Quaternion Averaging

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Introduction

Many applications require an algorithm that averages quaternions in an optimal manner. For example, when combining the quaternion outputs of multiple star trackers having this output capability, it is desirable to properly average the quaternions without recomputing the attitude from the the raw star tracker data. Other applications requiring some sort of optimal quaternion averaging include particle filtering [1] and multiple-model adaptive estimation [2], where weighted quaternions are used to determine the quaternion estimate.

For spacecraft attitude estimation applications, [1] derives an optimal averaging scheme to compute the average of a set of weighted attitude matrices using the singular value decomposition method [3]. Focusing on a 4-dimensional quaternion Gaussian distribution on

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the unit hypersphere, [4] provides an approach to computing the average quaternion by minimizing a quaternion cost function that is equivalent to the attitude matrix cost function in [1]. Motivated by [1] and extending its results, this Note derives an algorithm that determines an optimal average quaternion from a set of scalar- or matrix-weighted quaternions. Furthermore, a sufficient condition for the uniqueness of the average quaternion, and the equivalence of the minimization problem, stated herein, to maximum likelihood estimation, are shown.

For the scalar weighted case the goal is to find the average of a set of n attitude estimates, $\mathbf{q}_i$, in quaternion form with associated weights w_i. The simple procedure

$$\bar{\mathbf{q}}_{\text{bad}} \triangleq \left(\sum_{i=1}^{n} w_i\right)^{-1} \sum_{i=1}^{n} w_i\, \mathbf{q}_i \tag{1}$$

has two flaws. The first and most obvious flaw, that $\bar{\mathbf{q}}$ is not a unit quaternion, is easily fixed by the *ad hoc* procedure of dividing $\bar{\mathbf{q}}$ by its norm. The second flaw is subtler. It is well known that $\mathbf{q}$ and $-\mathbf{q}$ represent the same rotation, so that the quaternions provide a 2:1 mapping of the rotation group [5]. Thus changing the sign of any $\mathbf{q}_i$ should not change the average, but it is clear that Eq. (1) does not have this property.

The observation that we really want to average attitudes rather than quaternions, first presented in [1], provides a way to avoid both of these flaws. Following this observation, the average quaternion should minimize a weighted sum of the squared Frobenius norms of attitude matrix differences:

$$\bar{\mathbf{q}} \triangleq \arg\min_{\mathbf{q}\in\mathbb{S}^3} \sum_{i=1}^{n} w_i \left\|A(\mathbf{q}) - A(\mathbf{q}_i)\right\|_F^2 \tag{2}$$

where $\mathbb{S}^3$ denotes the unit 3-sphere.

The Average Quaternion

Using the definition of the Frobenius norm, the orthogonality of $A(\mathbf{q})$ and $A(\mathbf{q}_i)$, and some properties of the matrix trace (denoted by Tr) gives

$$\begin{aligned}\left\|A(\mathbf{q}) - A(\mathbf{q}_i)\right\|_F^2 &= \operatorname{Tr}\left\{\left[A(\mathbf{q}) - A(\mathbf{q}_i)\right]^T \left[A(\mathbf{q}) - A(\mathbf{q}_i)\right]\right\} \\ &= 6 - 2\operatorname{Tr}\left[A(\mathbf{q})A^T(\mathbf{q}_i)\right]\end{aligned} \tag{3}$$

This allows us to express Eq. (2) as

$$\bar{\mathbf{q}} = \arg\max_{\mathbf{q}\in\mathbb{S}^3} \operatorname{Tr}\left[A(\mathbf{q})\, B^T\right] \tag{4}$$

where

$$B \triangleq \sum_{i=1}^{n} w_i\, A(\mathbf{q}_i) \tag{5}$$

Equation (4) is in a form found in solving Wahba's Problem [6], so many of the techniques used for solving that problem [7] can be applied to finding the average quaternion. If computational efficiency is important, the well-known QUEST algorithm [8] can be recommended, as will be discussed later. The matrix B is known as the *attitude profile matrix* [9] since it contains all the information on the attitude.

