Small scale response and modeling of periodically forced turbulence

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Abstract

The response of the small scales of isotropic turbulence to periodic large scale forcing is studied using two-point closures. The frequency response of the turbulent kinetic energy and dissipation rate, and the phase shifts between production, energy and dissipation are determined as functions of Reynolds number. It is observed that the amplitude and phase of the dissipation exhibit non-trivial frequency and Reynolds number dependence that reveals a filtering effect of the energy cascade. Perturbation analysis is applied to understand this behavior which is shown to depend on distant interactions between widely separated scales of motion. Finally, the extent to which finite dimensional models (standard two-equation models and various generalizations) can reproduce the observed behavior is discussed.

I. INTRODUCTION

Statistical transients in turbulence remain a major challenge to both theory and modeling. The mechanisms by which a turbulent flow readjusts to new conditions, for example in boundary layers with sudden changes in wall roughness or pressure gradient[1], are not entirely understood and continue to resist prediction by models.

Another class of statistically time-dependent turbulent flows is defined by the presence of periodically oscillating forcing. The classic example is steady pipe flow with small superposed oscillations of the mean pressure gradient. This flow has been the subject of extensive experimental[2,3] theoretical[4,5] and numerical[6] investigation. There are two obvious limits: the 'static' limit of slow oscillations, in which the turbulence evolves through a sequence of local steady states, and a limit of 'frozen' turbulence in which the turbulence does not respond at all to the oscillations.

Analysis of oscillating pipe flow typically concentrates on the phase relations among the wall shear, centerline velocity, and pressure perturbation. These quantities prove remarkably difficult to predict at frequencies intermediate between the static and frozen limits even if the problem admits a linearized description, indicating unanticipated subtleties in the dynamics; indeed, the only entirely adequate predictions are by Large Eddy Simulation[6], which is very surprising in view of the apparent simplicity of the problem.

Recently, the problem of periodically forced homogeneous isotropic turbulence has been proposed[7] and investigated theoretically,[8] by numerical simulations,[9] and by experiments using time-dependent grids[10]. Because of the absence of complications like near-wall behavior, this problem provides an ideal setting in which to investigate the time-dependent spectral dynamics of turbulence.

Previous work on this problem has been motivated by a search for resonance-like energy response near a critical frequency proportional to the inverse large-eddy turnover time, and perhaps at integer multiples of this frequency as well. This paper focuses instead on the properties of the dissipation rate. At frequencies intermediate between the static and frozen turbulence limits, nontrivial Reynolds number dependent properties are found. The energy cascade acts as a filter that suppresses oscillations at small scales, but as in the oscillating pipe flow, the details are more complex than the simple problem statement would suggest.

The main results are obtained by the Eddy Damped Quasi-Normal Markovian (EDQNM)

closure[11,12]. The predictions of this closure for periodically forced turbulence are in reasonable qualitative agreement with existing results. Elementary arguments show that at forcing frequency ω, the amplitude of the energy and dissipation rate oscillations vary as ω^{-1} for large frequencies. But the calculations show that the dissipation rate modulation amplitude exhibits nontrivial ω^{-3} scaling in the intermediate frequency range, and the phase difference between the production and the dissipation rate has complex dependence on both ω and Reynolds number in this range.

To understand this behavior, we apply asymptotic analysis to two simpler models: the classical Heisenberg model[13,14] and a recent generalization[15]. In these models, the details of triad interactions are suppressed, but the essential idea of nonlocal interaction is retained. We show analytically how the energy cascade filters the oscillations, and that this filtering is responsible for the observations.

Some finite dimensional models of the two-equation type will be considered. The two-equation model is correct in both the static and frozen limits, but misses important features of the dynamics at intermediate frequencies, including the Reynolds number dependence of the dissipation. A more complex three-equation model allows for more complex phase relations, but is also incapable of capturing the Reynolds number dependence. It should be emphasized that this dependence is not a low Reynolds number effect.

A related problem involving periodic forcing is oscillatory homogeneous shear flow[16]. This problem has important, and even dominant linear effects; it therefore has a somewhat different character from periodically forced isotropic turbulence, in which only nonlinear mechanisms are important. Another related problem can be mentioned, in which turbulence is forced periodically at the boundary of the flow region[17,18]. This flow has many interesting similarities to periodically forced isotropic turbulence; although it is simpler in many respects than periodically modulated pipe flow, the dynamics of this problem may include effects of turbulent diffusion as well as energy transfer and may therefore not be entirely amenable to the present type of analysis.

II. DEFINITIONS AND ELEMENTARY PROPERTIES

The spectral evolution equation for time-dependent forced homogeneous isotropic turbulence is[14]

$$\dot{E}(\kappa,t) = P(\kappa,t) - T(\kappa,t) - 2\nu\kappa^2 E(\kappa,t) \tag{1}$$

where $E(\kappa,t)$ is the energy spectrum and $T(\kappa,t)$ is the energy transfer due to nonlinear interactions. The production spectrum $P(\kappa,t)$ is assumed to be localized near some wavenumber $\kappa_P(t)$. Consider a basic steady state, defined by the time-independent form of Eq. (1)

$$0 = \bar{P}(\kappa) - \bar{T}(\kappa) - 2\nu\kappa^2 \bar{E}(\kappa). \tag{2}$$

The problem of periodically forced turbulence is formulated by introducing a periodic perturbation of the production spectrum,

$$P(\kappa,t) = \bar{P}(\kappa) + \tilde{P}(\kappa)\cos(\omega t) \tag{3}$$

where we will assume

$$\tilde{P}(\kappa) = \varepsilon \bar{P}(\kappa) \tag{4}$$

with $\varepsilon \ll 1$, so that the problem can be analyzed by linearization about the steady state defined by Eq. (2). Then

$$E(\kappa,t) = \bar{E}(\kappa) + \delta E(\kappa,t) \tag{5}$$

with $\delta E(\kappa,t) \ll \bar{E}(\kappa)$. If at sufficiently long times, $E(\kappa,t)$ becomes periodic in time, linearity implies that the period is ω, hence

$$\delta E(\kappa,t) = \tilde{E}(\kappa)\cos(\omega t + \phi_E(\kappa)). \tag{6}$$

In terms of the quantities

$$\tilde{F}(\kappa) = \tilde{E}(\kappa)\cos\phi_E(\kappa) \quad \tilde{G}(\kappa) = \tilde{E}(\kappa)\sin\phi_E(\kappa) \tag{7}$$

$\delta E(\kappa,t)$ is written as

$$\delta E(\kappa,t) = \cos(\omega t)\tilde{F}(\kappa) - \sin(\omega t)\tilde{G}(\kappa). \tag{8}$$

