{\max})$$

where $R_n \equiv \| \frac{\Delta U_n}{\Delta t} \|_{\infty}$ and $r = 1.2$ is the geometric growth rate. The time step size is updated every $k = 5$ time steps.

Typically the maximum allowable time step for steady problems is $\Delta t_{\max} = 1$, which corresponds to the amount of time required for a fictitious point in the freestream to be convected one reference length.
After temporal & spatial discretization, Equation (13) can be written in residual form for the unknown nodal values $U_{n+1} \equiv U_h (t_{n+1})$ as the nonlinear algebraic system

$$R (U_{n+1}) = 0 \quad (26)$$

We seek to define a sequence of linear problems $\{U^l_{n+1}\}$ that converge to $U_{n+1}$, the solution of (26).
Newton Scheme

Expanding (26) with a Taylor series about iterate $U_{n+1}^l$ gives

$$R \left( U_{n+1}^{l+1} \right) = R \left( U_{n+1}^l \right) + \left[ \frac{\partial R \left( U_{n+1}^l \right)}{\partial U_{n+1}} \right] \delta U_{n+1}^{l+1} + O \left( (\delta U_{n+1}^{l+1})^2 \right)$$

(27)

where $\frac{\partial R}{\partial U}$ is the Jacobian matrix for the nonlinear system and

$$\delta U_{n+1}^{l+1} = U_{n+1}^{l+1} - U_{n+1}^l.$$

Truncating this expansion and setting $R \left( U_{n+1}^{l+1} \right) = 0$ yields Newton’s method:

$$0 = R \left( U_{n+1}^l \right) + \left[ \frac{\partial R \left( U_{n+1}^l \right)}{\partial U_{n+1}} \right] \delta U_{n+1}^{l+1}$$

$$\left[ \frac{\partial R \left( U_{n+1}^l \right)}{\partial U_{n+1}} \right] \delta U_{n+1}^{l+1} = -R \left( U_{n+1}^l \right)$$

(28)
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$$\left[ \frac{\partial R \left( U_{n+1}^l \right)}{\partial U_{n+1}^l} \right] \delta U_{n+1}^{l+1} = -R \left( U_{n+1}^l \right) \quad (28)$$
Newton’s method exhibits second-order *conditional* convergence

Even for a calorically perfect gas, full Newton implementation is complicated by the nonlinear transport properties and convective terms

Asymptotic convergence rate is rarely achieved for flows with strong shocks

Computational cost of full-Newton may be mitigated with an approximate Newton-Krylov technique which accounts for the action of the Jacobian matrix in (28) without explicitly forming it

This approach is especially attractive for “real gas” flows
Newton-Krylov Techniques

- The resulting sparse implicit linear systems are amenable to solution with iterative Krylov subspace techniques.
- The kernel operation for such methods is the action of a matrix-vector product. That is, solving
  \[ Ku = f \]
  requires repeated computations of \( w = Kv \).
- For the Newton linearization,
  \[
  \begin{bmatrix}
  \frac{\partial R}{\partial U}
  \end{bmatrix} \delta U = -R(U)
  \]
  and the required matrix-vector product is of the form \( w = \left[ \frac{\partial R}{\partial U} \right] v \), which is simply the derivative of \( R \) in the direction of \( v \).
- This may be approximated via differencing residual evaluations, yielding a so-called matrix-free method:
  \[
  w = \left[ \frac{\partial R}{\partial U} \right] v \approx \frac{R(U + \varepsilon v) - R(U)}{\varepsilon}
  \]
Implicit Solution Strategies

- Time-marching to steady-state is almost always used for high-speed flows
- Implicit techniques required for viscous problems with tight wall spacing (also for stiff chemistry in the case of nonequilibrium)
- For steady problems, at each time step the resulting nonlinear problem is usually solved only approximately (usually 1 Newton step)
- DOF coupling defined via standard finite element basis function support determines sparse matrix structure
- Matrix-free GMRES with block-diagonal preconditioning used in earlier work [5]
- This work uses matrix & matrix-free GMRES with full ILU-0 preconditioning – linearization is important

![Influence of linearization strategy on iterative convergence for Mach 3 flow over a cylinder](image)
Hypersonic Aerothermodynamics – Application Studies
The X-15 Experience
Edney’s Type IV Interaction Pattern [10]
An experimental test program was conducted in 1998 by France’s Office National d’Etudes et de Recherches Aérospatiales (ONERA) to investigate shock-shock interactions produced by an oblique shock impinging on the bow shock of a cylinder [11]. This configuration is examined here to assess the quality of surface heat transfer predictions.

Flow
M = 10
T = 52.5 K
Re/m = 166,000
Static temperature contours
Type IV Shock Interaction

- Normalized Surface Pressure ($P/P_{c,s}$)
- Normalized Heat Transfer ($q/q_{c,s}$)

Graph showing computed and measured pressure and heat transfer ratios with respect to angle ($\theta$) in degrees.
Hypersonic flow over a missile nose tip with a forward facing cavity has been observed to exhibit transient flowfield response in both experimental investigations and numerical simulations [12, 13]. The flowfield response characteristics are largely driven by the cavity length-to-diameter ratio (L/D).