A detailed review of quaternions can be found in [5], but we only need a few results for this paper. We denote the vector and scalar parts of a quaternion by $\mathbf{q} \triangleq \left[\boldsymbol{\varrho}^T \;\; q_4\right]^T$, which are assumed to obey the normalization condition $||\boldsymbol{\varrho}||^2 + q_4^2 = \mathbf{q}^T\mathbf{q} = 1$. The attitude matrix is related to the quaternion by

$$A(\mathbf{q}) = \left(q_4^2 - ||\boldsymbol{\varrho}||^2\right) I_{3\times3} + 2\,\boldsymbol{\varrho}\,\boldsymbol{\varrho}^T - 2\,q_4[\boldsymbol{\varrho}\times] \tag{6}$$

where $I_{3\times3}$ is a 3×3 identity matrix and $[\boldsymbol{\varrho}\times]$ is the cross-product matrix defined by

$$[\boldsymbol{\varrho}\times] \triangleq \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{7}$$

Equation (6) can be used to verify the identity

$$\operatorname{Tr}[A(\mathbf{q})\, B^T] = \mathbf{q}^T K \mathbf{q} \tag{8}$$

where K is the symmetric traceless 4×4 matrix

$$K \triangleq \begin{bmatrix} B + B^T - \operatorname{Tr}(B) I_{3\times3} & \mathbf{z} \\ \\ \mathbf{z}^T & \operatorname{Tr}(B) \end{bmatrix} \tag{9}$$

with $\mathbf{z}$ being defined by

$$[\mathbf{z}\times] \triangleq B^T - B \tag{10}$$

This is the basis of Davenport's q-method [7]. The case at hand admits considerable simplification, however. Substituting Eq. (6) for $A(\mathbf{q}_i)$ into Eq. (5) and then into Eq. (9) gives

$$K \triangleq 4\,M - w_{\text{tot}} I_{4\times 4} \tag{11}$$

where $w_{\text{tot}} \triangleq \sum_{i=1}^{n} w_i$, $I_{4\times 4}$ is a 4×4 identity matrix and M is the 4×4 matrix

$$M \triangleq \sum_{i=1}^{n} w_i\, \mathbf{q}_i\, \mathbf{q}_i^T \tag{12}$$

Thus the average quaternion can be found by the following maximization procedure:

$$\bar{\mathbf{q}} = \arg\max_{\mathbf{q}\in\mathbb{S}^3} \mathbf{q}^T M \mathbf{q} \tag{13}$$

The solution of this maximization problem is well known [10]. The average quaternion is the eigenvector of M corresponding to the maximum eigenvalue. This avoids both of the flaws of Eq. (1). The eigenvector is chosen to have unit norm to avoid the first flaw. The second flaw is obviously avoided because changing the sign of any $\mathbf{q}_i$ does not change the value of M. The averaging procedure only determines $\bar{\mathbf{q}}$ up to a sign, which is consistent with the 2:1 nature of the quaternion representation. If the QUEST algorithm is used to find the eigenvector associated with the maximum eigenvalue, then the K matrix in Eq. (11) must be used instead of the M matrix in Eq. (12), because the QUEST algorithm requires a traceless matrix.

A closer look at the attitude error matrix defined by $A(\boldsymbol{\delta}\mathbf{q}_i) \triangleq A(\mathbf{q})A^T(\mathbf{q}_i)$ gives a nice interpretation of the optimization problem. The error quaternion is the product of $\mathbf{q}$ and the inverse of $\mathbf{q}_i$, which can be written as [5]

$$\boldsymbol{\delta}\mathbf{q}_i \triangleq \mathbf{q} \otimes \mathbf{q}_i^{-1} = [\Xi(\mathbf{q}_i) \;\; \mathbf{q}_i]^T\, \mathbf{q} \tag{14}$$

where

$$\Xi(\mathbf{q}) \triangleq \begin{bmatrix} q_4 I_{3\times 3} + [\boldsymbol{\varrho}\times] \\ -\boldsymbol{\varrho}^T \end{bmatrix} \tag{15}$$

