Consistency of post-Newtonian waveforms with numerical relativity

John G. Baker,1 James R. van Meter,1,2 Sean T. McWilliams,3 Joan Centrella,4 and Bernard J. Kelly1
1Gravitational Astrophysics Laboratory, NASA Goddard Space Flight Center, 8800 Greenbelt Rd., Greenbelt, MD 20771, USA
2Center for Space Science & Technology, University of Maryland Baltimore County, Physics Department, 1000 Hilltop Circle, Baltimore, MD 21250
3University of Maryland, Department of Physics, College Park, MD 20742, USA
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General relativity predicts the gravitational radiation signatures of mergers of compact binaries, such as coalescing binary black hole systems. Derivations of waveform predictions for such systems are required for optimal scientific analysis of observational gravitational wave data, and have so far been achieved primarily with the aid of the post-Newtonian (PN) approximation. The quality of this treatment is unclear, however, for the important late inspiral portion. We derive late-inspiral waveforms via a complementary approach, direct numerical simulation of Einstein’s equations, which has recently matured sufficiently for such applications. We compare waveform phasing from simulations covering the last ~14 cycles of gravitational radiation from an equal-mass binary system of nonspinning black holes with the corresponding 3PN and 3.5PN orbital phasing. We find agreement consistent with internal error estimates based on either approach at the level of one radian over ~10 cycles. The result suggests that PN waveforms for this system are effective roughly until the system reaches its last stable orbit just prior to the final merger.

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Compact astrophysical binaries spiral together due to the emission of gravitational radiation. Calculating the dynamics of these systems, and the corresponding gravitational waveforms, has been a central problem in general relativity for several decades. With first-generation interferometric detectors such as LIGO, VIRGO, and GEO600 now operating, and development moving forward on the space-based LISA mission, the need for accurate and reliable waveforms for gravitational wave data analysis has become urgent.

Post-Newtonian (PN) methods, based on expansions in the parameter \( \varepsilon \sim v/c \), have been the major analytic tool used to calculate the system dynamics and waveforms during the early part of the inspiral, when the binary components are relatively widely separated and thus have a small orbital frequency [1]. Currently, gravitational wave data analysis for binary inspiral relies on waveforms derived from PN methods [2]. The current predicted orbital phase is available up to \( O(\varepsilon^5) \), which is referred to as 3.5PN order. However the convergence properties of the PN sequence are not well understood, and it is not yet clear how well PN predictions work late in the inspiral when frequencies are high.

Numerical relativity, in which the full set of Einstein’s equations is solved on a computer, is needed to handle the final stages of the binary evolution, when the components inspiral rapidly and merge. Recently, there has been dramatic progress in the use of numerical relativity to simulate the final inspiral and merger of black holes [3, 4, 5, 6, 7, 8, 9, 10, 11]. These breakthroughs have allowed numerical simulations with increasingly wider initial separations, producing longer waveforms in the late inspiral regime. The simulations begin at an angular gravitational wave frequency \( \omega \sim 0.051M^{-1} \). The frequency then sweeps upward through roughly an order of magnitude while the black holes undergo ~7 orbits, thus producing ~14 gravitational wave cycles before merger. For such long-lasting simulations, the primary consideration in providing a realistic initial data model is to set up the orbiting black holes with minimal eccentricity, as gravitating binary systems of comparable-mass objects are expected to...
FIG. 1: Gravitational strain waveforms from the merger of equal-mass Schwarzschild black holes. The solid curve is the waveform from the high resolution numerical simulation, and the dashed curve is a PN waveform with 3.5PN order phasing and 2.5PN order amplitude accuracy. Time \( t = 0 \) is the moment of peak radiation amplitude in the simulation.

circularize rapidly through the emission of gravitational radiation. We have selected an initial black hole configuration with relatively little eccentricity of less than one percent, as measured below.

Fig. 1 shows a comparison of gravitational wave strain generated by our highest-resolution numerical run and that predicted by the PN approximation with 3.5PN phasing and 2.5PN (beyond leading order) amplitude accuracy. The waves are based on the dominant \( \ell = 2, m = 2 \) spin-weighted spherical harmonic of the radiation, and represent an observation made on the system’s equatorial plane, where only one polarization component contributes to the measured strain. The initial phase and initial time of the waves have been adjusted so that the frequency and phase for each waveform agree early in the simulation, at \( t = -1000M \). We will quantitatively study the evident phase agreement below.