The basic time dependent single-point moments: total production $P(t)$, turbulent kinetic energy $k(t)$, and dissipation rate $\epsilon(t)$, are expressed in terms of their time averages $\bar{P}$, $\bar{k}$,

and $\bar{\epsilon}$ and their phase averages $\tilde{P}$, $\tilde{k}$, and $\tilde{\epsilon}$ as

$$P(t) = \bar{P} + \tilde{P}\cos(\omega t) \tag{9}$$

$$k(t) = \bar{k} + \tilde{k}\cos(\omega t + \phi_k) \tag{10}$$

$$\epsilon(t) = \bar{\epsilon} + \tilde{\epsilon}\cos(\omega t + \phi_\epsilon) \tag{11}$$

where

$$\begin{aligned} \bar{P} &= \int_0^\infty d\kappa\ \bar{P}(\kappa) \quad \tilde{P} = \int_0^\infty d\kappa\ \tilde{P}(\kappa) \\ \bar{k} &= \int_0^\infty d\kappa\ \bar{E}(\kappa) \quad \bar{\epsilon} = \int_0^\infty d\kappa\ 2\nu\kappa^2\bar{E}(\kappa) \end{aligned} \tag{12}$$

and in view of Eq. (8),

$$\tilde{k}\cos(\phi_k) = \int_0^\infty d\kappa\ \tilde{F}(\kappa) \quad \tilde{k}\sin(\phi_k) = \int_0^\infty d\kappa\ \tilde{G}(\kappa) \tag{13}$$

$$\tilde{\epsilon}\cos(\phi_\epsilon) = \int_0^\infty d\kappa\ 2\nu\kappa^2\tilde{F}(\kappa) \quad \tilde{\epsilon}\sin(\phi_\epsilon) = \int_0^\infty d\kappa\ 2\nu\kappa^2\tilde{G}(\kappa). \tag{14}$$

For simplicity of notation, the spectral densities $\bar{P}(\kappa)$ and $\tilde{P}(\kappa)$ are distinguished from the corresponding single-point moments $\bar{P}$ and $\tilde{P}$ by their arguments rather than by a new letter.

The simplest formulation of the problem seeks the dependence of the phase averaged amplitudes $\tilde{k}$ and $\tilde{\epsilon}$ and the phase shifts ϕ_k, ϕ_ϵ on the forcing frequence ω; $\tilde{k}$ will be called the *modulated energy* and $\tilde{\epsilon}$ the *modulated dissipation*. $P(t)$, $k(t)$, and $\epsilon(t)$ are related, independently of any closure hypothesis, by the energy balance, obtained by integrating Eq. (1) over all wavenumbers:

$$\dot{k}(t) = P(t) - \epsilon(t) \tag{15}$$

where energy conservation by nonlinear interactions implies that

$$\int_0^\infty d\kappa\ T(\kappa, t) = 0. \tag{16}$$

Substituting Eqs. (9)–(11) in Eq. (15) and subtracting the steady balance $\bar{P} = \bar{\epsilon}$ gives the relation for modulated quantities

$$-\omega\tilde{k}\sin(\omega t + \phi_k) = \tilde{P}\cos(\omega t) - \tilde{\epsilon}\cos(\omega t + \phi_\epsilon) \tag{17}$$

or equivalently

$$-\omega\tilde{k}\sin\phi_k = \tilde{P} - \tilde{\epsilon}\cos\phi_\epsilon \tag{18}$$

$$-\omega\tilde{k}\cos\phi_k = \tilde{\epsilon}\sin\phi_\epsilon. \tag{19}$$

Elementary trigonometric identities give the explicit relations

$$\begin{aligned}\tilde{k} &= \frac{1}{\omega}\sqrt{(\tilde{\epsilon}\sin\phi_\epsilon)^2 + \left(\tilde{P} - \tilde{\epsilon}\cos\phi_\epsilon\right)^2} \\ \tan\phi_k &= \frac{\tilde{P} - \tilde{\epsilon}\cos\phi_\epsilon}{\tilde{\epsilon}\sin\phi_\epsilon}\end{aligned} \tag{20}$$

and the equivalent relations

$$\begin{aligned}\tilde{\epsilon} &= \sqrt{\left(\omega\tilde{k}\cos\phi_k\right)^2 + \left(\tilde{P} + \omega\tilde{k}\sin\phi_k\right)^2} \\ \tan\phi_\epsilon &= -\frac{\omega\tilde{k}\cos\phi_k}{\tilde{P} + \omega\tilde{k}\sin\phi_k}.\end{aligned} \tag{21}$$

Although additional assumptions are obviously required to close the problem, explicit closure hypotheses are not required to reach some simple but useful conclusions about the limits of asymptotically high and low oscillation frequency. Linearity implies that the frequency of the perturbation at any scale of motion must be the imposed frequency ω, but in the inertial range, disturbances are damped on the Kolmogorov time-scale $(\epsilon^{1/3}\kappa^{2/3})^{-1}$; accordingly, we anticipate that if $\omega \gg \epsilon^{1/3}\kappa^{2/3}$, the perturbations must be over-damped, but that they are active and only weakly damped if $\omega \ll \epsilon^{1/3}\kappa^{2/3}$. This argument suggests that in the static limit $\omega \downarrow 0$, the turbulence follows the slow modulations at all scales of motion, so that also $\phi_\epsilon(\omega), \phi_k(\omega) \downarrow 0$. Then Eq. (18) gives $\tilde{P} \approx \tilde{\epsilon}$; Eq. (19) is not satisfied exactly, but is approximately true since $\omega \approx 0$. Assuming that for slow modulations, the relation $\epsilon(t) = C_\epsilon k(t)^{3/2}/L$ remains valid with time-independent L, and that the small perturbations $\tilde{k}$ and $\tilde{\epsilon}$ are nearly static, then $\tilde{\epsilon}/\epsilon = (3/2)\tilde{k}/k$. These observations suggest that in this limit, the single-point modulated quantities admit series expansions in positive powers of ω:

$$\tilde{k} = \frac{2}{3}\frac{\bar{k}}{\bar{\epsilon}}\tilde{P} + O(\omega^2) \tag{22}$$

$$\tilde{\epsilon} = \tilde{P} + O(\omega^2) \tag{23}$$

$$\phi_k = O(\omega) \tag{24}$$

$$\phi_\epsilon = O(\omega) \tag{25}$$

where the powers of ω are suggested by the parity properties of Eqs. (18) and (19) under a change of sign of ω. Equivalently, to lowest order, we have

$$\tilde{\epsilon} = \bar{\omega}\tilde{k}; \quad \bar{\omega} = \frac{3}{2}\frac{\bar{\epsilon}}{\bar{k}} \tag{26}$$

where the frequency $\bar{\omega}$ defined by this equation is the 'critical' frequency discussed by Lohse[7].