Note for future reference that $[\Xi(\mathbf{q}_i) \;\; \mathbf{q}_i]$ is an orthogonal matrix representing a norm-preserving rotation in quaternion space. The vector and scalar parts of the error quaternion are given by

$$\boldsymbol{\delta\varrho}_i = \hat{\mathbf{e}}_i \sin(\delta\phi_i/2) = \Xi^T(\mathbf{q}_i)\, \mathbf{q} \tag{16a}$$

$$\delta q_{4_i} = \cos(\delta\phi_i/2) = \mathbf{q}_i^T \mathbf{q} \tag{16b}$$

with $\hat{\mathbf{e}}_i$ being the unit Euler axis and $\delta\phi_i$ the rotation angle of the error. Substituting Eqs. (16) into Eqs. (2) and (3) and using the quaternion normalization condition shows that the average quaternion is given by

$$\bar{\mathbf{q}} = \arg\min_{\mathbf{q}\in\mathbb{S}^3} \sum_{i=1}^{n} w_i ||\boldsymbol{\delta\varrho}_i||^2 = \arg\min_{\mathbf{q}\in\mathbb{S}^3} \sum_{i=1}^{n} w_i \sin^2(\delta\phi_i/2) \tag{17}$$

The fact that the average quaternion minimizes the weighted sum of the squared lengths of the vector parts of the error quaternions, or the weighted sum of the squares of the sines of the half-error-angles, may be more intuitively pleasing than the argument based on the Frobenius norm. Equation (17) will be used in the sequel for the generalization of the results to non-scalar weights.

Averaging Two Quaternions

The two quaternion case can be solved in closed form. The optimal quaternion average is given by

$$\begin{aligned} \bar{\mathbf{q}} &= \pm\frac{[(w_1 - w_2 + z)\mathbf{q}_1 + 2w_2(\mathbf{q}_1^T\mathbf{q}_2)\mathbf{q}_2]}{||(w_1 - w_2 + z)\mathbf{q}_1 + 2w_2(\mathbf{q}_1^T\mathbf{q}_2)\mathbf{q}_2||} \\ &= \pm\frac{[2w_1(\mathbf{q}_1^T\mathbf{q}_2)\mathbf{q}_1 + (w_2 - w_1 + z)\mathbf{q}_2]}{||2w_1(\mathbf{q}_1^T\mathbf{q}_2)\mathbf{q}_1 + (w_2 - w_1 + z)\mathbf{q}_2||} \end{aligned} \tag{18}$$

where $z \triangleq \sqrt{(w_1 - w_2)^2 + 4w_1w_2(\mathbf{q}_1^T\mathbf{q}_2)^2}$. These two forms are equivalent if $\mathbf{q}_1^T\mathbf{q}_2 \neq 0$. If $\mathbf{q}_1^T\mathbf{q}_2 = 0$, then $z = |w_1 - w_2|$; the first form gives the correct limit $\bar{\mathbf{q}} = \mathbf{q}_1$ if $w_1 > w_2$, while the second form gives the correct limit $\bar{\mathbf{q}} = \mathbf{q}_2$ if $w_1 < w_2$. If $\mathbf{q}_1^T\mathbf{q}_2 = 0$ and $w_1 = w_2$, neither form has a well-defined limit. This is true because the maximum eigenvalue of M, which is equal to $w_1 = w_2$, is not unique in this case. This is the only two-observation case for which the average quaternion is not uniquely defined.

Uniqueness of the Average Quaternion

Because the average quaternion $\bar{\mathbf{q}}$ is the eigenvector associated with the maximum eigenvalue of M, $\bar{\mathbf{q}}$ is unique if and only if the two largest eigenvalues of M are not equal. A sufficient condition for the uniqueness of the average quaternion is shown here. It is assumed there is a reference frame in which every quaternion estimate $\mathbf{q}_i$ differs from the identity quaternion $\mathbf{q}_{\text{ref}} = [0\ \ 0\ \ 0\ \ 1]^T$ by a rotation of less than $\pi/2$. This section proves that the average quaternion $\bar{\mathbf{q}}$ minimizing Eq. (2) is unique with this assumption.