To conduct comparisons with PN calculations, we need to extract an instantaneous gauge-invariant polarization phase \( \Phi \) and angular frequency \( \omega \) from our simulations. These are derived from the first time-derivative of the gravitational wave strain, which is a robust quantity in the numerical data. This frequency corresponds to the sweep rate of the polarization angle of the circularly polarized gravitational wave that can be observed on the system’s rotation axis.

We define eccentricity as deviation from an underlying smooth, secular trend. We obtain a monotonic “secular” frequency-time relation by modeling the frequency \( \omega \) as a fourth-order monotonic polynomial \( \omega_c(t) \), plus an eccentric modulation in the waveform angular frequency, \( \omega(t) = \omega(t) - \omega_c(t) = A \sin(\Phi(t)) \), where \( \Phi(t) \) is a quadratic function of time. The fitting procedure is illustrated in Fig. 2, which shows the original numerical frequency \( \omega(t) \), the fit, and \( \omega_c(t) \). Fitting this data yields \( A = 8(\pm1) \times 10^{-4} \). For Keplerian systems, conserved angular momentum is proportional to \( r^2 \omega \) so the (radial) eccentricity corresponds to half the fractional amplitude of frequency modulations: \( e = A/(2\omega) \). In our case the eccentricity starts near 0.008, decreasing by a factor of three by the time \( \omega_c M \sim 0.15 \). We will compare our simulation with non-eccentric PN calculations, with the expectation that small eccentricities have minimal effect on the important underlying secular trend in the rate at which frequency sweeps up approaching merger.

The phasing of the waveform is critical for gravitational wave observation. For data analysis, the optimal methods for both detection and parameter estimation rely on matched filtering, which employs a weighted inner product that can be expressed in Fourier space as \( <h, s> = \int (\hat{h}^*(f)\hat{s}(f) + \hat{h}(f)\delta^*(f))/S_n(f)df \), where \( h \) is the template being used, \( s \) is the signal being analyzed, and \( S_n \) is the one-sided power spectral density of the detector’s noise [13]. A template that maximizes \( <h, s> \) will provide an optimal filter. While a template with time-varying amplitude can emphasize some portions of a signal and not others, the crucial factor is the relative phasing of the template and signal. The inner product will cease to accumulate in sweeping through frequency if the template and the signal evolve to be out of phase with each other by more than a half-cycle, decreasing the effectiveness of the procedure.

Our key objective is to compare phasing between nu-
merical and PN waveforms. To avoid issues with timing alignment, we will compare phases as a function of polariza-
tion frequency, which corresponds to twice the or-
bital frequency in the PN case. For circular inspiral this
frequency should grow monotonically in time, with the
frequency \( \omega_c \) providing a physical reference of the "hard-
ness" of the tightening binary.

Circular inspiral phasing information is typically de-

rived in PN theory by imposing an energy balance re-
duction to the rate at which \( \omega_c \) evolves from the ra-
diation rate at a specified value of \( \omega_c \) [1]. Though not
strictly derived in the PN context, this physically sensi-
tive condition currently allows the determination of the
chirp rate \( \omega_c (\omega_c) \) (or something equivalent) up to 3.5PN
order [14]. From such a relation information about phase
and time are determined by integrating \( dq_c/d\omega_c = \omega_c/\omega_c \)
and \( dt/d\omega_c = 1/\omega_c \). The phasing information can be re-

presented by any one of several relations between phase,
frequency and time. Various approaches take the PN-
expanded representation of one of these relations as the
PN "result" for waveform phasing [1, 3, 14]. We consider
both the PN expansion of the chirp rate, numerically in-
TEGRATED as needed, as well as the PN expansion of the
phase.