In the 'frozen turbulence' limit $\omega \uparrow \infty$, we see that Eq. (18) is satisfied if $\phi_k \approx -\pi/2$; then $\tilde{k} \approx \tilde{P}/\omega$. If, as the simple argument above suggests, the perturbations are overdamped throughout the inertial range, the only scales of motion at which the oscillating force can be effective are the forcing scales themselves. If so, the modulated dissipation will also take place in this range of scales, so that

$$\tilde{\epsilon} = \int_0^\infty d\kappa\ 2\nu\kappa^2\tilde{E}(\kappa) \approx 2\nu\kappa_P^2 \int_0^\infty d\kappa\ \tilde{E}(\kappa) \approx 2\nu\kappa_P^2\tilde{k} \approx 2\nu\kappa_p^2\frac{\tilde{P}}{\omega}. \tag{27}$$

As in the previous limit, Eq. (19) is not satisfied exactly, suggesting that the perturbation quantities should admit series expansions in negative powers of ω:

$$\tilde{k} = \frac{\tilde{P}}{\omega} + O(\omega^{-3}) \tag{28}$$

$$\tilde{\epsilon} = 2\nu\kappa_P^2\frac{\tilde{P}}{\omega} + O(\omega^{-3}) \tag{29}$$

$$\phi_k = -\frac{\pi}{2} + O(\omega^{-1}) \tag{30}$$

$$\phi_\epsilon = -\frac{\pi}{2} + O(\omega^{-1}). \tag{31}$$

III. NUMERICAL RESULTS FROM SPECTRAL CLOSURE

In this section, we apply the EDQNM spectral closure[11] to this problem. The exact formulation of the model and the numerical method is the same as in Touil *et al.*[19] and for details we refer to that work. In this closure, nonlinear interactions among wavenumber triads of different 'shapes' are considered explicitly, with a definite weighting derived perturbatively from the governing equations.

The energy spectrum was initialized by a von Kármán spectrum; however, the influence of the initial energy spectrum vanishes after a transient and the results reported are evaluated after reaching an asymptotic state. The large scale forcing is

$$P(\kappa, t) = \alpha\left(\bar{P} + \tilde{P}\cos(\omega t)\right) exp(-\gamma\kappa^2), \quad \kappa \geq \kappa_0 > 0 \tag{32}$$

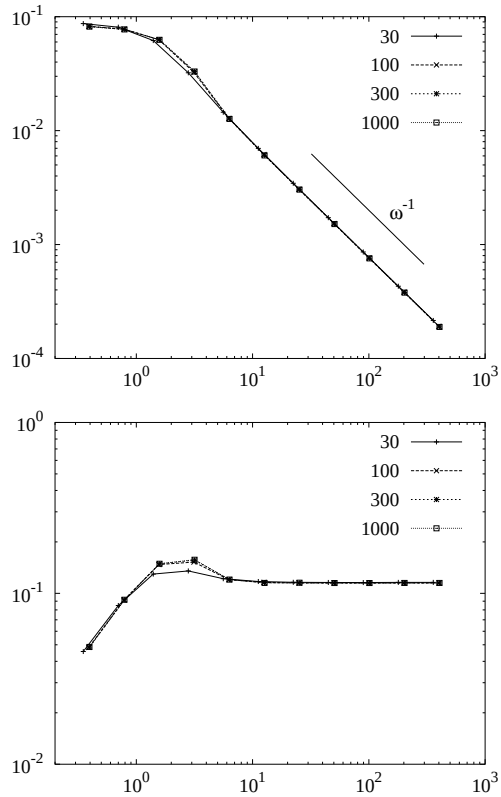

FIG. 1: top: $\tilde{k}$ as a function of $\omega/\bar{\omega}$ for R_λ varying from 30 to 1000; bottom: same for $\omega\tilde{k}$

where α is the normalization constant such that $\alpha \int_{\kappa_0}^{\infty} \exp(-\gamma\kappa^2) = 1$, with $\tilde{P}/\bar{P} = 0.125$ and $\gamma = 0.5$. The spectral resolution is approximately 20 wavenumbers per decade. The results are shown in Figures 1–4.

1. *Modulated kinetic energy $\tilde{k}$.*

Figure 1 shows a plateau for $\tilde{k}$ at low frequencies and ω^{-1} dependence for high frequencies as suggested by the elementary arguments leading to Eqs. (22) and (28): the static and frozen turbulence limits are well reproduced. In our calculations, no local maximum of $\tilde{k}$ is observed. In the DNS results of Kuczaj *et al.*[9] a small local maximum was present around the turbulent frequency $\bar{\omega}$ defined in Eq. (26), but in the shell model study by von der Heydt, Grossman and Lohse[20], this maximum was absent. The existence and explanation of this maximum remain open questions. However, all of the available data exhibits a clear response maximum of the compensated quantity $\omega\tilde{k}$ near $\bar{\omega}$. This maximum is also prominent in the

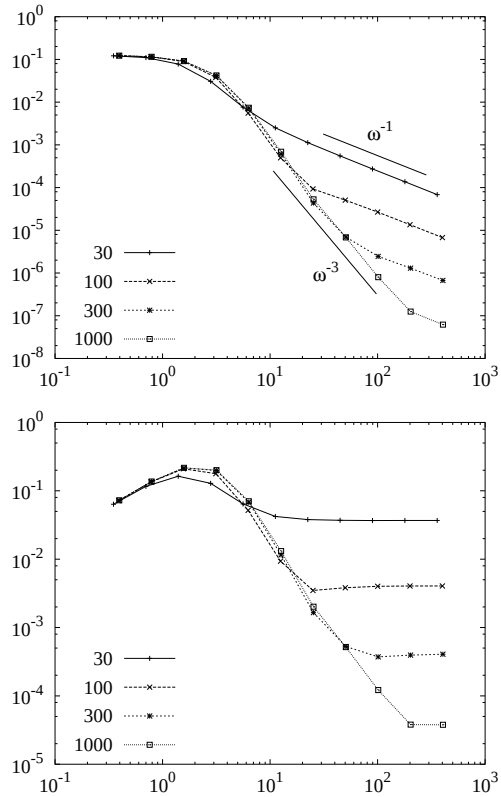

FIG. 2: top: $\tilde{\epsilon}$ as a function of $\omega/\bar{\omega}$ for R_λ varying from 30 to 1000; bottom: same for $\omega\tilde{\epsilon}$.

EDQNM results shown in Figure 1. We leave the question of whether a response maximum of $\tilde{k}$ itself is or is not consistent with closure unanswered for now. Conceivably, the answer is not universal, but may depend on the forcing scheme. The Reynolds number, or viscosity, does not seem to play an important role for $\tilde{k}$: for moderate and high Reynolds numbers, all the data collapses on a single curve.

2. *Modulated dissipation $\tilde{\epsilon}$.*

Figure 2 shows that $\tilde{\epsilon}$ also displays a plateau in the static limit as predicted by Eq. (23). Like the compensated quantity $\omega\tilde{k}$, the compensated data $\omega\tilde{\epsilon}$ shows a response maximum approximately near $\bar{\omega}$. Beyond this frequency, $\tilde{\epsilon}$ decreases sharply; at high Reynolds number, $\tilde{\epsilon} \sim \omega^{-3}$. But at even higher frequencies the ω^{-1} frequency dependence predicted in Eq. (29) is observed. It is interesting to note that the high frequency ω^{-1} range depends on the Reynolds number, and is indeed proportional to the viscosity, as suggested in Eq. (29).