The angle of rotation between $\mathbf{q}_i$ and $\mathbf{q}_{\text{ref}}$ is given by $2\arccos|q_{4_i}|$. When the angle is less than $\pi/2$, $q_{4_i}^2 > 1/2$, hence

$$q_{4_i}^2 > q_{1_i}^2 + q_{2_i}^2 + q_{3_i}^2 \tag{19}$$

Now consider an attitude quaternion that is orthogonal to the identity quaternion $\mathbf{q}_{\text{ref}}$, given by $\mathbf{q}^{\perp} = \begin{bmatrix} q_1^{\perp} & q_2^{\perp} & q_3^{\perp} & 0 \end{bmatrix}^T$. Define the gain function $g(\mathbf{q}) \triangleq \mathbf{q}^T M \mathbf{q}$. The gain functions of $\mathbf{q}^{\perp}$ and $\mathbf{q}_{\text{ref}}$ are

$$g(\mathbf{q}^{\perp}) = \sum_{i=1}^{n} w_i (q_1^{\perp}\, q_{1_i} + q_2^{\perp}\, q_{2_i} + q_3^{\perp}\, q_{3_i})^2 \tag{20}$$

and

$$g(\mathbf{q}_{\text{ref}}) = \sum_{i=1}^{n} w_i\, q_{4_i}^2 \tag{21}$$

We have

$$g(\mathbf{q}_{\text{ref}}) > g(\mathbf{q}^{\perp}) \tag{22}$$

because

$$q_{4_i}^2 > (q_1^{\perp}\, q_{1_i} + q_2^{\perp}\, q_{2_i} + q_3^{\perp}\, q_{3_i})^2 \tag{23}$$

To prove inequality (23), notice that, by the Cauchy inequality,

$$(q_1^{\perp}\, q_{1_i} + q_2^{\perp}\, q_{2_i} + q_3^{\perp}\, q_{3_i})^2 \le \left[(q_1^{\perp})^2 + (q_2^{\perp})^2 + (q_3^{\perp})^2\right] (q_{1_i}^2 + q_{2_i}^2 + q_{3_i}^2) = q_{1_i}^2 + q_{2_i}^2 + q_{3_i}^2 \tag{24}$$

where the fact that $\mathbf{q}^{\perp}$ has unit norm has been used. Inequality (23) then follows upon combining Eqs. (19) and (24).

Now, if the two largest eigenvalues of M are equal, the eigenvectors associated with the maximum eigenvalue span a 2D subspace. The intersection of this subspace and the orthogonal complement of the identity quaternion (the subspace spanned by all $\mathbf{q}^{\perp}$ quaternions) cannot be empty. A quaternion $\mathbf{q}$ in that intersection must satisfy $g(\mathbf{q}) = g(\bar{\mathbf{q}}) \ge g(\mathbf{q}_{\text{ref}})$ and $g(\mathbf{q}) < g(\mathbf{q}_{\text{ref}})$ simultaneously. By contradiction, the two largest eigenvalues cannot be equal, hence the average quaternion is unique.

Matrix Weighted Case

This section expands upon the scalar weighted case to include general matrix weights. For this case, a matrix weighted version of the minimization problem in Eq. (17) is assumed:

$$\bar{\mathbf{q}} \triangleq \arg\min_{\mathbf{q}\in\mathbb{S}^3} \sum_{i=1}^{n} \boldsymbol{\delta\varrho}_i^T R_i^{-1} \boldsymbol{\delta\varrho}_i^T = \arg\min_{\mathbf{q}\in\mathbb{S}^3} \sum_{i=1}^{n} \mathbf{q}^T \Xi(\mathbf{q}_i)\, R_i^{-1} \Xi^T(\mathbf{q}_i)\, \mathbf{q} \tag{25}$$

where R_i^{-1} is the i^{th} symmetric weighting matrix. In this case the average quaternion is the eigenvector corresponding to the maximum eigenvalue of the matrix

$$\mathcal{M} = -\sum_{i=1}^{n} \Xi(\mathbf{q}_i)\, R_i^{-1} \Xi^T(\mathbf{q}_i) \tag{26}$$