Experimental studies in conventional tunnels report oscillatory response even for relatively shallow cavities, suggesting a threshold L/D of 0.4. Numerical simulations predict a higher threshold L/D of $\approx 1.25$. Subsequent studies in a quiet wind tunnel verify the computational results, indicating freestream noise is the mechanism for driving unsteady response in shallow cavities [14].
Cavity base pressure versus time for a series of simulations used to assess time convergence. For $\text{CFL}_{\text{max}} = 20 \times 10^3$ there are $\approx 200$ time steps per oscillation cycle.
A sharp 25°–55° double cone was tested in N₂ at CUBRC. It was discovered that freestream vibrational nonequilibrium must be properly modeled for CFD to match experiment [15]. The AEDC Hypervelocity Wind Tunnel No. 9 also uses N₂ as its test gas. A series of tests were conducted at AEDC using the same model to investigate the presence of vibrational nonequilibrium in the freestream [16].
### Observations

- Four Reynolds numbers were tested in the nominally Mach 14 nozzle.
- No appreciable vibrational nonequilibrium effects observed.
- Highly unsteady flow observed for *all* Reynolds numbers tested.
- For a uniform freestream, CFD predicts steady flow for the two lowest Reynolds numbers.

<table>
<thead>
<tr>
<th>Run</th>
<th>M&lt;sub&gt;∞&lt;/sub&gt;</th>
<th>Re&lt;sub&gt;D&lt;/sub&gt;</th>
<th>ρ&lt;sub&gt;∞&lt;/sub&gt;</th>
<th>U&lt;sub&gt;∞&lt;/sub&gt;</th>
<th>T&lt;sub&gt;∞&lt;/sub&gt;</th>
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<td>2890</td>
<td>13.6</td>
<td>1.12 × 10&lt;sup&gt;6&lt;/sup&gt;</td>
<td>7.81 × 10&lt;sup&gt;-3&lt;/sup&gt;</td>
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<tr>
<td>2893</td>
<td>12.73</td>
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<tr>
<td>2894</td>
<td>12.63</td>
<td>5.86 × 10&lt;sup&gt;4&lt;/sup&gt;</td>
<td>3.98 × 10&lt;sup&gt;-4&lt;/sup&gt;</td>
<td>1682.6</td>
<td>42.7</td>
</tr>
</tbody>
</table>
AEDC Sharp Double Cone

Steady states, runs 2893 and 2894

\[ M_\infty = 12.63 \quad \text{Re}_D = 58,600 \]

\[ M_\infty = 12.73 \quad \text{Re}_D = 84,400 \]
Time Convergence, run 2894

Uniformly Refined Mesh
Baseline Mesh

$M_{\infty} = 12.63$
$Re_D = 58,600$
High speed schlieren, run 2890
Computed schlieren, run 2890
Possible Mechanism for Observed Unsteadiness

- For a uniform inflow, CFD converges to a steady-state for the two lowest Reynolds numbers tested.
- This is in contrast to the experimental results.
- My conjecture is that freestream noise drives the unsteady behavior at these low Reynolds numbers.
- Remaining analysis is focused on testing this theory.
Noise Characterization [17]
Noise Characterization [17]

\[ y = -0.0032 \ln(x) + 0.0709 \quad M_8 \]
\[ y = -0.0051 \ln(x) + 0.1102 \quad M_{10} \]
\[ y = -0.0086 \ln(x) + 0.1628 \quad M_{14} \]

**Variation of Pitot Pressure Fluctuation With Varying Reynolds Number**

![Graph showing variation of Pitot pressure fluctuation with Reynolds number for different Mach numbers, M8, M10, M14, with a bad data point in M10 historical data.](image)
Results – Flowfield

6.4% RMS, 25 KHz Noise
Results – Surface Pressure
25 kHz, 6% RMS Pitot Pressure Fluctuation
Frequency Influence

![Graph showing Frequency Influence with pressure coefficient $C_p$ vs. normalized distance from cone apex $x/D$. The graph includes lines for different frequencies: 6.5 kHz, 13 kHz, 25 kHz, and 50 kHz.](image-url)
*Computational Fluid Mechanics and Heat Transfer*. 

*Incompressible Flow*. 

A comparative study of different sets of variables for solving compressible and incompressible flows. 

The finite element computation of compressible flows. 

*Parallel Finite Element Computations in Aerospace Applications*. 

Parallel Fluid Dynamics Computations in Aerospace Applications. 

A new finite element formulation for computational fluid dynamics: IV. a discontinuity operator for multidimensional advective–diffusive systems. 

A new finite element formulation for computational fluid dynamics: X. the compressible Euler and Navier–Stokes equations. 

Improving convergence to steady state of implicit SUPG solution of Euler equations. 