What remains to be explained is the fast drop of $\tilde{\epsilon}$ at intermediate frequencies. Intu-

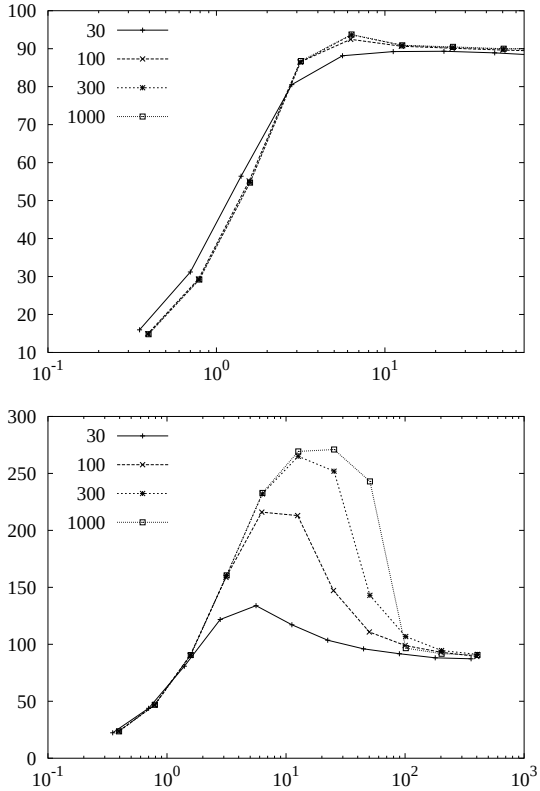

FIG. 3: top: Phase lags $-\phi_k$ as a function of $\omega/\bar{\omega}$ for R_λ varying from 30 to 1000; bottom: Phase lags $-\phi_\epsilon$ for different R_λ

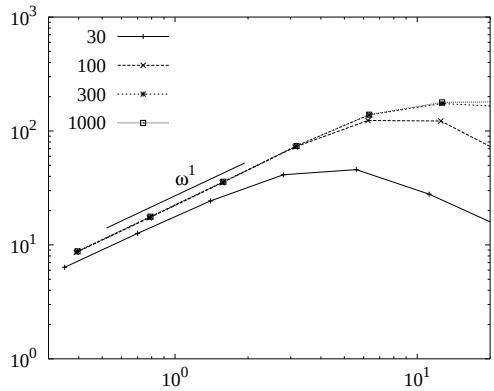

FIG. 4: $-(\phi_\epsilon - \phi_k)$ as a function of $\omega/\bar{\omega}$ for R_λ varying from 30 to 1000

itively it can be explained as follows: at low frequencies the energy cascade can follow the modulation. At high frequencies the cascade filters the modulated energy flux, since the the turbulent frequency is lower than the modulated frequency. The fast drop corresponds to the rate at which the energy cascade filters the modulated energy flux. Insights into this

process have important physical consequences as they clarify how small scales are influenced by large scale forcing.

3. Phase shifts.

Phase shift data is shown in Figures 3 and 4. The phase lags ϕ_k and ϕ_ϵ both go to zero for small ω. In this limit, everything is in phase as suggested by Eqs. (24) and (25). At high ω, ϕ_k and ϕ_ϵ go to 90 degrees, consistently with Eqs. (30) and (31). A slight overshoot in ϕ_k is observed around the turnover frequency. At intermediate values ϕ_ϵ shows a large overshoot with respect to 90 degrees and a very noticeable dependence on Reynolds number. This can be explained as follows: as long as the energy cascade can follow the modulation, *i.e.* at low frequencies, the modulated energy is transfered to the dissipation range through the energy cascade. The finite cascade time $\mathcal{T}_c$ introduces a phase shift between $\tilde{k}$ and $\tilde{\epsilon}$ proportional to $\omega\mathcal{T}_c$. This is illustrated in Figure 4. At low frequencies $\phi_\epsilon - \phi_k$ is a linear function of ω, which permits determining the cascade time.

IV. ANALYTICAL TREATMENT BY SPECTRAL CLOSURE

We supplement these numerical computations with analytical results. The complexity of the EDQNM transfer integral does not permit simple direct analysis, so we will consider much simpler models which embody certain features of nonlinear turbulence dynamics, but in a way that permits analytical conclusions to be drawn relatively easily.

A. General formulation

The general closure equation is found by introducing the closure hypothesis

$$T(\kappa, t) = \frac{\partial}{\partial \kappa}\mathcal{F}[E(\kappa, t)] \tag{33}$$

in Eq. (1). Eq. (33) expresses the energy transfer in terms of the energy flux $\mathcal{F}$, which is assumed to be a functional of the energy spectrum. In the problem of periodic forcing, the perturbation $\delta E(\kappa, t)$ defined by Eq. (5) satisfies

$$\delta\dot{E}(\kappa, t) = \tilde{P}(\kappa)\cos(\omega t) - \mathcal{L}[\delta E(\kappa, t)] - 2\nu\kappa^2\delta E(\kappa, t) \tag{34}$$

where $\mathcal{L}$ is the linear functional

$$\mathcal{L}[\Phi(\kappa,t)] = \frac{\partial}{\partial\kappa}\left(\frac{\delta\mathcal{F}}{\delta E}\right)_{\bar{E}}[\Phi(\kappa,t)] \tag{35}$$

and $(\delta\mathcal{F}/\delta E)_{\bar{E}}$ denotes linearization of $\mathcal{F}$ at the steady state $\bar{E}(\kappa)$.

Separating terms proportional to $\cos(\omega t)$ and $\sin(\omega t)$, Eq. (34) can be written as

$$-\omega\tilde{G}(\kappa) = \tilde{P}(\kappa) - \mathcal{L}[\tilde{F}(\kappa)] - 2\nu\kappa^2\tilde{F}(\kappa) \tag{36}$$

$$-\omega\tilde{F}(\kappa) = \mathcal{L}[\tilde{G}(\kappa)] + 2\nu\kappa^2\tilde{G}(\kappa). \tag{37}$$

In view of Eq. (35),

$$\int_0^\infty d\kappa\ \mathcal{L}[\Phi(\kappa)] = 0 \tag{38}$$

therefore integration of Eqs. (37) and (36) recovers the single-point relations Eqs. (18) and (19).

Before beginning the analysis, we note that substituting Eqs. (36) and (37) in Eq. (13) gives

$$\begin{aligned}\tilde{k}\sin(\phi_k) &= -\frac{1}{\omega}\tilde{P} + \frac{1}{\omega}\int_0^\infty 2\nu\kappa^2\tilde{F}(\kappa)d\kappa \\ \tilde{k}\cos(\phi_k) &= -\frac{1}{\omega}\int_0^\infty 2\nu\kappa^2\tilde{G}(\kappa)d\kappa.\end{aligned} \tag{39}$$

Ignoring the viscous terms recovers $\tilde{k}\sin\phi_k \approx -\omega^{-1}\tilde{P}$, which is equivalent to $\tilde{k} \approx \omega^{-1}\tilde{P}$ and $\phi_k \approx -\pi/2$, the approximations obtained by elementary arguments as Eqs. (28) and (30).