If $R_i^{-1} = w_i\, I_{3\times3}$, the identity $\Xi(\mathbf{q}_i)\, \Xi^T(\mathbf{q}_i) = I_{4\times4} - \mathbf{q}_i\, \mathbf{q}_i^T$ can be used to show that $\mathcal{M} = M - w_{\text{tot}}\, I_{4\times4}$, where M is given by Eq. (12). The traceless K matrix for the matrix-weighted case is

$$K = 4\,\mathcal{M} + \operatorname{Tr}\left(\sum_{i=1}^{n} R_i^{-1}\right) I_{4\times4} = \sum_{i=1}^{n} K_i \tag{27}$$

with

$$K_i \triangleq -4\,\Xi(\mathbf{q}_i)\, R_i^{-1} \Xi^T(\mathbf{q}_i) + \operatorname{Tr}\left(R_i^{-1}\right) I_{4\times4} = [\Xi(\mathbf{q}_i) \;\; \mathbf{q}_i]\, \tilde{K}_i\, [\Xi(\mathbf{q}_i) \;\; \mathbf{q}_i]^T \tag{28}$$

where

$$\tilde{K}_i = 2 \begin{bmatrix} 2\,\mathcal{F}_i - \operatorname{Tr}(\mathcal{F}_i) I_{3\times3} & \mathbf{0}_{3\times1} \\ \\ \mathbf{0}_{3\times1}^T & \operatorname{Tr}(\mathcal{F}_i) \end{bmatrix} \tag{29}$$

$$\mathcal{F}_i \triangleq \frac{1}{2} \operatorname{Tr}(R_i^{-1})\, I_{3\times3} - R_i^{-1} \tag{30}$$

and $\mathbf{0}_{3\times1}$ denotes a 3×1 vector of zeros.

The matrix $\tilde{K}_i$ has the same structure as the one in Eq. (9) with the corresponding attitude profile matrix given by $\tilde{B}_i = 2\mathcal{F}_i$. Because $[\Xi(\mathbf{q}_i) \;\; \mathbf{q}_i]$ is an orthogonal matrix, Eq. (28) shows that K_i and $\tilde{K}_i$ are related by a rotation. Their corresponding attitude profile matrices are related by the same rotation. The attitude profile matrix corresponding to K_i is given by $B_i = \tilde{B}_i\, A(\mathbf{q}_i)$, as is shown in the Appendix, and the attitude profile matrix corresponding to K is given by

$$B = \sum_{i=1}^{n} B_i = 2 \sum_{i=1}^{n} \mathcal{F}_i A(\mathbf{q}_i) \tag{31}$$

If $R_i^{-1} = w_i\, I_{3\times3}$, Eqs. (27) and (31) reduce to the corresponding quantities for scalar weights.

Relation to Maximum Likelihood Estimation

The relationship of the minimization problem in Eq. (25) to a maximum likelihood estimation problem is now shown. Reference [11] establishes a maximum likelihood problem for attitude matrices, which is related to the averaging problem in this paper. The maximum likelihood estimate of the attitude matrix, denoted $\hat{A}_{\text{ML}}$, is given as

$$\hat{A}_{\text{ML}} \triangleq \arg \min_{A \in \mathbb{SO}^3} \frac{1}{2} \sum_{i=1}^{n} \operatorname{Tr}\left[(A_i - A)^T \mathcal{F}_i (A_i - A)\right] \tag{32}$$

where A_i is the i^{th} given attitude matrix, $\mathbb{SO}^3$ denotes the (special orthogonal) group of rotational matrices and the matrix $\mathcal{F}_i$ defined by Eq. (30) is the Fisher information matrix of the small attitude matrix errors, with R_i being the covariance of the small attitude vector errors. Using properties of the matrix trace, we can write

$$J_i(A) \triangleq \frac{1}{2} \operatorname{Tr}\left[(A_i - A)^T \mathcal{F}_i (A_i - A)\right] = \operatorname{Tr}(\mathcal{F}_i) - \frac{1}{2} \operatorname{Tr}(A\, B_i^T) \tag{33}$$

where B_i is the attitude profile matrix defined by Eq. (31). Using $A = A(\mathbf{q})$ and the definition of $\mathcal{F}_i$ gives J_i as a function of the quaternion:

$$J_i(\mathbf{q}) = \operatorname{Tr}(\mathcal{F}_i) - \frac{1}{2} \operatorname{Tr}\left[A(\mathbf{q})\, B_i^T\right] = \frac{1}{2}\left[\operatorname{Tr}(R_i^{-1}) - \mathbf{q}^T K_i\, \mathbf{q}\right] = 2\, \boldsymbol{\delta\varrho}_i^T R_i^{-1} \boldsymbol{\delta\varrho}_i^T \tag{34}$$