For the purpose of comparison with our numerical sim-
ulations, we invert the monotonic function \( \omega_c(t) \) to obtain
the phase as a function of frequency: \( \phi(\omega_c) = \phi(t(\omega_c)) \).
Note that the effect of eccentricity is not removed from
\( \phi \), though the "circularized" frequency \( \omega_c \) does provide
the abscissa according to which phases are compared in
the different treatments.

Fig. 3 shows the wave phases; here the phases are ad-
justed by addition of a constant so that each phase van-
ishes at \( \omega_c M = 0.054 \), corresponding to the time 1000M
before the radiation merger peak in the high resolution
numerical simulation. The sequence of numerical results
converges at fourth-order (verified in Fig. 4), allowing us
to project by Richardson extrapolation to a fifth-order
accurate result (thick solid line in Fig.3).

In Fig. 3 we compare the numerical simulation re-
sults with two versions of PN-phasing, based on either
the PN expansion of the phase or on the numerically in-
TEGRATED PN expansion of the chirp rate. We show each
of these at 3PN and 3.5PN order. While the 3PN phase
seems to agree closely with the extrapolated numerical
phase, the 3.5PN phase moves away, and develops an
unphysical negative slope after \( \omega_c M \sim 0.15 \). The agree-
ment of the extrapolated numerical result with the in-
TEGRATED PN chirp rate improves from 3PN to 3.5PN, with
the 3.5PN result showing striking agreement up to about
\( \omega_c M \sim 0.15 \).

We look more quantitatively at the differences among
the phases in Fig. 4. The solid curve shows the differ-
ence between our medium and high resolution results,
while the dotted curve shows the difference between our
low and medium resolution results, scaled such that for
fourth-order convergence the curves should superpose.
This is indeed demonstrated to good approximation. A
good estimate for the error of the phase in the high resolu-
tion run is given by its difference from the phase obtained
by Richardson extrapolation; this comes out to \( \sim 93\% \)
of the med-high (solid) curve shown in the figure. Note
that the cumulative errors in the numerically-generated
waveforms accrue primarily before the orbital frequency
\( \omega_c M = 0.1 \). (This makes sense generally since the simu-
lations spend much more time at lower frequencies.)

Without monotonic convergence between the 2PN,
2.5PN and 3PN at the frequencies considered here, it is
difficult to estimate errors in the PN phase. Nonetheless
we tentatively take the difference between the integrated
3PN and 3.5PN chirp rates, shown by the dashed line in
Fig. 4, as a measure of PN errors.

The trend in the slope of these error curves indicates
the rate at which phase error accumulates, as independ-
ently estimated within each approach. Fig. 4 sug-
ests that phasing errors for our high resolution simu-
lation accumulate more quickly than PN phasing errors
for \( \omega_c M \leq 0.08 \) (at \( t < -300M \)), with the numerical simu-
lation phasing being more accurate than PN at higher
frequencies. In both cases, the phase error accumulates
at roughly two radians by \( \omega_c M \sim 0.2 \) as the black holes
begin to plunge together.

We now address the central objective of this letter,
a quantitative comparison of numerical and PN phasing
results. We compare the Richardson-extrapolated phase,
our best simulation estimate, with the integrated 3.5PN
chirp rate result. The difference is shown in the solid

![FIG. 3](image-url)
results of new long-lasting numerical simulations. These simulations have sufficient accuracy to provide a meaningful comparison with PN waveforms over the last $t \sim 1000M$ of the coalescence, specifically addressing a binary system of equal-mass non-spinning black holes. We find phase agreement consistent with internal phase-error estimates conducted in each approach, indicating that phase accuracies within a few radians are now achievable for this part of the coalescence waveform.

We emphasize, however, that there is still much important work to be done in improving and further assessing PN and numerical simulation waveforms. Certainly we have only addressed one case in a large parameter space of potential binaries, which will inevitably include systems, such as rapidly precessing unequal-mass spinning binaries, which are harder to treat with present PN and numerical techniques. With either approach, even for our simple case, a non-negligible amount of phasing error accumulates over the range studied, and more will have accumulated at lower-frequencies addressable through the PN approximation. We expect continuing developments in numerical simulations and the pursuit of higher-order PN treatments to be crucial for developing a refined understanding of coalescence waveforms, which will be crucial in some data analysis applications for gravitational wave observations.

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