The corresponding substitutions in Eq. (14) yield

$$\tilde{\epsilon}\cos\phi_\epsilon = -\frac{1}{\omega}\left[\int_0^\infty d\kappa\ 2\nu\kappa^2\mathcal{L}[\tilde{G}(\kappa)] + \int_0^\infty d\kappa\ 4\nu^2\kappa^4\mathcal{L}[\tilde{G}(\kappa)]\right] \tag{40}$$

$$\tilde{\epsilon}\sin\phi_\epsilon = -\frac{1}{\omega}\left[\int_0^\infty d\kappa\ 2\nu\kappa^2\tilde{P}(\kappa) - \int_0^\infty d\kappa\ 2\nu\kappa^2\mathcal{L}[\tilde{F}(\kappa)] - \int_0^\infty d\kappa\ 4\nu^2\kappa^4\mathcal{L}[\tilde{F}(\kappa)]\right]. \tag{41}$$

Obviously, very strong assumptions are needed to reach any conclusion about $\tilde{\epsilon}$ and ϕ_ϵ, demonstrating that the behavior of the oscillating dissipation rate is somewhat subtle. Thus, the elementary conclusion that $\tilde{\epsilon}$ can be approximated by taking only the first term in Eq. (41) requires arguing that the terms in $\nu\mathcal{L}$, which represent oscillatory vortex stretching, can be ignored, and that, despite the presence of κ^4 in the corresponding integrals, the terms in ν^2, which represent oscillatory enstrophy destruction, can also be neglected. These assumptions are much less convincing than those underlying the elementary approximation

for $\tilde{k}$. In fact, more careful analysis will reveal nontrivial features of the dynamics of the modulated dissipation. But these features can only be computed using a model; this issue will be considered in the next section.

B. Simplified integral closure models

Kraichnan[21] showed that if the correlation equation in a closure of the DIA family is simplified by restricting attention to distant interactions only, an energy transfer model close in structure to the classical Heisenberg model is obtained. We refer to[21] for detailed explanations. Following this observation, Rubinstein and Clark[15] constructed a generalized Heisenberg model by adding asymptotically local interactions to the transfer model. The result is the energy flux closure

$$\mathcal{F}[E(\kappa)] = \mathsf{C}\left\{\int_0^\kappa d\mu\ \mu^2 E(\mu)\int_\kappa^\infty dp\ E(p)\theta(p) - \int_0^\kappa d\mu\ \mu^4\int_\kappa^\infty dp\ \frac{E(p)^2\theta(p)}{p^2}\right\}. \tag{42}$$

where the time argument is not explicitly written. Here and subsequently C will denote some constant, but not necessarily the same constant each time it appears. This energy transfer model was supplemented[15] by an evolution equation for the time-scale $\theta(\kappa)$, but the present work will use the simple algebraic closure,

$$\theta(\kappa) = \left[\kappa^3 E(\kappa)\right]^{-1/2}. \tag{43}$$

Theoretical features of this model include the possibility of energy transfer from small to large scales (energy 'backscatter') due to the possibility of negative energy flux, and consistency with the existence of inviscid equipartition ensembles in which $E(\kappa) \propto \kappa^2$. We note that Canuto and Dubovikov[22] had already obtained a simple spectral model consistent with many of the properties of analytically much more complex models by supplementing a renormalization group model with a backscatter term.

The classical Heisenberg model[13] is obtained by discarding the negative term in Eq. (42), so that

$$\mathcal{F}[E(\kappa)] = \mathsf{C}\int_0^\kappa d\mu\ \mu^2 E(\mu)\int_\kappa^\infty dp\ E(p)\theta(p). \tag{44}$$

In this model, the energy flux is necessarily positive; hence energy is always transferred from large scales to small scales. This property is inconsistent with the possibility of inviscid equipartition. Despite these drawbacks, the Heisenberg closure models one feature

of turbulent energy transfer that will be crucial to the present analysis: the possibility of 'distant' interactions between modes with disparate wavenumbers.

The linearized transfer for the generalized Heisenberg model is

$$\mathcal{L}[\Phi(\kappa)] = \mathsf{C}\left\{\kappa^2\Phi(\kappa)\int_\kappa^\infty dp\,\sqrt{\frac{\bar{E}(p)}{p^3}} - \sqrt{\frac{\bar{E}(\kappa)}{\kappa^3}}\int_0^\kappa d\mu\;\mu^2\Phi(\mu) + \right.$$
$$\frac{1}{2}\kappa^2\bar{E}(\kappa)\int_\kappa^\infty dp\,\frac{\Phi(p)}{\{\bar{E}(p)p^3\}^{1/2}} - \frac{1}{2}\frac{\Phi(\kappa)}{\{\bar{E}(\kappa)\kappa^3\}^{1/2}}\int_0^\kappa d\mu\;\mu^2\bar{E}(\mu) -$$
$$\left.\frac{3}{2}\kappa^4\int_\kappa^\infty dp\,\frac{\bar{E}(p)^{1/2}\Phi(p)}{p^{7/2}} + \frac{3}{2}\frac{\bar{E}(\kappa)^{1/2}\Phi(\kappa)}{\kappa^{7/2}}\int_0^\kappa d\mu\;\mu^4\right\}. \tag{45}$$

and the linearized transfer for the classical Heisenberg model is

$$\mathcal{L}[\Phi(\kappa)] = \mathsf{C}\left\{\kappa^2\Phi(\kappa)\int_\kappa^\infty dp\,\sqrt{\frac{\bar{E}(p)}{p^3}} - \sqrt{\frac{\bar{E}(\kappa)}{\kappa^3}}\int_0^\kappa d\mu\;\mu^2\Phi(\mu) + \right.$$
$$\left.\frac{1}{2}\kappa^2\bar{E}(\kappa)\int_\kappa^\infty dp\,\frac{\Phi(p)}{\{\bar{E}(p)p^3\}^{1/2}} - \frac{1}{2}\frac{\Phi(\kappa)}{\{\bar{E}(\kappa)\kappa^3\}^{1/2}}\int_0^\kappa d\mu\;\mu^2\bar{E}(\mu)\right\}. \tag{46}$$

Returning to the analysis of Eqs. (36) and (37), we note that they can be decoupled to give

$$\begin{aligned}\left(\omega^2\mathcal{I} + (\mathcal{L} + 2\nu\kappa^2\mathcal{I})^2\right)\tilde{F}(\kappa) &= \mathcal{L}[\tilde{P}(\kappa)]\\ \left(\omega^2\mathcal{I} + (\mathcal{L} + 2\nu\kappa^2\mathcal{I})^2\right)\tilde{G}(\kappa) &= -\omega\tilde{P}(\kappa)\end{aligned} \tag{47}$$

so that inversion of the linear operators on the left gives the solution for $\tilde{F}$ and $\tilde{G}$. But since exact inversion is only possible numerically, we will seek asymptotic solutions for large ω using standard methods. A lowest order approximate solution of Eqs. (36) and (37) is obtained by balancing the leading order terms in ω, so that

$$\tilde{F}(\kappa) \approx 0 \quad \tilde{G}(\kappa) \approx \frac{1}{\omega}\tilde{P}(\kappa). \tag{48}$$

Since this result ignores nonlinearity, it might be called 'rapid distortion theory' for this problem.

A formal solution of Eqs. (36) and (37) can be constructed in powers of ω^{-1} by perturbing about the leading order solution Eq. (48); taking only the correction terms of the next order gives

$$\begin{aligned}\tilde{F}(\kappa) &= \omega^{-2}\mathcal{L}[\tilde{P}(\kappa)]\\ \tilde{G}(\kappa) &= -\omega^{-1}\tilde{P}(\kappa) + \omega^{-3}\mathcal{L}^2[\tilde{P}(\kappa)].\end{aligned} \tag{49}$$

This approximation can also be obtained by operator inversion in Eq. (47) by a Neumann series. The resulting series is divergent but asymptotic in ω; therefore, as usual in such cases, the truncated series can provide useful information.