where Eqs. (8) and (28) have been used. Using the invariance property of the maximum likelihood estimate [12],

$$\hat{A}_{\text{ML}} = A(\hat{\mathbf{q}}_{\text{ML}}) \tag{35}$$

where $\hat{\mathbf{q}}_{\text{ML}}$ is the maximum likelihood estimate of the quaternion. Hence, using Eq. (34) in Eq. (32) gives

$$\hat{\mathbf{q}}_{\text{ML}} = \arg \min_{\mathbf{q} \in \mathbb{S}^3} \sum_{i=1}^{n} \boldsymbol{\delta\varrho}_i^T R_i^{-1} \boldsymbol{\delta\varrho}_i^T \tag{36}$$

which is identical with Eq. (25). Thus, we conclude that the average quaternion defined by Eq. (25) is a maximum likelihood estimate.

The error-covariance associated with the small-angle attitude errors of the average quaternion is given by

$$\bar{R} = \left\{ \Xi^T(\bar{\mathbf{q}}) \left[\sum_{i=1}^{n} \Xi(\mathbf{q}_i) R_i^{-1} \Xi^T(\mathbf{q}_i) \right] \Xi(\bar{\mathbf{q}}) \right\}^{-1} \tag{37}$$

For small errors, this matrix is well approximated by

$$\bar{R} \approx \left(\sum_{i=1}^{n} R_i^{-1} \right)^{-1} \tag{38}$$

Equation (38) can be used to develop 3-sigma bounds for the attitude errors between the average and true quaternion.

Conclusions

An algorithm is presented for determining the average norm-preserving quaternion from a set of weighted quaternions. The solution involves performing an eigenvalue/eigenvector decomposition of a matrix composed of the given quaternions and weights. For both the scalar- and matrix-weighted cases, the optimal average quaternion can be determined by the computationally efficient QUEST algorithm. A uniqueness sufficient condition is presented for the scalar-weighted case. In the matrix-weighted case, when the matrix weight is given by the inverse of the covariance of the small attitude-vector errors, the average quaternion is shown to be a maximum likelihood estimate. Thus, in this case, the averaging procedure introduced here enjoys the well known desirable properties of maximum likelihood estimators.

Appendix

This appendix proves that $B_i = \tilde{B}_i\, A(\mathbf{q}_i)$. Here we let unsubscribed $\mathbf{q}$ denote a normalized, but otherwise completely arbitrary, quaternion. Then from Eqs. (8) and (28),

$$\mathrm{Tr}[A(\mathbf{q})B_i^T] = \mathbf{q}^T K_i\, \mathbf{q} = \mathbf{q}^T \left[\Xi(\mathbf{q}_i) \quad \mathbf{q}_i\right] \tilde{K}_i \left[\Xi(\mathbf{q}_i) \quad \mathbf{q}_i\right]^T \mathbf{q} \tag{A-1}$$

Using Eq. (14) and the fact that the attitude profile matrix of $\tilde{K}_i$ is $\tilde{B}_i$ gives

$$\mathrm{Tr}[A(\mathbf{q})B_i^T] = \boldsymbol{\delta}\mathbf{q}_i^T\, \tilde{K}_i\, \boldsymbol{\delta}\mathbf{q}_i = \mathrm{Tr}[A(\boldsymbol{\delta}\mathbf{q}_i)\tilde{B}_i^T] \tag{A-2}$$

It follows from Eq. (14) that $A(\boldsymbol{\delta}\mathbf{q}_i) = A(\mathbf{q} \otimes \mathbf{q}_i^{-1}) = A(\mathbf{q})A^T(\mathbf{q}_i)$, so

$$\mathrm{Tr}[A(\mathbf{q})B_i^T] = \mathrm{Tr}[A(\mathbf{q})A^T(\mathbf{q}_i)\tilde{B}_i^T] \tag{A-3}$$

Since this must be true for any quaternion $\mathbf{q}$, it follows that

$$B_i = [A^T(\mathbf{q}_i)\tilde{B}_i^T]^T = \tilde{B}_i\, A(\mathbf{q}_i) \tag{A-4}$$

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