The corrections in Eq. (49) depend on $\mathcal{L}[\tilde{P}(\kappa)]$. We note from Eqs. (46) and (45) that all contributions to $\mathcal{L}[\tilde{P}(\kappa)]$ are proportional to $\tilde{P}(\kappa)$, and therefore vanish when $\tilde{P}(\kappa)$ vanishes, or are proportional to $\int_\kappa^\infty \tilde{P}(p)dp$ and vanish for large κ since $\tilde{P}(\kappa)$ is nonzero only for small κ, with one exception, the term common to both models,

$$\mathcal{L}_{NL}[\tilde{P}(\kappa)] = \sqrt{\frac{\bar{E}(\kappa)}{\kappa^3}} \int_0^\kappa d\mu\ \mu^2 \tilde{P}(\mu), \tag{50}$$

where the subscript NL denotes that these are contributions from nonlocal interactions. Because this term pertains to forward transfer alone, we obtain it in the Heisenberg model; the backscatter term in the generalized Heisenberg model therefore plays no role in this particular analysis.

The contribution to linearized transfer in Eq. (50) shows that the oscillatory disturbance is not confined to the region where $\tilde{P}(\kappa)$ is nonzero, even at asymptotically large ω, contrary to the conclusion suggested by elementary considerations. Instead, the oscillatory disturbance can propagate into all scales of motion. The remaining terms in Eqs. (45) and (46) with $\Phi(\kappa)$ replaced by $\tilde{P}(\kappa)$ can be considered corrections to the leading order solution Eq. (48), and will be ignored in the following analysis; the term in Eq. (50) itself provides the leading order solution in the regions where $\tilde{P}(\kappa)$ vanishes.

For large κ,

$$\begin{aligned}
\mathcal{L}_{NL}[\tilde{P}(\kappa)] &= \sqrt{\frac{\bar{E}(\kappa)}{\kappa^3}} \int_0^\kappa d\mu\ \mu^2 \tilde{P}(\mu) \sim \kappa_P^2 \tilde{P} \bar{\epsilon}^{1/3} \kappa^{-7/3} \\
\mathcal{L}_{NL}^2[\tilde{P}(\kappa)] &\sim \sqrt{\frac{\bar{E}(\kappa)}{\kappa^3}} \int_0^\kappa d\mu\ \kappa_P^2 \tilde{P} \bar{\epsilon}^{1/3} \mu^{-7/3} \sim \kappa_P^2 \tilde{P} \bar{\epsilon}^{2/3} \kappa^{-5/3}.
\end{aligned} \tag{51}$$

Adding these nonlocal contributions to the leading order solution Eq. (48), the approximation Eq. (49) takes the form

$$\begin{aligned}
\tilde{F}(\kappa) &\sim \omega^{-2} \kappa_P^2 \tilde{P} \bar{\epsilon}^{1/3} \kappa^{-7/3} \\
\tilde{G}(\kappa) &\sim \omega^{-1} \tilde{P}(\kappa) + \omega^{-3} \kappa_P^2 \tilde{P} \bar{\epsilon}^{2/3} \kappa^{-5/3}.
\end{aligned} \tag{52}$$

Eq. (51) shows that $\mathcal{L}_{NL}^p[\tilde{P}(\kappa)] \sim \kappa^{-10/3+p}$; it follows that the series expansions for $\tilde{F}(\kappa)$ and $\tilde{G}(\kappa)$ proceed in positive powers of κ; the higher order approximations will eventually contain positive powers of κ, indicating the divergence of the series noted earlier.

Assuming that scaling ranges for $\tilde{F}$ and $\tilde{G}$ both begin at a scale of the order of κ_P, we will have

$$\tilde{k}\cos(\phi_k) = \int_0^\infty d\kappa\ \tilde{F}(\kappa) \sim \omega^{-2}\bar{\epsilon}^{1/3}\kappa_P^{2/3}\tilde{P}$$
$$\tilde{k}\sin(\phi_k) = \int_0^\infty d\kappa\ \tilde{G}(\kappa) \sim -\omega^{-1}\tilde{P} + \omega^{-3}\bar{\epsilon}^{2/3}\kappa_P^{4/3}\tilde{P}. \tag{53}$$

If ω is large, the terms of order larger than ω^{-1} can be ignored, and we again return to the elementary estimate $\tilde{k} \approx \tilde{P}/\omega$ and $\phi_k \approx \pi/2$ with corrections depending on $\bar{\epsilon}^{1/3}\kappa_P^{2/3}/\omega$.

The situation is quite different for the modulated dissipation rate, for which

$$\tilde{\epsilon}\cos(\phi_k) = \int_0^\infty d\kappa\ 2\nu\kappa^2\tilde{F}(\kappa) \sim \omega^{-2}\kappa_P^2\tilde{P}\bar{\epsilon}^{1/3}\nu\kappa_d^{2/3} = \omega^{-2}\kappa_P^2\tilde{P}\bar{\epsilon}^{1/2}\nu^{1/2}$$
$$\tilde{\epsilon}\sin(\phi_k) = \int_0^\infty d\kappa\ 2\nu\kappa^2\tilde{G}(\kappa) \sim \omega^{-1}2\nu\kappa_P^2\tilde{P} + \omega^{-3}\kappa_P^2\tilde{P}\bar{\epsilon} \tag{54}$$

where $\kappa_d = (\epsilon/\nu^3)^{1/4}$ is the Kolmogorov scale. Evidently, there is a competition between the limits $\omega \to \infty$ and $\nu \sim Re^{-1} \to 0$. The limit $\omega \to \infty$ at fixed Re will indeed recover the elementary result $\tilde{\epsilon} \sim \omega^{-1}$, but at fixed large ω, the limit $Re \to \infty$ gives instead $\tilde{\epsilon} \sim \omega^{-3}\kappa_P^2\tilde{P}\bar{\epsilon}$. The phase has the general approximate value $\tan\phi_\epsilon \approx \omega\nu^{1/2}\bar{\epsilon}^{-1/2} + \omega^{-1}\nu^{-1/2}\bar{\epsilon}^{1/2}$ indicating a complex joint dependence on ω and Re in general.

The main consequences of this analysis are the ω^{-3} range for $\tilde{\epsilon}$ and the complex dependence of ϕ_ϵ on Reynolds numbers; both are confirmed by the EDQNM calculations.

C. Scaling analysis for $\tilde{\epsilon}$

The analysis in the previous section shows how nonlocal interactions in the Heisenberg and generalized Heisenberg models can carry the oscillatory disturbance into the inertial range. These observations suggest a simple scaling analysis for the modulated energy flux. Assume, following the discussion in Sect. II that scales of motion for which the oscillations are overdamped, that is, scales satisfying $\bar{\theta}(\kappa)^{-1} > \omega$ do not transfer any modulated flux, but that modulated flux is transferred by scales of motion such that $\bar{\theta}(\kappa)^{-1} < \omega$. The crossover occurs at the scale κ_ω defined by $\bar{\theta}(\kappa)^{-1} = \omega$, or $\kappa_\omega = \sqrt{\omega^3/\bar{\epsilon}}$. In both the Heisenberg and generalized Heisenberg models, the transfer of modulated flux is then given approximately by

$$\tilde{\epsilon} \sim \int_0^{\kappa_P} d\mu\ \mu^2\tilde{E}(\mu)\int_{\kappa_\omega}^\infty dp\ \bar{E}(p)\bar{\theta}(p) \sim \kappa_P^2\tilde{k}\bar{\epsilon}^{1/3}\kappa_\omega^{-4/3} \sim \kappa_P^2\tilde{P}\omega^{-1}\bar{\epsilon}^{1/3}\omega^{-2}\bar{\epsilon}^{2/3} \sim \kappa_P^2\tilde{P}\omega^{-3}\bar{\epsilon}. \tag{55}$$

This result is consistent with the existence of a contribution to $\tilde{G}$ scaling as $\kappa^{-5/3}$ obtained more formally in Eq. (52).

The argument can be extended to the EDQNM closure as follows. Modulated kinetic energy is injected in the flow around the wavenumber κ_P. This energy will leave the large scales to enter the energy cascade at a rate $\epsilon^{in}(\kappa_P)$. Using classical reasoning, this rate can be estimated by:

$$\epsilon^{in}(\kappa_P) \sim \frac{\tilde{k}}{\overline{\theta}(\kappa_P)} \tag{56}$$

at high frequencies the modulated energy is:

$$\tilde{k} \sim \tilde{P}\omega^{-1} \tag{57}$$

and the timescale can be estimated by

$$\overline{\theta}(\kappa_P) \sim \overline{\epsilon}^{-1/3}\kappa_P^{-2/3} \tag{58}$$

so that

$$\epsilon^{in}(\kappa_P) \sim \tilde{P}\omega^{-1}\overline{\epsilon}^{1/3}\kappa_P^{2/3}. \tag{59}$$

The point is now that this energy will be overdamped if it passes through the scales $\kappa_P < \kappa < \kappa_\omega$. The only way to reach the zone that can transfer the modulated flux, $\kappa > \kappa_\omega$ is by nonlocal energy transfer. This transfer will involve, for $\kappa_P \ll \kappa_\omega$, triads with two legs of a length κ_ω and one leg equal to κ_P. The disparity parameter s defined as

$$s = \frac{\max(\kappa, p, q)}{\min(\kappa, p, q)} \tag{60}$$

with κ, p, q the norms of the wavevectors forming a triad, is for these triads

$$s \approx \frac{\kappa_\omega}{\kappa_P} \sim \frac{\omega^{3/2}}{\kappa_P\overline{\epsilon}^{1/2}}. \tag{61}$$

It was predicted by Kraichnan[23] (compare also the DNS study by Zhou[24]), that the nonlocal part of the energy transfer involving triads with a disparity around s, $\epsilon^f(\kappa, s)$ with respect to the total energy flux $\epsilon^f(\kappa)$ scales as:

$$\frac{\epsilon^f(\kappa, s)}{\epsilon^f(\kappa)} \sim s^{-4/3}. \tag{62}$$

In our case we identify the total flux of modulated energy, $\epsilon^f(\kappa, s)$ with $\epsilon^{in}(\kappa_P)$. The nonlocal flux $\epsilon^f(\kappa, s)$ corresponds to the modulated energy flux that manages to reach the

range $\kappa > \kappa_\omega$ and that will eventually be dissipated, and so is equal to $\tilde{\epsilon}$. One finds therefore combining Eqs. (56) and (62) that the modulated dissipation for high frequencies equals:

$$\tilde{\epsilon} \sim \frac{\tilde{k}}{\theta(\kappa_P)} s^{-4/3} \sim \kappa_P^2 \tilde{P} \overline{\epsilon} \omega^{-3} \tag{63}$$

in agreement with Eq. (55). The inviscid nature of this correction to the modulated dissipation is in agreement with the observation in Figure 2.

An important distinction between the classical and generalized Heisenberg models and EDQNM is that the power-law scaling of Eq. (62) applies for all s in the simple models, but is given by a more complex expression for EDQNM. This implies a difference in the detailed predictions when κ_ω/κ_P is of order one.

V. FINITE DIMENSIONAL MODELS

The problem of periodically forced turbulence has been investigated through properties of the single point moments $k(t)$ and $\epsilon(t)$; complete results for these quantities have been found from various spectral closure theories. Single-point modeling attempts to circumvent spectral modeling by constructing closed equations for the single point moments themselves. It is an important theoretical question whether such equations exist[25], and indeed, much stronger assumptions are needed to close the problem at this level. In this section, we will assess how much of the dynamics is accessible to single-point modeling.

In order to permit the underlying steady state, a two-equation model for periodically forced turbulence must take the form

$$\dot{k} = P - \epsilon \tag{64}$$

$$\dot{\epsilon} = C\frac{\epsilon}{k}(P - \epsilon) \tag{65}$$

where Eq. (64) is just the energy equation previously stated as Eq. (15). For forcing at a fixed length scale, it can be shown[26] that $C = 3/2$; the ϵ transport equation Eq. (65) then states that $L = k^{3/2}/\epsilon$ is constant, since Eqs. (64) and (65) imply $\dot{L}/L = (3/2)\dot{k}/k - \dot{\epsilon}/\epsilon = 0$, the same argument that gave Eq. (22).

Eqs. (64) and (65) admit a steady solution in which $P(t) = \bar{P} = \epsilon(t) = \bar{\epsilon}$. We consider the perturbation about this steady state due to oscillating forcing Eq. (9); linearization

about the steady state and using the value of the model constant $C = 3/2$ gives

$$-\omega\tilde{k}\sin(\omega t + \phi_k) = \tilde{P}\cos(\omega t) - \tilde{\epsilon}\cos(\omega t + \phi_\epsilon) \tag{66}$$

$$-\omega\tilde{\epsilon}\sin(\omega t + \phi_\epsilon) = \bar{\omega}\left[\tilde{P}\cos(\omega t) - \tilde{\epsilon}\cos(\omega t + \phi_\epsilon)\right] \tag{67}$$

where Eq. (66) restates Eq. (17). Divide Eqs. (66) and (67) to obtain

$$\frac{\sin(\omega t + \phi_\epsilon)}{\sin(\omega t + \phi_k)}\frac{\tilde{\epsilon}}{\tilde{k}} = \bar{\omega} \tag{68}$$

so that

$$\phi_k = \phi_\epsilon = \phi \tag{69}$$

and

$$\tilde{\epsilon} = \bar{\omega}\tilde{k} \tag{70}$$

The linearized equations reduce to

$$\begin{aligned} -\omega\tilde{k}\sin\phi &= \tilde{P} - \bar{\omega}\tilde{k}\cos\phi \\ -\omega\cos\phi &= \bar{\omega}\sin\phi. \end{aligned} \tag{71}$$

Note that this is just the general result of Eqs. (18)–(19) with the special closure hypothesis $\phi_k = \phi_\epsilon$. Indeed, this is perhaps even a rather plausible closure, since $\phi_k = \phi_\epsilon$ is certainly true in both limits $\omega \downarrow 0$ and $\omega \uparrow \infty$. It follows that

$$\tan\phi = -\frac{\omega}{\bar{\omega}}; \quad \tilde{k} = \frac{\tilde{P}}{\bar{\omega}}\cos\phi = \frac{\tilde{P}}{\bar{\omega}}\frac{1}{\sqrt{1 + (\omega/\bar{\omega})^2}}. \tag{72}$$

The limits

$$\begin{aligned} &\phi \sim -\pi/2; \;\; \tilde{k} \sim \tilde{P}/\omega \;\; \text{for } \omega \to \infty \\ &\phi \sim 0; \qquad \tilde{k} \sim \tilde{P}/\bar{\omega} \;\; \text{for } \omega \to 0 \end{aligned} \tag{73}$$

are consistent with the limiting results previously obtained as Eqs. (28) and (22). Whereas it is certainly expected that a two-equation model should be adequate in the static limit, it may be surprising that the frozen turbulence limit for $\tilde{k}$ is also predicted correctly, particularly in view of the suggestion[5] that in oscillating channel flow, predicting the frozen turbulence limit requires rapid distortion theory.

Despite these successes, the two-equation model has some important limitations. First of all, the phase shifts ϕ_k and ϕ_ϵ are equal for all ω, in disagreement with Figure 3. Second, Eq. (70) states that $\tilde{\epsilon}$ and $\tilde{k}$ are proportional for all ω. Recall that this proportionality was

found in Eq. (26) as a consequence of assuming a constant length scale. Comparison of Figures 1 and 2 shows that $\tilde{k}$ and $\tilde{\epsilon}$ are certainly not always proportional. This comparison demonstrates that the identification of the constant forcing scale κ_P^{-1} with a multiple of the ratio $k^{3/2}/\epsilon$ cannot be made for general values of ω; following the common terminology that 'equilibrium' turbulence is turbulence in which all dimensional arguments are valid, we can say that periodically forced turbulence is not in 'equilibrium.'

What is most striking is that the two-equation model cannot predict the ν-dependence of $\tilde{\epsilon}$ and ϕ_ϵ, which is not a low Reynolds number effect in this case. A fundamental observation of Speziale and Bernard[27] is that Reynolds number dependence in the dissipation rate dynamics is a manifestation of *unbalanced vortex stretching*, the absence of which underlies the classic formulation of the ϵ equation by Tennekes and Lumley[28]. Even if we were satisfied with a high Reynolds number model, it should predict $\tilde{\epsilon} \approx 0$ for large ω. We have noted that this limit is due to the filtering effect of the spectral cascade. Evidently, this effect cannot be captured at the level of a two-equation model.

Another way to understand the relation between spectral closure and the two-equation model is to note that Eq. (71) is obtained from the general closure model Eqs. (36) and (37) by making the single relaxation time approximation

$$\mathcal{L} + 2\nu\kappa^2\mathcal{I} \approx \bar{\omega}\mathcal{I}. \tag{74}$$

before integrating over κ. This type of simplification, by which the continuum of time scales in a turbulent flow is replaced by a single dominant time scale, is a mainstay of modeling, and is often very useful; however, in the problem of periodically forced turbulence, it suppresses the nontrivial features of the finite ω dynamics.

One remedy is, as always, to argue that the model constants should be functions. If we set $C = C(\omega/\bar{\omega})$, then if $C \downarrow 0$ for $\tilde{\omega} \uparrow \infty$, the correct behavior can be reproduced. However, this *ad hoc* model would have no validity apart from this very special problem and would merely amount to curve-fitting.

We would like to comment briefly on the modeling of this flow with a more complex finite dimensional model with two characteristic time scales; that is, a 'multiple-scale' model[26]. For example, consider a three-equation model in which energy flux f is distinguished from dissipation ϵ. A general form for such a model that is consistent with a steady state is

$$\dot{k} = P - \epsilon$$
$$\dot{f} = C_1 \frac{f}{k}(P - f)$$
$$\dot{\epsilon} = C_2 \frac{\epsilon}{k}(f - \epsilon). \tag{75}$$

The limits $C_1 \uparrow \infty$ and $C_2 \uparrow \infty$ both recover the two-equation model. The primary motivation for this model is that there are now two time scales k/ϵ and k/f instead of only one.

Linearizing as usual about the steady state

$$k(t) = \bar{k} + \tilde{k}\cos(\omega t + \phi_k)$$
$$f(t) = \bar{f} + \tilde{f}\cos(\omega t + \phi_f)$$
$$\epsilon(t) = \bar{\epsilon} + \tilde{\epsilon}\cos(\omega t + \phi_\epsilon). \tag{76}$$

Then

$$-\omega\tilde{k}\sin(\omega t + \phi_k) = \tilde{P}\cos(\omega t) - \tilde{\epsilon}\cos(\omega t + \phi_\epsilon) \tag{77}$$
$$-\omega\tilde{f}\sin(\omega t + \phi_f) = C_1\frac{\bar{f}}{\bar{k}}\left[\tilde{P}\cos(\omega t) - \tilde{f}\cos(\omega t + \phi_f)\right] \tag{78}$$
$$-\omega\tilde{\epsilon}\sin(\omega t + \phi_\epsilon) = C_2\frac{\bar{\epsilon}}{\bar{k}}\left[\tilde{f}\cos(\omega t + \phi_f) - \tilde{\epsilon}\cos(\omega t + \phi_\epsilon)\right]. \tag{79}$$

The intervention of the new quantity f in the dynamics means that k and ϵ are no longer constrained to be in phase. However, prediction of the high Reynolds number result $\tilde{\epsilon} \approx 0$ remains impossible: Eq. (79) then requires $\tilde{f} = 0$, which is inconsistent with Eq. (78). It is not difficult to evaluate both phase lags ϕ_f and ϕ_ϵ, but even without explicit results, it is evident that Reynolds number dependence of ϕ_ϵ remains inaccessible to this model. Although the three-equation model allows more complex phase relations and modeling of time delays in the spectral cascade, it, like the two-equation model, cannot take the Reynolds number dependence into account correctly. The addition of time scales to the dissipation rate dynamics does not solve all of the problems of two-equation modeling.

VI. CONCLUSIONS

The influence of periodic large scale forcing on isotropic turbulence was investigated by spectral closure theory. The asymptotic frequency dependence of the modulated energy and

modulated dissipation as observed in recent simulations[9] were recovered. It was pointed out that the asymptotic behavior of the modulated dissipation, which is proportional to ω^{-1}, corresponds to the viscous damping of the forced wavenumbers, which is local in wavenumber space. For high and moderate Reynolds numbers an intermediate ω^{-3} frequency dependence of the modulated dissipation was observed in the EDQNM calculations. This range characterizes the filtering properties of the energy cascade. Closures allowing for nonlocal interactions (EDQNM, classical or generalized Heisenberg) can reproduce this behavior as it corresponds to nonlocal energy transfer between the forced scales and a range of wavenumbers characterized by a crossover wavenumber $\kappa > \kappa_\omega \sim \sqrt{\omega^3/\epsilon}$. Finally it was argued that finite dimensional models can not correctly describe the problem of modulated turbulence.

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